WELLS PRODUCING LAYERED RESERVOIRS:
UNEQUAL FRACTURE LENGTH

by
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WELLS PRODUCING LAYERED RESERVOIRS:
UNEQUAL FRACTURE LENGTH

A THESIS
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ABSTRACT

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Response of Wells Producing Layered Reservoirs: Unequal Fracture Length (100 pp. - Chapter V)
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(158 words)

The response of fractured wells producing non-communicating layered reservoirs is the focus of this work. The conductivity of the fracture is assumed to be finite. The fracture length is assumed to vary from layer to layer. Two modes of production—constant wellbore pressure and constant rate—are considered.

In the first part of this work the fractures are assumed to communicate only at the wellbore. The results given in this section are intended to provide engineers with analytical capabilities to examine responses in wells where the layers that have been stimulated are separated by considerable distances. Procedures to interpret the results of pressure buildup and/or production tests (drawdown responses) in terms of layer properties are presented. Criteria to ensure maximum productivity are specified. The second part of this work examines well responses when the fractures are in communication at points other than the wellbore. All other things being identical, we show that communication between fractures increases productivity.
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CHAPTER I

INTRODUCTION

This work examines the response of fractured wells in commingled reservoirs, when the fracture lengths in each zone of the commingled reservoir are not equal. Recent work by Bennett\textsuperscript{1}, considered situations where the fracture lengths were equal. The Bennett study introduced the concept of dimensionless reservoir conductivity and demonstrated a simple way to correlate results with single-layer systems. Using the Bennett study as a foundation, this work extends his results to cases where fracture lengths are not equal.

Correspondents have also suggested that in many cases zones in layered reservoirs are stimulated individually, and the layers are then commingled. The results of this study should have application to these circumstances.

In some cases, single-layer, single-fracture, homogeneous reservoir models are used to analyze the response of fractured wells in multi-layer reservoirs\textsuperscript{2}. Under these circumstances, there is a need to be able to convert "estimated parameters" to the conditions that might exist in the field. This aspect is important if we wish to improve stimulation procedures. This work presents a rigorous, direct, and easy technique to fulfill this objective. In particular, we present a procedure for extending the empirical observations of Ref. 2 to a wide variety of conditions. This procedure is one of the accomplishments of this study. We also delineate situations when commingled
reservoir systems should not be analyzed by single-layer models.

As in the Bennett study, we have developed techniques to correlate the commingled reservoir solutions with existing single-layer solutions\(^3,4\) by redefining terms involving the fracture half-length and fracture conductivity in the definitions of dimensionless time and dimensionless fracture conductivity. For example, we show that multi-layer solutions may be correlated with single-layer solutions if the fracture half-length term in the single-layer solutions is replaced by the sum of the product of the dimensionless layer conductivity and fracture half-length of each layer. A similar modification to the dimensionless fracture conductivity term is presented. To our knowledge, these observations have not been presented until now. We also discuss conditions that must be specified to ensure maximum productivity. The final part of this study explores the influence of communication between the fractures. It is shown that the productivity of wells improved if the fractures are in communication.

New approximate analytical solutions derived to support the numerical results are also presented. The analytical solutions are approximate in the sense that additional assumptions are needed to solve the problem by Laplace transforms. As will be shown in the text, these analytical solutions provided information on the structure of the rigorous solutions and, thus were useful in developing the correlations presented here. The analytical solutions were also useful in verifying the accuracy of the numerical model (for identical conditions, see Ref. 5). However, it should be noted that all results presented here were obtained with a numerical model.
CHAPTER II
FORMULATION AND DEFINITIONS

In this section, we introduce the basic assumptions of this study and define the variables used in this work. All results are presented in dimensionless form.

2.1 Formulation

A schematic diagram of the system modeled in this work is shown in Fig. 2.1. Since we assume symmetry with respect to the wellbore, only one-fourth of the drainage region is shown here. For simplicity we assume that the reservoir consists of two layers. Each layer is a uniform and homogeneous porous medium of constant thickness. The characteristic properties of each layer (porosity, permeability and compressibility) are assumed to be different. The well is assumed to be at the center of a square drainage region. All reservoir boundaries are assumed to be closed. The reservoir is filled with a slightly compressible fluid of constant viscosity. The initial pressure is assumed to be independent of depth. A cross sectional view of the system considered here is shown in Fig. 2.2.

A hydraulic fracture of finite width and conductivity intercepts the wellbore and extends over the entire extent of the reservoir. The well is located at the center of the fracture; that is, each wing of the fracture (in a layer) is of equal length. As shown in Fig. 2.1, the fracture is assumed to be parallel to two of the sides of the square drainage region. The porosity, compressibility and permeability of the
Fig. 2.1 - Schematic Diagram
Fig. 2.2 - Cross Sectional View of the System
fracture in each layer are assumed to be constant; that is, the fracture in each layer is assumed to be a homogeneous porous medium. The fracture lengths and fracture properties in each layer are assumed to be different. We consider two modes of production--constant wellbore pressure and constant rate. The sandface pressure is assumed to be independent of depth. In both cases fluid can be produced only via the fracture. Communication between the layers occurs only via the sandface; that is, there is no communication between the fractures. Thus, fluid may not redistribute itself prior to being produced at the sandface.

2.2 Definitions

We consider the well response for both constant rate and constant pressure production. We present results in dimensionless form so that the results are general. The dimensionless wellbore pressure drop, \( p_{\text{WD}} \), (constant rate production), and the dimensionless rate, \( q_{\text{D}} \), (constant pressure production), are given, respectively, by the following equations:

\[
p_{\text{WD}}(t_{\Delta x_f}) = \frac{\bar{k} h [p_i - p_{\text{wf}}(t)]}{141.2 \; q_{\text{B}} u},
\]

and

\[
\frac{1}{q_{\text{D}}(t_{\Delta x_f})} = \frac{\bar{k} h (p_i - p_{\text{wf}})}{141.2 \; B_u q(t)}.
\]

In the above equations, \( p \) represents pressure, \( q \) denotes surface rate, \( h \) represents reservoir thickness, \( B \) is formation volume factor,
and \( \mu \) denotes the viscosity of the fluid. The subscripts, \( f \) and \( i \), denote the fracture and initial conditions, respectively, and the subscript \( \text{wf} \) represents well-flowing conditions. Equations 2.1 and 2.2 are expressed in conventional oil field units (psi, STB/D, md, feet, RB/STB, cp). If the well produces at a constant rate, \( p_{\text{wf}} \) is a function of time, \( t \), and if the well produces at a constant pressure, then \( q \) is a function of time.

The symbol \( \bar{k} \) represents the thickness averaged value of layer permeabilities and is given by:

\[
\bar{k} = \frac{1}{n} \left( \sum_{j=1}^{n} k_j h_j \right) / h.
\]  

(2.3)

Here \( k_j \) and \( h_j \) refer to the permeability and thickness of layer \( j \), and \( n \) is the total number of layers. For most of the results in this study, \( n = 2 \). The justification for defining \( p_{\text{WD}} \) and \( q_D \) on \( \bar{k} \) is given in Refs. 1, and 6-8.

Bennett\(^1\) introduced the concept of dimensionless layer conductivity, \( R_{\text{CD}j} \), and showed that the well performance can be characterized in terms of the dimensionless layer conductivity, \( R_{\text{CD}j} \), where \( R_{\text{CD}j} \) is defined by:

\[
R_{\text{CD}j} = \frac{k_j h_j}{k \ h \ \sqrt{\eta_j}}.
\]  

(2.4)

Here \( \eta_j \) is the diffusivity of layer \( j \), that is, \( \eta_j = k_j / (\phi_j c_j \mu) \) and \( \bar{\eta} = \bar{k} / (\bar{\phi} \bar{c} \mu) \).

The dimensionless reservoir conductivity, \( R_{\text{CD}} \), is the sum of the
dimensionless layer conductivities; that is,

$$R_{cD} = \sum_{j=1}^{n} R_{cDj} \quad . \quad (2.5)$$

Dimensionless time used in this work is based on the following relation:

$$t_{Dx_{f1}} = \frac{2.637 \times 10^{-4}}{\bar{\phi c}_t u L^2} \frac{k t}{x_{f1}} \quad . \quad (2.6)$$

Here, $L_{x_{f1}}$ is fracture half-length in layer 1, and $\bar{\phi c}_t$ is the thickness averaged value of the porosity-compressibility products of the individual layers. Thus, if $\phi_j$ is the porosity of layer $j$ and $c_{tj}$ is the compressibility of layer $j$, then

$$\bar{\phi c}_t = \left( \sum_{j=1}^{n} \phi_j c_{tj} h_j \right) / h \quad . \quad (2.7)$$

Justification for this definition of $\bar{\phi c}_t$ is also given in Refs. 1, and 6-8. The above definition of dimensionless time implies that the fracture length ratios $L_{x_{fj}} / L_{x_{f1}}$, $j=2,...,n$ are parameters of interest. (One objective of this work is to define a dimensionless group that will incorporate the influence of this parameter.) The dimensionless fracture conductivity for layer $j$, $\sigma_{fDj}$, is given by

$$\sigma_{fDj} = \frac{k_{fj} b_j}{k L_{x_{f1}}} \quad . \quad (2.8)$$
Here $k_{fj}$ represents the fracture permeability in layer $j$ and $b_j$ is the fracture width of layer $j$. The dimensionless layer thickness of layer $j$, $h_{Dj}$, is defined by

$$h_{Dj} = \frac{h_j}{h} = \frac{h_{fj}}{h}, \quad (2.9)$$

where $h_j$ is the thickness of layer $j$ and $h_{fj}$ is the fracture height in layer $j$. (Note $h_{fj} = h_j$.)

As shown in Ref. 1, the early-time response of a fractured well is influenced by the storage capacity of the fracture. The dimensionless storage capacity of layer $j$, $C_{fDj}$, is defined by the following equation:

$$C_{fDj} = \frac{\phi_{fj} b_j c_{ftj}}{\phi_c L X_{f1}}. \quad (2.10)$$

As in most studies\textsuperscript{1,3-6}, all results in this study assume that the influence of $C_{fDj}$ is negligible.
CHAPTER III
THEORETICAL CONSIDERATIONS

In this Chapter, we state the appropriate partial differential equations and associated auxiliary conditions. We then consider modifications to the problem in order to obtain approximate solutions. The approximate solutions are useful for the following reasons: (i) they provide information on the structure of the solutions, (ii) they suggest methods to correlate results by numerical means, and (iii) they provide a means to verify the structure of the numerical solutions for the time range in which the analytical solutions are valid (see Refs. 1 and 8). We emphasize that the analytical solutions are approximations and are used only for the purposes mentioned above. All results in Chapter IV were obtained by a finite difference model; that is, they incorporate none of the additional assumptions required to obtain an analytical solution.

3.1 The Governing Differential Equations

The partial differential equation for flow in the reservoir is given by:

\[
\frac{\partial^2 p_j}{\partial x^2} + \frac{\partial^2 p_j}{\partial y^2} = \frac{1}{\eta_j} \frac{\partial p_j}{\partial t}; \quad j=1,2,\ldots,n, \quad (3.1.1)
\]
on \( D_j = \{ (x,y,z) \mid 0 < x < L_{x_fj} \} \cup \{ (x,y,z) \mid b_j/2 < y < L_{y} \} \cup \{ (x,y,z) \mid 0 < y < L_{y} \} \cup \{ (x,y,z) \mid \Sigma_{\xi=0}^{j-1} \Sigma_{\xi=0}^{j} h_{e} \leq x < \Sigma_{\xi=0}^{j} h_{e} \} \cup \{ (x,y,z) \mid \Sigma_{\xi=0}^{j-1} h_{e} \leq x < \Sigma_{\xi=0}^{j} h_{e} \} \}

and \( t > 0 \). Here we have assumed that \( h_0 = 0 \). The diffusivity of each layer is denoted by \( \eta_j \); that is, \( \eta_j = \frac{k_j}{(\phi_j \cdot c_j \cdot u_j)} \). Note that we consider only two dimensional flow in the reservoir; that is, it is implicit that vertical gradients within each layer are zero (commingled reservoir behavior). As mentioned earlier we assume that the fracture height equals the formation height \( \Sigma_{j=1}^{n} h_{fj} = \Sigma_{j=1}^{n} h_{j} = h \).

The conditions to be satisfied on the outer boundary of the reservoir in each layer \( j, j=1,2,\ldots,n \) are

\[
\frac{\partial p_i}{\partial x} \bigg|_{x = L_{x_e}} = 0 ; \quad (3.1.2)
\]

for \( 0 \leq y \leq L_{y_e} \), and

\[
\frac{\partial p_i}{\partial x} \bigg|_{x = 0} = 0 , \quad (3.1.3)
\]

for \( b_j/2 \leq y \leq L_{y_e} \),

\[
\frac{\partial p_i}{\partial y} \bigg|_{y = L_{y_e}} = 0 ; \quad (3.1.4)
\]

for \( 0 \leq x < L_{x_e} \), and
\frac{\partial p_{fj}}{\partial y} \bigg|_{y = 0} = 0 \quad , \quad (3.1.5)

for \( L_{xfj} \leq x < L_{xe} \).

The pressure distribution in the vertical fracture is governed by the following equations:

\[
\frac{\partial^2 p_{fj}}{\partial x^2} + \frac{\partial^2 p_{fj}}{\partial y^2} = \frac{1}{\eta_{fj}} \frac{\partial p_{fj}}{\partial t} \quad ; \quad j=1,2,\ldots,n \quad (3.1.6)
\]

on \( F_j = \left\{ (x,y,z) \mid 0 < x < L_{xfj} \quad ; \quad 0 < y \leq b_j/2 \quad ; \quad \Sigma h_{l} < z < \Sigma h_{j} \right\} \)

and \( t = 0 \). Again, we assume that \( h_{o} = 0 \). It is to be noted that vertical gradients are neglected since the fracture does not communicate, except at the wellbore. The boundary conditions to be satisfied by Eq. (3.1.6) are considered next. At \( y = 0 \), we have

\[
\frac{\partial p_{fj}}{\partial y} \bigg|_{y = 0} = 0 \quad ; \quad (3.1.7)
\]

on \( 0 \leq x \leq L_{xfj} \), for each value of the index \( j \). This condition specifies that no fluid crosses the vertical plane that bisects the vertical fracture. (Note only one-fourth of the system is modelled.)

The conditions at the fracture-reservoir interface are given by:

\[
p_{j} (x,y = b_j/2, t) = p_{fj} (x,y = b_j/2, t) \quad , \quad (3.1.8)
\]
for \(0 \leq x \leq L_x f_j\), and

\[
p_j (x = L_x f_j, y, t) = p_{f_j} (x = L_x f_j, y, t); \quad 0 \leq y \leq b_j/2.
\]

\[\ldots (3.1.9)\]

The above equations are valid for each value of \(j, j=1,2,\ldots,n\), and state that the pressures are continuous at the fracture-reservoir interface. We also require that the fluxes be continuous at the reservoir-fracture interface. This condition is expressed by the following equations:

\[
k_j h_j \frac{\partial p_j}{\partial y} \bigg|_{y = b_j/2} = k_{f_j} h_j \frac{\partial p_{f_j}}{\partial y} \bigg|_{y = b_j/2}, \quad (3.1.10)
\]

for \(0 \leq x \leq L_x f_j\), and

\[
k_j h_j \frac{\partial p_j}{\partial x} \bigg|_{x = L_x f_j} = k_{f_j} h_j \frac{\partial p_{f_j}}{\partial x} \bigg|_{x = L_x f_j}, \quad (3.1.11)
\]

for \(0 \leq y \leq b_j/2\).

The above equations are valid for each layer \(j, j=1,2,\ldots,n\).

The initial condition is given by:

\[
p_j(x,y,0) = p_{f_j}(x,y,0) = p_i, \quad (3.1.12)
\]

for each layer \(j, j=1,2,\ldots,n\).
The boundary at the wellbore will be considered next. For both modes of production, the wellbore pressure is assumed to be independent of depth. That is,

\[ p_{fj} (x = 0, 0 \leq y \leq b_j/2) = p_w(t) \]  \hspace{1cm} (3.1.13)

for all \( j, j=1,2,\ldots,n \).

