ANALYSIS AND FORECASTING OF GAS WELL PERFORMANCE
A RIGOROUS APPROACH USING DECLINE CURVE ANALYSIS

A Thesis

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ABSTRACT

Analysis and Forecasting of Gas Well Performance
A Rigorous Approach Using Decline Curve Analysis
(August 1993)

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Decline curve analysis is an important component in the set of tools used in any reservoir management plan. Not only does decline curve analysis provide current information regarding the fluid content and the formation characteristics, but also permits a prediction of future performance. This is done at no cost in lost production.

The application of decline analysis for liquid flow in reservoirs has been thoroughly examined in the petroleum literature but in spite of this, physical constraints regarding the production practices may make it difficult to identify the model to which the performance must be matched. On the other hand, the study of gas well performance was first developed similar to that for liquid flow but ignoring the variation of gas properties with pressure. The variation of gas properties with pressure represents a very significant deviation from liquid flow theory. Our objective is to modify the gas flow relations to yield a liquid equivalent solution. The liquid equivalent variables are called "pseudovariables," and the application of these pseudovariables yields a rigorous equivalent solution.

Presently, the analysis of gas production data is such that we must make overidealized assumptions of a constant bottomhole pressure or a constant rate. These assumptions allow us to match the rate-time data to a theoretical model. The use of pseudovariables and, specifically, the use of pseudotime, requires knowledge of the gas-in-place, \( G \). Without knowledge of the gas-in-place, \( G \), we are left with a tedious and troublesome iterative procedure in order to obtain the accurate pseudotime function.
This thesis introduces new techniques that permit the analysis of single phase fluid flow, either liquid or gas, under actual production conditions of variable pressure drop and/or variable rate.

For liquid flow, actual pressure and a simple material balance time are required. The calculation of this material balance time is straightforward (instantaneous cumulative production divided by instantaneous flowrate). However, for gas flow, pseudopressure and a material balance pseudotime are necessary to accurately account for the actual production conditions as well as the variation in gas properties with pressure. The calculation of the material balance pseudotime requires that gas-in-place be known. Rather than using an iterative procedure to get gas-in-place, we introduce a new technique that allows the explicit computation of this volume.

For both liquid and gas flow, we will show that by using material balance time or pseudotime and pressure or pseudopressure drop normalization, a harmonic type of decline is always obtained regardless of the production scenario, whether constant or variable pressure drop, or constant or variable flow rate occurs.

We also introduce a time function called the "constant pressure analog time function." This time function was presented previously in the literature but its computation requires the knowledge of certain formation properties. The new approach avoids this requirement and allows direct use of the analog time function for "conventional" decline curve analysis. It is also shown that the constant pressure analog time should yield an exponential type of decline for any production conditions.

Finally, new analytical approximations for the constant rate and constant pressure solutions are developed, as well as a new type curve which includes both the liquid and gas flow production cases. These developments represent improvements over previous decline analysis methods and should enhance the ability of the engineer to perform a more accurate analysis.
ACKNOWLEDGMENTS

The first and most important of all thanks goes to you almighty God for having given me the greatest gift of all, faith. Faith to believe that everything is possible in you. I asked and received, knocked and was admitted, sought and found. Glory to you Lord who listened when I asked you to send me your Holy Spirit. Thanks Heavenly Father.

I would like to thank ECOPETROL and especially Dr. Hernan Gutierrez for his help in letting me pursue a Master's degree at Texas A&M University.

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To my loving parents Jesus Angel Palacio Lopez and Mariela Uran de Palacio, sisters Diana Maria Palacio and Olga Marcela Palacio and to my soon to be wife, Gloria Patricia Uribe P. In dedication of your love that has known no boundaries, and your giving without expecting. But most importantly, to your believing in me. May God bless you all.
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CHAPTER I

INTRODUCTION

The analysis of production data using decline curve analysis is essentially a modeling and forecasting technique. Actual production rate and time data are history matched to a theoretical model using either computer programs or specialized graphs known as type curves. The theoretical model is then used to estimate in-place oil or gas volumes as well as formation properties.

Decline curve analysis is one of the oldest reservoir engineering tools used in the petroleum industry and the wide use of decline curve analysis is supported by the fact that this method relies on principles that have a sound theoretical basis. Not only is rate-time analysis founded on fundamental flow principles, but this method is also capable of providing the engineer with the same results as conventional pressure transient analysis. In addition it allows a prediction to be made at no cost in lost production. This provides an advantage over pressure buildup testing, especially in those cases where shutting-in a well is economically unfeasible.

The present work introduces new theoretical developments along with implementation techniques for use in the analysis and interpretation of both liquid and gas production data using decline curve methods. In this work we give particular emphasis to the analysis of gas flow data where rate and pressure drop are not prescribed but allowed to vary in a general fashion.

There have been many articles that consider the analysis of rate-time production data. These references are discussed in Chapters II and III of this thesis in order to give the reader a perspective of the proposed new developments in decline curve analysis. Chapter II gives a brief description of the pertinent techniques and theoretical

This thesis follows the style of the Journal of Petroleum Technology.
developments related to decline curve analysis developed up to present time. Chapter III includes detailed theoretical explanations of the most important decline analysis methods currently available for the case of single phase flow, for either liquid or gas.

Chapters II and III also show that most of the previous references have focused on the single-phase liquid problem, treating the gas case as an approximation of the liquid flow case. Techniques have recently been developed which allow rigorous analysis of the performance of a gas reservoir. The most rigorous techniques that have been developed usually involve the use of iterative schemes to compute gas-in-place and pseudotime, which makes their application somewhat limited, especially when a quick hand-derived analysis is preferred.

The use of pseudovariables is required in order to account for the gas property variations as a function of pressure. Unfortunately, even the methods which use these pseudovariables are not entirely suitable to model and analyze actual field production data, mostly because they are usually limited by idealized constant bottomhole flowing pressure or constant rate assumptions. In addition to these limitations, the implementation of these rigorous methods has always included cumbersome iterative procedures to determine gas-in-place and pseudotime.

The most important characteristics of the problem of analyzing gas production data can be summarized as follows:

1. Analysis and interpretation of gas well production data can not be performed accurately using real time as a decline variable.
2. Current methods assume idealized conditions of constant bottomhole flowing pressure.

This thesis presents solutions to the gas flow problem in Chapter IV. Along with these solutions, we present several techniques for liquid analysis as well as new approximations that facilitate the analysis of reservoir performance data in general. These approximations are new forms of computing the constant rate and the constant pressure solution without the necessity of using Laplace transform solutions or inverse solutions.
Another important result of this work is a new decline type curve that is based on type curves developed for liquid and gas which have been presented previously in the petroleum literature. The new type curve gathers several possibilities for the analysis of either single phase gas or liquid flow. When this type curve is combined with the new theoretical developments for gas, it can be used to interpret data regardless of the production conditions, whether constant or variable bottomhole pressure, or constant or variable rate.

We will show verification of the analysis and interpretation techniques in Chapter V by means of constant rate and pressure cases generated using finite-difference simulation techniques. In Chapter VI we illustrate application of these methods to both simulated and field data cases.

The importance of the tools introduced in this thesis are summarized below:
1. For the problems involving gas flow, we provide a simple, direct, and fast way to compute gas-in-place and pseudotime.
2. These methods allow inexpensive production tests to replace more expensive transient tests.
3. In many cases these methods avoid the necessity of a long pressure build-up especially when dealing with tight, low permeability reservoirs.
4. These methods are not constrained by highly idealized assumptions regarding the production conditions of either liquid or gas.
5. We include practical solutions that although approximate, have proven very accurate when compared to the rigorous and more complicated analytical solutions.
6. Finally, the real gas pseudofunctions rigorously account for the change in gas properties with pressure.

We believe that production testing should become a major component of the reservoir management plan for a given field. The results obtained from using these methods would improve estimates of formation properties and in-place fluids, as well as provide a better representation of the reservoir for engineering field studies. In doing so, all the techniques proposed in this thesis should find a place for application in the industry.
CHAPTER II

LITERATURE REVIEW

In this section we review the techniques used to analyze production data. This review will range from the early developments of the 1900's to the current procedures used in the oil and gas industry. Many of the current methods are computer intensive and somewhat abstract, but we will show that these methods are relatively easy to apply and yield very accurate results and interpretations.

The earliest reported effort to study the production drop over time was that by Arnold and Anderson\(^1\) in 1908. They proposed that the production rates during equal time intervals formed a geometric series and stated that the production drop expressed as a fraction was approximately constant. Arnold and Anderson called this fraction "the decline." These authors were also the first to notice that the rate versus time curve exhibited a straight line on semilog paper.

Between that year and 1944 there was extensive research in this area and although too numerous to cite them all, a work performed by Cutler\(^2\) in 1924 must be highlighted. Cutler noticed that the geometric or exponential type of decline curve gives conservative estimates of volume as well as a conservative production forecast. This author also stated there that a hyperbolic relationship on log-log paper would better describe the production decline.

In 1927 Johnson and Bollens\(^3\) introduced the so called "loss ratio method." This was defined as the ratio between production rates and production drops at equal time intervals, which was seen to remain approximately constant, thus providing an easy method for extrapolation.

In 1944 Arps\(^4\) published a comprehensive review of the previous efforts regarding decline curve analysis. Based on his results, he was able to verify on an empirical basis the equations for exponential and hyperbolic decline behavior. The continuous use of these equations even presently is basically due to the ease of
application and their acceptance in the industry. Arps\(^4\) also showed how to extrapolate rate-time data following an exponential or hyperbolic decline. While the exponential decline is the simplest model to use, it also yields conservative estimates and remains the most popular method within the petroleum industry.

Not much new technology was added until 1968 when Slider\(^5\) demonstrated a curve matching technique to obtain hyperbolic extrapolations. Slider claimed that his method was superior to other decline analysis techniques, despite the large amount of graphing work required in the preparation of the data analysis curves.

A statistical study of Arps\(^4\) and three other empirical equations was performed by Ramsay and Guerrero\(^6\) in 1969. These authors obtained basically the same distribution of values of the decline exponent "\(b\)" as obtained by Arps, but stated that hyperbolic and harmonic declines had a more frequent occurrence than had been previously indicated. Ramsay and Guerrero also suggested that the ideal application of decline curves should include the full range of decline exponents from zero to one, having the exponential and harmonic cases as limits at the extremities.

A new set of type curves was presented by Gentry\(^7\) in 1972. Gentry rearranged Arps\(^4\) original equations in order to obtain two different type curves, one related to rate and the other to cumulative production. In each of these cases, there was a curve for every value of the decline exponent. Since the initial flow rate could be obtained from the production history, the combination of the two graphs allowed the engineer to solve for the other two unknowns, the decline exponent and the initial decline rate. Neither of Gentry's type curves was presented as an overlaying (or matching) technique at the time, although they could have proved useful as such.

In 1973, Fetkovich\(^8\) developed a significant improvement over Slider's\(^5\) original idea of using a curve matching procedure. Fetkovich showed that the solutions for a well in a bounded reservoir, producing at constant bottomhole pressure, and the Arps' solutions could be combined to yield a single type curve. The Fetkovich type curve includes both the transient and boundary dominated flow solutions. Fetkovich defined special dimensionless variables for rate and time as functions of the initial flow rate and the initial decline that allow the analytical and empirical (Arps) solutions to be combined.
The Fetkovich type curve combines analytical solutions for transient flow with Arps\textsuperscript{4} empirical equations (for boundary dominated flow) in terms of the decline exponent "b." Although it is possible to estimate reserves from either an exponential or a hyperbolic decline using the Fetkovich type curve, accurate matches can sometimes become awkward due to data scatter.

A sound mathematical basis for liquid exponential decline analysis was illustrated by Cox\textsuperscript{9} in 1978. Cox clearly showed that an exponential decline in production rate is the analog to pseudosteady state flow when production is at a fixed bottomhole flowing pressure. It was shown that the production rate for a constant well pressure does decline exponentially. The relationship of the constants in the decline equation (the extrapolated initial rate and the decline rate) to reservoir properties was also shown.

Also, in 1978 Gentry\textsuperscript{10} presented a study on the effect of reservoir and fluid properties on the decline exponent "b." Gentry concluded that reservoir heterogeneities can lead to "b" values greater than one. In addition, a new empirical equation was proposed, which was claimed to describe certain decline behaviors better than the equations presented by Arps.\textsuperscript{4}

A comprehensive theoretical study on rate decline analysis for wells produced at constant pressure was published by Economides and Ramey\textsuperscript{11} in 1981. It was shown that a well producing under constant flowing pressure conditions exhibits a transient rate decline which can be analyzed using techniques similar to those used for constant flowrate test data. The authors presented practical well test analysis methods, such as the step pressure testing, as reasonable alternatives to a buildup test. The most important advantage to their method is the short lived storage effect. Economides and Ramey also show that theoretically an exponential decline in rate occurs for a well that has produced long enough against a constant backpressure.

In an effort to determine the decline curve behavior of gas systems, Carter\textsuperscript{12,13} presented two complementary studies dealing with radial and linear flow of gas in finite systems for the case of production at a constant bottomhole pressure. Carter considered the problem of the variation of the viscosity-compressibility product over
the range of average reservoir pressure traversed during depletion. The results of this study were expressed in terms of a correlation parameter called $\lambda$, where $\lambda$ is defined as the quotient of the viscosity-compressibility product taken at initial conditions over the same product evaluated at average reservoir pressure. Carter showed that the average viscosity-compressibility product can be expressed in terms of the difference between initial and flowing pressure and pseudopressures. The concept of a single correlating variable was proven to yield satisfactory results by means of reservoir simulation. It was further shown that the solution for constant viscosity-compressibility product, which represents the liquid case, is a limiting case where the parameter $\lambda$ has a value of unity.

Fetkovich et al.\textsuperscript{14} presented a recent effort which focused on case histories using decline curve analysis by means of type curves. One of the most important conclusions of their work is the observation that \textit{transient data should not be interpreted with the Arps\textsuperscript{4}equations} for this effort would lead to apparent values of the decline exponent "$b$" greater than unity. It was also shown that such "false" decline exponents can lead to poor results and unwise exploitation decisions. Moreover, it was demonstrated by means of an example, that dual porosity reservoirs do not exhibit decline exponents greater than one, except when a match is attempted on transient data using Arps\textsuperscript{4} solutions.

Finally, as a means to ensure that the correct match is being made, Fetkovich et al.\textsuperscript{14} suggested that reservoir parameters and pore volume should not be computed from decline analysis before depletion is clearly indicated. These authors show that only after the onset of depletion is it possible to make a reasonable estimate of the decline exponent "$b$".

Meunier et al.\textsuperscript{15} illustrated the advantages of using normalized pseudopressure and pseudotime functions. In a study of gas well test analysis the most important advantage of using these functions is that we can describe gas flow in a porous medium by a liquid equivalent equation. Since both equations for liquid and gas flow have identical forms, even though these relations involve different variables, the solutions for liquid flow can be used for modeling and analyzing gas flow problems.
In 1987, Fraim and Wattenbarger\textsuperscript{16} used the concept of the normalized pseudovariables introduced by Meunier \textit{et al.}\textsuperscript{15} Fraim and Wattenbarger modified the pseudotime definition to be exact for boundary dominated flow. These authors evaluated the viscosity-compressibility product within the integral at average reservoir pressure rather than at the wellbore pressure as had been done for transient analysis. By assuming production against a constant backpressure it was shown that through this linearization, all gas results should fit the exponential decline stem (liquid solution) on the Fetkovich\textsuperscript{8} type curve. An iterative procedure was proposed to compute the original gas-in-place which is required to compute pseudotime. The gas-in-place is usually unknown but is required to determine the average reservoir pressure profile for the computation of pseudotime.

Fraim and Wattenbarger\textsuperscript{16} showed that no case of gas flow produced at constant bottomhole pressure exactly matches any stem (exponential, harmonic or general hyperbolic decline curves) when ordinary time is used as a plotting function. However, these authors proved that the use of normalized pseudopressure and pseudotime allows that the boundary dominated (pseudosteady state) flow data match the exponential decline. As a consequence, accurate values of gas-in-place and flow capacity can be determined from a normalized pseudopressure and pseudotime type curve analysis.

A comprehensive study of the application of decline curve analysis to determine formation properties was developed by Blasingame and Lee\textsuperscript{17}. These authors present analysis techniques for post-transient single phase liquid at constant bottomhole pressure. Analysis techniques are given for wells produced at a constant bottomhole flowing pressure in homogeneous, naturally fractured and vertically fractured reservoirs. In the case of gas flow, both pseudopressure and pseudotime must be substituted for actual pressure and time.

Blasingame and Lee\textsuperscript{17} showed that the reservoir characteristics that can be estimated using decline curve analysis depend on the reservoir type. The authors also gave a useful table that summarizes the different possibilities of analysis. It was emphasized that the reservoir must be at the post-transient state, otherwise the reservoir characteristics estimated from these methods will be incorrect.
Also, in 1986 Blasingame and Lee\textsuperscript{18} developed a general, variable-rate approximation for single phase flow of a liquid of small and constant compressibility. This approximation allows for the estimation of the reservoir pore volume and shape directly from production data, without shutting-in the well to obtain the average reservoir pressure.

These authors assumed that changes in flow rate cause transients to have a negligible effect compared to the influence of the outer boundary. This assumption was proven through simulation techniques. A graphical method was suggested which yields a straight line in which the slope and intercept can be used to estimate pore volume and shape. The method gives excellent results for cases where the rate changes are small. In particular, monotonic rate decline cases can be modeled very well using this method. The authors also illustrate a successful application for a random rate case.

The major contribution of Blasingame and Lee's\textsuperscript{18} work with regard to decline curve analysis is the development of a relation for the average reservoir pressure during stabilized flow for a variable-rate scenario. This relation has the same form as the result derived by Dietz\textsuperscript{19} for the constant rate case.

Schmidt \textit{et al.}\textsuperscript{20} noted that the original Fetkovich\textsuperscript{8} type curve neglects the effects of real gases such as the variations of properties with pressure. This type curve does not include non-Darcy flow either. Schmidt \textit{et al.} further noted that even though Fetkovich\textsuperscript{8} accounted for real gas flow behavior by normalizing actual rates in his later work,\textsuperscript{14} turbulent effects were still neglected. Schmidt \textit{et al.} noted the importance of turbulence on gas deliverability, especially at high flow rates around the wellbore. These authors introduced type curves that incorporate real gas flow behavior and non-Darcy flow during the pseudosteady state flow in closed drainage volumes. As an additional result, Schmidt \textit{et al.} showed that reservoir permeability and initial reservoir pressure significantly affect the type curve response.

Fraim, Lee, and Gatens\textsuperscript{21} introduced a new type curve to analyze long term production data from a vertically fractured well assuming that the well is produced at a constant bottomhole pressure. The well model assumes a finite conductivity vertical fracture and real gas properties as well as non Darcy flow existing in the fracture.
These characteristics are accounted for through an extension of the definition of normalized pseudotime. The solution results are represented by a modified Fetkovich type curve. This type curve includes the effect of a finite conductivity vertical fracture for ratios of drainage area length (for a square) to fracture half length, $x_e/x_f$, less than 10.

This type curve is a considerable improvement over the original Fetkovich type curve for the analysis of fractured wells. The Fetkovich type curve can be used only when the fracture has a high conductivity, in a medium to high permeability formation, and when $x_e/x_f$ is greater than 10. If these conditions are met, the existence of pseudoradial flow is probable and the assumptions made in the Fetkovich work are not violated.

Fraim, Lee, and Gatens also used a transformation of dimensionless rate and time similar to that used by Fetkovich, so that all the bounded reservoir solutions coincide. The methods proposed for the use of the type curve are iterative and can be used to predict the performance of fractured gas wells with or without non-Darcy flow.

Concerned that most decline type curve analysis methods assume isolated well patterns, Harrington et al. presented a method based on a computer solution that attempts to collect both the liquid and gas solutions into a single solution. The main interest of the authors was the variety of parameters that need be considered to perform a good type curve match. These authors noted that rate normalization requires a knowledge of gas-in-place, making an iterative procedure necessary, and that if numerous shut-ins or bottomhole pressure changes occur, the type curve must be reconstructed in time by means of superposition as shown by Fetkovich.

Making a strong emphasis on transient-flow data Harrington et al. developed a computer program that considers radial or linear transient flow and bounded reservoir flow. In doing so, and in contrast to previous authors Harrington et al. decided to update the gas viscosity-compressibility product at the average drainage reservoir pressure. That is, at the average of the area that has been contacted by the propagating pressure distribution rather than the overall average reservoir pressure.
A limitation of this technique is that parameters such as permeability, skin and other near wellbore properties must be known from previous well tests. Nevertheless, the authors claim this as a flexible procedure that overcomes the limitations caused by constant pressure or constant rate assumptions generally attached to type curve predictions. The authors also claim that the computer program is useful in actual field cases to forecast the gas well performance and backcalculate reserves based on both pressure and rate decline data.

For the case of a solution gas drive reservoir producing at conditions below the bubble point pressure with an immobile water saturation, and producing at a constant bottomhole pressure Chen and Poston\textsuperscript{23} developed an accurate analysis scheme using pseudofunctions. Normalized pseudovariables which incorporate two phase flow were proposed to account for the effects of system mobility and compressibility. These new pseudovariables involve the total mobility (which is a function of the oil, gas and water relative permeabilities and viscosities) as well as the oil saturation.

Even though Chen and Poston\textsuperscript{23} dealt with an oil reservoir, it was necessary to employ pseudovariables to account for the multiphase effect of producing below the bubble point pressure. Their analysis confirmed that relative permeability has the most significant effect on the decline characteristics of solution gas drive systems. Their results were presented in the form of a Fetkovich type curve. Based on the use of the redefined pseudovariables the authors proposed an iterative procedure to estimate oil-in-place and other reservoir parameters.