If the well produces at constant pressure, then the right-hand side of Eq. (3.1.13) is a constant. If the well produces at a constant rate, then \( p_w \) is a function of time and the following condition must also be satisfied. The production rate, \( q \), is given by:

\[ \frac{qB}{4} = \sum_{j=1}^{n} \frac{k_f j h_j}{\mu} \int_{0}^{b_j/2} \left. \frac{\partial p_{fj}}{\partial x} \right|_{x=0} \, dy \]  \hspace{1cm} (3.1.14)

The constant 4 on the left-hand side arises because only one-fourth of the reservoir is being modeled. If the well produces at a constant pressure, then Eq. (3.1.14) is used to determine the wellbore flow rate as a function of time.

3.2 Dimensionless Variables

In this section, we define dimensionless variables that are pertinent to the derivations that follow. We consider variables that are useful for the constant production rate first.

The dimensionless pressure drop at any location in the reservoir
is given by:

\[
p_{Dj} (x_D, y_D, t_{Dx_{f1}}) = \frac{k \cdot h}{141.2 \cdot q \cdot B \cdot u} \left[ p_i - p_{fj}(x, y, t) \right] ,
\]

(3.2.1)

for \(j=1,2,\ldots,n\). All quantities on the right-hand side of Eq. 3.2.1 are expressed in conventional oil field units. Here \(x_D\) and \(y_D\) are dimensionless distances based on the fracture half-length in layer 1, \(L_{x_{f1}}\), and \(t_{Dx_{f1}}\) is dimensionless time based on \(L_{x_{f1}}\) \(\) (see Eq. 2.6). Thus, \(x_D\) and \(y_D\) are defined by the following equations:

\[
x_D = \frac{x}{L_{x_{f1}}} ,
\]

(3.2.2)

and

\[
y_D = \frac{y}{L_{x_{f1}}} .
\]

(3.2.3)

The dimensionless pressure drop in the fracture is given by:

\[
p_{fDj} (x_D, y_D, t_{Dx_{f1}}) = \frac{k \cdot h}{141.2 \cdot q \cdot B \cdot u} \left[ p_i - p_{fj}(x, y, t) \right] ,
\]

(3.2.4)

for \(j=1,2,\ldots,n\). The dimensionless flow rate from fracture layer \(j\) into the well is given by:

\[
q_{wDj} = \frac{q_{wj}}{q} = -\frac{2}{\pi} \int_{0}^{b_{Dj}/2} \frac{k_{fj} \cdot h_{j}}{k \cdot h} \frac{\partial p_{fDj}}{\partial x_D} \bigg|_{x_D=0} \, dy_D .
\]

(3.2.5)

Note that \(\sum_{j=1}^{n} q_{wj} = q\). The dimensionless flow rate from layer \(j\) into the fracture that is in communication with layer \(j\), (fracture layer, \(j\))
is:

\[ q_{RDj} = \frac{q_{Rj}}{q} \tag{3.2.6} \]

where

\[ q_{Rj} = - \frac{2q}{\pi} \int_C \frac{k_j h_i}{kh} \left( \frac{\partial P_D}{\partial n} \right) \frac{ds}{s} \tag{3.2.7} \]

Here \( \partial n \) denotes differentiation along the outward-drawn normal at fracture-reservoir interface and the integration is along the boundary arcs of the fracture-reservoir interface. Note that this definition assumes the fluid enters the fracture along the length of the fracture \((0 \leq x \leq L_{x_{fj}})\) and via the fracture tip \((0 < y < b_j)/2\).

If the well produces at a constant pressure, \( p_{wf} \), then the dimensionless pressure drop in the reservoir and the fracture are given by:

\[ p_{Dj}(x, y, t_{Df1}) = \frac{p_i - p_j(x, y, t)}{(p_i - p_{wf})} \tag{3.2.8} \]

and

\[ p_{Dfj}(x, y, t_{Df1}) = \frac{p_i - p_{fj}(x, y, t)}{(p_i - p_{wf})} \tag{3.2.9} \]

The dimensionless flow rate from fracture layer \( j \) into the well is given by:

\[ q_{WDj} = \frac{q_{wj}(t) B \mu}{141.2 \frac{k h}{(p_i - p_{wf})}} \tag{3.2.10} \]
where
\[
q_{wj} = 4 \left(1.127 \times 10^{-3}\right) \int_0^{b_j/2} \frac{k_{fj}}{\mu} \frac{h_i}{h} \left( \frac{\partial p_{fj}}{\partial x} \right)_x = 0 \ dy. \tag{3.2.11}
\]

All quantities on the right-hand side of Eq. 3.2.11 are expressed in conventional oil field units. The dimensionless flow rate from reservoir layer \( j \) to fracture layer \( j \), is given by:
\[
q_{RDj} = \frac{q_{Rj}(t) \mu B}{141.2 \ k \ h \ (p_i - p_{wf})}, \tag{3.2.12}
\]

where \( q_{Rj}(t) \) is the flow rate from reservoir layer \( j \) to the fracture layer \( j \).

3.3 Analytical Considerations

Four modifications to the problem specified in Section 3.1 are needed to enable us to obtain approximate analytical solutions. The modifications discussed below are similar to those used by Bennett.1 We restrict our attention to one-dimensional flow in the reservoir; that is, we assume \( \partial p_j/\partial x \) to be zero. Second, we assume that the reservoir extends to infinity in the \( y \)-direction. Third, we assume that \( \partial p_{fj}/\partial x \) and \( \partial p_{fj}/\partial t \) are independent of \( y \). This assumption is similar to the pseudofunction assumption used in reservoir simulation studies. Fourth, we assume that the fracture tips are sealed (\( \partial p_{fj}/\partial x = 0 \) at \( x = L_{xfj} \)). Solutions that satisfy this boundary are useful, when solutions obtained by analytical and numerical techniques are compared1. Cinco-L. and Samaniego-V.4 presented the solution for the single-layer case. They also considered a solution for the case where \( L_{xf} \to \infty \) (\( p_f \to p_i \) as \( x \to \infty \)). It is possible to derive a solution for the problem under
consideration that incorporates the condition that $p_f \rightarrow p_i$ as $x \rightarrow \infty$. However, from the viewpoint of establishing asymptotic expressions during various flow regimes either boundary condition may be used.

As mentioned earlier, the solution that assumes that the fracture tips are sealed is more useful if we compare the analytical solutions with numerical solutions. We will now consider the modified problem.

If we assume that $\frac{\partial p_j}{\partial x} = 0$, then Eq. 3.1.1, in terms of the dimensionless variables defined in Section 3.2 may be written as:

$$\frac{\partial^2 p_{Dj}}{\partial y_D^2} = \frac{1}{\eta_{Dj}} \frac{\partial p_{Dj}}{\partial t_{Dx_{f1}}}$$

(3.3.1)

for $b_{Dj}/2 < y_D < \infty$, $j=1,2,\ldots,n$. Here $b_{Dj} = b_j/L_x f_1$ and $\eta_{Dj} = \eta_j/\eta$.

The solution of Eq. 3.3.1 should satisfy the following initial and boundary conditions:

$$p_{Dj}(y_D, t_{Dx_{f1}} = 0) = 0$$

(3.3.2)

$$p_{Dj}(y_D \rightarrow \infty, t_{Dx_{f1}}) = 0$$

(3.3.3)

and

$$p_{Dj}(y_D = b_{Dj}/2, t_{Dx_{f1}}) = p_{fDj}(y_D = b_{Dj}/2, t_{Dx_{f1}})$$

(3.3.4)

for $j=1,2,\ldots,n$. Equation (3.3.2) specifies that the initial pressure is uniform everywhere. Equation 3.3.3 states that $p_j \rightarrow p_i$ as $y_D \rightarrow \infty$ and Eq. (3.3.4) requires that the pressures be continuous at the
fracture-reservoir interface.

The equation governing flow in the fracture (Eq. 3.1.6) in dimensionless form is given by:

\[
\frac{k_{fj}}{k} \left\{ \frac{\partial^2 P_{fDj}}{\partial x_D^2} + \frac{\partial^2 P_{fDj}}{\partial y_D^2} \right\} = \frac{\phi_{fj} c_{ftj}}{\phi_c} \frac{\partial P_{fDj}}{\partial t_{Dx_f1}},
\]

for j=1,2,...,n. Integrating Eq. (3.3.5) with respect to y_D from y_D = 0, to y_D = b_{Dj}/2 and invoking the pseudofunction approximation we have:

\[
\frac{k_{fj}}{k} \frac{\partial^2 P_{fDj}}{\partial x_D^2} + \frac{2k_j L_{x_f1}}{k b_j} \left( \frac{\partial P_{Dj}}{\partial y_D} \right)_{y_D = b_{Dj}/2} = \frac{\phi_{fj} c_{ftj}}{\phi_c} \frac{\partial P_{fDj}}{\partial t_{Dx_f1}}.
\]

. . . . . (3.3.6)

Here we have used the interface condition (continuity of flux) specified by Eq. (3.1.10) and the no flow boundary condition given by Eq. (3.1.7). Equation (3.3.6) may be rewritten as follows:

\[
\frac{\partial^2 P_{fDj}}{\partial x_D^2} + \frac{2k_j}{k b_j} \frac{\partial P_{Dj}}{\partial y_D} \bigg|_{y_D = b_{Dj}/2} = \frac{1}{\eta_{fDj}} \frac{\partial P_{fDj}}{\partial t_{Dx_f1}},\]

where \( \eta_{fDj} = \eta_{fj} / \eta; \ j=1,2,...,n. \)

The solution to Eq. (3.3.7) must satisfy the following condition at
the fracture tip:

\[ \frac{\partial p_{fDj}}{\partial x_D} (x_D = L x_{fj} / L x_{f1}, t_{Dx_{f1}}) = 0 \quad . \quad (3.3.8) \]

The initial condition is given by:

\[ p_{fDj} (0 \leq x_D \leq L x_{fj} / L x_{f1}, 0) = 0 \quad . \quad (3.3.9) \]

The specification of the problem will be complete, when the wellbore condition is specified. (Thus far, the problem as stated is valid for both modes of production.) We consider production at a constant rate first. For this mode of production, the boundary condition at

\[ x_D = 0 \]

considering \( \partial p_{fDj} / \partial y_D \) at \( x_D = 0 \) to be independent of \( y_D \) (Eq. 3.1.14) is given by:

\[ l = -\frac{1}{n} \sum_{j=1}^{n} \sigma_{fDj} h_{fDj} \frac{\partial p_{fDj}}{\partial x_D} \bigg|_{x_D = 0} \quad . \quad (3.3.10) \]

and

\[ p_{fDj} (x_D = 0, t_{Dx_{f1}}) = p_{wD} (t_{Dx_{f1}}) \quad , \quad (3.3.11) \]

for \( j = 1, 2, \ldots, n \).
3.4 An Approximate Analytical Solution - Constant Production Rate

Here, we will derive a solution to the problem formulated in Section 3.3 by the method of Laplace Transforms. We will first obtain a solution to the pressure distribution in the reservoir by solving Eqs. 3.3.1 - 3.3.4. We then consider solution of Eq. 3.3.7, subject to the conditions outlined in Eqs. 3.3.8 - 3.3.11. Note that the solution to the pressure distribution in the fracture requires that we first obtain a solution to Eq. 3.3.1.

Using the method of Laplace Transforms, the pressure distribution in the reservoir is given by:

\[
\bar{p}_{Dj} = \bar{p}_{FDj} \exp \left[ -\left( y_D - b_{Dj}/2 \right) \sqrt{s/\eta_{Dj}} \right]. \tag{3.4.1}
\]

Here \( \bar{p}_{Dj} \) and \( \bar{p}_{FDj} \) are the Laplace Transforms of \( p_{Dj} \) and \( p_{FDj} \), respectively and \( s \) is the Laplace variable. The solution for the pressure distribution in the fracture may be obtained by substituting the value of \( \left[ \frac{\partial \bar{p}_{Dj}}{\partial y_D} \right] \) at \( y_D = b_{Dj}/2 \) obtained from Eq. 3.4.1 in Eq. 3.3.7. Using Laplace Transforms and performing this operation we may write Eq. 3.3.7 as:

\[
\frac{\partial^2 \bar{p}_{FDj}}{\partial x_D^2} \left[ \frac{2k}{\sigma_{FDj} \sqrt{\eta_{Dj} \eta_{FDj}}} \right] = 0. \tag{3.4.2}
\]

Here, we have used the initial condition given by Eq. 3.3.9. The
The general solution of Eq. 3.4.2 is:

$$P_{fDj} = A_j \exp (-\alpha_j x_D) + B_j \exp (\alpha_j x_D), \quad (3.4.3)$$

where

$$\alpha_j = \frac{s C_{fDj} + 2 R_{cDj} \sqrt{s/h_{fDj}}}{\sigma_{fDj}} \quad (3.4.4)$$

for j = 1, 2, ..., n.

Using the boundary condition at the fracture tip, we obtain $B_j \exp (\alpha_j L_{Dj}) = A_j \exp (-\alpha_j L_{Dj})$, where $L_{Dj} = L_{x_fj} / x_{f1}$. Thus, we may write Eq. 3.4.3 as:

$$P_{fDj} = 2 A_j \exp (-\alpha_j L_{Dj}) \cosh (\alpha_j (L_{Dj} - x_D)) \quad (3.4.5)$$

The constants $A_j$ may be determined from the boundary condition at the wellbore. Differentiating Eq. 3.4.5 with respect to $x_D$ and using the resulting expression for $\partial P_{fDj}/\partial x_D$ at $x_D = 0$, in Eq. 3.3.10, we obtain:

$$\frac{\pi}{s} = 2 \sum_{j=1}^{n} A_j \alpha_j h_{fDj} \sigma_{fDj} \exp (-\alpha_j L_{Dj}) \sinh (\alpha_j L_{Dj}) \quad (3.4.6)$$

The fact that the wellbore pressures are identical (Eq. 3.3.11), requires that

$$A_1 \exp (-\alpha_1) \cosh (\alpha_1) = A_j \exp (-\alpha_j L_{Dj}) \cosh (\alpha_j L_{Dj}) \quad (3.4.7)$$
for \( j = 2, 3, 4, \ldots, n \). Combining Eq. (3.4.6) and Eq. (3.4.7), we obtain expressions for all \( A_j \)'s. Using these expressions, the pressure distribution in the fracture is given by:

\[
\bar{p}_{FDj} = \frac{\pi}{s} \cosh[\alpha_j (L_{DJ} - x_D)] \left\{ \cosh(\alpha_j L_{DJ}) \right\} \sum_{\ell=1}^{n} a_\ell h_{FD\ell} \sigma_{FD\ell} \tanh(\alpha_\ell L_{D\ell}) \right\}^{-1} . \tag{3.4.8}
\]

Thus, the wellbore pressure drop is given by:

\[
\bar{p}_{WD} = \frac{\pi}{s} \left\{ \sum_{\ell=1}^{n} a_\ell h_{FD\ell} \sigma_{FD\ell} \tanh(\alpha_\ell L_{D\ell}) \right\}^{-1} . \tag{3.4.9}
\]

Note that the \( \alpha_j \)'s in Eq. 3.4.9 are a function of \( s \). If we use the Inversion Theorem for the Laplace Transformation, the wellbore pressure drop, \( p_{WD} \), is given by:

\[
p_{WD}(t_{DF_{f1}}) = \int_c \frac{\exp(st_{DF_{f1}})}{2\pi i} \frac{\pi}{s} \left\{ \sum_{\ell=1}^{n} a_\ell h_{FD\ell} \sigma_{FD\ell} \right\}^{-1} ds . \tag{3.4.10}
\]

Here, \( c \) denotes integration in the complex plane and \( i = \sqrt{-1} \). Unfortunately, analytical inversion using the Inversion Theorem for the Laplace Transformation does not appear to be feasible. To obtain \( p_{WD}(t_{DF_{f1}}) \), two options are available: Numerical Inversion and analytical inversion of asymptotic approximations of \( \bar{p}_{WD} \) (these will be valid for specific time periods). Both approaches are useful.
Numerical Inversion may be used to compare the analytical and finite difference solutions (see Chapter IV). Inversion of the Asymptotic Approximations to $P_{\text{wd}}$ are also useful in understanding the structure of solutions. As will be shown below, the solutions also suggest methods to combine the parameters of interest. We will now consider three asu, 3pt to sp; itopms.

3.4.1 The Early-Time Approximation

We proceed along the lines suggested by Fetter and Walecka. If we substitute $\zeta = st_{DX_{f1}}$ in Eq. 3.4.10, we obtain:

$$P_{\text{wd}}(t_{DX_{f1}}) = \int_{c} \frac{e^{\zeta}}{2\pi i \zeta} \left\{ \sum_{j=1}^{n} h_{\text{fDj}} \sigma_{\text{fDj}} \alpha_{j}(\zeta/t_{DX_{f1}}) \right\}^{-1} \tanh[\alpha_{j}(\zeta/t_{DX_{f1}})L_{Dj}] \right\} d\zeta \quad (3.4.11)$$

For small values of $t_{DX_{f1}}$, $\zeta/t_{DX_{f1}}$ is large and thus $\tanh[\alpha_{j}(\zeta/t_{DX_{f1}})L_{Dj}] = 1$. The summation term in Eq. 3.4.11 may be written as:

$$\left( \sum_{j=1}^{n} h_{\text{fDj}} \sigma_{\text{fDj}} \alpha_{j} \tanh[\alpha_{j}(\zeta/t_{DX_{f1}})L_{Dj}] \right)^{-1} = \left( \sum_{j=1}^{n} h_{\text{fDj}} \sigma_{\text{fDj}} \alpha_{j} \right)^{-1}$$

$$= \left\{ \sum_{j=1}^{n} \sqrt{\frac{h_{\text{fDj}} \sigma_{\text{fDj}} (\zeta/t_{DX_{f1}})}{h_{\text{fDj}} \sigma_{\text{fDj}} \sqrt{\zeta/t_{DX_{f1}}}}} \left[ 1 + \frac{2R_{\text{cDj}}}{h_{\text{fDj}} \sigma_{\text{fDj}} \sqrt{\zeta/t_{DX_{f1}}}} \right]^{1/2} \right\}^{-1} \quad (3.4.12)$$
If we consider time ranges where the second term in the square brackets is small, then Eq. 3.4.11 may be written as:

\[
P_{WD}(t_{DX_{f1}}) = \int_{C} \frac{e^{st_{DX_{f1}}}}{2\pi i s^{3/2}} \pi \left\{ \sum_{j=1}^{n} \frac{\sqrt{h_{FDj}^{2} C_{FDj} \sigma_{FDj}}}{h_{FDj}^{2} C_{FDj} \sigma_{FDj}} \right\}^{-1} ds. \quad (3.4.13)
\]

Using the Inversion Theorem for Laplace Transformation, we obtain:

\[
P_{WD}(t_{DX_{f1}}) = 2 \sqrt{\pi t_{DX_{f1}}} \left\{ \sum_{j=1}^{n} \left( \frac{h_{FDj}^{2} C_{FDj} \sigma_{FDj}}{h_{FDj}^{2} C_{FDj} \sigma_{FDj}} \right)^{1/2} \right\}^{-1}. \quad (3.4.14)
\]

This solution is similar to the early-time solutions derived by Cinco-L. and Samaniego-V. and Bennett for the fracture linear flow period. The Cinco-L.-Samaniego-V. solution can be obtained if we assume that \( \sigma_{FDj} \) and \( C_{FDj} \) are identical for all values of \( j \) and note that \( \frac{1}{\sigma_{FDj}} = 1 \).

3.4.2 An Intermediate-Time Approximation

The approximation derived below assumes that the storage capacity of the fractures is negligible. At the same time, we assume that the time ranges we consider are small enough for the approximation given by Eq. 3.4.12 to be valid; that is, we consider values of \( s \), such that \( \tanh(\alpha_{j} L_{Dj}) \approx 1 \), for \( j = 1, 2, \ldots, n \). If \( \tanh(\alpha_{j} L_{Dj}) \approx 1 \),
then we can write:

\[
\left\{ \sum_{j=1}^{n} h_{fDj} \sigma_{fDj} \alpha_j (\xi/t_{Dx_f1}) \tanh[\alpha_j (\xi/t_{Dx_f1}) L_{Dj}] \right\}^{-1} = \\
\left\{ \sum_{j=1}^{n} \left[ 2 h_{fDj} \sigma_{fDj} R_{cDj} \sqrt{\xi/t_{Dx_f1}} \right]^{1/2} \left[ 1 + (C_{fDj} h_{fDj}/2R_{cDj}) \sqrt{\xi/t_{Dx_f1}} \right]^{1/2} \right\}^{-1}.
\]

\ldots \ldots \text{(3.4.15)}

If we now assume that the second term in the second square bracket is small; that is, the influence of \( C_{fDj} \) is negligible, then we may write Eq. 3.4.10 as

\[
P_{wD}(t_{Dx_{f1}}) = \int_{c}^{-} \frac{\exp(st_{Dx_{f1}})}{2\pi is^{5/4}} \left\{ \sum_{j=1}^{n} \sqrt{2h_{fDj} \sigma_{fDj} R_{cDj}} \right\}^{-1} ds.
\]

\ldots \ldots \text{(3.4.16)}

Integrating, we obtain the following expression for \( P_{wD}(t_{Dx_{f1}}) \):

\[
P_{wD}(t_{Dx_{f1}}) = \frac{\pi}{\sqrt{2} \sqrt{t_{Dx_{f1}}} / \Gamma (5/4) \sum_{j=1}^{n} (2h_{fDj} \sigma_{fDj} R_{cDj})^{1/2}}.
\]

\ldots \ldots \text{(3.4.17)}

If all layer properties and fracture properties are identical, we obtain:

\[
P_{wD}(t_{Dx_{f1}}) = \frac{\pi}{\sqrt{2} \sigma_{fD} / \Gamma (5/4)} 0.25.
\]

\ldots \ldots \text{(3.4.18)}
This solution is the bilinear flow approximation obtained by Cinco-L. and Samaniego-V. For convenience, we refer to Eq. 3.4.17 as the bilinear flow approximation. Later we will compare the solution given by the right-hand side of Eq. 3.4.17 with the corresponding solution given by Bennett¹, and discuss the differences between the two solutions.

3.4.3 The Long-Time Approximation

In this section we consider the long-time approximation to $P_{WD}$. We examine the situation when times are large enough so that

$$\tanh(a_j L_{Dj}) = a_j L_{Dj}. \quad \text{Thus:}$$

$$a_j(s)\tanh[a_j(s)L_{Dj}] = a_j^2 L_{Dj} = L_{Dj}(s C_{fDj} h_{fDj} + 2 \sqrt{s} R_{cDj})/(\sigma_{fDj} h_{fDj}).$$

...(3.4.19)

Substituting the expression on the right side of the equality in Eq. 3.4.19 for $a_j(s)\tanh[a_j(s)L_{Dj}]$ in Eq. 3.4.10, we obtain:

$$P_{WD}(t_{Dx_{f1}}) = \int_c \frac{\exp(st_{Dx_{f1}})}{2\pi i} \frac{\pi}{2s^{3/2}} \left[ \sum_{j=1}^n \left( R_{cDj} L_{Dj} \right) \right]^{-1}$$

$$\left\{ \frac{2 \sum_{j=1}^n \left( R_{cDj} L_{Dj} \right)}{\sum_{j=1}^n \left( C_{fDj} h_{fDj} L_{Dj} \right)} \right\} \left\{ \sqrt{s} + 2 \sum_{j=1}^n \left( R_{cDj} L_{Dj} \right) \right\} ds. \quad (3.4.20)$$
Integrating the right-hand side of Eq. 3.4.20, we obtain:

\[ p_{WD}(t_{DX}^{f1}) = \left\{ \frac{\pi t_{DX}^{f1}}{2 \left( \sum_{j=1}^{n} R_{cDj} L_{Dj} \right)} \right\}^{1/2} \]

\[-\frac{\pi}{4} \sum_{j=1}^{n} \left( \frac{C_{fDj} h_{fDj} L_{Dj}}{R_{cDj} L_{Dj}} \right) \]

\[ \left[ 1 - \exp(\psi^2 t_{DX}^{f1} \text{erfc}(\psi \sqrt{t_{DX}^{f1}})) \right], \]

\[ \ldots \quad (3.4.21) \]

where

\[ \psi = 2 \sum_{j=1}^{n} \left( \frac{R_{cDj} L_{Dj}}{C_{fDj} h_{fDj} L_{Dj}} \right) \quad (3.4.22) \]

If we consider a single-layer system, then we obtain:

\[ p_{WD}(t_{DX}^{f}) = \sqrt{\pi t_{DX}^{f}} - \left( \frac{\psi}{4} \right) C_{fD} \left[ 1 - \exp \left( \frac{4t_{DX}^{f}}{C_{fD}} \text{erfc} \left( \frac{2\sqrt{t_{DX}^{f}}}{C_{fD}} \right) \right) \right]. \quad (3.4.23) \]

If \( t_{DX}^{f} \) is large, then \( \exp(\psi^2 t_{DX}^{f1} \text{erfc}(\psi \sqrt{t_{DX}^{f1}})) \to 0 \), and Eq. (3.4.21) becomes

\[ p_{WD}(t_{DX}^{f1}) = \sqrt{\frac{\pi t_{DX}^{f1}}{2 \left( \sum_{j=1}^{n} R_{cDj} L_{Dj} \right)}} \]

\[-\left( \frac{\psi}{4} \right) \sum_{j=1}^{n} \left( \frac{C_{fDj} h_{fDj} L_{Dj}}{R_{cDj} L_{Dj}} \right) \left[ 1 - \exp \left( \frac{4t_{DX}^{f1}}{C_{fD}} \text{erfc} \left( \frac{2\sqrt{t_{DX}^{f1}}}{C_{fD}} \right) \right) \right]. \quad (3.4.24) \]
The long-time approximation suggests that a linear flow region should be evident at long-times. The correspondence between Eq. 3.4.23 and Eq. 3.4.24 should be noted. We will consider the similarities between the two solutions in the Discussion Section.

3.4.4 Layer Rates

The dimensionless flow rate from reservoir layer $j$ into fracture layer $j$ in Laplace Space is given by the following relation:

$$
\tilde{q}_{RDj} = -\frac{2}{\pi} \int_{0}^{L_{Dj}} \frac{k_{j}h_{j}}{kh} \left. \frac{\partial \tilde{p}_{Dj}}{\partial y_{D}} \right|_{y_{D}=b_{Dj}/2} dx_{D}.
$$

(3.4.25)

Using the value of $\frac{\partial \tilde{p}_{Dj}}{\partial y_{D}}$ obtained by differentiating Eq. 3.4.1, we obtain:

$$
\tilde{q}_{RDj} = \frac{2\sqrt{s}}{\pi} R_{CDj} \int_{0}^{L_{Dj}} \tilde{p}_{fDj} dx_{D}.
$$

(3.4.26)

Substituting the right-hand side of Eq. 3.4.8 for $\tilde{p}_{fDj}$ in Eq. 3.4.26, the layer flow rate is given by

$$
\tilde{q}_{RDj} = 2 R_{CDj} \tanh(\alpha_{j} L_{Dj}) \left\{ \sum_{\ell=1}^{n} \frac{\alpha_{\ell} h_{f\ell} \sigma_{f\ell}}{\tanh(\alpha_{\ell} L_{D\ell})} \right\}^{-1}/(\sqrt{s} \alpha_{j}).
$$

(3.4.27)

Analytical inversion of Eq. 3.4.27 does not appear to be feasible. Asymptotic approximations for the layer rates can be derived if the approximations used in the previous section are used. In the following,
we consider the intermediate-time and late-time approximations. Let us consider the intermediate-time approximation first. If we may approximate \( \tanh(\alpha L_{Dj}) \) by unity, then Eq. 3.4.27 may be written as (using the definition of \( \alpha_j \), given by Eq. 3.4.4):

\[
q_{RDj} = \frac{2 R_{CDj} (h_{FDj} \sigma_{FDj})^{1/2}}{\sqrt{s} \left[ s h_{FDj} C_{FDj} + 2 R_{CDj} \right]^{1/2} \left[ \sum_{\ell=1}^n (h_{FD\ell} \sigma_{FD\ell}) (s h_{FD\ell} C_{FD\ell} + 2 R_{CD\ell} \sqrt{s}) \right]^{1/2}}.
\]

\[\ldots \text{(3.4.28)}\]

Simplifying the right-hand side of Eq. 3.4.28, we obtain

\[
q_{RDj} = \frac{2(R_{CDj} h_{FDj} \sigma_{FDj})^{1/2}}{2s[1 + \sqrt{s} C_{FDj} h_{FDj} / (2R_{CDj})]^{1/2} \left[ \sum_{\ell=1}^n (h_{FD\ell} \sigma_{FD\ell} R_{CD\ell}) (1 + \sqrt{s} C_{FD\ell} h_{FD\ell} / (2R_{CD\ell}) \right]^{1/2}}.
\]

\[\ldots \text{(3.4.29)}\]

If we now assume that the influence of the \( C_{FDj} \) is negligible so that \( 1 + \sqrt{s} C_{FDj} h_{FDj} / (2R_{CDj}) = 1 \), then we may use the Inversion Theorem for the Laplace Transformation to obtain \( q_{RDj} \) as follows:

\[
q_{RDj} = \sqrt{R_{CDj} h_{FDj} \sigma_{FDj}} / \sum_{\ell=1}^n \sqrt{R_{CD\ell} h_{FD\ell} \sigma_{FD\ell}}.
\]

(3.4.30)

The long-time approximation to the dimensionless layer rate, \( q_{RDj} \), may be obtained by using the approximation \( \tanh(\alpha_j L_{Dj}) \approx \alpha_j L_{Dj} \).
Using this approximation, Eq. 3.4.27 may be written as:

\[ q_{RDj} = 2 R_{CDj} L_{Dj} \left\{ \sum_{\xi=1}^{n} h_{FD\xi} a_{FD\xi} L_{D\xi} \alpha_{\xi} \right\}^{-1} / \sqrt{s} \]  

(3.4.31)

Substituting the right-hand side of Eq. 3.4.4 for \( \alpha_{\xi}^2 \), we have:

\[ q_{RDj} = 2 R_{CDj} L_{Dj} \left\{ \sqrt{s} \sum_{\xi=1}^{n} \left( s C_{FD\xi} h_{FD\xi} + 2 R_{CD\xi} \sqrt{s} \right) L_{D\xi} \right\}^{-1} \]

\[ = \left[ R_{CDj} L_{Dj} / (s \sum_{\xi=1}^{n} R_{CD\xi} L_{D\xi}) \right] \left[ 1 + \sqrt{s} \sum_{\xi=1}^{n} L_{D\xi} C_{FD\xi} h_{FD\xi} / (2 \sum_{\xi=1}^{n} R_{CD\xi} L_{D\xi}) \right]^{-1} \]  

(3.4.32)

If we expand the bracketed term on the right-hand side of Eq. 3.4.32 by the binomial theorem and neglect terms of \( O(\sqrt{s}) \) (note \( s \) is small) and higher order terms, and apply the Inversion Theorem for the Laplace Transformation, we have

\[ q_{RDj} = R_{CDj} L_{Dj} / \sum_{\xi=1}^{n} R_{CD\xi} L_{D\xi} \]  

(3.4.33)

The change in layer rates as a function of time can be determined by comparing the right-hand sides of Eqs. 3.4.30 and 3.4.33. The differences between the two expressions deserve mention. Suppose we consider a two-layer system. Equation 3.4.30 indicates that \( q_{R1}/q_{R2} \propto R_{CD1}/R_{CD2} \); whereas during the Linear Flow Period the layer rates are proportional to \( R_{CD1}/R_{CD2} \). The layer rates during the Bilinear Flow Period are independent of the fracture half lengths. The layer rates during the
Linear Flow Period are directly proportional to the fracture half-lengths in each layer.