In 1988, Fraim and Wattenbarger\textsuperscript{24} studied the production from a well in a solution gas drive reservoir with an immobile water saturation. As with their previous work for gas flow, Fraim and Wattenbarger also transformed this problem into an equivalent slightly compressible liquid scenario. In order to accomplish this task, Fraim and Wattenbarger introduced equivalent liquid variables that although different from those defined in Ref. 23, these new variables have the same effect of linearizing the diffusivity equation. This linearization allows the Fetkovich type curve or any other set of liquid depletion type curves to be used for analysis and performance prediction. Fraim and Wattenbarger also accounted for changes in bottomhole pressure by using superposition with the equivalent liquid variables.
In 1988, Blasingame and Lee\textsuperscript{25} presented a new method for the variable-rate reservoir limits testing of gas wells. Similar to the slightly compressible case, the goals were again to estimate reservoir size and shape from production data. However, in the case of gas wells the average reservoir pressure profile in time was required to compute the gas properties used in the pseudotime calculation. A strategy was developed to determine the gas-in-place and the average pressure profile simultaneously. This strategy consists of an iterative procedure using the gas equation for variable-rate boundary dominated flow, and the real gas material balance relation in terms of pseudotime. The new gas flow equation is put in terms of a pseudotime that is analogous to the material balance time used in the relation for the liquid case.\textsuperscript{18} It should also be noted that even though this method is only applicable to post-transient or boundary dominated flow data, the results obtained by simulation suggest that the calculated average pressure profile may be also valid for application to transient gas flow data.

In 1989, Ding \textit{et al.}\textsuperscript{26} presented a study of the problem of long term gas flow at both constant rate and constant pressure, neglecting wellbore storage, non-Darcy flow, and rock compressibility. These authors showed that an exponential decline can be applied to gas reservoirs, provided that the appropriate pseudotime functions can be generated.

Ding \textit{et al.}\textsuperscript{26} also showed that by combination of dimensionless variable definitions, the material balance equation and the pseudosteady-state flow equation can be used to estimate the pore volume, the average reservoir pressure, the gas viscosity-compressibility product and in some cases the shape factor without requiring an iterative procedure.

Even though the process is non-iterative, it has two significant disadvantages. First, it is necessary to obtain an estimate of the time for the onset of pseudosteady state and second, the method requires accurate computation of pressure derivatives for the constant rate case and rate derivatives for the constant pressure case. This makes the application of this method to actual field data very awkward because differentiation of field derived measurements can cause significant errors.
In an effort to couple previous theories and new computing facilities, Duong\textsuperscript{27} recently presented a numerical approach for decline curve analysis. Duong put Arps\textsuperscript{4} hyperbolic rate equation in terms of cumulative production and used a multiple linear regression procedure to obtain the optimal values for the coefficients of the hyperbolic relation. The resulting coefficients allow the computation of both initial rate and initial decline. This approach is useful for the computer-automated curve fitting of oil and gas decline analysis, but only from an empirical standpoint.

Aminian \textit{et al.}\textsuperscript{28} published a study where the objective was to derive a representative decline type curve for gas flow analysis that accounts for the pressure dependency of viscosity and compressibility and the pressure loss owed to non-Darcy flow, and also accounts for the existence of other producing wells in the same reservoir. The solution, as with many of the previous solutions, was based on the assumption of pseudosteady-state gas flow at a constant bottomhole pressure with no appreciable water drive. Aminian \textit{et al.} stated that all the type curves published previously required either difficult manipulation of data or were sensitive to formation characteristics as well as reservoir pressure variations. This fact makes the determination of a unique match very difficult, especially when the production decline history is limited.

One of the most recent studies on decline curve analysis was presented by Blasingame \textit{et al.}\textsuperscript{29} early in 1991. Based on McCray's\textsuperscript{30} original presentation, Blasingame \textit{et al.} introduced a new time function which was shown to transform the variable pressure drop/variable rate problem into an equivalent constant pressure system. This new function was termed the "constant pressure analog time". The theoretical basis for such a function was derived using previous results\textsuperscript{18,25} which indicated that a variable rate problem could be translated into a constant rate analog case for both gas and liquid flow.

This resulting constant pressure analog time makes it possible to analyze actual field data using type curves such as the one presented by Fetkovich\textsuperscript{8} ($b=0$ stem), which assumes constant backpressure.

Currently, no completely general solution exists for the decline curve analysis of production data from gas wells. Recently, Spivey \textit{et al.}\textsuperscript{31} presented a new type
curve whose objective was to improve the reliability of matching noisy field data. This type curve was built upon the direction of thought and efforts of Blasingame et al.\textsuperscript{29} and McCray.\textsuperscript{30} As with previous efforts, the Spivey et al. type curve was developed for wells producing a single phase liquid at constant flowing bottomhole pressure from a finite radial reservoir.

Spivey et al.\textsuperscript{31} type curve includes the plotting of dimensionless cumulative production and the product of dimensionless decline flow rate and dimensionless decline time versus dimensionless decline time. This type curve makes it possible to match rate and cumulative production simultaneously, which provides more confidence in the selection of the early and late time stems. In addition to this, Spivey et al. provided plotting functions to extend the use of this type curve to gas wells and also to variable flowing pressure conditions. In the case of gas flow, an iterative procedure on average reservoir pressure is necessary to compute the pseudotime functions. When the bottomhole pressure is not prescribed, the authors suggest the use of the constant pressure analog time introduced in Ref. 29.

In 1993, Palacio and Blasingame\textsuperscript{32} presented a study where the objective was to develop a solution for the general case of variable rate/variable pressure drop for the flow of either liquid or gas. These authors show that regardless of the production scenario, the use of pressure and a simple material balance time function in the case of liquid flow, or pseudopressure and a material balance pseudotime for gas flow, yields a harmonic ($b=1$ stem) rate decline. These authors provide the theoretical proof for their developments and introduce an algorithm that allows an explicit, one time solution for gas-in-place avoiding the use of iterative procedures. These developments are significant because they provide the means to analyze any single phase production data using decline type curves such as the one presented by Fetkovich,\textsuperscript{8} which was originally intended for the analysis of liquid produced at a constant bottomhole pressure.
CHAPTER III

CURRENT METHODS FOR LIQUID AND GAS PRODUCTION DATA ANALYSIS

In this chapter we discuss the most important decline curve analysis methods for the analysis of field production data and we provide derivations of the basic equations for decline curve analysis. These developments will serve as the background to prepare for the techniques that we introduce in Chapter IV. Some of the derivations presented in this Chapter involve the use of new approaches that allow us to arrive at some of the same equations presented in the literature, with considerably less effort and/or much more clarity. Even though the methods are presented from a qualitative point of view, this discussion creates a link between the new developments presented in Chapter IV and current technology.

We present techniques for both liquid and gas flow; however, we emphasize gas flow analysis strategies because they are the subject of this thesis. The liquid methods are presented in terms of actual pressure and time, whereas the methods for gas flow analysis are presented in terms of normalized pseudovariables. We introduce and discuss normalized pseudovariables in Appendix A, and we refer the reader who is not familiar with these variables to that section of this thesis to understand their meaning, importance and necessity.

3.1 LIQUID DEVELOPMENTS

There are two categories of techniques that have been developed to analyze liquid production data which are currently used in the industry. The first category is based on empirical observation and is represented by Arps' hyperbolic and exponential decline relations. The second category are analytical results and the most important work in this category is the work presented by Fetkovich almost twenty years ago.
3.1.1 Arps' Empirical Relations

We must remember that the Arps' equations for exponential and hyperbolic decline behavior are empirical. Arps showed no theoretical justification for these relations. We present these equations in two parts. The first part corresponds to the hyperbolic model, and the second part describes the more frequently used exponential model.

3.1.1.1 Hyperbolic Decline Model

The hyperbolic decline model is based on a simple differential equation that relates the production rate and time derivative of the production rate to a power law function of rate, in this case, the "decline rate," \( D \). This differential equation is given as

\[
\frac{1}{q_o} \frac{dq_o}{dt} = -\eta^b q_o^b = -D \quad \text{............................................. (3.1)}
\]

where \( b \) is the decline curve exponent and \( \eta \) is a positive constant. If Eq. 3.1 is integrated from time \( t=0 \) to \( t=t \) where \( q_{oi} \) is the initial flow rate,

\[
\int_{q_{oi}}^{q_o} q_o^{(b-1)}dq = \int_0^t -\eta^b d\tau \quad \text{............................................. (3.2)}
\]

or

\[
\frac{q_o^b}{-b} \bigg|_{q_{oi}}^{q_o} = -\eta^b \tau \bigg|_0^t \quad \text{............................................. (3.3)}
\]

since we assume that the decline curve exponent \( b \) is constant. Then,

\[
\frac{q_o^b - q_{oi}^b}{-b} = -\eta^b \tau
\]

or

\[
q_o^b = q_{oi}^b + b \eta^b \tau
\]

Multiplying through by \( q_{oi}^b \) we obtain

\[
\frac{q_{oi}^b}{q_o^b} = 1 + b \eta^b q_{oi}^b \tau \quad \text{............................................. (3.4)}
\]

From Eq. 3.1, we see that we can define the initial value of the decline rate, \( D_I \), as \( \eta^b q_{oi}^b \). Then, we can write Eq. 3.4 as
\[
\frac{q_{oi}^b}{q_i^b} = 1 + D_i bt 
\] .................................................. (3.5)

Eq. 3.5 can be rearranged to yield

\[
q_i^b = \frac{q_{oi}^b}{(1 + D_i bt)}
\]

or as another form in terms of the \(q_o/q_{oi}\) ratio

\[
\frac{q_o}{q_{oi}} = \frac{1}{(1 + D_i bt)^{1/b}} 
\] .................................................. (3.6)

For \(b=1\) we obtain the special type of decline called "harmonic decline." This relation is given as

\[
\frac{q_o}{q_{oi}} = \frac{1}{(1 + D_i t)} 
\] .................................................. (3.7)

The general shape of a hyperbolic decline profile with a "b" value between zero and one is shown in Fig. 3.1. This is a semilog plot of flow rate versus time. The extremes,

![Fig. 3.1 - Decline Shapes on a Semilog Plot of Rate versus Time](image)

\[ q = q_i (1 + bDt)^{-1/b} \quad (b > 0) \]

Exponential \( (b=0) \)

\[ q = q_i \exp(-Dt) \]

Harmonic \( (b=1) \)

Hyperbolic \( (b>0) \)
i.e., the decline models for \( b=0 \) and \( b=1 \), are known as the exponential and the harmonic models respectively. The characteristic shape of the lines in Fig. 3.1 can be used to determine the type of decline present in a specific situation. In particular, the exponential decline yields a straight line in a plot of the type of Fig. 3.1. Further discussion of these decline models is presented throughout this thesis for both liquid and gas flow.

3.1.1.2 Exponential Decline Model

The exponential decline arises whenever the decline rate, \( D \), given in Eq. 3.1 is assumed to be constant. Arps\(^4\) stated this concept as "a constant loss ratio." Arps observed that frequently, following an initial period during which the flow rate is steady, the flow rate follows an exponential decline trend.

In developing equations for this case we start with Eq. 3.1 which is the governing relation. Rewriting Eq. 3.1 as Eq. 3.8 we have

\[
\frac{1}{q_o} \frac{dq_o}{dt} = -D \quad \text{..............................................} (3.8)
\]

Separating variables for integration we get

\[
\frac{1}{q_o} dq_o = -D dt \quad \text{..............................................} (3.9)
\]

Integrating Eq. 3.8 we obtain

\[
\int_{q_{oi}}^{q_o} \frac{1}{q_o} dq_o = - \int_{0}^{t} D dt \quad \text{..............................................} (3.10)
\]

where \( q_{oi} \) is the flow rate at \( t=0 \). The result of this integration is therefore

\[
\ln\left(\frac{q_o}{q_{oi}}\right) = -Dt \quad \text{..............................................} (3.11)
\]

where \( q_{oi} \) is the flow rate at \( t=0 \). The last result has always been expressed as an exponential type of equation. Thus, exponentiating we have

\[
\frac{q_o}{q_{oi}} = \exp(-Dt) \quad \text{..............................................} (3.12)
\]

In this case the slope, \( D \), is always equal to \( D_i \) (the initial decline rate) because it is assumed to have a constant value throughout the production history. We note that \( D \) is always the decline rate \( \frac{1}{q} \frac{dq}{dt} \). The difference between the exponential and the hyperbolic
case lies in the fact that for the former, $D$ is constant and, in the latter, $D$ is rate dependent (by a power law function). It is important to note that the exponential decline is always referred to as the "$b=0$" case. This is just a name for the exponential decline case, since substitution of a $b$ value of zero would produce an infinite power of the denominator of Eq. 3.6.

It is obvious from the developments shown above that an exponential decline must yield a straight line in a plot of the type of Fig. 3.1. The simplicity involved in this result has prompted engineers to use a plot like Fig. 3.1 to try to analyze just about every type of production data set in the hope that a simple straight line could be obtained, so that an exponential decline may be assumed for reserves and predictions. This thesis shows that such an analysis technique is an oversimplification of the problem at best and completely incorrect at worst.

3.1.2 Analytical Decline Curve Developments

In order to give decline curve analysis a sound theoretical basis many authors have provided rigorous proofs to support the simplified techniques we currently use. This and the following sections illustrate the most important principles that have been proven to be theoretically sound for decline curve analysis.

We will demonstrate two different proofs that the exponential decline relation is the correct analytical solution for the behavior of the flow rate when a single phase liquid is being produced at constant bottomhole flowing pressure. The first proof is based on a relation presented by van Everdingen and Hurst in Laplace space, and the second proof is an approach in real space that is only different in algebraic manipulation from the result presented by Fetkovich. Functional results in this thesis are written in terms of field units unless otherwise indicated. The reader is referred to the nomenclature section and Appendix A for clarity in this matter.

3.1.2.1 Development of the Exponential Decline Using van Everdingen and Hurst Approach

We will verify the theoretical validity of the exponential solution for a well producing at a constant bottomhole pressure using the constant rate solution that describes
the pressure response once a pseudosteady-state behavior has been reached. This relation is given by

\[ p_{wD,CR}(t_{DA}) = 2\pi t_{DA} + \frac{1}{2} \ln \left( \frac{4}{e^r C_A r_w^2} \frac{A}{\phi \mu_o C_f A} \right) \] .................................(3.13)

where \( p_{wD,CR} \) is the dimensionless pressure response for the constant rate case at the wellbore, and

\[ t_{DA} = t_D \frac{r_w^2}{A} = \frac{0.00633 k_o T}{\mu_o C_f A} \]

where \( r_w = r_w e^{-s} \), with \( s \) being the skin factor. Eq. 3.13 can also be written as

\[ p_{wD,CR}(t_{DA}) = a t_{DA} + b_{D, pss} \] .................................(3.14)

where the following definitions are used

\[ a = 2\pi = \text{constant} \]

and

\[ b_{D, pss} = \frac{1}{2} \ln \left( \frac{4}{e^r C_A r_w^2} \right) = \text{constant} \]

Because decline curves are usually thought of as representations of constant flowing bottomhole pressure situations, it is clear that we need a link between the constant rate solution and the constant bottomhole pressure solution. The link between the two solutions was provided by van Everdingen and Hurst\(^{33}\) in 1949. This derivation, as well as the superposition principle on which it is based, are given in Appendix C of this thesis, using the current petroleum industry nomenclature. The final result shown in Appendix C is used to obtain the constant pressure response, \( q_{D, CP} \), from the constant rate response given by Eq. 3.13 or Eq. 3.14.

Taking the Laplace transform of Eq. 3.14, it follows that

\[ \bar{p}_{wD, CR}(u) = \frac{a}{u^2} + \frac{b_{D, pss}}{u} \]

where \( u \) represents the Laplace transform. Eq. C.4 from Appendix C is

\[ \bar{q}_{D, CP}(u) \bar{p}_{wD, CR}(u) = \frac{1}{u^2} \] .................................(C.4)

If we combine these relations and solve for \( \bar{q}_{D, CP}(u) \), we obtain
\[ \bar{q}_{D,CP} (u) = \frac{1}{u^2} \left[ \frac{1}{a + \frac{b_{D, pss}}{u}} \right] \]

Then, after canceling common terms in numerator and denominator this is left as

\[ \bar{q}_{D,CP} (u) = \frac{1}{a + u b_{D, pss}} \]

or even further, as

\[ \bar{q}_{D,CP} (u) = \frac{1}{b_{D, pss}} \left[ \frac{a}{b_{D, pss}} + u \right] \]

The inversion of this relation into real space is obtained by noting that in general,

\[ \mathcal{L}^{-1}[\alpha \exp (-\beta t)] = \frac{\alpha}{\beta + u} \]

Thus,

\[ \mathcal{L}^{-1}[\bar{q}_D(u)] = q_D(t_{DA}) = \frac{1}{b_{D, pss}} \exp \left( -\frac{a}{b_{D, pss}} t_{DA} \right) \]

Writing the last expression back in terms of dimensionless variables using the previous definitions of \( a \) and \( b_{D, pss} \), we have

\[ q_{D,CP}(t_{DA}) = \frac{1}{2} \ln \left( \frac{4}{\pi} \frac{A}{C_A r_w^2} \right) \exp \left[ -\frac{4\pi t_{DA}}{\ln \left( \frac{4}{\pi} \frac{A}{C_A r_w^2} \right)} \right] \] ........................(3.15)

Eq. 3.15 is the exponential decline relation that describes the behavior of the liquid rate after the onset of boundary dominated (or pseudosteady state) flow, provided the bottomhole flowing pressure remains fixed.

Later in this chapter we will present a comparison of Eq. 3.15 to the relation derived by Fetkovich. We note that although the equations do not seem to be identical at first glance, these results are exactly the same when the reservoir is assumed to have a circular geometry with the well in the center of the reservoir.

3.1.2.2 Fetkovich Exponential Approach

We now present the derivation given by Fetkovich for the exponential decline case of a well producing a single liquid phase at constant flowing bottomhole pressure. We
believe that the approach presented here is not only more detailed, but also easier to follow and understand than either of the methods used by either van Everdingen and Hurst or Fetkovich.

It has been shown in the literature\textsuperscript{34} that for the scheme of constant rate production of single phase liquid, the flow equation representing the pressure response once a pseudosteady state has been reached is given by

\[
\bar{p} = p_{wf} + \frac{141.2}{k_{o}h} \frac{q_{o}B_{o}u_{o}}{ \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right) } \quad \text{...................... (3.16)}
\]

We must realize that Eq. 3.16 is only valid for the case of single phase liquid flow. Eq. 3.16 may be rearranged as

\[
\frac{k_{o}h}{141.2} \frac{q_{o}B_{o}u_{o}}{ \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right) } (\bar{p} - p_{wf}) = \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right)
\]

It is also known that by definition

\[
p_{D, \text{avg}} = \frac{k_{o}h}{141.2} \frac{q_{o}B_{o}u_{o}}{ \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right) } (\bar{p} - p_{wf})
\]

If the previous definition of \( b_{D,ps} \) is used again, it follows that since

\[
b_{D, ps} = \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right)
\]

then

\[
p_{D, \text{avg}} = \frac{k_{o}h}{141.2} \frac{q_{o}B_{o}u_{o}}{ \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right) } (\bar{p} - p_{wf}) = \frac{1}{2} \ln\left(\frac{4}{e^{\gamma} C_{A} r_{w}^{2}}\right) = b_{D, ps} \quad \text{...... (3.17)}
\]

Since \( \bar{p} \) is seldom known, every attempt must be made to get a relation in terms of \( p_{i} \) and \( p_{wf} \). This is attained by coupling Eq. 3.17 with a material balance relation which is in turn derived from the compressibility definition. This latter definition may be given as

\[
c_{i} = \frac{1}{V} \frac{dV}{dp}
\]

where \( c_{i} \) involves both fluid and rock compressibilities, and \( V \) is the total fluid volume. In this case \( V \) is assumed to be all the pore volume since oil saturation is taken as unity. Using the chain rule, the compressibility definition can be rewritten as

\[
c_{i} = \frac{1}{V} \frac{dV}{dt} \frac{dt}{dp}
\]
but since \( V = \frac{A \phi h}{5.615} \) and \( \frac{dV}{dt} = B_o q_o \), with \( B_o \) in RB/STB, then

\[
c_i = - \frac{5.615}{A \phi h} (q_o B_o) \frac{dt}{d \bar{p}} \tag{3.18}
\]

This relation can be solved for the liquid flow rate as

\[
q_o = - \frac{A \phi h c_i}{5.615 B_o} \frac{d \bar{p}}{dt} \tag{3.19}
\]

Eq. 3.19 is valid for any time because it is a rigorous material balance equation. Now, Eq. 3.17 may be rearranged as

\[
\bar{p} - p_{wf} = \frac{141.2 q_o B_o \mu_o}{k_o h} b_{D, pss} \tag{3.20}
\]

Substituting Eq. 3.19 into Eq. 3.20 it follows that

\[
\bar{p} - p_{wf} = - \left( \frac{A \phi h c_i}{5.615 B_o} \right) \left( \frac{141.2 B_o \mu_o}{2 \pi k_o h} \right) b_{D, pss} \frac{d \bar{p}}{dt}
\]

or

\[
\bar{p} - p_{wf} = - \frac{141.2}{5.615} \frac{\phi \mu_o c_i A}{k_o} b_{D, pss} \frac{d \bar{p}}{dt}
\]

For convenience we define the constant \( \delta \) as

\[
\delta = \frac{141.2}{5.615} \frac{\phi \mu_o c_i A}{k_o} b_{D, pss} = \text{constant}
\]

Using the definition of \( \delta \) we obtain

\[
\bar{p} - p_{wf} = - \delta \frac{d \bar{p}}{dt} \tag{3.21}
\]

At this point, it must be noted that Eq. 3.16 was derived for a constant rate production scheme. However this equation has been proven\(^{18} \) to yield excellent results when the flow rate is variable, and specifically for a constant bottomhole flowing pressure. Therefore, Eq. 3.21 can be integrated assuming constant bottomhole flowing pressure, \( p_{wf} \). Given that \( \delta \) is also constant we integrate Eq. 3.21 as follows

\[
\int_0^t dt = - \delta \int_{p_i}^{\bar{p}} \frac{1}{p - p_{wf}} d \bar{p}
\]

\[
(t - 0) = - \delta [\ln (\bar{p} - p_{wf}) - \ln (p_i - p_{wf})]
\]

or
\[
\frac{1}{p_i - p_{wf}} = \frac{1}{\bar{p} - p_{wf}} \exp \left( -\frac{t}{\delta} \right) \tag{3.22}
\]

Combining Eq. 3.21 and 3.22 and substituting for \( \delta \) we obtain

\[
\frac{1}{p_i - p_{wf}} = \frac{k_{of}}{141.2 q_o B_o \mu_o} \frac{b_{D, pss}}{k_{oh}} \exp \left( -\frac{5.615 k_o}{141.2 \phi \mu_o c_i A} \frac{t}{b_{D, pss}} \right)
\]

or

\[
\frac{k_{oh}}{k_{oh} (p_i - p_{wf})} = \frac{1}{b_{D, pss}} \exp \left( -\frac{5.615 k_o}{141.2 \phi \mu_o c_i A} \frac{t}{1} \ln \left[ \frac{4}{e^\gamma C_A r_w^2} \right] \right)
\]

The left hand side of this relation is the definition of \( q_D \), and upon slight rearrangement we have

\[
q_D b_{D, pss} = \exp \left( -\frac{5.615}{141.2 (0.00633)} \right) \frac{0.00633 k_{of}}{\phi \mu_o c_i A} \frac{1}{2} \ln \left[ \frac{4}{e^\gamma C_A r_w^2} \right]
\]

Using the definition of \( t_{DA} \) which is

\[
t_{DA} = \frac{0.00633 k_{of}}{\phi \mu_o c_i A}
\]

The final result is

\[
q_{D, CP} (t_{DA}) = \frac{1}{2} \ln \left[ \frac{4}{e^\gamma C_A r_w^2} \right] \exp \left( -\frac{2\pi t_{DA}}{2} \ln \left[ \frac{4}{e^\gamma C_A r_w^2} \right] \right)
\]

This is exactly the same equation derived by using the van Everdingen and Hurst\(^3\)\(^3\) approach for the case of liquid flow at boundary dominated flow conditions, assuming a constant bottomhole pressure at the well.