3.4.5 Sand Face Rates

The dimensionless flow rate from fracture layer \( j \) into the wellbore is given in Laplace Space by the following relation:

\[
q_{\text{WDj}} = \frac{q_{\text{wj}}}{q} = - \frac{\sigma_{\text{FDj}} h_{\text{FDj}}}{\pi} \frac{\partial p_{\text{FDj}}}{\partial x_D} \bigg|_{x_D=0}
\]

Differentiating the right-hand side of Eq. 3.4.8 with respect to \( x_D \) and determining the value of \( \partial p_{\text{FDj}}/\partial x_D \) at \( x_D = 0 \), we have:

\[
q_{\text{WDj}} = \sigma_{\text{FDj}} h_{\text{FDj}} \alpha_j \tanh(\alpha_j L_{\text{Dj}}) \left\{ \sum_{k=1}^{n} \alpha_k h_{\text{FDk}} \sigma_{\text{FDk}} \tanh(\alpha_k L_{\text{Dk}}) \right\}^{-1} /s.
\]

Asymptotic solutions to \( q_{\text{WDj}} \) may be obtained by considering approximations to \( \tanh(\alpha_j L_{\text{Dj}}) \). If we assume that \( \tanh(\alpha_j L_{\text{Dj}}) = 1 \), then \( q_{\text{WDj}} \) may be written as:

\[
q_{\text{WDj}} = \left\{ \sigma_{\text{FDj}} h_{\text{FDj}} (sC_{\text{FDj}} h_{\text{FDj}} + 2 R_{\text{CDj}} \sqrt{s}) \right\}^{1/2} \left\{ \sum_{k=1}^{n} (h_{\text{FDk}} \sigma_{\text{FDk}}) \right\}^{1/2} \nonumber \\
\left[ sC_{\text{FDk}} h_{\text{FDk}} + 2 R_{\text{CDk}} \sqrt{s} \right]^{1/2} \left\{ \right\}^{-1} /s.
\]

(3.4.36)
Simplifying the right-hand side of Eq. 3.4.36, we obtain

\[
\tilde{q}_{\text{WD}j} = (\sigma_{\text{FD}} h_{\text{FD}j} R_{\text{CD}j})^{1/2} \left[ 1 + \sqrt{s} C_{\text{FD}j} h_{\text{FD}j}/(2 R_{\text{CD}j}) \right]^{1/2} \left\{ \sum_{\ell=1}^{n} \left( \sigma_{\text{FD}\ell} h_{\text{FD}\ell} R_{\text{CD}\ell} \right)^{1/2} \left[ 1 + \sqrt{s} C_{\text{FD}\ell} h_{\text{FD}\ell}/(2 R_{\text{CD}\ell}) \right]^{1/2} \right\}^{-1} / \text{s.}
\]

(3.4.37)

If we now assume that the influence of \( C_{\text{FD}} \) is negligible, then we may use the Inversion Theorem to obtain the following result:

\[
q_{\text{WD}j} = (\sigma_{\text{FD}j} h_{\text{FD}j} R_{\text{CD}j})^{1/2} / \sum_{\ell=1}^{n} \left( \sigma_{\text{FD}\ell} h_{\text{FD}\ell} R_{\text{CD}\ell} \right)^{1/2},
\]  

(3.4.38)

for \( j = 1, 2, \ldots, n \). Comparing the right-hand sides of Eq. 3.4.30 and Eq. 3.4.38, we find that they are identical. This result should be expected since we have assumed that the influence of the \( C_{\text{FD}j} \)'s is negligible. This result may be expected on intuitive grounds for if the influence of the \( C_{\text{FD}j} \)'s is negligible, then the influx into a fracture layer should be equal to the production rate from that layer.

A long-time approximation for \( q_{\text{WD}j} \) can be derived by assuming that \( \tanh(\alpha_j L_{Dj}) = \alpha_j L_{Dj} \). Making this approximation, Eq. 3.4.35 may be written as:

\[
\tilde{q}_{\text{WD}j} = L_{Dj} \left( s C_{\text{FD}j} h_{\text{FD}j} + 2 R_{\text{CD}j} \sqrt{s} \right) \sum_{\ell=1}^{n} L_{D\ell} \left( s C_{\text{FD}\ell} h_{\text{FD}\ell} + 2 R_{\text{CD}\ell} \sqrt{s} \right)^{-1} / \text{s.}
\]

(3.4.39)

Again assuming that the influence of the \( C_{\text{FD}j} \)'s is negligible, and
using the Inversion Theorem, we obtain:

\[
q_{DJ}^{(t_{Dx_{f1}})} = \frac{R_{DJ}}{\sum_{j=1}^{n} R_{CD_{j}} L_{D_{j}}} \cdot \frac{t_{Dx_{f1}}^{1/4}}{\Gamma(5/4)}.
\]  
(3.4.40)

3.4.6 Discussion

In addition to providing information on the structure of the numerical solution, the analytical solutions derived in the preceding section serve an important purpose in that they suggest methods to correlate the well response. Consider the approximations suggested by the Bilinear and Linear flow periods. The pressure response during the Bilinear and Linear flow periods are given by (Eqs. 3.4.17 and 3.4.24):

\[
P_{WD}^{(t_{Dx_{f1}})} = \frac{\pi^{1/2}}{\sum_{j=1}^{n} (2 h_{fD_{j}} c_{fD_{j}} R_{CD_{j}})^{1/2}} \cdot t_{Dx_{f1}}^{1/4}
\]  
(3.4.17)

and

\[
P_{WD}^{(t_{Dx_{f}})} = \sqrt{\pi t_{Dx_{f}}^{1/2}} / \left[ \sum_{j=1}^{n} R_{CD_{j}} L_{D_{j}} \right]^{2}
\]  
(3.4.24)

Equation 3.4.24 may be rewritten as:

\[
P_{WD}^{(t_{Dx_{f}})} = \sqrt{\pi t_{Dx_{f}}}
\]  
(3.4.41)
where

\[ t_{DX_f} = t_{DX_f1} \left( \frac{L_{x_{f1}}^2}{\sum_{j=1}^{n} R_{CDj} x_{fj}} \right)^2 \]

\[ = \frac{2.637 \times 10^{-4}}{K} \frac{1}{\phi c_{t} u} \left( \sum_{j=1}^{n} R_{CDj} x_{fj} \right)^2 \]  \hspace{1cm} (3.4.42)

This result suggests that if we flow the well for a long enough period, then we should be able to correlate the commingled reservoir solutions with the single layer solution at long times, provided that we express the commingled reservoir solutions in terms of the dimensionless time defined by Eq. 3.4.42. Note that \( t_{DX_f} \) is the dimensionless time based on the sum of the product of the dimensionless reservoir conductivity of layer \( j \) and the fracture half-length of layer \( j \). This result suggests that the "effective" fracture length of the system is \( \sum_{j=1}^{n} R_{CDj} x_{fj} \).

If we use the definition of dimensionless time defined by Eq. 3.4.42, then the pressure drop during the Bilinear flow period is given by:

\[ P_{WD}(t_{DX_f}) = \frac{t_{DX_f}^{1/4}}{\sqrt{2} \left\{ \sum_{j=1}^{n} \left( \frac{h_{fj}}{k_{fj}} b_{j} R_{CDj}/k \right)^{1/2} \right\}^{2}} \left( \frac{2}{\sum_{j=1}^{n} R_{CDj} x_{fj}} \right)^{1/2} \left( \frac{\pi}{\Gamma(5/4)} \right). \]  \hspace{1cm} (3.4.43)
Thus, if we define an average dimensionless conductivity, $\sigma_{FD}$, by the relation

$$\sigma_{FD} = \left\{ \sum_{j=1}^{n} \left[ \frac{h_{Dj} k_{fj} b_j R_{C Dj}}{R_{C D j}} \right]^{1/2} \right\}^2 \left/ \left( \sum_{j=1}^{n} R_{C D j} L_{x fj} \right) \right\}, \quad (3.4.44)$$

then we may write Eq. 3.4.44 as follows:

$$p_{wD}(x_{Df}) = \frac{\pi t_{Df}^{1/4}}{\sqrt{2} \sigma_{FD} \Gamma(5/4)} \quad (3.4.45)$$

This result suggests that if we define an average conductivity given by the right-hand side of Eq. 3.4.44, then we may correlate the commingled reservoir solutions with the corresponding single-layer solutions during the Bilinear flow period.

In essence, the above expressions suggest the following result: A well producing commingled reservoir, via non-communicating fractures should behave as if it were producing a single homogeneous reservoir with reservoir properties given by $\bar{k}$ and $\bar{\phi c_t}$ (Eq. 2.3 and 2.7) and the fracture properties given by $\bar{L}_{x_f}$ and $\bar{k_f b}$, where $\bar{L}_{x_f}$ and $\bar{k_f b}$ are given by the following relations:

$$\bar{L}_{x_f} = \sum_{j=1}^{n} R_{C Dj} L_{x fj}, \quad (3.4.46)$$

and

$$\bar{k_f b} = \left\{ \sum_{j=1}^{n} \left[ \frac{h_{Dj} k_{fj} b_j R_{C Dj}}{R_{C D j}} \right]^{1/2} \right\}^2 \quad (3.4.47)$$
This result is an important result, for it suggests a procedure to correlate the numerical solutions for a commingled reservoir during the transient flow period, with the solutions available in the literature.\textsuperscript{3,4} The main question that needs to be answered is: Can the solutions be correlated along the lines suggested above if the assumptions used to derive the analytical solutions are relaxed? This answer can be obtained only if we consider the numerical solutions which do not incorporate the restrictions used to obtain the analytical solutions (see Chapter IV).

3.5 An Approximate Analytical Solution - Constant Pressure Production

The general solution to the problem given by Eq. 3.4.5 may be used in this case also. However, the dimensionless pressure drop in this case is given by Eq. 3.2.9. The coefficients $A_j$ in Eq. 3.4.5 would have to be determined from the boundary condition that the wellbore pressure is constant. In Laplace Space this condition is given by:

$$
\bar{p}_{fDj}(x_D=0,s) = \bar{p}_{wD} = 1/s \quad ,
$$

(3.5.1)

for $j = 1,2,\ldots,n$. The coefficients $A_j$ are given by

$$
A_j = (1/2) \exp(\alpha_jL_{Dj}) [s \cosh(\alpha_jL_{Dj})]^{-1} \quad .
$$

(3.5.2)

Thus, the pressure distribution in fracture layer $j$ is given by:

$$
\bar{p}_{fDj}(x_D,s) = \cosh[\alpha_j(L_{Dj} - x_D)]/[s \cosh(\alpha_jL_{Dj})] \quad .
$$

(3.5.3)
The production rate from layer $j$ in Laplace space, $\bar{q}_{Dj}$, may be obtained from Eq. 3.2.10 after evaluating $\partial p_{FDj}/\partial x_D$ at $x_D = 0$. The production rate is given by:

$$
\bar{q}_{Dj} = \sigma_{FDj} h_{FDj} \alpha_j \tanh(\alpha_j L_{Dj})/(\pi s)
$$

(3.5.4)

The total production rate from all layers may be obtained from the relation $\bar{q}_D = \sum_{j=1}^{n} \bar{q}_{Dj}$, and is given by:

$$
\bar{q}_D = (1/\pi s) \sum_{j=1}^{n} h_{FDj} \sigma_{FDj} \alpha_j \tanh(\alpha_j L_{Dj})
$$

(3.5.5)

At this stage, two important points should be noted. First, the solution to obtain $\bar{q}_D$ involves the solution of $n$-single-layer problems. Specifically, the requirement that the wellbore pressure is constant, decouples the problem. Second, the solution to the problem may also be obtained if we follow the suggestion of van Everdingen and Hurst. They showed that the product of $\bar{p}_{WD}$ (for constant rate production) and $\bar{q}_D$ (for constant pressure production) is equal to $1/s^2$. If we compute the product of $\bar{p}_{WD}$ and $\bar{q}_D$ by using the right-hand sides of Eq. 3.4.9 and Eq. 3.5.1 respectively, we find that this relationship is valid.

As in Section 3-4, asymptotic expressions for the wellbore rate may be derived by considering approximations to $\tanh(\alpha_j L_{Dj})$. In the following we omit the detailed steps involved in obtaining the appropriate expressions and state only the principal results.

If we consider time ranges such that $\tanh(\alpha_j L_{Dj}) \approx 1$, and
neglect the influence of $C_{FDj}$, then $q_D(t_{DX_{f1}})$ may be obtained from
the following relation:

$$q_D(t_{DX_{f1}}) = \int_C \exp(st_{DX_{f1}}) \left\{ \frac{n}{2\pi i} \sum_{j=1}^{\infty} \frac{[2h_{FDj} \sigma_{FDj} R_{CDj}]}{\pi s^{3/4}} \right\} ds \quad (3.5.6)$$

Integrating the right-hand side of Eq. 3.5.6, we obtain:

$$q_D(t_{DX_{f1}}) = \frac{t_{DX_{f1}}}{\pi r(3/4)} \sum_{j=1}^{\infty} \frac{n}{(2h_{FDj} \sigma_{FDj} R_{CDj})^{1/2}} \quad (3.5.7)$$

The long-time approximation to $q_D(t_{DX_{f1}})$, is obtained by using the
approximation $\tanh(\alpha_j L_{Dj}) = \alpha_j L_{Dj}$. Thus, $q_D(t_{DX_{f1}})$ is given by:

$$q_D(t_{DX_{f1}}) = \int_C \frac{\exp(st_{DX_{f1}})}{2\pi i} \sum_{j=1}^{\infty} \frac{2 R_{CDj} L_{Dj}}{\pi s^{1/2}} ds \quad (3.5.8)$$

Using the Inversion Theorem for the Laplace Transformation we obtain:

$$q_D(t_{DX_{f1}}) = 2 \sum_{j=1}^{\infty} \frac{R_{CDj} L_{Dj}}{\sqrt{\pi} t_{DX_{f1}}} \quad (3.5.9)$$

Expressions for individual layer rates are obtained by considering only
one of the summation terms on the right-hand sides of Eqs. 3.5.7 and
3.5.9. This result follows immediately if we note that the total rate
is the sum of the individual layer rates. Thus the production rates
from layer j during the Bilinear and Linear Flow Periods are given by:

\[ q_{WDj} = \frac{(2 h_{fDj} c_{fDj} R_{cDj})^{1/2}}{\pi \Gamma(3/4) t_{Dx_f}^{1/4}}, \]  

(3.5.10)

and

\[ q_{WDj} = \frac{2 R_{cDj} L_{Dj}}{\sqrt{\pi} t_{Dx_f}}. \]  

(3.5.11)

Since Eqs. 3.5.7 and 3.5.9 were derived by assuming that the storage capacities of the layers are negligible, the intermediate and long-time approximations to the dimensionless layer rates are also given by Eq. 3.5.10 and Eq. 3.5.11, respectively.

3.5.1 Discussion

The intent of this section was to show that the observations regarding the equivalence of multilayer and single-layer systems are also valid if the well produces at a constant pressure. That is, we may express the right-hand sides of Eqs. 3.5.7 and 3.5.9 in terms of \( t_{Dx_f} \) as follows:

\[ q_D(t_{Dx_f}) = \frac{\sqrt{2} \sigma_{fD}}{\pi \Gamma(3/4) t_{Dx_f}^{-1/4}}. \]  

(3.5.12)

and

\[ q_D(t_{Dx_f}) = \frac{2}{\sqrt{\pi} t_{Dx_f}}. \]  

(3.5.13)

where \( t_{Dx_f} \) is defined by Eq. 3.4.42.
CHAPTER IV
RESULTS AND DISCUSSION

In this section, we will first discuss the steps used to ensure that the solutions are accurate. Next, we consider the well response for two-layer systems. For this case, we show that if we express the two-layer solutions in terms of \( p_wD \) (or \( 1/q_D \)) vs. \( t_{Dx_{f1}} \), then the well response can be characterized by two groups, \( \beta \) and \( \gamma \), where \( \beta \) and \( \gamma \) are given by

\[
\beta = \left( \sum_{j=1}^{n} \sqrt{\frac{k_j b_j h_j R_{cDj}}{k h \sum L_{x_f j}}} \right)^2,
\]

and

\[
\gamma = \sum_{j=1}^{n} \frac{R_{cDj} L_{x_f j}}{L_{x_f j}} = \frac{L_{x_f}}{L_{x_f j}}.
\]

Note that \( \beta \) and \( \gamma \) are defined in terms of \( t_{Dx_{f1}} \) and that \( \sigma_{FD} = \beta/\gamma \); that is, \( \sigma_{FD} \) is given by:

\[
\sigma_{FD} = \left( \sum_{j=1}^{n} \sqrt{\frac{k_j b_j h_j R_{cDj}}{k h \sum L_{x_f j} R_{cDj}}} \right)^2.
\]
This result also implies that we can express results in terms of $\sigma_{FD}$ and $\gamma$. Second, we examine the conditions under which the commingled reservoir solutions may be expressed in terms of an equivalent single-layer system. The results of this section provide the theoretical basis for the findings of Bixel and Christiansen. The procedure given here provides a rigorous, direct and easy technique to convert commingled reservoir solutions to that of an equivalent single-layer system. That is, the need for an iterative procedure (trial and error) is eliminated. More importantly, we have considered a wider range of parameters than those considered in Ref. 2. Third, we examine well responses in systems where the number of layers is greater than two. Fourth, we examine the influence of boundaries. Finally we examine the consequences of allowing communication between fractures within the reservoir and extend the results given by Bennett.

4.1 Verification of Numerical Solutions

The finite difference model used in this study is identical to the one used by Bennett. The numerical model is a three-dimensional, block-centered finite difference model. As in most studies, extensive tests were conducted to verify the accuracy of our solutions. The procedures used in this work were similar to those used in Refs. 1 and 9. Bennett showed that the size of the grid blocks in the x and y directions depend on the fracture dimensions $L_x$ and $b$ and the reservoir dimensions $L_x$, $L_y$, and $L_e$. In our case $L_x = L_y$. He developed an empirical procedure to obtain the appropriate spatial grid to solve the problem numerically. This method involves using fine grids near the sand face and near the fracture tip in the x-direction and adjacent to the fracture surface in the y-direction.
The Bennett recommendations were followed to obtain the grid block dimensions for each layer. If we use the Bennett procedure, the grid to be used in each layer will be different, since the fracture half lengths in the layers are different. The finite difference model, however, requires that the spatial grid be identical in all layers. To accommodate this requirement the grid spacing used for all layers is such that the requirements of each layer are satisfied. Thus, the overall scheme results in a finer grid than that specified by the Bennett procedure. (Obviously, this will increase computation time.) Figure 4.1 presents a schematic diagram of the procedure described above. Here we consider a two-layer system and assume that fracture width is identical in both layers. Thus we only consider variation in the grid block sizes in the x-direction. Part a in Fig. 4.1 is a schematic of the grid block sizes for the layer with the longer fracture, and part b of Fig. 4.1 is the corresponding schematic for the grid in the layer with the shorter fracture. Part c is the grid that is used to obtain the finite difference solution. This grid satisfies the requirements of the grids in part a and part b. If the fracture widths are different, then this procedure is also used to obtain the grid blocks for the y-direction.

In addition to using the procedure outlined above we used the following steps to insure that the numerical solutions are accurate:

1. Checked to ensure that straight lines with proper slopes are obtained during the bilinear, linear and pseudo-radial flow periods. (Note that all flow regimes may not be evident for a given set of conditions; however, the pseudo-radial flow period should be evident in all cases provided that boundary effects are non-existent.)
Fig. 4.1 - Procedure to Obtain Grid (x-Direction)
2. Compared the solutions obtained from the finite difference model and those obtained by numerically inverting the analytical solutions (not asymptotic approximations) derived in Chapter III. Stehfest's\textsuperscript{12} algorithm was used to numerically invert the analytical solutions. Figure 4.2 compares the responses predicted by the finite difference solution and the solution obtained by numerically inverting the analytical solution in Laplace space. Agreement between the two solutions is excellent for times until \( t_{DX_{fl}} \approx 10^{-2} \). At later times, as expected, agreement between the two solutions is not good. The analytical solutions assume: (i) one-dimensional flow in the reservoir and, (ii) that the fracture tip is sealed. These assumptions were not used to obtain the finite difference solutions.

3. Material balance checks were conducted, and solutions with grids finer than that given by the Bennett procedure were used to ensure that numerical convergence was obtained.

4.2 Well Responses in Two-Layer Systems

Figure 4.3 examines the well response (constant production rate) for a number of combinations of reservoir properties and fracture properties. The properties used to obtain the solutions given here are listed in Table 4.1 and results are graphed in terms of \( t_{DX_{fl}} \). The intent of this figure is to show that the well response can be characterized in terms of the parameters \( \beta \) and \( \gamma \) that are defined by the right-hand sides of Eqs. 4.1 and 4.2, respectively. For the values of \( \beta \) and \( \gamma \) considered here \( \sigma_{FD} = 500 \). The solid line in this figure is the well response of a fractured well in a homogeneous reservoir for the \( \sigma_{FD} = 500 \).
Fig. 4.2 - Comparison Between Analytical and Numerical Solutions

\[ \frac{h_1}{h_2} = 32.333, \frac{k_1}{k_2} = 0.1 \]
\[ \phi_1 / \phi_2 = 1, \frac{L_{x_{f1}}}{L_{x_{f2}}} = 0.7 \]
\[ \sigma_{fD} = 500, \sigma_{fDj} = 527.8 (j=1,2) \]
Fig. 4.3 - Correlation of Well Responses in Terms of $\beta$ and $\gamma$ - High Conductivity Case
case. The results given in this figure assume that fracture conductivities are equal; that is, 
\[ k_{f1}b_{1} = k_{f2}b_{2}. \]

The solutions shown by the circles and squares correspond to the first four cases listed in Table 4.1. All these solutions were obtained by assuming that \( \beta = 421.05 \) and \( \gamma = 0.8421 \). The open and solid circle data points show that although the permeability ratio, porosity ratio and compressibility ratio are changed, we obtain identical solutions provided that \( \beta \) and \( \gamma \) are fixed. The square data points demonstrate the same result for a different value of layer thickness, \( h_{j} \), fracture half-length, \( L_{x_{fj}} \), and fracture conductivity, \( k_{fj}b_{j} \).

The triangular points in Fig. 4.3 (cases 5, 6 and 7 of Table 4.1) demonstrate the same point for different values of \( \beta \) and \( \gamma \). The open and closed triangles examine the influence of the porosity ratio and the permeability ratio. The solutions given by the open triangles and the inverted triangles indicate that different values of \( h_{1} \) and \( h_{2} \) will result in the same solution provided that \( \beta \) and \( \gamma \) are fixed.

Figure 4.4 presents identical information to that presented in Fig. 4.3. Again, the solid line shown here is the homogeneous reservoir solution. For all solutions given here, \( \sigma_{fD} = 1 \). The main differences are the following: (i) the values of \( \beta \) (and \( \sigma_{fD} \)) considered here are much lower than the values of \( \beta \) considered in Fig. 4.3, and (ii) we show that \( \beta \) and \( \gamma \) can be used to characterize the well response, even if values of \( (k_{x}b) \) change from layer to layer (solid inverted triangles and diamond shaped data points; see last two lines of Table 4.1).

Figure 4.5 considers the well response in a two-layer reservoir when the permeability in layer 2 is varied, and all other parameters are held constant. Here \( h_{D} = h_{2}/h_{1} \) and \( L_{D} = L_{x_{f2}}/L_{x_{f1}} \). If we compute \( \sigma_{fD} \)
Fig. 4.4 - Correlation of Well Responses in Terms of $\beta$ and $\gamma$ - Low Conductivity Case
# Table 4.1

Properties used to simulate results in Figs. 4.3 and 4.4

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<tr>
<th>Fracture Half-Length Feet</th>
<th>Layer Thickness Feet</th>
<th>Permeability Ratio $k_2/k_1$</th>
<th>Porosity Ratio $\phi_2/\phi_1$</th>
<th>Compressibility Ratio $c_{t2}/c_{t1}$</th>
<th>Fracture Conductivity* md ft, $k_f b_j$</th>
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</table>

*When only one value is given fracture conductivities are assumed equal.
The dimensionless wellbore pressure drop, $p_{WD}$, is shown as a function of dimensionless time, $t_{Dx_{f1}}$, with different permeability ratios $k_2/k_1$, values of $\beta$, $\gamma$, and $\sigma_{fD}$. The parameters are:

- $h_D = 7.58$, $\phi_2/\phi_1 = 2$, $C_{f2}/C_{f1} = 0.5$,
- $L_D = 2$, $k_1 = 0.5$ md, $L_{x_{f2}} = 1000$ ft,
- $\sigma_{fD}$ values: 100, 13, 1.4,
- $k_2/k_1$ values: 0.02, 1, 10,
- $\beta$ values: 100, 24.479, 2.668,
- $\gamma$ values: 1, 1.883, 1.906,
- $\sigma_{fD}$ values: 100, 13, 1.4.

The graph shows the influence of permeability ratio on well response. The single-layer solution is indicated by a solid line.

Fig. 4.5 - Influence of Permeability Ratio on Well Response
values using the values of $\beta$ and $\gamma$ for the cases given here, we find that the $c_{f\Delta}$ values are 100, 13 and 1.4. The solid lines shown here are the single-layer solutions for $c_{f\Delta}$ values of 100, 13 and 1.4. The important point to note in this figure is that the two-layer response is identical to the single-layer response for the $\gamma = 1$, $\beta = 100$ case (data points represented by squares). If we consider the results in Figs. 4.3 and 4.4, we find that the difference between the single-layer solutions and the corresponding two-layer solutions is negligible if $\gamma = 1$. This result appears to be a characteristic feature of many of the solutions we have examined in this study. More will be said regarding this observation in the next section. For the present, it suffices to note that $t_{Dx_{f1}} / \gamma^2 = t_{Dx_f}$, where $t_{Dx_f}$ is given by:

$$t_{Dx_f} = \frac{0.0002637 \kappa t}{c_{f\Delta} u L_{x_f}}.$$  \hspace{1cm} (4.4)

Here $L_{x_f}$ is defined in Eq. 4.2. It should be noted that none of the solutions in Fig. 4.3 through 4.4 assume that $L_{x_{f1}} = L_{x_{f2}}$. In this study we have considered $L_{x_{f2}} / L_{x_{f1}}$ values as small as 0.05.

On the basis of the results shown above, we conclude the following: First, if results are expressed in terms of $p_{WD}$ (or $1/q_D$) and $t_{Dx_{f1}}$, then the well response is dependent on the parameters $\beta$ and $\gamma$. Second, if $\gamma = 1$, then the differences in the well response between the two-layer and single-layer solutions for identical values of $c_{f\Delta}$ are negligible. We now consider procedures to generalize the solutions further.
4.3 Correlation with Single-Layer Solutions

The approximate analytical solutions in Chapter III suggest that it should be possible to correlate the multilayer solutions with single-layer solutions, provided that we express the multilayer solutions in terms of \( t_{DX_f} \) and \( \sigma_{FD} \), where \( t_{DX_f} \) and \( \sigma_{FD} \) are defined by Eqs. 4.3 and 4.4, respectively.

We have also seen that if \( \gamma \approx 1 \), then the difference in the well response between the single-layer solution and the two-layer solution is negligible (note that \( t_{DX_f} = t_{DX_{f1}} / \gamma^2 \)). We replotted many of the two-layer solutions in terms of \( t_{DX_f} \) and \( \sigma_{FD} \). Figure 4.6 presents the well response for several cases where results are graphed in terms of \( t_{DX_f} \) and \( \sigma_{FD} \). Similar results are obtained if the well produces at a constant pressure. The solid lines in Fig. 4.6 are the single-layer solutions for the appropriate value of \( \sigma_{FD} \). The dashed line denotes the start of the pseudo-radial flow period. Although results are shown only up to \( t_{DX_f} = 10 \), similar results are obtained as long as boundary effects are negligible. The results given in Fig. 4.6 establish the fact that the parameters \( \beta \) and \( \gamma \) may be combined further, and that the well responses can be expressed in terms of the groups \( \sigma_{FD} \) and \( t_{DX_f} \). (Note that some of the cases considered in Fig. 4.4 are replotted in Fig. 4.6.) In other words, these results show that two-layer solutions may be expressed as an equivalent single layer system, provided that \( \overline{k_f b_f} \) and \( \overline{\phi c_{ef}} \) are used for the permeability and porosity-compressibility product and the equivalent conductivity and fracture half-length are given, respectively, by:

\[
\overline{k_f b_f} = \left\{ \frac{2}{\sum_{j=1}^{2} \sqrt{\left( k_{fj} b_j h_{Dj} R_{CDj} \right)^2}} \right\}^2, \tag{4.5}
\]
Approximate Start of Pseudo-Radial Flow

Single-Layer Solution

Fig. 4.6 - Correlation of Two-Layer Reservoir Solutions with Single-Layer Solutions
and

\[ \bar{L}_{xf} = \frac{2}{\sum_{j=1}^{c} R_{CDj} L_{xfj}}. \] (4.6)

Here \( h_{Dj} = h_j / h \). Figure 4.7 demonstrates that the correlations considered in Fig. 4.6 may be extended to systems with more than two layers. Data used to obtain the results given here are presented in Table 4.2. Again, we obtain excellent agreement with the appropriate single-layer solution.

4.4 Comparison with the Bixel-Christiansen\(^2\) Solutions

As mentioned earlier, Bixel and Christiansen conducted nine numerical experiments and attempted to match commingled reservoir responses using a single-layer model. They concluded that it is possible to match two-layer responses with a single-layer model, provided that the dimensionless conductivity of each layer (based on the permeability and fracture half-length of that layer) were equal; that is, \( (k_{f1} b_1) / (k_1 L_{xf1}) = (k_{f2} b_2) / (k_2 L_{xf2}) \). They also concluded that for those cases where it was possible to match two-layer responses by a single-layer model, the equivalent fracture length should be the arithmetic average of the fracture lengths in each layer.

Table 4.3 summarizes many of the cases considered by Bixel and Christiansen\(^2\). Cases 1, 2, 3 and 4 correspond to cases C, D, F and G of Ref. 2 respectively, and cases 5, 6 and 7 correspond to the three "No-Match Possible" cases of Ref. 2. The nomenclature given in Ref. 2 is given in parenthesis in column 1. Columns 5 and 6 of Table 4.3 present the final values of equivalent fracture conductivity and equivalent fracture half-length obtained in Ref. 2. Columns 7 and 8 present the
Fig. 4.7 - Correlation of Multi-Layer Reservoir Solutions with Single-Layer Solutions (Four and Six Layers)
### Table 4.2

Properties Used to Simulate Results in Figure 4.7

<table>
<thead>
<tr>
<th>Layer</th>
<th>Thickness Ratio $h_j/h_1$</th>
<th>Permeability Ratio $k_j/k_1$</th>
<th>Porosity-Compressibility Product Ratio ($\phi_j c_t j / (\phi_1 c_t 1)$)</th>
<th>Fracture Length Ratio $l_{xf j}/l_{xf 1}$</th>
<th>Dimensionless Layer Conductivity $R_{CDj}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Four-Layer Reservoir, $\sigma_{fD} = 50$, $R_{CD1} = 0.94321$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>0.000889</td>
<td>0.15</td>
<td>0.7</td>
<td>0.005167</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.000011</td>
<td>0.1</td>
<td>0.3</td>
<td>0.000497</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>0.00011</td>
<td>0.2</td>
<td>0.5</td>
<td>0.00111</td>
</tr>
<tr>
<td>(b) Six-Layer Reservoir, $\sigma_{fD} = 500$, $R_{CD1} = 0.04181$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.143</td>
<td>3.333</td>
<td>1.5</td>
<td>0.7</td>
<td>0.01336</td>
</tr>
<tr>
<td>3</td>
<td>1.714</td>
<td>21.667</td>
<td>4.5</td>
<td>0.3</td>
<td>0.70770</td>
</tr>
<tr>
<td>4</td>
<td>0.286</td>
<td>2.292</td>
<td>8</td>
<td>0.5</td>
<td>0.05115</td>
</tr>
<tr>
<td>5</td>
<td>0.571</td>
<td>1</td>
<td>4</td>
<td>0.4</td>
<td>0.04778</td>
</tr>
<tr>
<td>6</td>
<td>0.428</td>
<td>0.0033</td>
<td>2</td>
<td>0.2</td>
<td>0.00146</td>
</tr>
</tbody>
</table>
corresponding values of fracture conductivity and equivalent fracture half-length predicted by Eqs. 4.5 and 4.6, respectively. Comparing these columns, we obtain good agreement for \( \frac{L_{x_f}}{L_{x_{f1}}} \), (columns 6 and 8). The agreement is not as good for estimates of equivalent fracture conductivity, \( \frac{k_f b}{k_f b} \). Figure 4.8 presents well responses for some of the cases considered by Bixel and Christiansen.\(^2\) These results were obtained for the properties listed in Table 4.3 using the values of \( \frac{k_f b}{k_f b} \) and \( \frac{L_{x_f}}{L_{x_{f1}}} \), given in Eqs. 4.5 and 4.6, respectively (columns 7 and 8). All results shown here are gas well simulations, which have been converted to dimensionless form using the pseudo pressure concept of Al-Hussainy et al.\(^13\). The solid lines are the equivalent single-layer solutions that were obtained by using Eqs. 4.5 and 4.6. The circular data points represent results of the two-layer simulations. We were able to correlate all three "No Match Possible" cases considered in Ref. 2. The responses of the two-layer system and the equivalent single-layer system for two of the three "No Match Possible" cases are shown in Fig. 4.8. All three such cases assume that \( \frac{(k_{f1} b_1)/(k_{f1} L_{x_{f1}})}{\neq} \frac{(k_{f2} b_2)/(k_{f2} L_{x_{f2}})}{\) Thus, the restriction given in Ref. 2 that two-layer systems may be simulated by an equivalent single-layer system if the ratio \( \frac{(k_{fj} b_j)/(k_{fj} L_{x_{fj}})}{\) for \( j=1,2 \) is equal appears to be unnecessary.