In order to verify that Eq. 3.15 compares exactly with Fetkovich's\(^8\) exponential decline equation, it must be noted that Fetkovich assumed a circular reservoir, whereas Eq. 3.24 is written for an arbitrary reservoir geometry. Therefore, this comparison requires that the shape factor, \( C_A \), for a circular reservoir with a well in the center be substituted into Eq. 3.24. Using the Dietz\(^9\) result that \( C_A = \frac{4\pi e^{3/2}}{e^\gamma} \) gives

\[
q_{D, CP} (t_{DA}) = \frac{141.2 q_o B_o \mu_o}{k_{oh} (p_i - p_{wf})} = \frac{1}{\ln \left[ \frac{r_e}{r_w} \right]} \frac{3}{4} \exp \left( -\frac{2\pi t_{DA}}{2} \ln \left[ \frac{r_e}{r_w} \right] \right)
\]

\tag{3.25}
We now recall that Fetkovich type curve variables are defined as:

\[ q_{Dd} = \frac{q(t)}{q_i} = q_D \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] = \exp (-D_d t) = \exp (-t_{Dd}) \] ..........................(3.26)

and

\[ t_{Dd} = \frac{t_D}{\frac{1}{2} \left[ \ln \left( \frac{r_e}{r_w} \right)^2 - 1 \right] \ln \left( \frac{r_e}{r_w} - \frac{1}{2} \right)} \] ..........................(3.27)

It is necessary to note at this stage that Fetkovich\textsuperscript{8,14} used the constant \( \frac{1}{2} \) rather than \( \frac{3}{4} \) because "a better correlation was obtained particularly at small \( r_e/r_w \) stems" using this value. Therefore, when comparing Eq. 3.25 to Eqs. 3.26 and 3.27 this fact must be kept in mind. Then, looking at equations 3.25 and 3.26, it can be inferred that \( q_{Dd} \) should be defined as

\[ q_{Dd} = \frac{141.2 q_o B_o \mu_o}{k_o h (p_i - p_{wf})} b_{D, pss} = q_D \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] \]

Unfortunately, we must accept the use of \( \frac{1}{2} \) in the light of convention. On the other hand, \( t_{Dd} \) should be defined as

\[ t_{Dd} = \frac{2\pi t_{DA}}{b_{D, pss}} \]

We also note that

\[ t_{DA} = t_D \frac{r_w^2}{A} = t_D \frac{r_w^2}{\pi r_e^2} \]

Combining the last two expressions we have

\[ t_{Dd} = \frac{2\pi}{b_{D, pss}} \frac{r_w^2}{\pi r_e^2} t_D \]

or, since \( b_{D, pss} \) in this case is given by \( \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \), it may be stated that

\[ t_{Dd} = \frac{2}{\left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] r_e^2} t_D \]

or

\[ t_{Dd} = \frac{2t_D}{\left( \frac{r_e^2}{r_w^2} \right) \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right]} \] ..........................(3.28)
For Eq. 3.27, if \( r_e \) is very large compared to \( r_w \), we can write

\[
t_{Dd} = \frac{2tp}{r_e^2 \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2}}
\]

The last expression is exactly the same as Eq. 3.28, except for the difference between using \( \frac{1}{2} \), or \( \frac{3}{4} \). It must be clear now that, \( q_{Dd} \) and \( t_{Dd} \) as derived by Fetkovich are exactly the same variables involved in Eq. 3.25.

Summarizing these results it can be stated that for single phase liquid flow an exponential decline is the exact long time solution when a constant backpressure is held during production.

Based not only on the analytical \( q_{Dd} \) and \( t_{Dd} \), but also making use of Arps' empirical hyperbolic relations (in terms of \( q_{Dd}, t_{Dd} \) and \( b \)) Fetkovich\(^8\) developed a type curve that is probably the second most popular tool used within the industry to interpret production data. This type curve is second in use only to the oversimplified exponential model that is almost always assumed regardless of the production conditions.

The "Fetkovich" type curve is presented in Fig. 3.2. The relevant facts regarding its use and development are summarized next. This plot is one of dimensionless decline flowrate \( q_{Dd} \) versus dimensionless decline time \( t_{Dd} \). The curve itself can be divided into two regions. The left stems correspond to the transient flow regime and are prescribed by the dimensionless drainage radius, \( r_eD \). The stems on the right hand side of the plot correspond to the boundary dominated flow and are prescribed by the "b" decline curve exponent. The connecting point of all the stems can be understood as the transition between transient and pseudosteady-state flow. By means of reservoir simulation techniques Fetkovich\(^8\) obtained all the transient stems and the exponential stem for depletion, \( i.e., \) the \( b=0 \) stem that corresponds to an exponential decline for constant bottomhole pressure production at boundary dominated flow conditions. The other depletion stems for a \( b \) value different from zero, were obtained by using Arps\(^4\) empirical hyperbolic decline equation.
Fig. 3.2 - Fetkovich Composite of Analytical and Empirical Type Curves
It must be noted for future reference that a value of \( b=0 \) corresponds to the exponential decline, \( b=1 \) corresponds to the harmonic type of decline, and finally for a \( 0<b<1 \) all the stems are represented by a hyperbolic type of decline.

This type curve was intended for liquid analysis, but it will be seen in this work that pseudovariables can be used to apply this curve for the analysis of single phase gas production data.

3.1.2.3 Material Balance Approach for Liquid Flow

The purpose of this section is to introduce an analytical expression that will be used extensively in Chapter IV. This expression allows variable rate/variable pressure drop data to be transformed to match the \( b=1 \) stem on a Fetkovich\(^8\) type curve. This transformation will be done using a time function that represents material-balance (depletion) performance. The original material balance approach was developed by Blasingame and Lee\(^{18}\) and all the pertinent mathematical relations presented in this section are based on their work.

In order to illustrate the handling of the mathematical expressions involved in this process, the previous identity given by Eq. 3.19 is repeated here for clarity as Eq. 3.29

\[
q_o = - \frac{A \phi h c_i}{5.615 B_o} \frac{d \bar{p}}{d t} \tag{3.29}
\]

Since the single phase liquid case is the one under consideration, the total compressibility of the reservoir, \( c_i \), is assumed to be constant. Integrating Eq. 3.29 we obtain

\[
\int_0^t q_o \, dt = - \frac{A \phi h c_i}{5.615 B_o} \int_{\bar{p}_i}^{\bar{p}} \frac{d \bar{p}}{\rho_i}
\]

We note that the integral on the left-hand-side corresponds to the cumulative production. This is written as

\[
N_p = \int_0^t q_o \, dt
\]

Completing the integration on the right-hand-side and combining with our cumulative production result on the left hand side we have
\[ N_p = \frac{A \phi h c_i}{5.615 B_o} (p_i - \bar{p}) \]

Solving this relation for the pressure difference we obtain

\[ (p_i - \bar{p}) = \frac{5.615 N_p B_o}{\phi h A c_i} = \frac{N_p}{N c_i} \]

which may be also rearranged as

\[ (p_i - \bar{p}) \frac{h}{B_o} = \frac{5.615 N_p}{\phi A c_i} \]

If this relation is multiplied through by \( \frac{k_o}{141.2 q_o \mu_o} \), it may be written that

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{5.615 k_o}{141.2 \phi \mu_o c_i A} \frac{N_p}{q_o} \]

Defining \( \bar{t} = \frac{N_p}{q_o} \) to be the material balance time, and multiplying and dividing by 0.00633 on the right-hand-side term, it follows that

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{5.615 \left( 0.00633 \right) k_o}{141.2 \left( 0.00633 \right) \phi \mu_o c_i A} \bar{t} \]

or

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{2 \pi \left( 0.00633 \right) k_o}{\phi \mu_o c_i A} \bar{t} \]

Using the definition of the dimensionless time based on the drainage area we have,

\[ \bar{t}_{DA} = \frac{0.00633 k_o}{\phi \mu_o c_i A} \bar{t} \]

Combining this definition with the previous relation we obtain

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = 2 \pi \bar{t}_{DA} \]

\[ \] (3.31)

The most important characteristic of Eq. 3.31 is that this relation is always valid, regardless of time, flow regime, or production scenario (whether constant or variable bottomhole flowing pressure or constant or variable flow rate). Eq. 3.31 is the material balance equation for our system and is therefore always valid.

In the following section, Eq. 3.31 is coupled with the pseudosteady-state flow relation that relates \( \bar{p} \) and \( p_{wf} \). Although this relation is derived for constant flowrate
(variable \( p_{wf} \)), it has been shown\(^{18} \) to be a good approximation when the flowing bottomhole pressure is fixed. The result is an equation that is strictly valid for pseudosteady-state flow regime, for any rate or pressure profile. The pseudosteady-state flow equation to be used is Eq. 3.17 which is

\[
(\bar{p} - p_{wf}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{1}{2} \ln \left( \frac{4}{e^2} \frac{A}{C_A r_w^2} \right) = b_{D,pss} \text{..........................(3.17)}
\]

Adding Eqs. 3.31 and 3.17 it follows that

\[
(p_i - p_{wf}) \frac{k_o h}{141.2 q_o B_o \mu_o} = 2\pi DA + b_{D,pss} \text{.........................(3.32)}
\]

Eq. 3.32 can be also written in field variables as

\[
\frac{(p_i - p_{wf})}{q_o} = m \tilde{t} + b_{pss} \text{..........................(3.33)}
\]

where

\[
m = \frac{2\pi(0.00633)k_o}{\phi \mu_o C_i A} \frac{141.2 B_o \mu_o}{k_o h} = \frac{5.615 B_o}{\phi h C_i A} = \frac{1}{Nc_i}
\]

and

\[
b_{pss} = \frac{141.2 B_o \mu_o}{k_o h} b_{D,pss}
\]

Eq. 3.33 clearly represents the equation of a straight line. Thus, a Cartesian plot of the normalized pressure drop \( \frac{(p_i - p_{wf})}{q_o} \) versus the material balance time \( \tilde{t} \), which is computed as \( \tilde{t} = \frac{N_p}{q_o} \), should help in the estimation of the pore volume as well as the approximate geometry of the reservoir. The slope of the resulting straight line should be used to compute the approximate value of the oil-in-place, and the intercept of this plot should allow the engineer to estimate the shape factor of the reservoir, provided that the permeability value and the skin factor are known from previous well tests.


Even though this straight line plot is useful, it is not obvious that this plot or its originating equations have particular utility for decline analysis. However, the presentation of these developments was necessary in order to provide background for the application of Eq. 3.33 in decline analysis. This, will be shown in Chapter IV.
3.2 GAS DEVELOPMENTS

When dealing with gas, we must take particular care of the treatment of the fluid properties. Particularly, neither the compressibility nor the viscosity of a real gas can rigorously be considered to have constant values, as has always been the case when we deal with the liquid flow problem. As we discussed in our literature review in Chapter II, various ways of treating the variation of gas properties have been introduced, ranging from approaches intended simply to treat the gas flow problem as a liquid flow problem, to rigorous analyses that not only account for variations in gas properties but also for the existence of effects like non-Darcy and multiphase flow. In this section, we will review useful methods to analyze gas production data, in much the same way as we did for liquid flow data.

Two different categories of decline curve analysis methods for gas flow will be presented in this section. The first one of them has been called semi-analytical or conventional. This category considers the semi-analytical developments given by Carter.\textsuperscript{12,13} The second category is called rigorous and includes two major approaches. These approaches are the Fraim and Wattenbarger\textsuperscript{16} treatment of the constant backpressure case, and the rigorous analysis of the general variable rate/variable pressure drop case given by Blasingame and Lee.\textsuperscript{18} An additional development presented by Ding \textit{et al.}\textsuperscript{26} which also belongs in the second category will be only briefly reviewed. This is done so because even though Ding \textit{et al.} method is mathematically sound and rigorous, it is also very awkward to apply to actual data for this method relies on the accurate computation of flow rate or pressure drop derivatives.

We must state that all the gas flow methods discussed in this thesis neglect rock compressibility, non-Darcy effects and geopressed reservoir effects. The consideration of rock compressibility as negligible is almost always true because gas compressibility is usually 100 times larger than rock compressibility.

Before introducing the current techniques available to analyze gas production data it is worth commenting on the variables that will be required in order to have a liquid equivalent problem, even though there are several approximate methods for the gas flow problem that do not make use of them. These variables are called pseudopressure and pseudotime. We introduce new pseudovariables and techniques to compute them. These
pseudovariables as well as the complete mathematical derivations are given in Appendix A of this thesis. The reader is referred to this section for details and application relations. In addition, we present the following comments to introduce the reader to the concepts and utility of pseudovariables for gas flow analysis.

In 1984 Meunier et al.\textsuperscript{15} introduced the definition of normalized pseudopressure and pseudotime, in contrast to the previous definitions of pseudopressure\textsuperscript{35,36} and pseudotime.\textsuperscript{37,38} However, Russell et al.\textsuperscript{39} introduced normalized pseudopressure in 1966. Russell et al.\textsuperscript{39} formulated a mathematical analog of gas flow in a reservoir using normalized pseudopressure, in order to facilitate the numerical solution of the diffusivity equation. Meunier et al.\textsuperscript{15} were not then the first to introduce the normalized pseudopressure and pseudotime although they may have been the first authors to show the real advantages of these variables in gas well test analysis. Meunier et al.\textsuperscript{15} state that an effective linearization using normalized pseudopressure and pseudotime is acceptable for many practical situations and the resulting gas flow equation can be put in identical form as the liquid equation. This "makes possible the direct use of liquid solutions to solve gas problems and the retention of pressure and time units." Moreover, "the existing liquid solutions are directly applicable without any modification provided that gas volumes are expressed in liquid equivalent units."

3.2.1 Semi-Analytical Methods

3.2.1.1 Carter's Decline Parameter and the Carter Type Curve

The work by Carter\textsuperscript{12,13} constitutes the development of one of the most important tools for the decline curve analysis of gas well production data. This work presents type curves based on dimensionless variables (similar to those variables defined by Fetkovich\textsuperscript{8}) as well as the depletion parameter, \( \lambda \). The results proved useful to estimate formation permeability and reservoir size. The most important assumptions made involve constant bottomhole flowing pressure, negligible liquid and rock compressibility and also negligible non-Darcy flow effects.

The variables and parameters presented by Carter\textsuperscript{12,13} were obtained from exact liquid solutions which corresponded to \( \lambda=1 \) and the results of this study were found to be valid for gas flow (\( \lambda<1 \)) by use of numerical gas flow solutions. A shortened description
of the procedure given by Carter for developing the correlation parameter \( \lambda \), is given below. The only difference between our development and the original one given by Carter is that we use normalized pseudopressure rather than the conventional pseudopressure as defined by Al-Hussainy and Ramey.\textsuperscript{35} This translates into slight modifications of the original equations that do not affect the outcome of the analysis in any manner.

The normalized pseudopressure definition is given by

\[
\frac{p_p}{p_i} = \frac{\mu_i}{\mu_{zi}} \int_0^p \frac{p}{\mu_g(p)z(p)} \, dp 
\]

(3.34)

The differentiation and inversion of this definition leads to

\[
\frac{\partial p}{\partial p_p} = \frac{p_i}{\mu_{zi}} \frac{\mu_i}{p} 
\]

(3.35)

Also,

\[
\frac{d}{dp} \frac{p}{z} = \frac{1}{z} - \frac{p}{z^2} \frac{\partial z}{\partial p} 
\]

(3.36)

Using the chain rule we can write the following

\[
\frac{\partial (\frac{p}{z})}{\partial p_p} = \frac{\partial (\frac{p}{z})}{\partial p} \frac{\partial p}{\partial p_p} 
\]

(3.37)

The combination of Eqs. 3.35, 3.36 and 3.37 yields

\[
\frac{\partial (\frac{p}{z})}{\partial p_p} = \left(1 - \frac{p}{z} \frac{\partial z}{\partial p}\right) \frac{p_i}{\mu_{zi}} \frac{\mu_g}{p} 
\]

(3.38)

The real gas compressibility definition is given as

\[
c_g = \frac{1}{p} - \frac{1}{z} \frac{\partial z}{\partial p} 
\]

(3.39)

Comparing Eqs. 3.38 and 3.39 we have

\[
\frac{\partial (\frac{p}{z})}{\partial p_p} = c_g p \frac{p_i}{\mu_{zi}} \frac{\mu_g}{p} 
\]

which upon rearranging we have
\[ \mu_g c_g = \frac{\mu_{gi}}{p_i} \frac{\partial (\frac{P}{z})}{\partial p_p} \]  \hspace{2cm} (3.40)

Eq. 3.40 suggested to Carter that an average value for the viscosity-compressibility product over the interval from \( p_i \) to \( p_{wf} \) could be defined. Using a finite-difference approximation to the derivative term on the right-hand-side of Eq. 3.40, Carter obtained an expression which was shown by simulation to account for the variation of \( \mu_g c_g \) over the range of average reservoir pressures traversed during depletion. This result is

\[ \overline{\mu_g c_g} \equiv \frac{\mu_{gi}}{p_i} \left[ \frac{(\frac{P}{z})_{i} - (\frac{P}{z})_{wf}}{p_{pi} - p_{pwf}} \right] \]  \hspace{2cm} (3.41)

Eq. 3.41 led to the definition of the correlation parameter \( \lambda \) as

\[ \lambda = \frac{\mu_{gi} c_g}{\mu_g c_g} \]  \hspace{2cm} (3.42)

The combination of Eqs. 3.41 and 3.42 permits us to express \( \lambda \) as

\[ \lambda = \frac{p_{ci} c_g}{z_i} \left[ \frac{p_{pi} - p_{pwf}}{(\frac{P}{z})_{i} - (\frac{P}{z})_{wf}} \right] \]  \hspace{2cm} (3.43)

It was clear then from Eq. 3.42 that for the liquid case where the viscosity compressibility product remains basically constant, \( \lambda \) is unity. The same is true for gas flow whenever \( \frac{p_{i} - p_{wf}}{p_i} \) is much less than unity. That is, when the pressure drop within the reservoir is so small, that the \( \mu_g c_g \) product remains approximately constant.

For an ideal gas, assumed here to be one with constant viscosity, and a \( z \)-factor of unity, it follows that since \( \frac{dz}{dp} = 0 \), Eq. 3.39 becomes

\[ c_g = \frac{1}{p} \]  \hspace{2cm} (3.44)

and from Eq. 3.34 we have

\[ p_{pi} = \frac{\mu_g}{p_i} \frac{p_i^2}{2\mu_g} = \frac{p_i^2}{2p_i} \]  \hspace{2cm} (3.45)

\[ p_{pwf} = \frac{\mu_g}{p_i} \frac{p_{wf}^2}{2\mu_g} = \frac{p_{wf}^2}{2p_i} \]  \hspace{2cm} (3.46)
Substituting the last two equations into Eq. 3.43, setting \( z = 1 \) and \( c_{gi} = \frac{1}{p_i} \) as indicated from Eq. 3.44, gives us

\[
\lambda = \frac{P_i^2 - P_{wf}^2}{2p_i(p_i - P_{wf})}
\]

or finally

\[
\lambda = \frac{1}{2p_i}(p_i + P_{wf})
\]

(3.47)

From this last relation it can be readily seen that \( \lambda \) ranges from 0.5 to 1 for an ideal gas as assumed by Carter.\textsuperscript{12,13} The upper limit is obtained letting \( P_{wf} = 0 \) and the upper limit taking \( P_{wf} = p_i \).

Carter\textsuperscript{12,13} used dimensionless time and rate variables defined similar to those given by Fetkovich\textsuperscript{8} as well as a flow geometry parameter in addition to \( \lambda \), to create his type curves. This geometric parameter was necessary to identify the curves during transient flow, whereas \( \lambda \) is the representative parameter for boundary dominated flow conditions. In this work, the main interest lies in the depletion part of the Carter type curve for boundary dominated radial flow conditions. A typical Carter type curve is shown in Fig. 3.3.

In order to keep consistency with what will be presented in Chapter IV regarding the development and application of a Fetkovich-Carter type curve, we will use the plotting functions taken from Fetkovich.\textsuperscript{8} Thus, the definitions that will be suitable for defining the Fetkovich-Carter type curve are

\[
q_{Dd} = 141.2 \frac{q_s B_s \mu_i b_{D,p_{ss}}}{k_i h (p_i - P_{wf})}
\]

(3.48)

and

\[
t_{Dd} = 0.00633 \frac{k_i}{\phi \mu_i c_i \rho_i r_i^2} \frac{1}{\frac{A}{2 \pi r_i^2}} b_{D,p_{ss}}
\]

(3.49)

and also

\[
b_{D,p_{ss}} = \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A r_{i w}^2} \right)
\]

(3.50)
Using a finite-difference radial gas simulator, Carter\textsuperscript{12,13} generated numerical solutions for both ideal and real gases. It was observed that the point of departure from the infinite-acting region is about the same for all the values of $\lambda$ and that during boundary dominated flow the curves divert from one another.