It should also be noted that the observation in Ref. 2, that the equivalent fracture length will be the arithmetic average of the fracture lengths in each layer results because the \( R_{cDj} \) values for cases considered in Ref. 2 is either 0.5 or approximately 0.5. If \( R_{cDj} = 0.5 \), then Eq. 4.6 predicts that \( \frac{L_{x_f}}{L_{x_{f1}} + L_{x_{f2}}} = 0.5 \). On the basis of the results given here, the following observations regarding the conclusions given in Ref. 2 are pertinent. First, there
### Table 4.3

**Summary of Case Studies Considered in Ref. 2**

<table>
<thead>
<tr>
<th>Case</th>
<th>Fracture Permeability md.</th>
<th>Fracture Half-Length feet</th>
<th>Fracture Width inches</th>
<th>Results of Reference 2</th>
<th>Results of This Study</th>
<th>Dimensionless Layer Conductivity $R_{coJ}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(C)</td>
<td>992100 (both layers)</td>
<td>100;40</td>
<td>0.038;0.016</td>
<td>3141.65</td>
<td>2135.4</td>
<td>70</td>
</tr>
<tr>
<td>2(D)</td>
<td>992100 (both layers)</td>
<td>100;40</td>
<td>0.038;0.016</td>
<td>3141.65</td>
<td>2166.4</td>
<td>74.06</td>
</tr>
<tr>
<td>3(F)</td>
<td>1654 (both layers)</td>
<td>100;40</td>
<td>0.038;0.016</td>
<td>3.86</td>
<td>3.56</td>
<td>70</td>
</tr>
<tr>
<td>4(G)</td>
<td>992100 (both layers)</td>
<td>624;40</td>
<td>0.238;0.016</td>
<td>10747.75</td>
<td>7800.76</td>
<td>332</td>
</tr>
<tr>
<td>5(1)</td>
<td>10,000;1,000 (both layers)</td>
<td>100;100</td>
<td>0.038;0.038</td>
<td></td>
<td>13.725</td>
<td>100</td>
</tr>
<tr>
<td>6(2)</td>
<td>10,000;100</td>
<td>100;624</td>
<td>0.038;0.238</td>
<td>No matches were possible</td>
<td>12.375</td>
<td>362</td>
</tr>
<tr>
<td>7(3)</td>
<td>1654 (both layers)</td>
<td>100;40</td>
<td>0.038;0.016</td>
<td></td>
<td>3.612</td>
<td>74.06</td>
</tr>
</tbody>
</table>

**Other Reservoir Properties:**
- Initial pressure, $p_i = 3514.7$ psi
- Reservoir temperature, $T_r = 180^\circ F$
- Layer porosity, $\phi_1=\phi_2 = 0.05$, Layer thickness, $h_1 = h_2 = 10$ ft
- Gas gravity, $G = 0.650$, Compressibility at initial pressure = 0.0002331 psi$^{-1}$
- Layer permeability: $k_1 = 0.1$ md. in all cases; $k_2 = 0.5$ for cases 2 and 7; all others $k_2 = 0.1$ md.

Results in Fig. 4 are correlated using the liquid flow analog of Al Hussaibi et al. $R_{coJ}$'s are computed in a similar manner.
Fig. 4.8 - Correlation of the Bixel-Christiansen Solutions
is a direct method to obtain equivalent single-layer parameters, empiricism appears to be unnecessary, or at the very least the number of simulations may be minimized. Second, the observation that the equivalent fracture half-length is the average of the fracture half-lengths in each layer applies only to the specific cases considered in Ref. 2. It is not a general result and it should not be extended to other cases. Third, the restriction that 
\[
\frac{k_{f1} b_1}{k_1 x_{f1}} = \frac{k_{f2} b_2}{k_2 x_{f2}}
\]
in order to use homogeneous models to simulate two-layer systems, appears to be unnecessary.

The most important conclusion of the results we have obtained thus far is that it is possible to obtain estimates of equivalent fracture half-length and equivalent fracture conductivity by using the single-layer solutions. The estimate of the fracture half-length, (regardless of the approach used to obtain the result) represents the arithmetic average of the product of fracture half-length in each layer and dimensionless reservoir conductivity of that layer. The estimate of fracture conductivity represents the expression given by the right-hand side of Eq. 4.5. The above results also establish the fact that idealization based on the bilinear, linear and pseudo-radial flow regimes are also valid for the problem considered in this paper.

4.5 Discussion - Criteria for Maximum Productivity

We conducted over 1000 individual numerical experiments to ensure that the results given in Figs. 4.6-4.8 are generally valid. In these experiments, we computed values of \( p_{wD} \) (or \( l/q_D \)) vs. \( t_{DF} \) for fixed values of \( \sigma_{FD} \) and compared responses for a wide range of reservoir and fracture properties. We found that we can correlate well
responses in terms of the dimensionless variables \( p_{wd} \) (or \( 1/q_d \)), \( t_{dx_f} \) and \( \sigma_{fd} \). However, we found that for a given set of reservoir conditions, there is a range of fracture properties that will result in maximum productivity. This information is presented in Fig. 4.9. Our computations (see also Appendix A) suggests that maximum productivity will be achieved if

\[
\frac{\tilde{\sigma}_{fD2}}{\tilde{\sigma}_{fD1}} \frac{h_2}{h_1} = \frac{R_{cD2}}{R_{cD1}} \frac{L_x_{f2}}{L_x_{f1}} = (R_{cD})_r \frac{L_D}{L_{cD}} . \tag{4.7}
\]

Here \( \tilde{\sigma}_{fDj} \) \((j = 1,2)\) is defined by the following relation:

\[
\tilde{\sigma}_{fDj} = \frac{k_{fj} b_{kj}}{k L_x_{fj}} , \tag{4.8}
\]

\((j = 1,2)\) \( R_{cDr} = R_{cD2}/R_{cD1} \), \( L_D = L_{x_f2}/L_{x_f1} \) and \( h_D = h_2/h_1 \). Note that this definition is based on the fracture properties of each layer and on the average reservoir permeability, \( \bar{k} \). Eq. 4.7 is shown as a dashed line on Fig. 4.9. In the following, we shall first discuss the utility of the information given in Fig. 4.9. Later, we consider the characteristics of the curves in Fig. 4.9.

Fundamentally, the information in this figure should enable the engineer to determine conditions to ensure maximum productivity. The information given here serves the same function as the results given in Fig. 11 of Ref. 14 for single-layer systems. If values of \( \tilde{\sigma}_{fDj}, h_j, \)

\( R_{cDj} \) and \( L_{x_fj} \) are such that Eq. 4.7, (dashed line on the figure) is satisfied, then this combination of reservoir and fracture properties will result in maximum productivity—the optimum condition. The shaded
Fig. 4.9 - Criteria for Maximum Productivity
area demarcated by the curves A and B depict responses, which would be within five percent of the optimum condition. We consider an example to demonstrate this point. Let us consider a system with reservoir properties given by $R_{cD2}/R_{cD1} = 0.05$, $k = 0.333$ and $h_2/h_1 = 4$. Suppose that we now wish $L_{x_{f2}}/L_{x_{f1}} = 4$. For this case, Fig. 4.9 indicates that the productivity loss will be within five percent if $0.007 \leq \tilde{\sigma}_{fD2}/\tilde{\sigma}_{fD1} \leq 0.225$. The optimum condition suggests that if we require $\sigma_{fD}$ to be 500, then $\tilde{\sigma}_{fD1}$ and $\tilde{\sigma}_{fD2}$ should be 2083.3 and 104.2, respectively. On the other hand if $\sigma_{fD} = 50$, then $\tilde{\sigma}_{fD1}$ and $\tilde{\sigma}_{fD2}$ should be 208.3 and 10.4, respectively.

In essence, the above results suggest that if we wish to obtain a specific overall effective conductivity, for fixed values of fracture length and for fixed values of reservoir properties, then there is a specific value for the fracture conductivity ratio that will result in maximum productivity. Equation 4.7 provides information to obtain the ratio, and Fig. 4.9 provides information regarding conductivity ratios which will enable us to have productivities within five percent of this ideal value. We believe that the information given here is an important contribution to this work.

We now consider the characteristics of the curves in Fig. 4.9. Curves A and B were obtained by examining responses for a wide range of conditions, and noting the range of parameters for which the well response would be within five percent of the optimum case. It should be noted that if we consider point (1,1) as a reference point, then an odd-type symmetry exists between curves A and B. For example, the point (10, 1.6) is on curve A and the point (1/10, 1/1.6) is on the curve B. That is, the top branch may be obtained from the bottom
branch by reading off corresponding coordinates on the bottom branch and taking their reciprocals. Appendix A presents a rationale for the results given in Fig. 4.9 on the basis of the analytical solutions derived in Chapter III. If reservoir and fracture properties are such that \( R_{CD} L_D \) or \( R_{CD} L_D > 10^2 \), then for all practical purposes only one layer contributes to the productivity of the well.

In a practical sense, the restrictions suggested above merely state that if fracture conductivities and fracture lengths (for a fixed value of overall conductivity and equivalent length) are such that tip effects become evident at the wellbore at the same time, then we will obtain maximum productivity (see Appendix A). Note that Figure 4.9 makes no mention of the appropriate values of \( k_f b \) and \( L_x f \) for a given set of reservoir conditions, it applies equally well to high and low dimensionless conductivities. The optimum values of \( k_f b \) and \( L_x f \) would have to be obtained from Fig. 11 of Ref. 14 or a similar source. This point may become evident if we note that the abscissa in Fig. 4.9 reflects the ratio of the reservoir layer rates and the ordinate reflects the ratio of sandface rates. That is, the ratio \( (R_{CD2} L_x f_2)/(R_{CD1} L_x f_1) \) reflects the ability of the reservoir layers to deliver fluids to the corresponding fracture layers and the ratio \( (\bar{\sigma}_{FD2} h_2)/(\bar{\sigma}_{FD1} h_1) \) reflects the ability of the fracture layers to deliver fluid to the sandface.

Figure 4.10 presents well responses which serve to amplify the above remarks. Here values of \( R_{CDj} L_x f_j \), \( h_j \) and \( k_f b \) are fixed and only the ratio \( (k_f b_1)/(k_f b_2) \) is varied. We have considered two values of \( \sigma_{FD} \) (500 and 5). The solid lines shown here are the well responses for the maximum productivity condition. (They are also the single layer solution for the appropriate value of \( \sigma_{FD} \).) The responses shown by the
Fig. 4.10 - Well Responses Which Do Not Satisfy Criteria in Fig. 4.9
circular and square data points assume that the \((\tilde{c}_{fD}, h)\) ratios are 0.016 and 256, respectively. The deviations that result in Fig. 4.10 are a result of the fact that reservoir and fracture properties are such that we are outside the shaded area of Fig. 4.9. The productivity losses for the cases shown here are well over thirty percent. In fact, the deviations shown for the top curve in Fig. 4.8 are a direct consequence of the fact that the conditions specified in Fig. 4.9 are violated. From Fig. 4.10 we note that if \(\sigma_{fD}\) is large, then the productivity loss is significant at early times, and if \(\sigma_{fD}\) is small, then the productivity loss will not be evident until much later times. Systems where \(\sigma_{fD}\) is small are usually low permeability reservoirs with very long fractures. Finally, it should be noted that the results of Fig. 4.10 imply that if values of \(R_{cDj}, h_{Dj}, \tilde{c}_{fDj}\), and \(L_{xfj}\) are such that their ratios suggest a combination of reservoir and fracture properties outside the shaded area of Fig. 4.9, then analyses based on bilinear and/or linear flow approximations may not be applicable to commingled reservoirs produced via non-communicating fractures.

4.6 Influence of Communication; Comparison with the Bennett Solutions

Bennett assumed that the fracture lengths in each layer are equal and that the fractures are in communication at all points. He showed that the well response for this system is governed by three parameters: Dimensionless reservoir conductivity, \(R_{cD}\), dimensionless fracture height, \(H_{fD}\), and the dimensionless conductivity, \(\sigma_{fD,B}\). He defined \(H_{fD}\) and \(\sigma_{fD,B}\) by the following relations:

\[
H_{fD} = \frac{h}{L_{xf}}, \quad \sigma_{fD,B}.
\]  \hspace{1cm} (4.9)
and

\[
\sigma_{fD,B} = \frac{k_f b}{k L x_f} .
\]  

(4.10)

where \( k_f b \) is the thickness averaged fracture permeability width, which is defined by the relation:

\[
\bar{k}_f b = \sum_{j=1}^{n} k_{fj} b_j h_{Dj}.
\]

Here we use the subscript \( B \) in order to distinguish between our definition and Bennett's definition. Equation 4.10 assumes that the fracture is a homogeneous porous medium. He showed that if \( H_{fD} \) was small, then the commingled reservoir solutions may be correlated with the single-layer solutions, if the commingled reservoir solutions are graphed in terms of \( t_{DX_{f,B}}^{2} / R_{cD}^{2} \), where \( t_{DX_{f,B}}^{2} \) is given by:

\[
t_{DX_{f,B}}^{2} = \frac{0.0002637 \bar{K}}{\phi c_t \mu L^2 x_f} .
\]

(4.11)

If we now compare the definition of \( t_{DX_{f}}^{2} \) (Eq. 4.4) and the definition of \( t_{DX_{f,B}}^{2} \), and restrict our attention to equal fracture length conditions, then we find that \( t_{DX_{f}} = t_{DX_{f,B}}^{2} / R_{cD}^{2} \). That is, as far as the time scale is concerned, the correlating procedure suggested in this study is identical to the correlating procedure suggested by Bennett. (Note that our discussion is now restricted to the equal fracture length cases.) There is, however, a difference as far as the effective fracture conductivity is concerned. Consider the definition of \( \sigma_{fD} \) given by the right-hand side of Eq. 4.3. If we use the Cauchy-Buniakowsky-Schwartz inequality for sums (page 15 of Ref. 16), then we can show that

\[
\sum_{j=1}^{n} \left( k_{fj} b_j h_{Dj} R_{cDj} \right)^{1/2} \leq \left( \sum_{j=1}^{n} k_{fj} b_j h_{Dj} \right)^{1/2} \left( \sum_{j=1}^{n} R_{cDj} \right)^{1/2} .
\]
Noting that $\Sigma R_{cDj} = R_{cD'}$, we have:

$$\sum_{j=1}^{n} (k_{fj} b_j h_{Dj} R_{cDj})^{1/2} \leq \left( \sum_{j=1}^{n} k_{fj} b_j h_{Dj} \right)^{1/2} R_{cD}^{1/2}.$$

Using the above inequality, Eq. 4.3 gives the following result:

$$\sigma_{FD} \leq \left( \frac{\left( \sum_{j=1}^{n} k_{fj} b_j h_{Dj} \right) R_{cD}}{k \Sigma_{j=1}^{n} R_{cDj} L x_{fj}} \right)^{1/2}.$$

If we now consider equal fracture half-lengths, we have

$$\sigma_{FD} \leq \frac{k_{f} b}{k L x_{f}} ;$$

that is,

$$\sigma_{FD} \leq \sigma_{FD,B} .$$

This result implies that all other things being equal, communication between the fractures (i.e., vertical flow within fractures) will result in a higher effective conductivity. Here, we have, of course neglected the influence of $H_{FD}$. This result is important since it implies that communication in the fracture enhances productivity.