It was also noted that ideal and real gases correlated very well and the analytical liquid solutions compared favorably with the ideal gas solutions for $\lambda = 1$. In addition, the case of $\lambda = 1$ exhibited a straight line on semilog paper, being consistent with the late time rate behavior of a liquid which follows an exponential decline.

In order to generate useful solutions for gases of different compositions, Carter\textsuperscript{12,13} noted that when the pressure paths from $p_i$ to $p_{wf}$ for two systems that have the same geometry but different gas composition are selected so that each system has the
same lambda, then, the rate-time curves expressed in dimensionless form yield the same results. From this Carter concluded that sufficient accuracy can be obtained using type curves based on a specific gas composition to solve problems for different gas compositions.

During infinite-acting radial flow, \( \lambda = 1 \) represents virtually all the gas cases of interest because the viscosity-compressibility product remains almost constant. It was also shown that good approximations were obtained for reservoirs which could even be asymmetrical with flow regimes different from linear or radial.

In the application of these type curves for the analysis of gas well data it was suggested that though it is possible to obtain \( \lambda \) from the type curve match, this parameter should be determined prior to the type-curve matching itself, using Eq. 3.43. Late time data should be matched in preference to early data due to the requirement of boundary dominated flow for the \( \lambda \) concept to be valid for analysis and interpretation.

As a final point we note that all of Carter's developments require a constant bottomhole flowing pressure. These methods should not be applied for cases of variable bottomhole flowing pressure.

3.2.2 Rigorous Analysis Methods

The application of the methods in this section involve not only the use of pseudopressure, but also pseudotime. This permits an effective (although not entirely rigorous) linearization of the diffusivity equation. In Appendix A, we present a complete derivation of the general diffusivity equation in terms of normalized pseudovariables. The reader is referred to this section in order to clearly understand the statement made regarding the linearization process obtained using pseudovariables.

3.2.2.1 Method of Fraim and Wattenbarger for Constant Pressure Gas Flow

The work of Fraim and Wattenbarger\(^{16}\) is the most important publication on the practical application of pseudovariables for the analysis of gas production data. The authors introduced a normalized pseudotime where the viscosity and compressibility terms inside the integrand are taken at the average reservoir pressure rather than at the wellbore
flowing pressure as Lee and Holditch\textsuperscript{38} had done before for pressure buildup tests. This procedure turned out to be an exact linearization of the diffusivity equation for boundary dominated flow in a constant backpressure production scenario.

The assumption of constant $p_{wf}$ is somewhat limiting in regard to the analysis and interpretation of field data. Because of this limitation, we have undertaken this study to provide approaches for any pressure or rate profile. The results of these efforts are presented in the next chapter of this work.

In the following work we present a slight variation of the derivation of the exponential decline equation for gas flow systems presented by Fraim and Wattenbarger\textsuperscript{16} when the backpressure is fixed. The variation has to do with the fact that in our work both pseudovariables (pseudopressure and pseudotime) are normalized, whereas the original work was presented in terms of conventional pseudopressure and normalized pseudotime.

Our derivation starts from the gas material balance equation for a volumetric dry gas reservoir which is given by

$$\frac{\bar{p}}{\bar{z}} = \frac{p_i}{z_i} \left[ 1 - \frac{G_p}{G} \right]$$

This relation may be rearranged to yield

$$G_p = G \left[ 1 - \frac{\bar{p}}{\bar{z}} \frac{z_i}{p_i} \right]$$

Differentiating Eq. 3.51 with respect to time we obtain

$$q_g = \frac{dG_p}{dt} = -G \left( \frac{\bar{z} i}{p_i} \right) \frac{d}{dt} \left( \frac{\bar{p}}{\bar{z}} \right)$$

From the definition of gas compressibility we find that at average reservoir pressure we have

$$\bar{c}_g = \frac{1}{\bar{p}} \frac{d\bar{p}}{d\bar{p}} = \frac{\bar{z} R T}{\bar{p} M} \frac{d}{d\bar{p}} \left( \frac{\bar{p} M}{\bar{z} R T} \right) = \frac{\bar{z}}{\bar{p}} \frac{d\bar{p}}{d\bar{p}} \left( \frac{\bar{p}}{\bar{z}} \right)$$

which upon rearranging leads to

$$\frac{d}{d\bar{p}} \left( \frac{\bar{p}}{\bar{z}} \right) = \frac{\bar{p}}{\bar{z}} \bar{c}_g$$

Applying the chain rule in Eq. 3.52 we can write
\[
q_g = \frac{dG_p}{dt} = -G \left( \frac{z_i}{p_i} \right) \frac{d}{dp} \left( \frac{p}{z} \right) \frac{dp}{dt} \tag{3.54}
\]

Combining Eqs. 3.53 and 3.54 we produce the following
\[
q_g = -G \frac{\bar{p} z_i}{p_i \bar{c}_g} \frac{dp}{dt} \tag{3.55}
\]

Using these relations and the definitions of the normalized pseudovariables it will be shown how a gas system that is being produced at constant bottomhole flowing pressure can be represented by an exponential decline relation. The results presented by Al-Hussainy et al.\textsuperscript{35} and Al-Hussainy and Ramey\textsuperscript{36} for pseudosteady-state flow show that

\[
\frac{\bar{p}_p - p_{p,wf}}{q_g} = 141.2 \frac{B_{gi} \mu_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A \bar{r}'_w^2} \right) \right] \tag{3.56}
\]

where \( p_p \) is given by Eq. 3.34
\[
p_p = \frac{\mu_{gi} z_i}{p_i} \int_{p_{base}}^p \frac{p}{\mu_g(p) z(p)} dp \tag{3.34}
\]

and \( b_{D,pss} \) is given as
\[
b_{D,pss} = \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A \bar{r}'_w^2} \right)
\]

Combining these relations we have
\[
q_g = \frac{\bar{p}_p - p_{p,wf}}{141.2 \frac{B_{gi} \mu_{gi}}{k_g h} b_{D,pss}} \tag{3.57}
\]

Equating Eqs. 3.55 and 3.57 we obtain
\[
\frac{\bar{p}_p - p_{p,wf}}{141.2 \frac{B_{gi} \mu_{gi}}{k_g h} b_{D,pss}} = -G \frac{\bar{p} z_i}{\bar{c}_g} \frac{dp}{dt}
\]

Rearranging gives
\[
\frac{\bar{p}}{\bar{c}_g} \frac{dp}{\bar{p} - p_{p,wf}} = -\frac{p_i}{z_i G} \frac{1}{141.2 \frac{B_{gi} \mu_{gi}}{k_g h} b_{D,pss}} dt \tag{3.58}
\]

Also, from the normalized pseudopressure definition we have
\[ d\bar{P} = \frac{p_i}{\mu g_iz_i} \frac{\overline{\mu g}}{p} d\bar{P}_p \] \hspace{1cm} (3.59)

Combining Eqs. 3.58 and 3.59 produces

\[ \frac{\overline{c_g}}{p_p - p_{pwf}} \frac{\mu g}{c_g} d\bar{P}_p = -\frac{1}{141.2 G \frac{\overline{B_{gi}}}{{\mu g}} \frac{\lambda_{gi}}{k_g h} b_{D,pss}} dt \]

Multiplying through by \( \frac{c_g}{c_g} \) and integrating

\[ \int_{p_i}^{p} \frac{d\bar{P}_p}{\bar{p}_p - p_{pwf}} = -\frac{1}{c_g G \left( 141.2 \frac{\overline{B_{gi}}}{{\mu g}} \frac{\lambda_{gi}}{k_g h} b_{D,pss} \right)} \int_{0}^{t} \frac{\mu g}{c_g} c_g \ dt \] \hspace{1cm} (3.60)

The integral on the right hand side of this relation represents the conventional normalized pseudotime, \( t_a \) as defined by Fraim and Wattenbarger.\(^{16}\) That is,

\[ t_a = \int_{0}^{t} \frac{\mu g}{c_g} c_g \ dt \]

Therefore, if the integral on the left-hand-side of Eq. 3.60 is evaluated assuming constant flowing pressure, \( p_{pwf} \), and the \( t_a \) definition is used we obtain

\[ \ln \left( \frac{\bar{p}_p - p_{pwf}}{\bar{p}_i - p_{pwf}} \right) = -\frac{t_a}{c_g G \left( 141.2 \frac{\overline{B_{gi}}}{{\mu g}} \frac{\lambda_{gi}}{k_g h} b_{D,pss} \right)} \] \hspace{1cm} (3.61)

which upon exponentiation gives us

\[ \left( \frac{\bar{p}_p - p_{pwf}}{\bar{p}_i - p_{pwf}} \right) = \exp \left( -\frac{t_a}{c_g G \left( 141.2 \frac{\overline{B_{gi}}}{{\mu g}} \frac{\lambda_{gi}}{k_g h} b_{D,pss} \right)} \right) \] \hspace{1cm} (3.62)

Rearranging Eq. 3.57 we have

\[ \bar{p}_p - p_{pwf} = q_g \left( 141.2 \frac{\overline{B_{gi}}}{{\mu g}} \frac{\lambda_{gi}}{k_g h} b_{D,pss} \right) \] \hspace{1cm} (3.63)

Substituting Eq. 3.63 into Eq. 3.62 we obtain
\[
\left( q_g \frac{141.2 \frac{B_{gi} \mu_{gi}}{k_g h} b_{D,pss}}{p_{pi} - p_{pwf}} \right) = \exp \left\{ - \frac{t_a}{c_{gi} G \left[ 141.2 \frac{B_{gi} \mu_{gi}}{k_g h} b_{D,pss} \right]} \right\} \quad (3.64)
\]

Substituting the definition of \( b_{D,pss} \) into Eq. 3.64 we have

\[
\frac{q_g}{p_{pi} - p_{pwf}} \left[ 141.2 \frac{B_{gi} \mu_{gi}}{k_g h} \right] \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) = \exp \left( - \frac{k_g h}{141.2 B_{gi} \mu_{gi} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right]} \frac{t_a}{c_{gi} G} \right) \quad (3.65)
\]

The definition of the dimensionless flow rate is

\[
q_D = 141.2 \frac{q_g B_{gi} \mu_{gi}}{(p_{pi} - p_{pwf}) k_g h} \quad (3.66)
\]

Substituting this definition into Eq. 3.65 and rearranging, it follows that

\[
q_D \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right] = \exp \left( - \frac{k_g}{141.2 B_{gi} \mu_{gi} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right]} \frac{t_a}{G c_{gi} B_{gi} h} \right) \quad (3.67)
\]

This relation may be further simplified using two other definitions. The first definition relates gas-in-place, \( G \), with the dimensions of the reservoir and is given as

\[
\frac{V_p}{h} = \frac{B_{gi}}{h} G = \frac{\phi A}{5.615}
\]

The second definition is simply the dimensionless decline flow rate (in its general geometry form) given by

\[
q_{Dd} = q_D \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right]
\]

Combining the three last expressions we obtain

\[
q_{Dd} = \exp \left( - \frac{5.615}{141.2} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right] \phi_{gi} \mu_{gi} c_{gi} A \right)
\]

\[
q_{Dd} = \exp \left( - \frac{5.615}{141.2(0.00633)} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right] \frac{0.00633}{\phi_{gi} \mu_{gi} c_{gi} A} \right)
\]

or finally we have
\[ q_{Dd} = \exp \left[ - \frac{2\pi}{2 \ln \left( \frac{4 \cdot A}{e^\gamma C_A r_w^2} \right)} \right] t_{a,DA} \] .................................................. (3.68)

where

\[ t_{a,DA} = \frac{0.00633 \cdot k_g t_a}{\phi \mu_{gi} c_{gi} A} \] .................................................. (3.69)

The original equation derived by Fraim and Wattenbarger\textsuperscript{16} is obtained by a simple rearrangement of Eq. 3.68. Substituting Eq. 3.69 into Eq. 3.68 we have

\[ q_{Dd} = \exp \left[ - \frac{2\pi}{2 \ln \left( \frac{4 \cdot A}{e^\gamma C_A r_w^2} \right)} \frac{0.00633 \cdot k_g t_a}{\phi \mu_{gi} c_{gi} r_w^2} \frac{r_w^2}{A} \right] \] .................................................. (3.70)

Rearranging we obtain the following

\[ q_{Dd} = \exp \left[ - \frac{0.00633 \cdot k_g t_a}{\phi \mu_{gi} c_{gi} r_w^2} \frac{1}{2} \frac{A}{\pi r_w^2} \ln \left( \frac{2.2458 \cdot A}{C_A r_w^2} \right) \right] \] .................................................. (3.71)

Eq. 3.71 is exactly the Fraim and Wattenbarger's\textsuperscript{16} result. Using the modified decline dimensionless variable definitions we can write

\[ q_{Dd} = \exp (- t_{a,DD}) \] .................................................. (3.72)

where the dimensionless decline time is now defined as

\[ t_{a,DD} = \frac{0.00633 \cdot k_g t_a}{\phi \mu_{gi} c_{gi} r_w^2} \frac{1}{2} \frac{A}{\pi r_w^2} \ln \left( \frac{2.2458 \cdot A}{C_A r_w^2} \right) \] .................................................. (3.73)

These results suggest that gas well production data which are obtained under constant bottomhole flowing pressure conditions can be analyzed using the Fetkovich\textsuperscript{8} type curve. This should be done using the b=0 stem on Fetkovich\textsuperscript{8} type curve provided that the correct pseudopressure and pseudotime relations are used. This is the most significant contribution of the work by Fraim and Wattenbarger.\textsuperscript{16} It must be noted that this is still a limited solution, and the actual problem of variable rate and/or pressure drop remains unsolved.

In addition to the limiting assumption of a constant bottomhole pressure we also require to compute the gas-in-place and pseudotime using this method. The gas-in-place,
$G$, is needed in order to compute the average reservoir pressure for evaluating the fluid properties involved in the pseudotime integral. Gas-in-place is almost always unknown and is in fact an objective of decline curve analysis. This forces us to use an iterative procedure for the computation of gas-in-place and pseudotime. In this process, the first value assumed for the gas-in-place, $G$, is suggested to be the value obtained by matching the log-log plot curve of flow rates versus actual time to the Fetkovich\textsuperscript{8} type curve.

This constitutes the second condition that in addition to considering a fixed backpressure, makes us need a more direct means of computing the pseudotime function. We will find that the next method we describe was developed as an attempt to take care of more realistic situations where rate and/or pressure drop change arbitrarily with time.

3.2.2.2 Method of Blasingame and Lee for Variable Rate Gas Flow

Blasingame and Lee\textsuperscript{25} proposed a semi-analytical pseudotime approach for the general variable rate/variable pressure drop case. This work originated from previous efforts regarding a general variable-rate expression developed for boundary dominated flow of a liquid. Based on this development it was shown that for the liquid case, a Cartesian plot of the pressure difference divided by the flow rate versus the material balance time (cumulative production divided by flowrate), allows us to obtain estimates of fluid-in-place and gives a rough approximation of the shape of the reservoir. This approach requires that certain parameters (permeability and skin factor for example) be known in advance.

In order to develop a similar concept for the gas case, a new pseudotime function as well as a modified gas flow equation were proposed. These new tools were verified in the original study\textsuperscript{25} by the use of reservoir simulation techniques. In the next chapter, the analytical proofs for both the new pseudotime and the modified gas flow equation are presented.

We will now reproduce the method proposed by Blasingame and Lee.\textsuperscript{25} In doing so, we must start with the liquid case. The material balance equation for liquid is given by

$$\frac{(p_i - p)}{q_o} = \frac{1}{N} \frac{N \dot{p}}{c_t q_o} = \frac{1}{N} \frac{1}{c_t} \ddot{t} \quad \text{................................................................. (3.74)}$$
Blasingame and Lee\textsuperscript{25} proposed that in the gas case, a "modified" material balance equation should have exactly the same form as the liquid relation

\[
\frac{(p_{pi} - \bar{p}_p)}{q_g} = \frac{1}{G c_i} \tilde{t}_a \tag{3.75}
\]

As we will prove rigorously later in this thesis, Eq. 3.75 is valid when the new pseudotime variable is given by

\[
\tilde{t}_a = \mu_g c_{ii} \int_0^t \frac{q_g}{\mu_g(\bar{p}) c_i(\bar{p})} \, dt \tag{3.76}
\]

As stated before, these relations were proposed empirically, by comparison to the work of Fraim and Wattenbarger\textsuperscript{16} and were only proved to be valid by simulation. The authors then coupled this relation with the pseudosteady-state flow equation for gas, to create a new relation that was used to determine gas-in-place and reservoir shape from production data. Specifically, Eq. 3.75 was coupled with the pseudosteady-state flow relation given by Al-Hussainy et al.\textsuperscript{35} for the flow of single phase gas. This relation is given as

\[
\frac{(p_{pi} - \bar{p}_p)}{q_g} = 141.2 \frac{\mu_g B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right] \tag{3.77}
\]

Combining Eqs. 3.75 and 3.77 produces the following relation for variable-rate, post transient gas-flow

\[
\frac{(p_{pi} - p_{pwf})}{q_g} = 141.2 \frac{\mu_g B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_A r_w^2} \right) \right] + \frac{1}{G c_{ii}} \tilde{t}_a \tag{3.78}
\]

Using shorthand notation similar to that used before for the liquid case, we can write Eq. 3.78 as

\[
\frac{\Delta p_p}{q_g} = m_a \tilde{t}_a + b_{a,pss} \tag{3.79}
\]

where

\[
\Delta p_p = (p_{pi} - p_{pwf}) \tag{3.80}
\]

\[
m_a = \frac{1}{G c_{ii}} \tag{3.81}
\]

\[
b_{a,pss} = 141.2 \frac{\mu_g B_{gi}}{k_g h} - b_{D,pss} \tag{3.82}
\]

and
\[ b_{D,pss} = \frac{1}{2} \ln \left( \frac{4}{e^7} \frac{A}{C_A \nu_w^2} \right) \] (3.83)

Eq. 3.79 is identical in form to Eq. 3.33 which applies to the liquid case. Eq. 3.79 illustrates that the techniques presented to analyze liquid production data may be extended to the gas case with no significant variation. The computation of pseudotimes however, requires significantly more effort than the simple calculation of the cumulative production divided by the flow rate, which is the material balance time function required for the liquid case. As with the work of Fraim and Wattenbarger, this approach also requires an estimate of gas-in-place in order to obtain the correct pseudotime function. Once more however, \( G \) is an objective rather than a known value and we are again left with an iterative procedure. Blasingame and Lee\(^{25} \) suggest that the first value of gas-in-place with which to start the iterative process be somewhat higher than the last cumulative production value reported. A convergence criteria must be met in relation to the value of gas-in-place to ensure that the correct pseudotimes have been obtained.

An important remark made by Blasingame and Lee\(^{25} \) is that since the equations proposed only model boundary dominated flow, the gas-in-place estimated by this means ignores the effect of transient production. The authors showed that the "extra" gas production can be obtained as the difference between the actual observed production, and that calculated when the boundary dominated flow model is extrapolated into the transient flow region. This extra production must be added to the gas-in-place obtained from the graphical analysis proposed by Blasingame and Lee. This computation also requires that the correct start of stabilized flow be chosen.

At this point, application of these relations for decline curve analysis was not undertaken. However, an algebraic rearrangement of Eq. 3.79 shows that a harmonic type of decline will result for \( \frac{q_g}{\Delta P_p} \) versus \( t_o \) on a log-log plot. We will demonstrate the development of these harmonic decline relations for single phase liquid and gas flow in Chapter IV.

3.2.2.3 Method of Ding et al. for Constant Rate and Constant Pressure Gas Flow

Ding et al.\(^{26} \) present non-iterative methods for the analysis of gas well production data for production at a constant rate or constant bottomhole pressure. Since the constant
pressure technique is the one more closely related to decline curve analysis, we will only discuss that case in this thesis. This method permits us to estimate average reservoir pressures, pseudotime functions, and gas reservoir pore volume directly from boundary dominated flow data.

Although the methods of Ding et al.\textsuperscript{26} do not require iteration, these methods exhibit at least two major drawbacks in the light of practical applications. The first and most important of these drawbacks is the requirement of the accurate computation of gas flow rate derivatives. It is well known that field production data are hardly smooth, much less differentiable, especially since flow rates are usually measured much less precisely than pressures. The second disadvantage of this method is the assumption of a constant backpressure condition, which is rarely found in actual field operations.

In short, while this method does avoid the use of iterative schemes its application to actual field data is very awkward. Nevertheless, the method deserves further discussion, if for no other reason than as a comparison tool. Again, our developments are made in terms of normalized pseudovariables in order to keep consistency throughout this thesis.

This process starts by combining Eq. 3.68 with Eqs. 3.64 and 3.66. This combination yields a different form of Eq. 3.68 which is given as

\[
141.2 \frac{q_g B_g i \mu g_i}{k_g h(p_{pi} - p_{pwf})} = \frac{1}{b_{D, pss}} \exp \left[ - \frac{2\pi}{b_{D, pss} I_{a, DA}} \right] \ldots \ldots (3.84)
\]

where

\[
I_{a, DA} = \frac{0.00633 k_g t_a}{\phi \mu g_i c_g A}
\]

\[
t_a = \mu g_i c_g \int_0^t \frac{1}{\mu g(\bar{p}) c_g(\bar{p})} \, dt
\]

and

\[
b_{D, pss} = \frac{1}{2} \ln \left( \frac{4 A}{\varepsilon \gamma C_A r_w^2} \right)
\]

Solving Eq. 3.84 for rate and taking the natural logarithm of the result gives us

\[
\ln(q_g) = -\frac{2\pi}{b_{D, pss} I_{a, DA}} + \ln \left[ \frac{k_g h(p_{pi} - p_{pwf})}{141.2 B_g i \mu g_i b_{D, pss}} \right] \ldots \ldots (3.85)
\]
The time derivative of Eq. 3.85 may be written as

$$\frac{d}{dt} \left[ \ln (q_g) \right] = \frac{d}{dt} \left[ -\frac{2\pi}{b_{D,pss}} t_{a,DA} \right] \tag{3.86}$$

Recalling Eq. 3.62 we have

$$\left( \frac{\bar{p}_p - p_{pwf}}{p_{pi} - p_{pwf}} \right) = \exp \left( -\frac{t_a}{c_{gi} G \left( 141.2 \frac{B_{g}\mu_{gi}}{k_{gh}} b_{D,pss} \right)} \right) \tag{3.62}$$

which can also be written as

$$\left( \frac{\bar{p}_p - p_{pwf}}{p_{pi} - p_{pwf}} \right) = \exp \left[ -\frac{2\pi}{b_{D,pss}} t_{a,DA} \right] \tag{3.87}$$

Taking the natural logarithm of Eq. 3.87 we obtain

$$\ln \left( \frac{\bar{p}_p - p_{pwf}}{p_{pi} - p_{pwf}} \right) = \left[ -\frac{2\pi}{b_{D,pss}} t_{a,DA} \right] \tag{3.88}$$

Differentiating Eq. 3.88 with respect to time gives

$$\frac{d}{dt} \left[ \ln \left( \frac{\bar{p}_p - p_{pwf}}{p_{pi} - p_{pwf}} \right) \right] = \frac{d}{dt} \left[ -\frac{2\pi}{b_{D,pss}} t_{a,DA} \right] \tag{3.89}$$

Combining Eqs. 3.86 and 3.89 we have

$$\frac{d}{dt} \left[ \ln \left( \frac{\bar{p}_p(t) - p_{pwf}}{p_{pi} - p_{pwf}} \right) \right] = \frac{d}{dt} \left[ \ln (q_g) \right] \tag{3.90}$$

where Eq. 3.90 is only valid for boundary dominated (or pseudo-steady state) flow. If Eq. 3.90 is integrated from the time at the start of pseudosteady state, \( t_{pss} \), to some arbitrary time, \( t \), larger than \( t_{pss} \), it follows that

$$\ln \left( \frac{\bar{p}_p(t) - p_{pwf}}{p_{pi} - p_{pwf}} \right) - \ln \left( \frac{\bar{p}_p(t_{pss}) - p_{pwf}}{p_{pi} - p_{pwf}} \right) = \ln \left( \frac{q_g(t)}{q_g(t_{pss})} \right) \tag{3.91}$$

where \( q_g(t_{pss}) \) and \( \bar{p}_p(t_{pss}) \) represent the gas flow rate and the normalized pseudopressure both taken at the time of the onset of pseudosteady-state (or boundary dominated) flow. Taking the logarithmic time derivative of Eq. 3.85 we obtain

$$\frac{d}{dt} \left[ \ln \left( \frac{q_g(t)}{q_g(t_{pss})} \right) \right] = \frac{d}{dt} \left[ \frac{\ln (q_g(t))}{\ln (q_g(t_{pss}))} \right] = t \left[ \frac{d}{dt} \ln (q_g(t)) \right] = -\frac{2\pi}{b_{D,pss}} t_{a,DA} \tag{3.92}$$

Eqs. 3.88 and 3.92 are only valid during pseudosteady-state flow. Therefore, these relations can be evaluated at the start of pseudosteady state, \( t_{pss} \), and then equated. Performing these operations we obtain
\[
\ln \left( \frac{\bar{p}_g(t_{pss}) - p_{pwf}}{p_{pi} - p_{pwf}} \right) = t_{pss} \left[ \frac{d}{dt} \ln \left( q_g(t) \right) \right]_{t_{pss}} \tag{3.93}
\]

Substituting Eq. 3.93 into Eq. 3.91 gives us
\[
\ln \left( \frac{\bar{p}_g(t) - p_{pwf}}{p_{pi} - p_{pwf}} \right) = t_{pss} \left[ \frac{d}{dt} \ln \left( q_g(t) \right) \right]_{t_{pss}} + \ln \left( \frac{q_g(t)}{q_g(t_{pss})} \right) \tag{3.94}
\]

where Eq. 3.94 is only valid for \( t \geq t_{pss} \) only. This is the relation from which Ding et al.\(^{26}\) propose to obtain the average reservoir pressure profile. It is clear that our ability to estimate \( \bar{p}(t) \) depends on the derivative of the flow rate at pseudosteady state, and the time for the start of that flow regime.

Substituting the definition of \( t_{a,Dd} \) (Eq. 3.69) into Eq. 3.85 gives us
\[
\ln \left[ q_g(t) \right] = - \frac{2\pi}{b_{D,pss}} 0.00633k_g \frac{h}{\phi h \mu_i c_g} t_a + \ln \left[ \frac{k_g h (p_{pi} - p_{pwf})}{141.2B_{gi} \mu_i} \right] \tag{3.95}
\]

Eq. 3.95 implies that a cartesian plot of \( \ln [q_g(t)] \) versus \( t_a \), should give a straight line with a slope \( m_d \) equal to
\[
m_d = - \frac{2\pi}{b_{D,pss}} 0.00633k_g \frac{h}{\phi h \mu_i c_g} \tag{3.96}
\]
and an intercept at \( t_a=0 \) equal to \( \ln(q_{ini}) \), where
\[
q_{ini} = \frac{k_g h (p_{pi} - p_{pwf})}{141.2B_{gi} \mu_i} \tag{3.97}
\]

The gas-in-place, \( G \), is estimated from the relation that results from dividing Eq. 3.96 by Eq. 3.97 which is given as
\[
\frac{m_d}{q_{ini}} = - 5.615 \frac{B_{gi}}{\phi h \mu_i (p_{pi} - p_{pwf})} \tag{3.98}
\]

Thus, since \( G = \frac{\phi h}{5.615B_{gi}} \), with \( B_{gi} \) in [RB/Mscf], Eq. 3.98 can be solved directly for \( G \). This gives us
\[
G = - \frac{1}{c_i (p_{pi} - p_{pwf})} \frac{q_{ini}}{m_d} \tag{3.99}
\]

It is obvious that \( G \) has to be a positive value, and should be taken as the absolute value of Eq. 3.99. If estimates of porosity and reservoir thickness are available, then the reservoir drainage area, \( A \), can be computed from the gas-in-place value that was just
obtained. Also, if estimates of permeability and skin factor are available, then the shape factor can be estimated from Eq. 3.97 noting that
\[
\frac{1}{2} \ln \left( \frac{4}{e^y} \frac{A}{C_A r_w^2} \right) = \frac{k_h h (p_{pi} - p_{pwi})}{141.2 B_g \mu_g q_{ini}}
\]
which when we solve for \( C_A \) gives us
\[
C_A = \frac{4}{e^y} \frac{A}{r_w^2} \exp \left[ -\frac{k_h h (p_{pi} - p_{pwi})}{141.2 B_g \mu_g q_{ini}} \right]
\]
These relations conform the basis of the constant pressure analysis approach proposed by Ding et al.\(^{26}\) The analysis procedure may be summarized as follows:

1. An estimate of the time at which boundary dominated flow starts \( t_{pss} \) is required in order to use Eq. 3.94. This time must be obtained from a preliminary type curve match of the \( q_g \) versus actual time plot with constant pressure production type curves of \( q_{Dd} \) versus \( t_{Dd} \) for the liquid case. This time is approximated as the time at which the rate data begin departing from the infinite acting solution.

2. Next, the time to pseudosteady state, \( t_{pss} \), and the flow rate derivative at \( t_{pss} \) are used to estimate the average reservoir pseudopressures from Eq. 3.94. These \( \bar{p}_p \) values are then used in a table lookup process to determine the actual average reservoir pressures, \( \bar{p} \). The average reservoir pressures in turn permit us to estimate the normalized pseudotime functions, \( t_a \).

3. Finally, a Cartesian plot of \( \ln(q_g) \) versus \( t_a \) must be made. The slope and intercept of this plot allow the computation of \( G \) and \( C_A \) from Eqs. 3.99 and 3.101 respectively.
CHAPTER IV

DEVELOPMENT OF NEW DECLINE CURVE ANALYSIS TECHNIQUES

This chapter summarizes the new contributions presented in this thesis. In particular we will present and discuss the following:

1. An approximate solution for the constant pressure dimensionless rate solution, $q_D$, will be developed using Laplace transform methods and the constant rate dimensionless pressure solution, $p_D$. We verify this development for $q_D$ by comparison with known rate responses for certain types of reservoirs.

2. We present the analytical proof for the validity of the pseudotime function for variable rate/variable pressure drop type of analysis proposed by Blasingame and Lee.$^{25}$

3. We present the derivation of a harmonic type of decline for both gas and liquid flow. These derivations are based on rigorous material balance time functions that have been already presented in the literature, but not posed for decline curve analysis.

4. We present an algorithm that combines the gas material equation and the pseudosteady-state gas flow equation. The purpose of this algorithm is to estimate the average reservoir pressure profile and the gas-in-place with little or no iteration. The new coupling leads to a method where only a few successive substitutions are required to obtain the correct average reservoir pressure profile. Specifically, the computation of gas-in-place is independent of the decline curve analysis. The estimate of gas-in-place and the computed average reservoir pressures are then used to estimate the pseudotime function which is necessary for the rigorous interpretation of gas production data.

5. We also present a method that can be used to compute the "constant pressure analog time" as proposed by Blasingame et al.$^{29}$ This new method avoids the need for knowledge of the value of formation parameters in advance that was previously required.
6. Finally, we present a hybrid type curve that contains solutions for both gas and liquid flow. This type curve is called the "Fetkovich-Carter type curve."

Another focus of this thesis is to develop analysis and interpretation techniques that allow us to consider production data where the flowrate and pressure vary in an arbitrary fashion. These methods will be discussed in this Chapter and verified in Chapter V.

4.1 REAL SPACE APPROXIMATION TO THE CONSTANT PRESSURE RATE SOLUTION

4.1.1 Development of an Approximate Analytical Solution

In Appendix B of this thesis we present the assumptions and the mathematical developments that we used to obtain the approximate solution to the constant well pressure, rate solution. We show that we can assume a simple approximate model for the constant rate dimensionless pressure solution, \( p_D \), and obtain an accurate expression for the constant pressure dimensionless rate solution, \( q_D \).

We assume that the \( p_D \) solution can be accurately modeled as a simple linear function that is given by

\[
p_{D, CR} = a_1 + a_2 t_D
\]

(4.1)

We then use the superposition formula in Laplace space for relating constant rate and constant pressure solutions given by van Everdingen and Hurst\(^{33}\) (and rederived in a general fashion in Appendix C). Leaving the details of this development in Appendix B, we have the final result for \( q_D \) which is given as

\[
q_{D, CP} (r_D = 1, t_D) = \frac{1}{a_1} \exp \left( -\frac{a_2}{a_1} t_D \right)
\]

(4.2)

The parameters \( a_1 \) and \( a_2 \) can be written in terms of \( p_D \) and \( p'_D \), which gives

\[
q_{D, CP} = \frac{1}{p_{D, CR(t_D)} - p'_{D, CR(t_D)}} \exp \left[ \frac{1}{p_D, CR(t_D)} \left( 1 - \frac{p_{D, CR(t_D)}}{p'_{D, CR(t_D)}} \right) \right]
\]

(4.3)

where the \( p'_D \) term is the pressure derivative defined as
\[ p_{D,C} = \frac{dP_{D,C}}{dt} \] ..........................................................(4.4)

It is important to note that Eq. 4.3 is exact for pseudosteady-state flow and only an approximation for transient flow. This is because we assumed that the dimensionless pressure response is a linear function of dimensionless time. However, as we will show, Eq. 4.3 has been verified by comparison to the analytical solution to give accurate results even for the transient flow regime.

We will verify this approximation concept by coupling Eq. 4.3 with the constant rate dimensionless pressure responses and their derivatives for a bounded circular reservoir with no flow and constant pressure outer boundary conditions. The \( p_D \) and \( p'_D \) required for this comparison are obtained from developments that are also presented in Appendix B. For the case of a bounded circular reservoir with a no flow outer boundary we have derived the following solutions for the constant rate dimensionless pressure solution and its well testing derivative

\[
p_D(r, t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) - \frac{1}{2} E_1 \left( \frac{r_{ad}^2}{4t_D} \right) + \frac{2t_D}{r_D^2} \exp \left( \frac{-r_{ad}^2}{4t_D} \right) \\
+ \left( \frac{r_D^2}{r_{ad}^2} - \frac{1}{4} \right) \exp \left( \frac{-r_{ad}^2}{4t_D} \right) \hspace{1cm} ......................(4.5)
\]

\[
p'_D(r, t_D) = \frac{1}{2} \exp \left( \frac{-r_D^2}{4t_D} \right) + \frac{2t_D}{r_{ad}^2} \exp \left( \frac{-r_{ad}^2}{4t_D} \right) \hspace{1cm} + \frac{1}{2t_D} \left( \frac{r_D^2}{4} - \frac{r_{ad}^2}{8} \right) \exp \left( \frac{-r_{ad}^2}{4t_D} \right) \hspace{1cm} ......................(4.6)
\]

For the case of a bounded circular reservoir with a constant pressure outer boundary we have derived the following solutions for the constant rate dimensionless pressure response and its well testing derivative. These solutions are given as

\[
p_D(r, t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) - \frac{1}{2} E_1 \left( \frac{r_{ad}^2}{4t_D} \right) - \frac{(r_D^2 - r_{ad}^2)}{8t_D} \exp \left( \frac{-r_{ad}^2}{4t_D} \right) \hspace{1cm} ......................(4.7)
\]

and
\[ p'_D(r_D, t_D) = \frac{1}{2} \left[ \exp \left( -\frac{r_D^2}{4t_D} \right) - \exp \left( -\frac{2r_D^2}{4t_D} \right) \right] \\
+ \left( r_D^2 - \frac{r_D^2}{8t_D} \right) \left( \frac{r_D^2}{4t_D} - 1 \right) \exp \left( -\frac{r_D^2}{4t_D} \right) \] ............................................(4.8)

We note that these four equations alone constitute valuable new tools for reservoir engineering applications for they allow the accurate computation of \( p_D \) and \( p'_D \) directly in real space for the type of reservoir and conditions described above.

Using Eqs. 4.5-4.8 coupled with Eq. 4.3, we will generate and compare values with the analytical solution obtained by numerical inversion. The rigorous solutions for these cases are derived in Appendix B. The Laplace transform solution for a well in a circular reservoir with a no flow outer boundary is

\[ \bar{q}_{D, CP} (r_D=1, u) = \frac{I_1(\sqrt{u} r_D)}{u [I_0(\sqrt{u}) K_1(\sqrt{u} r_D) + K_0(\sqrt{u}) I_1(\sqrt{u} r_D) ]} \] ............................................(4.9)

and the Laplace transform solution for a constant pressure outer boundary is

\[ \bar{q}_{D, CP} (r_D=1, u) = \frac{I_0(\sqrt{u} r_D)}{u [K_0(\sqrt{u}) I_0(\sqrt{u} r_D) - K_0(\sqrt{u} r_D) I_0(\sqrt{u}) ]} \] ............................................(4.10)

We believe that it is unlikely that a direct inversion of Eqs. 4.9 and 4.10 into real space can be found in terms of known functions. This is why we use numerical inversion of these solutions for comparison with the approximate solutions discussed above.

4.1.2 Graphical Verification

In this section we provide the verification plots for the equations presented in the previous section. The first of these plots, Fig. 4.1, corresponds to the case of the constant rate dimensionless pressure response and its derivative for a bounded circular reservoir with a no flow outer boundary. The \( p_D \) and \( p'_D \) variables in this case are given by Eqs. 4.5 and 4.6, respectively. The verification plot also includes the response obtained from the numerical inversion of the rigorous solution in Laplace space.
Fig. 4.1 - Analytical and Approximate Values of $p_D$ and $p'_D$ for an Unfractured Well Centered in a Bounded Circular Reservoir Producing at Constant Flow Rate ($r_D = 1, s = 0$) with a No Flow Outer Boundary
We note excellent agreement between the approximate and numerical inversion solution in Fig. 4.1. We find that the proposed solutions give very good comparisons for all values of $r_{eD}$ and we can conclude that the proposed real space relations for $p_D$ and $p_D'$ given by Eqs. 4.5 and 4.6 respectively, are accurate approximations; certainly accurate enough for field applications.

Fig. 4.2 represents the solution for a bounded circular reservoir with a constant pressure outer boundary rather than a no flow outer boundary as shown in Fig. 4.1. The proposed solutions for $p_D$ and $p_D'$ in this case are given by Eqs. 4.7 and 4.8, respectively, and the derivation of these relations can be found in Appendix B.

We again note excellent agreement between the numerical inversion solutions and the proposed approximations. This suggests that the approximations given by Eqs. 4.7 and 4.8 are also sufficiently accurate for application in the field. Nevertheless, we note that the behavior of the $p_D'$ function in Fig. 4.2 does show some deviation from the numerical inversion solution as the $p_D'$ function undergoes transition from transient to steady-state flow. Although we developed more complex expansions as an attempt to solve this problem, no solution gave better performance than Eq 4.8. The overall behavior as the reservoir approaches steady-state is very difficult to model due to the tendency of the $p_D$ solution to flatten horizontally, and the $p_D'$ function to decay toward zero.

In order to ensure completeness we present a semilog plot of the $p_D$ functions for both cases, no flow outer boundary and constant pressure outer boundary, in Fig. 4.3. Once more we recognize that the agreement between the approximate and numerical inversion solutions is excellent, further confirming the validity of Eqs. 4.5 and 4.7.

The next verification plot, Fig. 4.4, shows the comparison of the numerical inversion solution and the real space approximation for the constant pressure dimensionless rate solution presented in the previous section. Specifically, Fig. 4.4 verifies the use of Eq. 4.3 for computation purposes. Here, the $p_D$ and $p_D'$ values are obtained from the relations for a bounded circular reservoir with a no-flow outer boundary (Eqs. 4.5 and 4.6). The result of that computation is compared to the numerical inversion from Laplace space of Eq. 4.9. We again note excellent agreement in the computed solutions.

Fig. 4.5 is our final verification plot. We note that the results shown in Fig. 4.5
Fig. 4.2 - Analytical and Approximate Values of $p_D$ and $p'_D$ for an Unfractured Well Centered in a Bounded Circular Reservoir Producing at Constant Flow Rate ($r_D = 1, s = 0$) with a Constant Pressure Outer Boundary
Fig. 4.3 - Semilog Plot of Analytical and Approximate Values of $p_D$ for an Unfractured Well Centered in a Bounded Circular Reservoir Producing at Constant Flow Rate ($r_D = 1, s = 0$) with either No Flow Outer Boundary or Constant Pressure Outer Boundary
Fig. 4.4 - Analytical and Approximate Values of $q_D$ for an Unfractured Well Centered in a Bounded Circular Reservoir Producing at Constant Flow Rate ($r_D = 1, s = 0$) with a No Flow Outer Boundary
Fig. 4.5 - Analytical and Approximate Values of $q_D$ for an Unfractured Well Centered in a Bounded Circular Reservoir Producing at Constant Flow Rate ($f = 1, z = 0$) with a Constant Pressure Outer Boundary
represent the case of a bounded circular reservoir that has a constant pressure outer boundary rather than a no flow exterior boundary as illustrated in Fig. 4.4. Fig. 4.5 shows the results of Eq. 4.3 solved in combination with Eqs. 4.7 and 4.8, where these results are compared to those obtained from the numerical inversion of the Laplace space solution given by Eq. 4.10.

Again the agreement between the approximate solution and the results obtained from the numerical inversion of Eq. 4.10 is excellent, confirming the validity of Eq. 4.3 as an accurate means of computing the constant pressure dimensionless rate solution.

In summary, even though we note that the assumptions of Eq. 4.3 are somewhat limiting, there is excellent agreement between the approximate and the numerical inversion solutions for the no-flow and constant pressure outer boundary cases. Therefore, we conclude that the approximations presented in this section can be used for the prediction, analysis, and interpretation of reservoir performance data.

4.2 ANALYTICAL PROOF OF THE PSEUDOTIME FUNCTION FOR VARIABLE RATE/VARIABLE PRESSURE DROP ANALYSIS

As we stated earlier in this chapter, we will provide a rigorous proof that the empirical "material balance pseudotime" function proposed by Blasingame and Lee\textsuperscript{25} is correct and has a sound theoretical basis. Blasingame and Lee suggest that

\[ \frac{p_{pi} - p_p}{q_g} = m_a \tilde{t}_a \] .................................(4.11)

where

\[ m_a = \frac{1}{G c_{ii}} \] .................................(4.12)

\[ \tilde{t}_a = \mu g_i c_{ii} \int_0^t \frac{q_g}{\mu g c_t} \, dt \] .................................(4.13)

Blasingame and Lee\textsuperscript{25} provide a verification of Eq. 4.11 using numerical simulation. However, we need a rigorous proof that Eq. 4.13 is the correct general variable rate pseudotime definition, like the proof that Fraim and Wattenbarger\textsuperscript{16} developed for the
constant pressure case. We begin this process by recalling the definition of isothermal gas compressibility given as

\[ c_g = \frac{1}{\rho} \left( \frac{dp}{dp} \right)_T \]

or

\[ c_g = \frac{zRT}{pM} \frac{dp}{dp} \left( \frac{pM}{zRT} \right) = \frac{z}{p \frac{dp}{dz}} \] \hspace{1cm} (4.14)

Evaluating this definition at the average reservoir pressure we have

\[ \frac{d}{dp} \left( \frac{p}{z} \right) = \frac{\overline{p}}{z} \overline{c}_g \] \hspace{1cm} (4.15)

Also, if we solve the gas material balance equation for cumulative production and then differentiate the result we obtain

\[ q_g = - \frac{G}{p_i} \frac{d}{dt} \left( \frac{p}{z} \right) \] \hspace{1cm} (4.16)

Using the chain rule on Eq. 4.16 we have

\[ q_g = \frac{dG_p}{dp} \frac{dp}{dt} \]

then

\[ q_g = - \frac{G}{p_i} \frac{zi}{z} \frac{\overline{p}}{z} \frac{d\overline{p}}{dt} \] \hspace{1cm} (4.17)

Combining Eqs. 4.15 and 4.17 gives

\[ q_g = - \frac{G}{p_i} \frac{zi}{z} \frac{\overline{p}}{z} \frac{d\overline{p}}{dt} \] \hspace{1cm} (4.18)

Substituting Eq. 4.18 into Eq. 4.13 we obtain the following expression for normalized pseudotime

\[ i_a = \frac{\mu g_i c_{li}}{q_g} \left( - \frac{G}{p_i} \frac{zi}{z} \right) \int_0^t \frac{\overline{p}}{z} \overline{c}_g \frac{1}{\mu g_i c_i} \frac{dp}{dt} \hspace{1cm} (4.19) \]

When we assume that the rock compressibility and water saturation-compressibility product are negligible compared to the gas compressibility, then \( c_i = c_g \). If we use this approximation and change the variable of integration from time to pressure, Eq. 4.19 may be written as
\[ \dot{t}_a = \frac{G}{q_g} \left( \frac{\mu_{gi} z_i}{p_i} \right) \int_{p_i}^{\bar{p}} \frac{p}{\mu_{gz}} \, dp \]  

...(4.20)

Furthermore, using the definition of normalized pseudopressure we can represent the pseudopressure difference between the values obtained at initial and average reservoir by

\[ p_{pi} - \bar{p}_p = \left( \frac{\mu_{gi} z_i}{p_i} \right) \int_{p_i}^{\bar{p}} \frac{p}{\mu_{gz}} \, dp \]  

...(4.21)

Combining Eq. 4.20 and Eq. 4.21 gives us

\[ \dot{t}_a = G \, c_{zi} \frac{p_{pi} - \bar{p}_p}{q_g} \]

which can be rearranged to yield

\[ \frac{p_{pi} - \bar{p}_p}{q_g} = \frac{\dot{t}_a}{G \, c_{zi}} \]  

...(4.22)

We note that Eq. 4.22 is identical to equation 4.11. This observation provides the proof that Eq. 4.13 is the correct definition of pseudotime for boundary dominated gas flow, for any production conditions.