We have conducted several simulation runs and examined the influence of communication between fractures. We found that the effective
dimensionless conductivity of the system, $\sigma_{FD,c}$, is given by

$$
\sigma_{FD,c} = \frac{\sum_{j=1}^{n} \left( \sum_{j} k_{fj} b_j h_{Dj} \right) R_{CD}}{\sum_{j=1}^{n} \frac{k}{\sum_{j} R_{CDj} L_{x_{fj}}}}.
$$

(4.12)

Here we use the subscript $c$ to denote that we are considering situations where there is communication. The definition of $t_{fx}^D$ is identical to Eq. 4.4. Figure 4.11 demonstrates the observations in the preceding paragraphs. The results reflected by the circles, triangles and squares represent three distinct conditions. The shaded symbols represent responses when there is no communication between the fractures. The unshaded symbols represent responses when there is communication between the fractures. The only difference between the responses given by the open and closed circles is that in one case the fracture layers are in communication and in the other case the layers are not in communication—the reservoir and fracture properties are identical. The same is true for the responses represented by the squares and triangles. Note that in all cases the two-layer solutions follow the appropriate $\sigma_{FD}$ single-layer solution—Eq. 4.3 for the "no communication case" and Eq. 4.12 for the "communication case". However, for identical reservoir and fracture properties and identical times, the $p_{WD}$ values for cases where the fractures are in communication are smaller than the corresponding $p_{WD}$ values for the no communication case. That is, the effective conductivity $k_{f}^{b}$ of the system is higher if the fractures are in communication.

Figure 4.12 presents information identical to that of Fig. 4.11. Here we graph results along the lines suggested by Rosato et al. The principal objective of presenting results in this manner is that the
Fig. 4.11 - Well Responses With and Without Communication Between Fractures
Fig. 4.12 - Comparison of Responses With and Without Communication Between Fractures
influence of communication for high $\sigma_{fD}$ values is better evident when results are graphed in this manner (care should be taken when results are interpreted on this graph since the abscissa involves $\sigma_{fD}$).

The solutions shown in Figs. 4.11 and 4.12 establish the following points. First, if there is communication between the fractures, then the dimensionless fracture conductivity is given by $\sigma_{fD,c}$ (the right-hand side of Eq. 4.12). Second, all other things being equal, communication between fractures enhances productivity since the effective conductivity is higher. This result is a direct consequence of interlayer communication--one of the fractures feeds the other, instead of being a "choke" in the system.

It should be noted that the above discussion regarding communication is restricted to cases where the additional pressure drops created due to crossflow are not significant. Bennett, et al. have shown that if gradients in the vertical direction are large, then this effect will manifest itself in terms of $h/L_{xf}$ for the equal fracture length case. It is beyond the scope of this work to examine this aspect.

4.7 Influence of Boundary Effects

The principal objective of this work is to examine the response of fractured wells in non-communicating reservoirs during the transient flow period in order to analyze short time transient data and predict deliverabilities during the transient period. In low permeability reservoirs transient effects may last for several tens of years.

Yeh et al. have examined the long-time response of fractured wells in layered reservoirs. Their work assumed that the fracture lengths were equal. They showed that in general one cannot model the response of commingled reservoirs by an equivalent single-layer system.
The results given there may be extended to the system considered here. Numerical computations suggest that the pressure response during the pseudo steady state flow period is given by:

$$P_{WD} = 2^n t_{DA} + \tau$$  \hspace{1cm} (4.13)

where

$$\tau = \frac{\bar{k}h}{2(\sum_{j=1}^{n} \phi_j c_t j h_j)^2} \frac{n}{\sum_{j=1}^{n} (\phi_j c_t j h_j)^2/(k_{h_j})} \ln \left[ \frac{4(L_{x_f j} / r'_w)^2 A}{e^\gamma \left( C_A L_{x_f j} \right)^2} \right]$$  \hspace{1cm} (4.14)

and $t_{DA}$ is dimensionless time based on the drainage area, $A$; that is

$$t_{DA} = \frac{0.0002637 \bar{k}t}{\phi c_t u A}$$  \hspace{1cm} (4.15)

In Eq. 4.14, $C_A$ is the shape factor, \(^{19}\) (for a well at the center of a square drainage region the value is 30.88), $r'_w$ is the effective wellbore radius, $\gamma$ is Euler's constant and equals 0.5772. Values of $L_{x_f j} / r'_w$ as a function of $\sigma_{fD}$ are given in Ref. 1 and may also be derived from the tables given in Ref. 20.

If the well produces at a constant pressure, then the well response during the time dominated by boundary effects is given by

$$q_p = \exp \left( -2\pi t_{DA}/\tau \right) / \tau$$  \hspace{1cm} (4.16)

If Eq. 4.16 is used to predict the long-time response of gas wells, then the modification suggested by Carter \(^{21}\) should be incorporated.
The results given above imply that it will not be possible to predict well performance during the time influenced by boundary effects using the values of $\frac{L_{xf}}{x_f}$ and $k_i b$. This point is demonstrated in Refs. 2 and 15 and in Fig. 4.13. The solid line in Fig. 4.13 is the two-layer response and the dashed line is the response predicted using equivalent values of fracture half-length and fracture conductivity. It is clear that the equivalent single-layer predictions will be pessimistic. (Ref. 15 presents information on the conditions for which the equivalent single-layer model may be used to predict the deliverability of wells producing two-layer reservoirs.)

From the above discussion it is clear the information on fracture half-lengths in each layer will be needed to predict long-time deliverability. A study yet to be reported presents several methods to determine fracture half-lengths in each layer.\(^{21}\)

If the ratio of the layer rates at the instant of shut in is known, then it is possible to determine individual fracture properties provided that layer properties are known. Figure 4.14 presents an example to demonstrate this point. The solid line corresponding to $\sigma_{fD} = 12.57$ is the two-layer drawdown solution. The circles represent the dimensionless buildup response ($\tilde{p}_{DS}$ vs. $\Delta t_D$) for the two layer system. As expected, the buildup response follows the drawdown response. The buildup responses follow the drawdown solution for all times shown here since $t \gg \Delta t$.\(^{23}\) Here $\tilde{p}_{DS}$ and $\Delta t_{DX_f}$ are defined by the following equations:

$$\tilde{p}_{DS} = \frac{-kh}{141.2qBu} (p_{ws} - p_{wf,s})$$

(4.17)
Fig. 4.13 - Comparison of Two-Layer Response with Corresponding Equivalent Single-Layer Response at Late Times
Fig. 4.14 - Pressure Buildup Responses for a Two-Layer System
and

\[ \Delta t_{DX_f} = \frac{0.0002637 \ k \ \Delta t}{\phi c_t \mu L x_f} \]  

(4.18)

where \( p_{ws} \) is the shut-in pressure, \( p_{wf,s} \) is the pressure at the instant the well is shut in, and \( \Delta t \) is the shut-in time. The solid lines corresponding to \( c_{FD} = 5 \) and \( a_{FD} = 100 \) are the drawdown responses that would result if each layer of the reservoir is produced individually; that is, they are the responses of single-layer systems. The square data points are the buildup responses, when the buildup data are graphed in terms of layer properties and the layer rates at the instant the well is shut in. That is, the dimensionless variables used to graph the buildup response are based on individual layer properties. The symbols \( \tilde{p}_{DSj} \) and \( \Delta t_{DX_{fj}} \) are given by:

\[ p_{DSj} = \frac{k_j h_j (p_{ws} - p_{wfj})}{141.2 q_j (\Delta t=0) B \mu} \]  

(4.19)

and

\[ \Delta t_{DX_{fj}} = \frac{0.0002637 \ k \ \Delta t}{\phi c_t j \mu L x_{fj}} \]  

(4.20)

Justification for using \( q_j (\Delta t=0) \) in Eq. 4.19 is based on the results in Ref. 23. Note that the buildup responses shown by the square data points assume that the layers are in communication throughout the buildup period.
Based on the results presented above and those given in Refs. 6 and 17, detailed documentation of buildup responses for wells producing appears to be unnecessary. Buildup responses in fractured wells can be analyzed along lines identical to those used for single-layer systems.

4.8 Example Application

Here we examine one of the examples given in Ref. 20 and extend the analysis given there. We consider the test results of Hamaweide Z2. An analysis similar to that given below may also be conducted for the other two cases considered in Ref. 24. The two zones of Hamaweide Z2 had been fractured individually and the layers had then been commingled. The authors of Ref. 24 were able to match the buildup response with the Agarwal et al. type curves. Results of the type curve match indicated that $\bar{L}_{xf} = 788$ feet and $k_b f_f = 43.46$ md-ft ($\sigma_{FD} = 3.75$). (The latter result is different from the result of Ref. 24, the authors use $\sigma_{FD} = 5.67$ which does not agree with Fig. 8. This difference is not germane to this discussion.) We assume that $\sigma_{FD} = 3.75$. If we compute $R_{CDj}$ for each layer we find that $R_{CD1} = 0.51$ and $R_{CD2} = 0.49$. For simplicity, let us assume that $R_{CD1} = R_{CD2} = 0.5$. Thus, the above results suggest that $L_{xf1} + L_{xf2} = 1576$ feet, $\sqrt{k_{f1} b_1 h_1} + \sqrt{k_{f2} b_2 h_2} = 113.7 \sqrt{\text{md-ft}^2}$. Since these authors used the single-layer solutions, we may also assume that $(k_{f1} b_1 h_1)/(k_{f2} b_2 h_2) = L_{xf1}^2 /L_{xf2}^2$. Thus, the analysis of the buildup tests suggests that we should be able to model the performance of this well as an equivalent single-layer system if we use $\bar{k}h = 2.186$ md-ft, $\bar{\phi}h = 9.2766$ feet, $k_b f = 43.46$ md-ft and $L_{xf} = 788$ feet. We may use these values to predict deliverability throughout the transient flow period. Based on the information given in Ref. 24 the transient period
should last for nine years. If we wish to model the performance of this well as a two-layer system, then we should model the response under the following restrictions: 

\[
\sqrt{k_{f2}} \ b_{2} \ h_{2} = 113.7 \ \sqrt{md-ft^2} \text{ and } \frac{(k_{f1} \ b_{1}) / (k_{f2} \ b_{2})}{0.89 \ (L_{x1} / L_{x2})^2}.
\]

Unfortunately, this information does not provide individual layer values.

However, information of this type is extremely useful if we wish to use a two-layer model to "history-match" buildup or flow data. In their simulations the authors of Ref. 24 used 

\[
L_{x_{f1}} + L_{x_{f2}} = 1099.08 \text{ feet}.
\]
CHAPTER V
CONCLUSIONS

The finite difference model developed by Bennett was used to examine the well response of fractured wells in layered reservoirs. Both constant pressure production and constant rate conditions were examined. The influence of variations in reservoir properties, fracture properties, fracture length and communication between fractures for a wide range of conditions was examined. Generalized rules were developed to correlate the response of fractured wells with available single-layer solutions. Rules to simulate commingled reservoir behavior by an equivalent single-layer model have been developed. This phase of our study generalizes the work of Bixel and Christiansen\(^2\) and also the results of Bennett\(^1\). Criteria to ensure maximum productivity have been developed.

Although all results presented in this study were based on numerical solutions, analytical solutions were also derived. These analytical solutions, although approximate, served several important functions. First, they identified important groups that influence the well response, suggested methods to combine solutions, and provide information on the structure of the solutions. Second, they suggested procedures to correlate the commingled reservoir responses with the single-layer solutions. Third, they enabled us to obtain a measure of the accuracy of the numerical solutions. Fourth, they were useful in developing the criteria for maximum productivity.

This study suggests several general conclusions that should be
useful for practical field applications:

1. The dimensionless layer conductivity, \( R_{CDj} \), characterizes the performance of fractured wells producing commingled reservoirs.

2. Definitions of dimensionless pressure drop (or dimensionless rate) and dimensionless time based on thickness averaged values of the layer permeability, and the porosity-compressibility product of each layer may be used to describe the well response.

3. It is possible to correlate the response of fractured wells producing commingled reservoirs with the corresponding single-layer solutions in terms of an equivalent fracture half-length and equivalent fracture conductivity. The appropriate value for the equivalent fracture half-length should be the sum of the product of the dimensionless layer conductivity and fracture half-length. The appropriate value for the equivalent fracture conductivity is a function of fracture conductivity, layer thickness and dimensionless layer conductivity product. The appropriate expression is given by the right-hand side of Eq. 4.5. If buildup or drawdown data are analyzed it is possible to estimate the equivalent fracture half-length and the equivalent fracture conductivity. If we wish to model a multi-layer system with an equivalent single-layer system during the transient period, then the above values of equivalent fracture half-length and equivalent fracture conductivity should be used.

4. For a given set of reservoir properties, there is a specific combination of fracture properties that will result in maximum productivity. Information to ensure that maximum productivity will be obtained is given in Fig. 4.9.

5. For a given set of reservoir and fracture properties,
communication between the fractures should improve the effective conductivity of the fracture system.
NOMENCLATURE

Symbols

A = Drainage area, ft²
B = Formation volume factor, RB/STB
b_j = Fracture width of layer j, feet
C_A = Geometric shape factor
C_{fDJ} = Dimensionless fracture storage of layer j, (Eq. 2.10)
c_{tj} = Total system compressibility of layer j, psi⁻¹
h = Formation height, feet
h_j = Height of layer j, feet
h_{fj} = Fracture height in layer j, (h_{fj} = h_j), feet
h_{DJ} = Dimensionless layer thickness based on total thickness 
(\frac{h_j}{h})
h_{fDJ} = Dimensionless fracture height based on total thickness 
(\frac{h_{fj}}{h})
h_D = Thickness ratio, two-layer reservoir
H_{fD} = Dimensionless fracture height, (see Eq. 4.7)
k_j = Permeability of layer j, md
k_{fj} = Permeability of fracture layer j, md
\bar{k} = Thickness averaged reservoir permeability, md
L_{xfj} = Fracture half-length in layer j, feet
\bar{L}_{xf} = Equivalent fracture half-length,(Eq. 3.4.46), feet
L_D = Fracture half-length ratio, \frac{L_{xf2}}{L_{xf1}}
L_{Dj} = Fracture half-length ratio, \frac{L_{xfj}}{L_{xf1}}
\( L_x \) = Distance to external boundary in the x-direction, feet
\( L_y \) = Distance to external boundary in the y-direction, feet
\( n \) = Number of layers
\( p_j \) = Pressure of layer j, psi
\( p_{fj} \) = Pressure of fracture layer j, psi
\( p_{w_f} \) = Wellbore flowing pressure, psi
\( p_{w_s} \) = Wellbore shut-in pressure, psi
\( p_{w_f, s} \) = Wellbore pressure at instant of shut-in, psi
\( p_{Dj} \) = Dimensionless pressure of reservoir layer j
\( p_{fDj} \) = Dimensionless pressure of fracture layer j
\( p_{wD} \) = Dimensionless wellbore pressure drop
\( \tilde{p}_{Ds} \) = Dimensionless pressure rise
\( \tilde{p}_{Dsj} \) = Dimensionless pressure rise based on properties of layer j
\( q \) = Total surface rate, STB/D
\( q_{Rj} \) = Flow rate from reservoir layer j into the fracture, STB/D
\( q_{wj} \) = Flow rate from fracture layer j into the wellbore, STB/D
\( q_D \) = Dimensionless flow rate when well is produced at constant pressure
\( q_{RDj} \) = Fractional flow rate from reservoir layer j into the fracture \( \left( \frac{q_{Rj}}{q} \right) \)
\( q_{Dj} \) = Fractional flow rate from fracture layer j into the wellbore \( \left( \frac{q_{wj}}{q} \right) \)
\( r'_w \) = Effective wellbore radius, (Eq. 4.14), feet
\( R_{cD} \) = Dimensionless reservoir conductivity, (Eq. 2.5)
\( R_{cDj} \) = Dimensionless reservoir layer j conductivity, (Eq. 2.4)
\( R_{cDr} \) = Ratio of dimensionless layer conductivity, \( \frac{R_{cD2}}{R_{cD1}} \)
\( s \) = Laplace Transform variable
\( t \) = Time, hours

\( t_{\text{Dx}_{f1}} \) = Dimensionless time based on \( L_{x_{f1}} \), (Eq. 2.6)

\( t_{\text{Dx}_{f}} \) = Dimensionless time based on \( L_{x_{f}} \), (Eq. 3.4.42)

\( t_{\text{Dx}_{f},B} \) = Dimensionless time used by Bennett (Equal Fracture Length Case)

\( t_{\text{DA}} \) = Dimensionless time based on drainage area, (Eq. 4.15)

\( \Delta t \) = Shut-in time, hours

\( \Delta t_{\text{Dx}_{f}} \) = Dimensionless shut-in time, (Eq. 4.18)

\( a_{j} \) = Dimensionless coefficient for layer \( j \), (Eq. 3.4.4)

\( \beta \) = Dimensionless group, (Eq. 4.1)

\( \phi_{j} \) = Porosity of reservoir layer \( j \), fraction

\( \phi_{fj} \) = Porosity of fracture layer \( j \), fraction

\( \bar{\phi} \) = Thickness averaged reservoir porosity, fraction

\( \mu \) = Viscosity, cp

\( \eta \) = Reservoir diffusivity, \( \text{md-psi}/\text{cp} \)

\( \eta_{f} \) = Fracture diffusivity, \( \text{md-psi}/\text{cp} \)

\( \bar{\eta} \) = Reservoir diffusivity based on thickness averaged reservoir properties

\( \bar{\eta}_{f} \) = Fracture diffusivity based on thickness averaged reservoir properties

\( \eta_{FD} \) = Dimensionless fracture diffusivity

\( \gamma \) = Dimensionless group, (Eq. 4.2); Euler's constant, 0.5772...