We now recall the pseudosteady-state gas flow equation as given by Al-Hussainy and Ramey\textsuperscript{35} which is

\[ \frac{\bar{p}_p - P_{pwf}}{q_g} = 141.2 \frac{\mu_{gi} B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_{A r^2}} \right) \right] \]  

...(4.23)

Combining Eq. 4.22 and Eq. 4.23 results in the total flow equation developed by Blasingame and Lee\textsuperscript{25} for a gas well at pseudosteady-state conditions. This equation is

\[ \frac{p_{pi} - P_{pwf}}{q_g} = \frac{1}{G \, c_{zi}} \dot{t}_a + 141.2 \frac{\mu_{gi} B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma C_{A r^2}} \right) \right] \]  

...(4.24)

We note that Eq 4.24 (which is the result for gas flow) and Eq. 3.33 (which is the result for liquid flow) have identical forms. This implies that the same techniques of analysis can be used for both gas and liquids, provided that the correct pseudopressure and pseudotime variables are used for the gas flow case.
4.3 DERIVATION OF A HARMONIC DECLINE RELATION FOR VARIABLE RATE/
VARIABLE PRESSURE DROP FLOW CONDITIONS

The basis for a harmonic decline relation that can be used to model pressure drop
and rate data is Eq. 3.33. Recalling Eq. 3.33 we have

\[
\frac{(p_i - p_{wf})}{q_o} = m\tilde{t} + b_{pss} \tag{3.33}
\]

We have already demonstrated in Chapter III that Eq. 3.33 can be very useful as a
production data analysis tool. Specifically, we showed that if certain reservoir parameters
such as permeability and skin factor are known in advance, Eq. 3.33 can be used to
estimate the volume of fluid-in-place and the approximate reservoir geometry.

We will develop a harmonic relation for liquid flow using a simple rearrangement of
Eq. 3.33. A similar effort applied to Eq. 4.24 will produce a harmonic decline relation for
gas flow. The advantage of obtaining a harmonic decline is that this function will provide
the basis for matching general rate and pressure data onto the Fetkovich type curve.

As we know, the liquid and gas flow relations (Eq. 3.33 and Eq. 4.24, respectively)
are identical in form and the relations differ only in the sense of which variables are used
for each case. The liquid case considers pressure and the time function which is defined
as cumulative production divided by rate. In contrast, the gas case requires that
normalized pseudopressure and normalized pseudotime be used. Normalized pseudotime
is defined by Eq. 4.13 and is analogous to the \( \tilde{t} \) function that is used for the liquid case.

We note that the most important characteristic of the two following results is that
these relations are not constrained by the limitations imposed by the assumption of a
constant flow rate or a constant bottomhole flowing pressure. In fact, our objective for
developing these relations is to analyze liquid or gas production data obtained under
variable rate and/or variable pressure drop conditions.
4.3.1 Development of a Harmonic Decline Relation for Liquid Flow

The development of relations for both liquid and gas flow appears redundant since the fundamental equations are of exactly the same form. However, we will present these results separately to provide individual analysis relations, as well as continuity.

We will first recall Eq. 3.33 which is given as

\[
\frac{(p_i - p_{wf})}{q_o} = m\bar{t} + b_{pss} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (3.33)
\]

where the "slope" term is defined as

\[
m = \frac{5.615 \ B_o}{\phi c_i \ A} = \frac{1}{Nc_t}
\]

and the "intercept" term is defined as

\[
b_{pss} = \frac{141.2 \ B_o \mu_o}{k_o \ h} \cdot b_{D,pss}
\]

where

\[
b_{D,pss} = \left[ \frac{1}{2} \ln \left( \frac{4}{e} \cdot \frac{A}{C_A \ r_i^2} \right) \right]
\]

Rearranging Eq. 3.33 for rate normalized by pressure drop we have

\[
\frac{q_o}{(p_i - p_{wf})} = \frac{1}{m\bar{t} + b_{pss}}
\]

Factoring out the \( \frac{1}{b_{pss}} \) term on the right-hand-side gives us

\[
\frac{q_o}{(p_i - p_{wf})} = \frac{1}{b_{pss}} \cdot \left( \frac{1}{1 + \frac{m\bar{t}}{b_{pss}}} \right)
\]

Multiplying through by \( b_{pss} \) gives

\[
\frac{q_o}{(p_i - p_{wf})} \cdot b_{pss} = \frac{1}{1 + \frac{m\bar{t}}{b_{pss}}} \cdot \left( \frac{1}{b_{pss}} \right) \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (4.25)
\]

Applying the definitions of \( q_D \) and on the left-hand-side of Eq. 4.25 gives

\[
q_{Dd} = \frac{q_o}{(p_i - p_{wf})} \cdot \frac{141.2 \ B_o \mu_o}{k_o \ h} \cdot b_{D,pss} = q_D \cdot b_{D,pss} = q_D \cdot \left[ \frac{1}{2} \ln \left( \frac{4}{e} \cdot \frac{A}{C_A \ r_i^2} \right) \right]
\]

where this relation corresponds exactly to \( q_{Dd} \) as defined by Fetkovich.\(^8\) For terms on the right-hand-side of Eq. 4.25 we can apply the definitions of \( t_{DA} \) and \( b_{D,pss} \) to yield
\[ t_{Dd} = \left( \frac{m}{b_{D, pss}} \right) \bar{t} = \frac{5.615 B_o \phi C_A}{141.2 B_o \mu_o b_{D, pss}} \cdot \frac{0.00633}{0.00633} = \frac{2\pi (0.00633) k_{o,f}}{b_{D, pss}} = \frac{2\pi t_{DA}}{b_{D, pss}} \]

We note that the only difference between the last relation and the Fetkovich\(^8\) definition of \( t_{Dd} \) is that now, \( \bar{t} \) is substituted for real time \( t \), and we recall that \( \bar{t} \) is the material balance time obtained from dividing the cumulative production by the flow rate.

Rewriting Eq. 4.25 in terms of \( q_D \), \( b_{D, pss} \) and \( t_{DA} \) we have

\[ q_D b_{D, pss} = \frac{1}{1 + 2\pi t_{DA}} \]

or if we express Eq. 4.25 in terms of the decline variables (\( q_{Dd} \) and \( t_{Dd} \)) from the Fetkovich\(^8\) type curve we obtain

\[ q_{Dd} = \frac{1}{1 + t_{Dd}} \]

(4.26)

We note that Eq. 4.26 is a "hyperbolic" decline model and in particular, this is a "harmonic" decline case since the power law exponent of the denominator is unity. Eq. 4.26 shows that as long as \( \bar{t} \) is used rather than ordinary time, we can model (hence analyze) variable rate and/or variable pressure production data using the Fetkovich harmonic decline type curve for \( b=1 \). We note that Eq. 4.26 is derived for the single phase case and proves that the harmonic decline stem on the Fetkovich type curve is a rigorous solution. For clarity we recall the hyperbolic decline relation given by Arps.\(^4\) This expression is given as

\[ \frac{q_o}{q_{oi}} = \frac{1}{(1 + b D_{f,t})^{1/b}} \]

When the decline exponent is taken as unity, this expression becomes the "harmonic" decline. Setting \( b=1 \) and reducing we have

\[ q_{Dd} = q_o = \frac{1}{(1 + D_{f,t})} = \frac{1}{(1 + t_{Dd})} \]

where we have used the definitions\(^8\) \( t_{Dd} = D_{f,t} \), and \( q_{Dd} = \frac{q_o}{q_{oi}} \).

It is important to note that Eq. 4.26 is derived from Eq. 3.33 which is only rigorously valid for pseudosteady-state (or boundary dominated) flow.
At this point we present the modifications required to apply the "conventional" decline analysis using type curves\(^8\) to variable rate/variable pressure drop data. The most important distinction of our analysis is that we require a scaled log-log plot of \(\frac{q_o}{(p_i - p_{wf})}\) versus \(\bar{t}\) which must be overlaid on the \(q_{Da}\) versus \(t_{Da}\) trend for a harmonic \((b=1)\) decline on the Fetkovich\(^8\) type curve.

Once the match has been obtained, the following relations should be used to estimate reservoir properties. First we solve for the oil-in-place, \(N\), which is obtained using

\[
N = \frac{1}{c_{ti}} \frac{(\bar{t})_{M.P.}}{(q_{Da})_{M.P.}} \left( \frac{q_o}{(p_i - p_{wf})}_{M.P.} \right) \left( \frac{q_o}{(q_{Da})_{M.P.}} \right) 
\]

Then we solve for the reservoir drainage area

\[
A = 5.615 \frac{NB_o}{\phi h} 
\]

and finally we solve for the formation permeability

\[
k_o = 141.2 \frac{B_o \mu_o}{h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^\gamma} \frac{A}{C_A r_w^2} \right) \right] \left( \frac{q_o}{(p_i - p_{wf})}_{M.P.} \right) \left( \frac{q_o}{(q_{Da})_{M.P.}} \right) 
\]

where \(r_w = r_w e^{-s}\) in Eq. 4.29 and this implies that the skin factor, \(s\), is known or can be reasonably estimated. Similarly, the shape factor, \(C_A\), also in Eq. 4.29 should be known, although for many applications we can assume that \(C_A=31.62\) for a well centered in a circular reservoir.

4.3.2 Development of a Harmonic Decline Relation for Gas Flow

As we have shown in earlier developments, the gas flow equation for pseudosteady-state (or boundary dominated) flow conditions has identical form as the equation for liquid flow, assuming pseudopressure and pseudotime are used. Recalling the gas flow relation, Eq. 4.24, we have

\[
\frac{p_{pi} - p_{pwf}}{q_g} = \frac{1}{G c_{ti}} \bar{t}_a + 141.2 \frac{\mu_gi B_gi}{k_gh} \left[ \frac{1}{2} \ln \left( \frac{A}{e^\gamma C_A r_w^2} \right) \right] 
\]

which can also be rewritten using shorthand notation as

\[
\frac{p_{pi} - p_{pwf}}{q_g} = m_{g, a} \bar{t}_a + b_{a, pss} 
\]
where

\[ m_a = \frac{1}{Gc_i \bar{t}_a} \]

\[ b_{a,pss} = \frac{141.2 B_{gi} \mu_{gi}}{k_g h} b_{D,pss} \]

where

\[ b_{D,pss} = \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A r_w^2} \right) \]

We have added the "\(a\)" subscript to distinguish these gas coefficients from those for the liquid case. We are now using the material balance pseudotime \(\bar{t}_a\) as the time function and the normalized pseudopressure as the pressure function.

Because the liquid and gas flow equations have identical forms, we can use the same techniques for both gas and liquid flow data. Rearranging Eq. 3.79 we have

\[ \frac{q_g}{(p_{pi} - p_{pwf})} = \frac{1}{m_a \bar{t}_a + b_{a,pss}} \]

which can be rearranged even further to yield

\[ \frac{q_g}{(p_{pi} - p_{pwf})} b_{a,pss} = \frac{1}{1 + \left( \frac{m_a}{b_{a,pss}} \right) \bar{t}_a} \]

(4.30)

Applying the definitions of \(q_D\) and \(b_{a,pss}\) to the left-hand-side (LHS) of Eq. 4.30 we obtain

\[ q_{Dd} = \frac{q_g}{(p_{pi} - p_{pwf})} \frac{141.2 B_{gi} \mu_{gi}}{k_g h} b_{D,pss} = q_D b_{D,pss} = q_D \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A r_w^2} \right) \]

(4.31)

where this identity is equal in form to the \(q_{Dd}\) variable as defined by Fetkovich.\(^8\) The only difference is that this relation uses normalized pseudopressures rather than actual pressures. For terms on the right-hand-side of Eq. 4.30 we can apply the definitions of \(\bar{t}_a\) and \(b_{a,pss}\) to give us

\[ \bar{t}_{a,Dd} = \left( \frac{m_a}{b_{a,pss}} \right) \bar{t}_a = \frac{1}{Gc_i \bar{t}_a} \frac{k_g h}{141.2 \mu_{gi} B_{gi} \left[ \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A r_w^2} \right) \right] \bar{t}_a} \]

Combining the definition of gas-in-place, \(G = \frac{\phi A h}{5.615 B_{gi}}\) (\(B_{gi}\) in RB/Mscf), with the previous relation we obtain
\[ \tilde{t}_{a, Dd} = \frac{5.615}{141.2(2\pi)} \frac{k_g}{\mu_i c_i r_{i, D}^2} \phi \left( \frac{1}{2} \frac{A}{\pi r_{w, D}^2} \right) \left( \frac{1}{2} \ln \left( \frac{4}{e^7} \frac{A}{C_{At, D} r_{w, D}^2} \right) \right) \tilde{t}_a \]

or finally, we have

\[ \tilde{t}_{a, Dd} = 0.00633 \frac{k_g}{\phi \mu_i c_i r_{i, D}^2} \tilde{t}_a \left( \frac{1}{2} \frac{A}{\pi r_{w, D}^2} \right) \left( \frac{1}{2} \ln \left( \frac{4}{e^7} \frac{A}{C_{At, D} r_{w, D}^2} \right) \right) \] .................................(4.32)

We note that the definition of \( \tilde{t}_{a, Dd} \) given by Eq. 4.32 differs only in the use of \( \tilde{t}_a \) in place of time, \( t \), in the definition of \( t_{Dd} \) given by Fetkovich.\(^8\) The material balance pseudotime, \( \tilde{t}_a \), is given by Eq. 4.13. Similar to our efforts for the liquid case we show that Eq. 4.30 can be rearranged in terms of the \( q_{Dd} \) and \( \tilde{t}_{a, Dd} \) definitions to yield

\[ q_{Dd} = \frac{1}{1 + \tilde{t}_{a, Dd}} \] .................................................(4.33)

where for the gas case we have

\[ q_{Dd} = \frac{q_g}{(p_{pi} - p_{p,w})} b_{a, pss} \]

and

\[ \tilde{t}_{a, Dd} = \left( \frac{m_a}{b_{a, pss}} \right) \tilde{t}_a \]

As expected, Eq. 4.33 which is for gas flow, has exactly the same form of Eq. 4.26, which is for liquid flow. Therefore we recognize that Eq. 4.33 is also a harmonic decline equation, where the decline exponent is unity. This means that as long as \( \tilde{t}_a \) is used rather than ordinary time, it is possible to model variable rate and/or variable pressure production schemes for the single phase gas case using the Fetkovich harmonic decline type curve for \( b=1 \).

We now recall that for the gas case the pressure function is normalized pseudopressure and the time function is the normalized pseudotime as given by Eq. 4.13. The computation of the pseudotime function requires an iterative procedure to simultaneously estimate the gas-in-place, \( G \). In the next section of this chapter we introduce a semi-explicit method to estimate gas-in-place using the gas material balance equation and the pseudosteady state flow equation for gas flow. The method is "semi-explicit" because we have found that the method requires a few iterations per time point, rather than iterating over all of the data several times for gas-in-place convergence.
As with the liquid case, we now present the decline curve methodology for gas flow data using type curves. Eq. 4.33 suggests that if \( \tilde{t}_a \) is correctly calculated then a scaled log-log plot of \( \frac{q_g}{(p_{pi} - p_{pwf})} \) versus \( \tilde{t}_a \) will overlay the \( q_{DD} \) versus \( t_{DD} \) trend for a harmonic decline on the Fetkovich\(^8\) type curve.

Once the match has been obtained, the following relations are used to estimate the properties of gas reservoirs. We first solve for the gas-in-place, \( G \), using both time and rate match points

\[
G = \frac{1}{c_{ii}} \left( \frac{\tilde{t}_M}{(t_{DD})_{M.P.}} \right) \left( \frac{q_g}{(p_{pi} - p_{pwf})} \right)_{M.P.} \left( \frac{\tilde{t}_M}{(q_{DD})_{M.P.}} \right) \tag{4.34}
\]

We then solve for the reservoir drainage area as

\[
A = 5.615 \frac{GB_{gi}}{\phi_h} \tag{4.35}
\]

and finally we solve for the formation permeability as

\[
k_g = 141.2 \frac{B_{gi} \mu_{gi}}{h} \left[ \frac{1}{2} \ln \left( \frac{A}{C_A r_w^2} \right) \right] \left( \frac{q_g}{(p_{pi} - p_{pwf})} \right)_{M.P.} \left( \frac{\tilde{t}_M}{(q_{DD})_{M.P.}} \right) \tag{4.36}
\]

### 4.4 NEW PROCEDURE TO ESTIMATE GAS-IN-PLACE AND THE AVERAGE RESERVOIR PRESSURE PROFILE

In this section we introduce a new approach which couples the gas material balance equation and the pseudosteady-state gas flow relation. This coupling produces an algorithm where only a few successive substitutions are required to estimate the average reservoir pressure and to compute the value of the gas-in-place.

We start this approach by recalling the material balance equation for a volumetric dry gas reservoir

\[
\frac{\bar{p}}{z} = \frac{p_i}{z_i} \left( 1 - \frac{G_p}{G} \right) \tag{4.37}
\]

Rearranging Eq. 4.37 for the reciprocal of the original gas-in-place, \( G \), we have
\[
\frac{1}{G} = \frac{1}{G_p} \left[ 1 - \frac{\bar{p}_i}{z_i} \right] \tag{4.38}
\]

Given the fact that the gas-in-place is a fixed value, Eq. 4.38 can be written for two different points in time (i.e., for two different values of average reservoir pressure). For times \(j\) and \(j+1\), we can write

\[
\frac{1}{G} = \frac{1}{G_{p,j+1}} \left[ 1 - \frac{\bar{p}_{j+1}}{z_{j+1}} \right] \frac{z_i}{p_i} = \frac{1}{G_{p,j}} \left[ 1 - \frac{\bar{p}_j}{z_j} \right] \frac{z_i}{p_i} \tag{4.39}
\]

We note that Eq. 4.39 can be rearranged to yield

\[
\frac{\bar{p}_j}{z_j} \frac{z_i}{p_i} = 1 - \frac{G_{p,j}}{G_{p,j+1}} \left( 1 - \frac{\bar{p}_{j+1}}{z_{j+1}} \right) \frac{z_i}{p_i}
\]

or

\[
\left( \frac{\bar{p}}{z_j} \right) = p_i \left[ 1 - \frac{G_{p,j}}{G_{p,j+1}} \left( 1 - \frac{\bar{p}_{j+1}}{z_{j+1}} \right) \frac{z_i}{p_i} \right]
\]

Upon expanding terms on the right-hand-side, we obtain

\[
\left( \frac{\bar{p}}{z_j} \right) = p_i - \frac{G_{p,j}}{G_{p,j+1}} \left( \frac{p_i}{z_i} - \frac{\bar{p}_{j+1}}{z_{j+1}} \right) \tag{4.40}
\]

From pseudosteady-state flow theory,\(^{35}\) the gas flow equation is given by

\[
\bar{p}_p - p_{pwf} = q_g \, b_{a,pss} \tag{4.41}
\]

where

\[
b_{a,pss} = 141.2 \frac{\mu_{gj} B_{gi}}{k_{gh}} \left[ \frac{1}{2} \ln \left( \frac{4}{e} \frac{A}{C_A \rho_w^2} \right) \right] = \text{constant}
\]

Eq. 4.41 can also be written for two different points in time and equated for \(b_{a,pss}\) since this term is constant. For the \(j+1\) time level we have

\[
\bar{p}_{p,j+1} - p_{pwf,j+1} = q_{g,j+1} \, b_{a,pss}
\]

whereas for the \(j\) time level we have

\[
\bar{p}_p - p_{pwf,j} = q_{g,j} \, b_{a,pss}
\]

Solving for \(b_{a,pss}\) and equating the resulting relations

\[
b_{a,pss} = \frac{\bar{p}_{p,j+1} - p_{pwf,j+1}}{q_{g,j+1}} = \frac{\bar{p}_p - p_{pwf,j}}{q_{g,j}} \tag{4.42}
\]

where we can rearrange Eq. 4.42 to yield
\[
\frac{\bar{p}_{p,j+1} - p_{p\text{wf},j+1}}{\bar{p}_{p,j} - p_{p\text{wf},j}} = \frac{q_{g,j+1}}{q_{g,j}}
\]

or in terms of the average reservoir pressure at the \(j+1\) time level we obtain

\[
\bar{p}_{p,j+1} = p_{p\text{wf},j+1} + \frac{q_{g,j+1}}{q_{g,j}} (\bar{p}_{p,j} - p_{p\text{wf},j}) \quad \cdots \quad (4.43)
\]

We propose to couple Eqs. 4.40 and 4.43 to obtain the average pressure profile as a function of time which can be used to estimate the gas-in-place volume. The procedure for estimating \(\bar{p}(t)\) and \(G\) using gas well production data is given by the following steps.