\( \sigma_{fdj} \) = Dimensionless fracture conductivity of layer \( j \) based on \( L_{x_{f1}} \), (Eq. 2.8)

\( \tilde{\sigma}_{fdj} \) = Dimensionless fracture conductivity of layer \( j \) based on \( L_{x_{fj}} \), (Eq. 4.8)

\( \sigma_{FD} \) = Dimensionless fracture conductivity of a multilayer commingled reservoir, (Eq. 3.4.44)
\( \sigma_{fD,B} \) = Dimensionless fracture conductivity used by Bennett\(^1\) (equal half fracture length)

\( \sigma_{fD,c} \) = Dimensionless fracture conductivity for a multilayer reservoir with communication between fractures, (Eq. 4.12)

**Subscripts**

- \( D \) = Dimensionless
- \( f \) = Fracture
- \( i \) = Initial
- \( j \) = Index number of layer in multilayer reservoir
- \( w \) = Well

**Superscripts**

- \( - \) = Laplace Transform of appropriate variable; average value
REFERENCES


21. Thompson, L. G., Reynolds, A. C. and Raghavan, R.: "Use of Variable Rate Well Test Pressure Data Using Duhamel's Principle", to be submitted to the SPE.


APPENDIX A

BASIS FOR RESULTS PRESENTED IN FIGURE 4.9

The basis for the correlation developed in Fig. 4.9 is given below. Although the correlation given below is based on the analytical solutions derived in Chapter III, the results given in Fig. 4.9 are based on the numerical solutions obtained in this study. The solution presented here is based on an asymptotic approximation to the analytical solution derived in Chapter III. If we consider a two-layer system, then this solution should be applicable in the time interval after one fracture tip influences the well response.

The starting point of our analysis is the expression for the wellbore pressure drop given in Chapter III. This expression is:

\[
\overline{p}_{wD} = \frac{n}{s} \left[ \sum_{j=1}^{n} h_{FDj} \frac{\sigma_{FDj}}{\alpha_{j} \tan h(\alpha_{j} L_{Dj})} \right]^{-1}, \quad (A-1)
\]

where \(\alpha_{j}\) is given by:

\[
\alpha_{j}^2 = \frac{C_{FDj} h_{FDj} s + 2 R_{CDj} \sqrt{s}}{h_{FDj} \sigma_{FDj}}, \quad (A-2)
\]

for \(j=1,2,\ldots,n\).

We will now assume that we will consider time ranges large enough so that we may represent \(\tan h(\alpha_{j} L_{Dj})\) by the following relation:

\[
\tan h(\alpha_{j} L_{Dj}) = \alpha_{j} L_{Dj} - \left(\alpha_{j} L_{Dj}\right)^3/3 \quad . \quad (A-3)
\]
Thus, we may write the following expression:

\[
\sigma_{FDj} h_{FDj} \alpha_j \tanh(\alpha_j L_{Dj}) = L_{Dj} \left( s C_{FDj} h_{FDj} + 2 \sqrt{s} R_{CDj} \right)
- L_{Dj}^3 \left( s C_{FDj} h_{FDj} + 2 \sqrt{s} R_{CDj} \right)^2 /
(3 \sigma_{FDj} h_{FDj}) .
\]  

(A-4)

Simplifying further, we may write:

\[
\sum_{j=1}^{n} \alpha_j \sigma_{FDj} h_{FDj} \tanh(\alpha_j L_{Dj}) = 2 \sqrt{s} \left\{ \sum_{j=1}^{n} R_{CDj} L_{Dj} (1 + \omega_j) \right\}
- \left\{ 1 - \sum_{j=1}^{n} \omega_j^2 \delta_j - 2 \sum_{j=1}^{n} \delta_j \omega_j - \sum_{j=1}^{n} \delta_j \right\}
= 2 \sqrt{s} \left\{ \sum_{j=1}^{n} R_{CDj} L_{Dj} (1 + \omega_j) \right\} \left\{ 1 - \sum_{j=1}^{n} \delta_j (1 + \omega_j)^2 \right\} ,
\]  

(A-5)

where

\[
\omega_j = \frac{\sqrt{s} C_{FDj} h_{FDj}}{2 R_{CDj}} ,
\]  

(A-6)

and

\[
\delta_j = \frac{2}{3} \frac{\sqrt{s} R_{CDj}^2 L_{Dj}}{n \sum_{\ell=1}^{n} \sigma_{FD\ell} h_{FD\ell} R_{CD\ell} L_{D\ell} (1 + \omega_\ell)} .
\]  

(A-7)
We will now restrict our attention to conditions for which \( \omega_j \ll 1 \), which means long-time approximation when the influence of \( C_{fDj} \) is negligible.

Thus, we have

\[
\sum_{j=1}^{n} \sigma_{fDj} h_{fDj} \alpha_j \tanh(\alpha_j L_{Dj}) = 2 \sqrt{s} \left( \sum_{j=1}^{n} \sigma_{cDj} L_{Dj} \right) (1 - \sum_{j=1}^{n} \delta_j) \quad (A-8)
\]

Substituting the right-hand side of Eq. A-8 for \( \sum_{j=1}^{n} \sigma_{fDj} h_{fDj} \alpha_j \tanh(\alpha_j L_{Dj}) \) in the right-hand side of Eq. A-1, we obtain:

\[
\frac{-P_{WD}}{2 s^{3/2}} \left( \sum_{j=1}^{n} \sigma_{cDj} L_{Dj} \right)^{-1} \left\{ 1 - \frac{2 \sqrt{s}}{3} \sum_{j=1}^{n} \frac{R_{cDj}^2 L_{Dj}^3}{(\sigma_{fDj} h_{fDj})} \right\}^{-1} \quad (A-9)
\]

If we now expand the term in the second bracket on the right-hand side of Eq. A-9, we obtain:

\[
\frac{-P_{WD}}{2 s^{3/2}} \frac{1}{\sum_{j=1}^{n} \sigma_{cDj} L_{Dj}} + \frac{\sigma_{fDj} h_{fDj} \sum_{j=1}^{n} R_{cDj}^2 L_{Dj}^3}{2 \sum_{j=1}^{n} \left( \sigma_{cDj} L_{Dj} \right)^2} \quad (A-10)
\]
Using the Inversion theorem, the following expression for $P_{wD}$:

$$P_{wD} = \sqrt{\pi} \frac{t_{Dx_f} f_1}{n \sum_{j=1}^{n} R_{cDj} L_{Dj}} + \frac{\pi}{3} \frac{n}{\Sigma R_{cDj} L_{Dj}} \left[ \frac{\Sigma R_{cDj} L_{Dj}}{j=1} \right] \left( \frac{\Sigma (\sigma_{fDj} \ h_{fDj} / R_{cDj} L_{Dj})}{j=1} \right)^2 . \quad (A-11)$$

The expression for the wellbore pressure drop in a single-layer system that is equivalent to Eq. A-11 is given by:

$$P_{wD}(t_{Dx_f}) = \sqrt{\pi} \frac{t_{Dx_f}}{f_D} + \frac{\pi}{3} \frac{1}{\sigma_{fD}} . \quad (A-12)$$

Equation A-11 may be rewritten in terms of $t_{Dx_f}$ and $\sigma_{fD}$ for the multi-layer system (Eqs. 4.3 and 4.4). If we express results in terms of $t_{Dx_f}$ and $\sigma_{fD}$, (Eqs. 4.3 and 4.4) we obtain

$$P_{wD} = \sqrt{\pi} \frac{t_{Dx_f}}{f_D} + \frac{\pi}{3} \frac{1}{\sigma_{fD}} \left( \frac{\Sigma (k_{fj} b_j h_{fDj} R_{cDj})^{1/2}}{j=1} \right)^2 \frac{n}{\Sigma R_{cDj} L_{Dj}} \left( \frac{\Sigma R_{cDj} L_{Dj}}{j=1} \right)^3 \left( \frac{\Sigma (k_{fj} b_j h_{fDj})}{j=1} \right) \quad (A-13)$$

We shall now discuss the procedure to obtain the correlation developed in Fig. 4.9. If we subtract Eq. A-12 from Eq. A-13, consider identical values for $t_{Dx_f}$, and denote the difference by $f$, then we
obtain the following expression for $f$:

$$f = \frac{\pi}{3} \frac{\sigma_{FD}}{f_{D}^{2}} \left\{ \left[ 1 + \sum_{j=2}^{n} \frac{(k_{fj}b_{j})}{(h_{j}/h_{1})(k_{j}c_{tj})^{1/4}} \right]^{2} \right. $$

$$\left. \left[ 1 + \sum_{j=2}^{n} \frac{(h_{j}/h_{1})(k_{j}c_{tj})^{1/2}}{(h_{j}^{3/2}L_{Dj}^{2})} \right]^{3} \right\}. \quad \text{(A-14)}$$

Here $(k_{fj}b_{j}) = (k_{fj}b_{j})/(k_{f1}b_{1})$, $(k_{j}c_{tj}) = (k_{j}c_{tj})/(k_{1}c_{t1})$ and $L_{Dj} = L_{x_{fj}}/L_{x_{f1}}$. If we now restrict our attention to two-layer systems and assume that the fracture conductivities are equal, then the expression for $f$ is:

$$f = \frac{\pi}{3} \frac{\sigma_{FD}}{f_{D}^{2}} \left\{ \left[ 1 + (h_{2}/h_{1})(k_{f}c_{t})^{1/4} \right]^{2} \right. $$

$$\left. \left[ 1 + (h_{2}/h_{1})(k_{f}c_{t})^{1/2} \right]^{3} \right\} - 1 \right\}, \quad \text{(A-15)}$$

where $(k_{f}c_{t}) = (k_{f}c_{t})_{2}/(k_{f}c_{t})_{1}$. Note that $f = 0$ if $(k_{f}c_{t})_{1}^{1/4} = 1/L_{D2}$.

For this particular case, there is no difference between the commingled reservoir-cominged fracture solutions and the corresponding single-layer solution. Figure A-1 (obtained from Eq. A-15) presents three-dimensional views of the variation in $\tilde{f}$ as a function of $(k_{f}c_{t})_{r}$ and $h_{2}/h_{1}$ for $L_{D2} = 1.43$, where $\tilde{f} = f_{D}^{2}/\sigma_{FD}$. These solutions assume that $(k_{f}b_{1})_{r} = (k_{f}b_{2})_{r}$. Figures (b), (c) and (d) of Fig. A-1 are views of Fig. (a) from locations 1, 2 and 3, respectively. These figures have been drawn by assuming that the base of each figure represents $\tilde{f} = 0$. 
Fig. 4-1 - Three-Dimensional Views of Function $\tilde{F}$, for $L_2 = 1.43$ (Analytical Results)
Figure A-2 is a map of the intersection of planes reflecting fixed values of \( \tilde{f} \) and Fig. A-1a. From Figures A-1 and A-2, we note that for the conditions considered here, the productivity will decrease considerably if \( (k\phi c_t)_r \) is large and \( (h_2/h_1) \) is small or vice versa. We also noted that if both \( (k\phi c_t)_r \) and \( h_2/h_1 \) are large or both \( (k\phi c_t)_r \) and \( h_2/h_1 \) are small, then \( \tilde{f} = 0 \). This result should be expected since these ranges reflect conditions where the influence of one of the layers is negligible. We constructed graphs similar to Fig. A-2 for other values of \( L_{D2} \). These graphs were used to determine conditions where they may be differences between the two solutions.

We then conducted numerical simulations to verify whether differences in dimensionless pressure are greater than 5 percent. Figure A-3 presents this information for which the differences in pressure drop between the commingled reservoir solutions and the single-layer solutions will be less than 5 percent in the time range \( 10^{-5} \leq t_{Dx_f} \leq 10 \). Results are shown for two values of \( \sigma_{FD} \) and three values of \( L_{D2} \). We found that if we are on or outside the envelopes shown, then the difference in solutions will be less than 5 percent. Note that the results in Fig. A-3 are based on numerical solutions—not the analytical approximations. The results shown here depend on \( \sigma_{FD} \) primarily because the duration of the bilinear flow period depends on \( \sigma_{FD} \). Also note that the results shown here are valid only for the time range \( 10^{-5} \leq t_{Dx_f} \leq 10 \).

The above discussion assumed that \( (k_f b)_1 = (k_f b)_2 \). Results may be generalized to include unequal \( (k_f b) \) values. For example, if we consider a two-layer system, then the wellbore pressure drop, \( p_{WD} \) is
Fig. A-2 - Map of Function $\tilde{\xi}$, for $L_{D2} = 1.43$ (Analytical Results)
Fig. A-3 - Curves Representing 5% Relative Difference Between the Commingled System and the Single-Layer Solutions, for Time Range: \(10^{-5} \leq t_{Dx_f} \leq 10\)
given by:

\[
P_{WD} = \sqrt{\frac{\pi}{3}} \frac{t_{DF}}{x_f} + \frac{\pi}{3} \frac{(1 + \sqrt{\kappa \lambda})^2}{\sigma_{FD} (1 + \lambda)^3} \left[ 1 + \frac{\lambda^2}{\kappa} \right], \tag{A-16}
\]

where

\[
\kappa = \frac{k_{f2} b_{f2} h_2}{k_{f1} b_{f1} h_1} \frac{L_{x_{f1}}}{L_{x_{f2}}} = \frac{\tilde{\sigma}_{FD2}}{\tilde{\sigma}_{FD1}} \frac{h_2}{h_1}, \tag{A-17}
\]

and

\[
\lambda = \frac{R_{CD2}}{R_{CD1}} \frac{L_{x_{f2}}}{L_{x_{f1}}}. \tag{A-18}
\]

Equation A-16 indicates that the well response is a function of two groups, \( \kappa \) and \( \lambda \). Equations A-16 through A-18 were the basis for the correlation developed in Fig. 4.9.

Finally, it should be noted that approximation for \( \tanh(x) \) given by the right-hand side of Eq. A-3 appears to be superior to that given by \( \tanh(x) \approx x \) (c.f. Section 3.4.3). Figure A-4 compares the well responses given by numerical inversion and the two approximations for \( \tanh(x) \) for a single-layer system. The dashed lines are the numerical solutions. The symbols \( t_{DE_{bf}} \) and \( t_{b_{bf}} \) mark the end of bilinear and beginning of linear flow periods, respectively, for the appropriate \( \sigma_{FD} \) cases. It is clear that the approximation given by Eq. A-3 is preferable to the approximation \( \tanh(a_j L_{DJ}) \approx a_j L_{DJ} \).
Fig. A-4 - Graph Comparing Solutions With Two Approximations for \( \tanh \alpha \)

- \( \Delta \) \( \tanh(\alpha) = \alpha - \alpha^3/3 \)
- \( \bullet \) \( \Delta \) \( \tanh(\alpha) = \alpha \)