1. We first set up a table of \(p, z, p/z\) and \(p_p\) for the particular gas lab data or correlations can be used to establish the \(\mu, z\) data.

2. For the first timestep \((j=1)\), we assume that \(\bar{p}_{p1} = \bar{p}_{pi}\). That is, we assume that the first average pseudopressure is equal to the initial reservoir pseudopressure. Then, \(\bar{p}_{p,j+1} = \bar{p}_{p2}\) can be obtained from Eq. 4.43 as

\[
\bar{p}_{p2} = p_{p\text{wf}2} + \frac{q_{g2}}{q_{g1}} (\bar{p}_{p1} - p_{p\text{wf}1}) \quad \cdots \quad (4.43)
\]

where again \(\bar{p}_{p1} = \bar{p}_{pi}\).

3. \(\bar{p}_{j+1}\) and \(\bar{z}_{j+1}\) (\(\bar{p}_2\) and \(z_2\) for the first timestep) are obtained using a table look-up procedure in the fluid property table that was set up in Step 1. \(\frac{\bar{p}_{j+1}}{\bar{z}_{j+1}}\) is then estimated using the interpolated values.

4. Using \(\frac{\bar{p}_{j+1}}{\bar{z}_{j+1}}\) (\(\bar{p}_2\) for the first case) obtained in Step 3, we can compute \(\frac{p_j}{z_j}\) (\(\frac{p_1}{z_1}\) for the first case) from Eq. 4.40 as

\[
\frac{p_j}{z_j} = p_i \cdot \frac{G_{p,j}}{G_{p,j}} \left( \frac{p_i}{z_i} - \frac{\bar{p}_{j+1}}{\bar{z}_{j+1}} \right) \quad \cdots \quad (4.40)
\]

For the first timestep, this calculation would be

\[
\frac{p_1}{z_1} = p_i \cdot \frac{G_{p,1}}{G_{p,2}} \left( \frac{p_i}{z_i} - \frac{\bar{p}_2}{\bar{z}_2} \right)
\]
\( \overline{p}_j \) and \( \overline{p}_{p,j} \) are obtained from \( \frac{p_j}{z_j} \) using the fluid property table that was set up in Step 1.

5. We recompute \( \overline{p}_{p,j+1} \) with the most recent value of \( \overline{p}_{p,j} \) using Eq. 4.40. For the first timestep \( \overline{p}_{p,2} \) would be recomputed using the last \( \overline{p}_{p,1} \).

6. Steps 2 through 5 must be repeated until convergence is obtained (0.01 psi in \( \overline{p}_j \)).

7. Once \( \overline{p}_j \) (\( \overline{p}_1 \) for the first case) has converged, the same procedure is applied to obtain the \( \overline{p} \) values at the next time level. For instance, to extend this procedure from \( j=1 \) to \( j=2 \) the pertinent equations will be

\[
\overline{p}_{p,3} = \overline{p}_{wfr} + \frac{q_s}{q_g} \left( \overline{p}_{p,2} - \overline{p}_{wfr} \right)
\] ...............................(4.44)

and

\[
\left( \frac{\overline{p}}{z_i} \right)_2 = \frac{p_i}{z_i} - \frac{G_{p,2}}{G_{p,3}} \left[ \frac{p_i}{z_i} - \left( \frac{\overline{p}}{z_i} \right)_3 \right]
\] ...............................(4.45)

where the initial value for \( \overline{p}_{p,2} \) in Eq. 4.44 is the value obtained in the last substitution of the loop used to compute the correct \( \overline{p}_1 \).

In order to ensure that the results obtained through this analysis technique are directionally correct, we propose that a plot of gas-in-place versus time should be made and that this plot should yield a horizontal straight line during pseudosteady state. The gas-in-place values used for plotting should be obtained using the calculated values of average reservoir pressure in the reciprocal of Eq. 4.38. The value read at the horizontal straight line represents the approximate gas-in-place volume which we have shown to be sufficient for estimating the pseudotime function, for use in decline curve analysis. The implementation of this procedure along with the illustration of the concept of the harmonic decline for gas are clearly explained in Chapter VI using both simulated data and field examples.
4.5 CONSTANT PRESSURE EQUIVALENT ANALYSIS USING THE FETKOVIICH TYPE CURVE FOR VARIABLE RATE/VARIABLE PRESSURE DROP FLOW DATA

This section presents the development of an equivalent time function that allows us to analyze variable rate/variable pressure drop data using the exponential decline stem \((b=0)\) on the Fetkovich\(^8\) type curve. This new tool is based on previous developments made by Blasingame, McCray and Lee.\(^{29}\) This time function allows variable rate/variable pressure drop data to be analyzed as though the well were produced at a constant bottomhole pressure. This can simplify both the analysis and interpretation of these data for those analysts who are used to the "conventional" Fetkovich style of analysis using decline type curves.

As a review, we will present the major developments of the constant pressure equivalent time function. We showed earlier that the relations for liquid and gas flow have identical forms. These relations are

\[
\frac{p_i - p_{wf}}{q_o} = \frac{1}{Nc_t} \tilde{t} + 141.2 \frac{\mu_o B_o}{k_o h} \left[ \frac{1}{2} \ln \left( \frac{4}{e \gamma C_A r_w^2} \right) \right] \tag{3.33}
\]

and

\[
\frac{p_{pi} - p_{pwf}}{q_g} = \frac{1}{Gc_{ti}} \tilde{t}_a + 141.2 \frac{\mu g_i B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{e \gamma C_A r_w^2} \right) \right] \tag{4.24}
\]

In order to present a general technique for any type of fluid (tof), these two relations are expressed using shorthand notation as

\[
\frac{\Delta p_{tof}}{q_{tof}} = m_{tof} \tilde{t}_{tof} + b_{tof} \tag{4.46}
\]

where the definitions according to the type of fluid (tof) are given in Table 4.1.

Another fundamental concept that we need is the general long term solution for a well produced at a constant bottomhole flowing pressure which is given by

\[
\frac{q_{tof}}{\Delta p_{tof}} = \frac{1}{b_{tof}} \exp \left( -\frac{m_{tof}}{b_{tof}} t_{dum} \right)
\]

where \(t_{dum}\) is taken as actual time for liquid, or conventional pseudotime for gas (as defined by Fraim and Wattenbarger\(^{16}\)).
Table 4.1
Definitions by Type of Fluid for Eq. 4.46

<table>
<thead>
<tr>
<th>Type of Fluid (tof)</th>
<th>Liquid</th>
<th>Gas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta p_{tof}$</td>
<td>$p_i - p_{wf}$</td>
<td>$p_{pi} - p_{pwf}$</td>
</tr>
<tr>
<td>$q_{tof}$</td>
<td>$q_o$</td>
<td>$q_g$</td>
</tr>
<tr>
<td>$m_{tof}$</td>
<td>$\frac{1}{Nc_t}$</td>
<td>$\frac{1}{Gc_{ti}}$</td>
</tr>
<tr>
<td>$b_{tof}$</td>
<td>$141.2 \frac{\mu_o B_o}{k_o h} \left[ \frac{1}{2} \ln \left( \frac{4 e \gamma A}{C A r_w^2} \right) \right]$</td>
<td>$141.2 \frac{\mu_g B_g}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4 e \gamma A}{C A r_w^2} \right) \right]$</td>
</tr>
<tr>
<td>$\bar{t}_{tof}$</td>
<td>$\bar{t} = \frac{N_p}{q_o}$</td>
<td>$\bar{t}<em>a = \frac{\mu g c</em>{ti}}{q_g} \int_0^t \frac{q_g}{\mu g c_t} , dt$</td>
</tr>
</tbody>
</table>

As we mentioned earlier Blasingame et al. set out to develop a "constant pressure analog" time variable, $t_{cp}$, that would permit the analysis of variable rate/variable pressure drop processes using a well model that assumes a constant bottomhole flowing pressure. The solution for a constant bottomhole flowing pressure modified for the variable rate/variable pressure drop function is given as

$$\frac{q_{tof}}{\Delta p_{tof}} = \frac{1}{b_{tof}} \exp \left( \frac{m_{tof} \ t_{cp}}{b_{tof}} \right) \tag{4.47}$$

Blasingame et al. give the following definition for the "constant pressure analog" time, $t_{cp}$

$$\frac{Q(t)_{tof}}{\Delta p_{tof}} = \int_0^{t_{cp}} \frac{q_{tof}}{\Delta p_{tof}} \, dt \tag{4.48}$$

Originally, this result was proposed via induction from the constant pressure solution. However, Blasingame et al. provided a rigorous proof that Eq. 4.48 is the constant pressure analog solution for the variable rate/variable pressure drop case.

Considering the liquid flow case we start with the definition of normalized time which is given as
\[ t = \frac{Q}{q} \] ................................................................. (4.49)

Rewriting Eq. 4.46 for the liquid case we have
\[ \frac{\Delta p}{q} = \frac{m Q}{q} + b_{pss} \] ......................................................... (4.50)

Rearranging again we obtain
\[ \frac{Q}{\Delta p} = \frac{1}{m} \left[ 1 - b_{pss} \frac{q}{\Delta p} \right] \] ......................................................... (4.51)

In order to prove the validity of the \( t_{cp} \) variable for general use, we must show that the right-hand sides of Eqs. 4.48 and 4.51 are identical. This is done by integrating Eq. 4.47 using the definition given by Eq. 4.48. Performing the integration we have
\[ \int_{0}^{t_{cp}} \frac{q}{\Delta p} d\tau = \frac{1}{b_{pss}} \left[ \int_{0}^{t_{cp}} \exp \left( -\frac{m}{b_{pss}} \tau \right) d\tau \right] \] ......................................................... (4.52)

or completing the integration
\[ \int_{0}^{t_{cp}} \frac{q}{\Delta p} d\tau = \frac{1}{m} \left[ 1 - \exp \left( -\frac{m}{b_{pss}} t_{cp} \right) \right] \] ......................................................... (4.53)

Where the combination of Eqs. 4.47 and 4.53 produces our desired result
\[ \int_{0}^{t_{cp}} \frac{q}{\Delta p} d\tau = \frac{1}{m} \left[ 1 - b_{pss} \frac{q}{\Delta p} \right] \] ......................................................... (4.54)

We see that the right-hand-sides of Eqs. 4.51 and 4.54 are identical, which validates the concept of \( t_{cp} \) for variable-rate/variable pressure drop flow analysis.

Blasingame et al.\textsuperscript{29} went on further to derive an explicit relation to compute \( t_{cp} \). However, the result requires that the formation properties and reservoir geometry be known \textit{a priori}, preventing the use of this relation for decline curve analysis.

We now introduce a new approach to compute \( t_{cp} \) directly, without knowledge of any formation properties. We start this process by noting that the combination of Eqs. 4.48 and 4.49 gives us
\[ \frac{\Delta q}{\Delta p} = \int_{0}^{t_{cp}} \frac{q}{\Delta p} \, d\tau \] ..............................................(4.55)

Differentiating Eq. 4.55 with respect to \( t_{cp} \) we obtain

\[ \frac{d}{dt_{cp}} \left[ \frac{\Delta q}{\Delta p} \right] = \frac{d}{dt_{cp}} \left[ \int_{0}^{t_{cp}} \frac{q}{\Delta p} \, d\tau \right] = \frac{q}{\Delta p} \] ..............................................(4.56)

Defining two temporary variables, \( x \) and \( w \) for convenience, we have

\[ x = \frac{\Delta q}{\Delta p} \] ..............................................(4.57)

and

\[ \frac{1}{w} = \frac{q}{\Delta p} \] ..............................................(4.58)

Substituting Eqs. 4.57 and 4.58 into Eq. 4.56 we obtain

\[ \frac{dx}{dt_{cp}} = \frac{1}{w} \]

where upon integration we have

\[ t_{cp} = \int_{0}^{x} w \, dx \] ..............................................(4.59)

This simple derivation provides a rigorous and straightforward means to compute \( t_{cp} \), but it is important to note that the integrand variable, \( w \), could be distorted by data noise. Any match of data to a decline type curve must overlay the \( b=0 \) stem (which is the exponential solution) using the \( t_{cp} \) approach. These methods are valid for both liquid and gases, provided the liquid equivalent pseudovariables are used for the gas case (as defined in Table 4.1).

4.6 FETKOVICH-CARTER TYPE CURVE

This section describes a new type curve that has been constructed using the two most popular type curves for decline curve analysis of gas wells. The first type curve is the case presented by Fetkovich\(^8\) which considers rigorous liquid flow solutions as well as the empirical hyperbolic solutions given by Arps.\(^4\) The second type curve used in this
development is the case presented by Carter\textsuperscript{12,13} which includes rigorous solutions for liquid flow and numerical solutions for gas flow. A detailed explanation of these two type curves was already provided in Chapter III of this thesis.

We have termed the new type curve as the "Fetkovich-Carter Type Curve." In the following sections we will discuss the development of this type curve and present a brief summary of the possible applications of this type curve to decline curve analysis.

4.6.1 Description of the New Type Curve

The Fetkovich-Carter type curve is shown in Fig. 4.6. The new type curve combines the original solutions presented by Fetkovich\textsuperscript{8} and Carter\textsuperscript{12,13} into a single plot using the Fetkovich decline variables. Carter type curve uses pseudopressure in the dimensionless decline rate axis, but uses actual time for the dimensionless decline time axis. Due to the assumption of a constant bottomhole pressure for the gas cases, Carter showed that ordinary time, not pseudotime, would yield accurate decline curve results for wells produced at a constant bottomhole pressure. Carter derived a single parameter, \( \lambda \), that characterizes the variation in gas properties essentially based on the relative ratio of producing pressure to initial reservoir pressure. The concepts and applications of the Carter type curve are not valid for changing producing pressures.

The variables used for the generation of the new type curve are specifically given by the following relations which use nomenclature that avoids specifying the type of fluid

\[
q_{D_d} = \frac{141.2B_i \mu_i}{k h} \frac{q}{(p_{fun,i} - p_{fun,wf})} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right]
\]

\[
t_{D_d} = 0.00633 \frac{k}{\phi \mu_i c_{i_0} r_w^2} \frac{1}{\frac{1}{2} \left[ \frac{r_e^2}{r_w^2} - 1 \right]} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] t_{fun}
\]

We note that the relations represent a circular reservoir geometry. The \( p_{fun} \) variable represents the appropriate pressure function in each case, and \( t_{fun} \) is the appropriate time function. For the Fetkovich\textsuperscript{8} part of the type curve, \( p_{fun} \) is taken as actual pressure and \( t_{fun} \) is actual time. Fetkovich\textsuperscript{8} original stems are the solid lines in the Fetkovich-Carter type curve.
For the Carter part of the type curve, \( p_{fun} \) is taken as pseudopressure rather than actual pressure, but \( t_{fun} \) is still actual time.

In the new type curve we have added the \( q_{Dd} \) versus \( t_{Dd} \) stems for several different constant rate cases generated using a gas well simulator. These new solutions are shown as the dashed lines on the new type curve. Carter\(^{12,13} \) did not include these cases for he was only concerned with the constant bottomhole flowing pressure case. Since the Carter type curve was intended to solve gas problems, these new additional curves should prove useful in obtaining at least an approximate answer for data taken at a constant gas flow rate. The constant rate stems are correlated by a modification of the \( q_{Dd} \) definition that we give as

\[
q_{Ddc} = \frac{141.2B \mu_i}{k h} \left( \frac{q}{p_{p,i} - p_{p,abn}} \right) \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2}
\]

where \( p_{p,abn} \) is the pseudopressure at which the reservoir would be abandoned.

We have shown rigorously from theory that the \( b=1 \) stem is the constant rate liquid solution (\( \tilde{t}=t \)). We should expect that these additional type curves for constant rate flow would lie on the Fetkovich harmonic stem. We see from Fig. 4.6 that this is not the case. Although the constant rate stems begin pseudosteady-state flow lying on the harmonic \( (b=1) \) trend, these curves diverge from the liquid solution as time increases. This behavior is due to the use of time rather than pseudotime, but this performance does not distract from this curve being used to analyze constant rate data.

Because we analyze both liquid and gas flow data using the new Fetkovich-Carter type curve, the guidance as to what variables to use for analysis are given in Table 4.2.

Table 4.2

<table>
<thead>
<tr>
<th>Type Curve</th>
<th>( p_{fun} )</th>
<th>( t_{fun} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fetkovich</td>
<td>( p )</td>
<td>( t )</td>
</tr>
<tr>
<td>Carter (original)</td>
<td>( p_p )</td>
<td>( t )</td>
</tr>
<tr>
<td>Carter (constant rate)</td>
<td>( p_p )</td>
<td>( t )</td>
</tr>
</tbody>
</table>
4.6.2 Type Curve Analysis Methods

To ensure completeness we have prepared a summary of the techniques used in conventional decline curve analysis and given in this thesis for both liquid and gas data. We recall the new approaches presented in this thesis, where these new methods account for variable rate/variable pressure drop conditions. These methods include the rigorous harmonic decline for both liquid and gas flow, and also, the rigorous exponential decline for both liquid and gas flow.

As references for analysis we present Tables 4.3 and 4.4. These tables include the descriptions of the data plotting functions that must be used in a particular production scenario. We also include the pseudovariables required for rigorous analysis of gas flow data. The pertinent definitions for Table 4.3 are

\[ p_p = \left( \frac{\mu_{g_1} Z_1}{\mu_i Z_i} \right) \int_{p_{base}}^{p} \frac{p}{\mu_g Z} \, dp \]

and

\[ t_a = \mu_{g_1} c_i \int_{0}^{t} \frac{1}{\mu_g Z_i} \, d\tau \]

Table 4.3

Decline Curve Analysis Methods Currently Available

<table>
<thead>
<tr>
<th>Case</th>
<th>Pressure Function</th>
<th>Time function</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>General liquid case</td>
<td>Pressure, ( p )</td>
<td>Time, ( t )</td>
<td>Fettovich ( b ) stems as necessary</td>
</tr>
<tr>
<td>Liquid constant ( p_{wf} ) (rigorous)</td>
<td>Pressure, ( p )</td>
<td>Time, ( t )</td>
<td>Fettovich ( b=0 ) stem</td>
</tr>
<tr>
<td>General gas case (not rigorous)</td>
<td>Pseudopressure, ( p_p )</td>
<td>Time, ( t )</td>
<td>Fettovich ( b ) stems as necessary</td>
</tr>
<tr>
<td>Gas constant ( p_{wf} ) (semi-rigorous)</td>
<td>Pseudopressure, ( p_p )</td>
<td>Time, ( t )</td>
<td>Carter ( \lambda ) stems</td>
</tr>
<tr>
<td>Gas constant ( p_{wf} ) (rigorous)</td>
<td>Pseudopressure, ( p_p )</td>
<td>Conventional Pseudotime, ( t_a )</td>
<td>Fettovich ( b=0 ) stem</td>
</tr>
</tbody>
</table>
For Table 4.4, the definitions of the material balance pseudotime, $\tilde{t}_a$, and the constant pressure analog time, $t_{cp}$, for gas are given as

$$\tilde{t}_a = \frac{\mu_g c_t}{\rho_g} \int_0^t \frac{q_g}{\mu_g c_t} \, d\tau$$

and

$$t_{cp} = \int_0^x \frac{p_{pi} - p_{pwf}}{q_g} \, dx$$

where $x$ is calculated from

$$x = \frac{\tilde{t}_a q_g}{p_{pi} - p_{pwf}}$$

The first of these two variables, $\tilde{t}_a$, not only accounts for the gas properties variation in a rigorous manner, but also permits the analysis of production data from a variable rate/variable pressure drop scenario for gas flow using the harmonic decline stem ($b=1$) on the Fetkovich type curve. The second variable, $t_{cp}$, also permits variable rate and pressure analysis but on an exponential decline stem ($b=0$). The examples presented in Chapter VI will illustrate the general usefulness of $\tilde{t}_a$ and some of the problems associated with the $t_{cp}$ function in the presence of significant data noise.

### Table 4.4

Decline Curve Analysis Methods Introduced in This Work

<table>
<thead>
<tr>
<th>Case</th>
<th>Pressure Function</th>
<th>Time function</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid variable pressure/rate (rigorous)</td>
<td>Pressure, $p$</td>
<td>Material balance time, $\tilde{t} = N_p/q_o$</td>
<td>Fetkovich $b=1$ stem</td>
</tr>
<tr>
<td>Gas variable pressure/rate (approximation)</td>
<td>Pseudopressure, $p_p$</td>
<td>Material balance time, $\tilde{t} = G_p/q_g$</td>
<td>Fetkovich $b=1$ stem</td>
</tr>
<tr>
<td>Gas variable pressure/rate (rigorous)</td>
<td>Pseudopressure, $p_p$</td>
<td>Material Balance pseudotime, $\tilde{t}_a$</td>
<td>Fetkovich $b=1$ stem</td>
</tr>
</tbody>
</table>
Table 4.4 (Continued)

<table>
<thead>
<tr>
<th>Case</th>
<th>Pressure Function</th>
<th>Time function</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid variable pressure/rate</td>
<td>Pressure, $p$</td>
<td>Constant pressure analog time, $t_{cp}$,</td>
<td>Fetkovich $b=0$ stem</td>
</tr>
<tr>
<td>(rigorous)</td>
<td></td>
<td>(for liquid)</td>
<td></td>
</tr>
<tr>
<td>Gas variable pressure/rate</td>
<td>Pseudopressure, $p_p$</td>
<td>Constant pressure analog time, $t_{cp}$,</td>
<td>Fetkovich $b=0$ stem</td>
</tr>
<tr>
<td>(rigorous)</td>
<td></td>
<td>(for gas)</td>
<td></td>
</tr>
<tr>
<td>Gas constant rate</td>
<td>Pseudopressure, $p_p$</td>
<td>Actual time, $t$,</td>
<td>$q_{dc}$ stems on Fetkovich</td>
</tr>
<tr>
<td>(semi-rigorous)</td>
<td></td>
<td></td>
<td>Carter Type Curve</td>
</tr>
</tbody>
</table>
CHAPTER V

VERIFICATION OF FLOW RELATIONS BY SIMULATION TECHNIQUES

The purpose of this chapter is to validate the definitions of the dimensionless pressure, flow rate, and time variables discussed previously. Although this research focuses on gas flow, we validate the dimensionless variables for both liquid and gas production conditions. The producing conditions that we modeled include not only the simple constant bottomhole flowing pressure and flow rate cases but also the more realistic variable pressure and flow rate conditions.

Before starting the validation process we should state that both the oil and gas runs were made using "Sabre," which is a commercial simulator. The solutions were obtained using the "simultaneous solution" (SS) method which solves for all the unknowns implicitly and simultaneously. The dimensional results we obtained from the simulator were then transformed into dimensionless variables and compared to analytical solutions for accuracy.

5.1 VERIFICATION OF LIQUID FLOW RELATIONS

All the computer runs for the liquid case were made using the formation and reservoir property values given in Table 5.1. The plotting functions we use for this verification process are the following definitions of the dimensionless variables for liquid flow:

\[ p_D = \frac{k_o h}{141.2 \ q_o B_o \mu_o} (p_i - p_w) \] .................................(5.1)

\[ t_{DA} = 0.00633 \frac{k_o}{\phi \mu_o C_i A} t \] .................................(5.2)

\[ q_D = \frac{141.2 \ q_o B_o \mu_o k_o h}{k_o h (p_i - p_w)} \] .................................(5.3)
Table 5.1
Summary of Reservoir and Fluid Properties for the Liquid Case

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Reservoir Pressure</td>
<td>4800.0  [psia]</td>
</tr>
<tr>
<td>Formation Permeability</td>
<td>1.0 [md]</td>
</tr>
<tr>
<td>Formation Thickness</td>
<td>30.0 [ft]</td>
</tr>
<tr>
<td>Formation Volume Factor (at 4800 psia)</td>
<td>1.0 [RB/STB]</td>
</tr>
<tr>
<td>Oil Viscosity (at 4800 psia)</td>
<td>0.4 [cp]</td>
</tr>
<tr>
<td>Formation Porosity</td>
<td>0.3 [fraction]</td>
</tr>
<tr>
<td>Total Compressibility (at 4800 psia)</td>
<td>4.179E-06 [1/psia]</td>
</tr>
<tr>
<td>Reservoir Drainage Radius</td>
<td>745.0 [ft]</td>
</tr>
<tr>
<td>Wellbore Radius</td>
<td>0.2 [ft]</td>
</tr>
</tbody>
</table>

We note that the dimensionless time definition given by Eq. 5.2 has been defined in terms of the drainage area rather than the wellbore radius. A more complete derivation of these dimensionless variables is given in Appendix A.

5.1.1 Constant Rate Production

The first case we studied is that of liquid production at constant flow rate. Using the fluid and reservoir properties presented in Table 5.1, we made several runs for different oil flow rates. The results of these runs are compared to the analytical solution which is obtained by numerically inverting the long term solution given by Eq. B-19. We recall that Eq. B-19 (derived in Appendix B) is

$$
\bar{p}_D(r_D,u) = \frac{1}{u} K_0(\bar{\nu}_D) + \frac{1}{u} \frac{K_1(\bar{\nu}_D)}{I_1(\bar{\nu}_D)} I_0(\bar{\nu}_D) \quad \text{.................................(B-19)}
$$

The results are presented in Fig. 5.1 which is a log-log plot of dimensionless pressure, $p_D$, versus dimensionless time, $t_{DA}$. The solid line represents the analytical solution, while the solid data points represent the simulated results. The excellent comparison between the simulated and analytical solutions suggests us that the liquid dimensionless variables can be defined in terms of actual pressure and time.
Fig. 5.1 - Verification of Liquid Flow Relations for Several Cases of Constant Flow Rate
5.1.2 Constant Pressure Production

The second case studied is that of liquid production at constant bottomhole flowing pressure. Using the same fluid and reservoir properties given in Table 5.1 we made several runs for different ratios of the initial reservoir and bottomhole flowing pressures, \( \frac{p_i}{p_{wf}} \). We then plotted the results of these runs along with the analytical solution in Fig. 5.2 which is a log-log graph of \( q_D \) versus \( t_{DA} \). The analytical solution was obtained by numerically inverting Eq. B-71 (derived in Appendix B). We recall that Eq. B-71 is given as

\[
q_{D,CP} (r_D = 1, \mu) = \frac{I_1(\sqrt{\mu} \cdot r_{ed})}{\mu \left[ I_0(\sqrt{\mu}) K_1(\sqrt{\mu} \cdot r_{ed}) + K_0(\sqrt{\mu}) I_1(\sqrt{\mu} \cdot r_{ed}) \right]} \quad \text{.........(B-71)}
\]

We recognize the excellent agreement obtained between the simulated results and the analytical solution in Fig. 5.2. This agreement provides additional evidence of the validity of the dimensionless variables proposed by Eqs. 5.2 and 5.3.

5.2 VERIFICATION OF GAS FLOW RELATIONS

All the computer runs for the gas case were made using the formation and reservoir property values given in Table 5.2. The dimensionless variables we use for the gas flow cases are defined by Eqs. 5.4-5.7. These variables are similar to those used for the liquid case except pseudopressure (Eq. 5.8) and pseudotime (Eq. 5.9 and 5.10) are used rather than actual pressure and time.

\[
p_D = \frac{k_g h}{141.2 \cdot q_g B_S \mu_g i} (p_{pi} - p_{wf}) \quad \text{.........................................(5.4)}
\]

\[
q_D = \frac{141.2 \cdot q_g B_S \mu_g i}{k_g h (p_{pi} - p_{wf})} \quad \text{.........................................(5.5)}
\]

\[
t_{DA} = 0.00633 \frac{k_g}{\mu_g i c_{ui} A} t_a \quad \text{.........................................(5.6)}
\]

\[
\bar{t}_{DA} = 0.00633 \frac{k_g}{\mu_g i c_{ui} A} \bar{t}_a \quad \text{.........................................(5.7)}
\]
Fig. 5.2 - Verification of Liquid Flow Relations for Several Cases of Constant Bottomhole Flowing Pressure
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Reservoir Pressure</td>
<td>4800.0 [psia]</td>
</tr>
<tr>
<td>Formation Permeability</td>
<td>1.0 [md]</td>
</tr>
<tr>
<td>Formation Thickness</td>
<td>30.0 [ft]</td>
</tr>
<tr>
<td>Formation Volume Factor (at 4800 psia)</td>
<td>0.69246 [RB/Mscf]</td>
</tr>
<tr>
<td>Gas Viscosity (at 4800 psia)</td>
<td>0.023485 [cp]</td>
</tr>
<tr>
<td>Formation Porosity</td>
<td>0.3 [fraction]</td>
</tr>
<tr>
<td>Total Compressibility (at 4800 psia)</td>
<td>1.4775E-04 [1/psia]</td>
</tr>
<tr>
<td>Gas Deviation Factor (at 4800 psia)</td>
<td>1.00093 [1/psia]</td>
</tr>
<tr>
<td>Reservoir Drainage Radius</td>
<td>745.0 [ft]</td>
</tr>
<tr>
<td>Wellbore Radius</td>
<td>0.2 [ft]</td>
</tr>
</tbody>
</table>

where the pseudopressure and pseudotime variables are defined as

\[
p_p = \frac{\mu_{g} c_i}{P_i} \int_{P_{base}}^{P} \frac{P}{\mu_{g} c_i} \, dp \quad \text{(5.8)}
\]

\[
t_a = \mu_{g} c_i \int_{0}^{t} \frac{1}{\mu_{g} c_i} \, dt \quad \text{(5.9)}
\]

\[
\bar{t}_a = \frac{\mu_{g} c_i q_i}{q_{g}} \int_{0}^{t} \frac{q_{g}}{\mu_{g} c_i} \, dt \quad \text{(5.10)}
\]

5.2.1 Constant Rate Production

Similar to the constant-rate liquid case, we made computer runs for several values of gas flow rate. The simulated results in terms of dimensionless variables are compared to the analytical solution in Fig. 5.3 which is a log-log graph of \(P_D\) versus \(t_{DA}\). Again, we obtained the analytical solution by numerically inverting Eq. B-19. The definitions of the dimensionless plotting functions (\(P_D\) and \(t_{DA}\)) are given by Eqs. 5.4 and 5.6.
Fig. 5.3 - Verification of Gas Flow Relations for Several Cases of Constant Flow Rate
We observe excellent agreement in Fig. 5.3 between the simulated results and the analytical solution. This agreement suggests that, for the range of reservoir and formation properties investigated, the dimensionless variables defined by Eqs. 5.4 and 5.6 are valid.

5.2.2 Constant Pressure Production

Proceeding in the same manner as for the constant pressure liquid case, we made computer runs for several values of the ratio between the initial and bottomhole flowing pressures, \( \frac{P_i}{P_{wf}} \). The results of these runs are compared to the analytical solution in Fig. 5.4 which is a log-log graph of \( q_D \) versus \( t_{DA} \). We note that the definitions of the dimensionless plotting functions (\( q_D \) and \( t_{DA} \)) are given in this case by Eqs. 5.5 and 5.6. Also, we obtained the analytical solution by numerically inverting Eq. B-71.

We see in Fig. 5.4 that the simulated results and the analytical solution match. This match again suggests that the dimensionless variables defined by Eqs. 5.5 and 5.6 are valid for constant pressure gas production.

5.2.3 Variable Rate Production

In order to analyze the effects of a variable rate profile we made several runs for gas flow using a stepwise changing flow rate history. We present the results of these runs as log-log plots of \( q_D \) versus \( \tilde{t}_{DA} \). The plotting variables, \( q_D \) and \( \tilde{t}_{DA} \), are defined by Eqs. 5.5 and 5.7, respectively. We show the individual cases in Figs. 5.5-5.7 and we also present a plot that summarizes all the dimensionless results in Fig. 5.8. In addition, we superimpose the gas flow rate profile used to generate the simulated case as well as the bottomhole flowing pressure response on each corresponding log-log plot.

In Figs. 5.5-5.8 we can see the comparison of each simulated case to the constant pressure analytical solution obtained from Eq. B-71. The results shown in these figures clearly illustrate that a variable-rate gas production history plotted in terms of the dimensionless rate (Eq. 5.5) versus dimensionless time (Eq. 5.7) can be matched with the analytical solution for a constant pressure case. It is important to note that the dimensionless time (Eq. 5.7) is now a function of the material balance pseudotime defined by Eq. 5.10 and not by Eq. 5.9 as for the constant pressure or rate cases.
Fig. 5.5 - First Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Flow Rate
Fig. 5.6 - Second Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Flow Rate
Fig. 5.7 - Third Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Flow Rate
Fig. 5.8 - Summary of the Comparisons of Analytical and Simulated Responses for Various Cases of Gas Production at Variable Flow Rate
The ability to match a variable-rate gas production history using these plotting functions allows us to obtain formation and reservoir properties from type curve plotting techniques. We must also note that in Figs. 5.5-5.8 several points from the variable-rate gas production history do not align with the analytical solution. These points occur at the beginning of each rate change and, therefore, represent transient conditions. Since the analytical solution was developed for pseudosteady-state flow, we should not expect points from the transient flow regime to match. In fact, we observe that when stabilized reservoir conditions are reached following a rate change, the simulated data again match the analytical solution, thus validating our definitions of dimensionless variables (Eqs. 5.4, 5.5 and 5.7).

5.2.4. Variable Pressure Production

The final case we studied is the production of gas with a changing bottomhole flowing pressure. We made several computer runs for this case using several stepwise changes in $p_{wf}$. The simulated results plotted in log-log graphs of $q_D$ versus $\tilde{t}_{DA}$ are shown in Figs. 5.9-5.11. The definitions for the dimensionless plotting variables, $q_D$ and $\tilde{t}_{DA}$, are given by Eqs. 5.5 and 5.7, respectively. In addition, we present a plot that summarizes the results from these computer runs in Fig. 5.12. In Figs. 5.9-5.12 we again compare each simulated case to the analytical solution obtained by numerically inverting Eq. B-71. Further, the appropriate bottomhole flowing pressure history used to generate the simulator results and the rate profile obtained are presented on the corresponding log-log plot.

We conclude from our results that the production data obtained with a variable bottomhole flowing pressure history can be matched with the analytical solution. This match requires that the production data be plotted in terms of dimensionless rate given by Eq. 5.5 against the dimensionless time given by Eq. 5.7. Finally, we note that similar to the variable rate-case, the data deviating from the analytical solution represent transient conditions corresponding to a change in bottomhole flowing pressure. Furthermore, the simulated data do match the analytical solution when pseudosteady-state flow is reached. These matches confirm and validate the dimensionless variable definitions given by Eqs. 5.4, 5.5 and 5.7.
Fig. 5.9 - First Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Bottomhole Flowing Pressure
Fig. 5.10 - Second Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Bottomhole Flowing Pressure
Fig. 5.11- Third Comparison of Analytical and Simulated Responses for the Case of Gas Production at Variable Bottomhole Flowing Pressure
Fig. 5.12: Summary of the Comparisons of Analytical and Simulated Responses for Various Cases of Gas Production at Variable Bottomhole Flowing Pressure

Legend: (P_i = 4800 psia) Variable Pressure Gas Cases:
- Analytical Solution
- BHP (P_wf = 4000, 3000, 2000, 1500, 1000, 5000)
- Step Wise Changes:
  - P_wf = 4100, 3500, 1000, 2000
  - P_wf = 1200, 600

\( q_b \)
In summary, the work we present in this Chapter proves that liquid flow can be accurately modeled with dimensionless variables in terms of actual pressure and time. In order to model gas flow conditions, however, we require the use of pseudovariables. For constant rate or constant bottomhole flowing pressure conditions, we can use conventional pseudopressure (Eq. 5.8) and pseudotime (Eq. 5.9). For variable rate or variable bottomhole flowing pressure conditions, however, we require pseudopressure (Eq. 5.8) and the material balance pseudotime (Eq. 5.10).
CHAPTER VI

APPLICATION OF DECLINE CURVE TECHNOLOGY TO SIMULATED AND FIELD DATA

This chapter summarizes the usefulness of the new analysis techniques we illustrated in previous Chapters. These techniques are coupled with existing technology to obtain the desired information in the analysis and interpretation of both simulated and field gas cases. We begin this process by illustrating the type of information that is required to perform the analysis, as well as the general procedure that we propose to accomplish this task in a rigorous manner.

6.1 DATA REQUIRED FOR ANALYSIS

The type of information we need to undertake a decline analysis for gas as we propose in this thesis, involves two primary requirements. The first and vital requirement is a periodic record of time, gas flow rates, and bottomhole flowing pressures. The second requirement is PVT data, specifically, a data table containing corresponding values of pressures, gas deviation factors or formation volume factors, viscosities and compressibilities. This table is usually obtained from either a direct laboratory report of the fluid properties, or the computation of all pertinent fluid parameters by means of published correlations, which make use of basic fluid information usually available.

6.2 NEW PROCEDURE FOR GAS DECLINE CURVE ANALYSIS

The general procedure for analysis we provide below includes two options which represent two different forms of estimating the value of gas-in-place. The first option is based on the use of the simple material balance time function \( \tilde{t} \). We have already proven that the \( \tilde{t} \) function is rigorous to obtain a harmonic decline for the case of liquid flow. We suggest the use of the \( \tilde{t} \) function because we have observed that even though the use of this function is not rigorous for gas flow analysis by any means, we can still obtain an
approximate value of gas-in-place. The second option makes direct use of the new algorithm to estimate the value of gas-in-place. We clearly explained this algorithm in Chapter IV. However, we recall that the algorithm makes use of an average reservoir pressure profile in time obtained through a back substitution process between the material balance relation and the boundary dominated flow equation. All the examples we present in this Chapter make use of this algorithm so that the reader may become familiar with it.

We now outline the proposed procedure to analyze gas flow production data using decline type curves. This procedure is intended for actual field conditions, i.e., for variable pressure drop/variable flow rate conditions. The following are the basic steps:

1. **Computation of the Basic Plotting Function for the Vertical Axis**

   We must convert pressures, \( p \), to normalized pseudopressures, \( p_p \), and then calculate the group \( \frac{q_g}{p_{pi} - p_{pwf}} \). This group constitutes the simplest plotting function for the vertical axis of the data plot needed to perform type curve matching to decline curves.

2. **Computation of Gas-in-Place**

   **Option A :**

   We calculate the \( \tilde{t} \) values for each time recorded in the input data set. This is done by using the \( \tilde{t} \) definition given as

   \[
   \tilde{t} = \frac{1}{q_g} \int_0^t q_g \, dt = \frac{G_p}{q_g}
   \]

   In addition, we compute the complementary plotting functions for the vertical axis based on \( \tilde{t} \). The complementary functions are the integral of \( \frac{q_g}{p_{pi} - p_{pwf}} \) with respect to \( \tilde{t} \), and the derivative of that integral with respect to \( \tilde{t} \). These two additional plotting functions have mathematical forms\(^{30}\) that follow the well testing practice of keeping the units for both the integral, and the integral derivative, the same as those of the original function. The appropriate definition for the flow rate integral is given by
\[ \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_i = \frac{1}{t} \int_{0}^{i} \frac{q_g}{p_{pi} - p_{pwf}} \, dt \]

whereas the flow rate integral derivative is obtained as

\[ \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_{id} = \frac{1}{t} \frac{d}{dt} \left( \frac{1}{t} \int_{0}^{i} \frac{q_g}{p_{pi} - p_{pwf}} \, dt \right) \]

We refer the reader to the original publication (Ref. 30), where a thorough discussion of these definitions and the appropriate type curves for matching are presented. We give a brief explanation of the basic dimensionless variables involved in the type curves\(^8,30\) that we use for analysis in the last section of Appendix D.

After doing these computations, we plot the three different vertical functions versus \( \bar{t} \) and match these plots with the liquid decline type curves presented by Fetkovich\(^8\) and McCray\(^30\) to estimate the best possible value of gas-in-place. The relation to make the gas-in-place computation is given as

\[ G = \frac{1}{c_{ti}} \left( \frac{\left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_{fun}}{(q_{Dd})_{M.P.}} \right)_{M.P.} \]

The subscript "fun" in the above relation is a dummy group that can be either deleted or substituted with "i" or "id". This means that we may take the group with the "fun" subscript as the normalized rate value, \( \frac{q_g}{p_{pi} - p_{pwf}} \), its integral with respect to \( \bar{t} \), \( \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_{i} \), or the flow rate integral derivative with respect to \( \bar{t} \), \( \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_{id} \).

Option B:

This option makes use of the new algorithm (presented in Section 4.4) to obtain an approximated value of gas-in-place. A plot of gas-in-place, \( G \), versus time must be made and we determine the best possible value for \( G \) from the pseudosteady-state region of this plot which yields a horizontal straight line.
3. Computation of the Rigorous Plotting Function for the Horizontal Axis

Using the computed value of gas-in-place in the material balance equation, we obtain the average reservoir pressure profile in time. This profile allows us to calculate the material balance pseudotime functions. These calculations are done using the definition of the material balance pseudotime given by

\[ \tilde{t}_a = \frac{\mu_{gi}}{\mu_g} \int_0^t \frac{q_g}{\bar{c}_i} \, dt \]

4. Computation of the Additional Plotting Functions for the Vertical Axis

We first compute the complementary plotting functions for the vertical axis based on \( \tilde{t}_a \). These functions are the integral of \( \frac{q_g}{p_{pi} - p_{pwf}} \) with respect to \( \tilde{t}_a \), and the derivative of that integral with respect to \( \tilde{t}_a \). These functions are mathematically expressed\(^{30}\) following the common well testing practice which keeps the units for both the integral and the integral derivative the same as those of the original function. In this case, we obtain the flow rate integral as

\[ \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_i = \frac{1}{\tilde{t}_a} \int_0^{\tilde{t}_a} \frac{q_g}{p_{pi} - p_{pwf}} \, d\tilde{t}_a \]

and the flow rate integral derivative as

\[ \left( \frac{q_g}{p_{pi} - p_{pwf}} \right)_{id} = \tilde{t}_a \frac{d}{d\tilde{t}_a} \left[ \frac{1}{\tilde{t}_a} \int_0^{\tilde{t}_a} \frac{q_g}{p_{pi} - p_{pwf}} \, d\tilde{t}_a \right] \]

We refer the reader once again to the original publication (Ref. 30) where these definitions were first presented for a thorough discussion of this matter. We also recall that the dimensionless variables used in the generation of the McCray\(^{30}\) type curves that we will use for analysis are discussed in Appendix D of this thesis.

The second stage of this step is analogous to the first one, however, the additional plotting functions for the vertical axis are now based on \( t_{cp} \). This latter variable is the "constant pressure analog time" which we introduced in Chapter IV. We obtain the \( t_{cp} \) function as
\[ t_{cp} = \int_{0}^{x} \frac{p_{pi} - p_{pwf}}{q_{g}} \, dx \]

where \( x \) is given by

\[ x = \frac{\tilde{t}_a q_{g}}{p_{pi} - p_{pwf}} \]

Then, we compute the flow rate integral as

\[ \left( \frac{q_{g}}{p_{pi} - p_{pwf}} \right)_{i} = \frac{1}{t_{cp}} \int_{0}^{t_{cp}} \frac{q_{g}}{p_{pi} - p_{pwf}} \, dt_{cp} \]

and the flow rate integral derivative as

\[ \left( \frac{q_{g}}{p_{pi} - p_{pwf}} \right)_{id} = t_{cp} \frac{d}{dt_{cp}} \left[ \frac{1}{t_{cp}} \int_{0}^{t_{cp}} \frac{q_{g}}{p_{pi} - p_{pwf}} \, dt_{cp} \right] \]

We note that \( t_{cp} \) depends on \( \tilde{t}_a \), therefore we recommend that the material balance pseudotime functions be used as the first solution approach for gas analysis. In addition, we recognize that the use of the \( t_{cp} \) function is a resource and not a requirement for this analysis, i.e., we can obtain an accurate analysis with the \( \tilde{t}_a \) related functions alone. We present the \( t_{cp} \) function as a means of simplifying the analysis and interpretation of production data for those analysts used to the exponential style of analysis using type curves.

5. Matching of the Different Plotting Functions to Liquid Type Curves

With the results obtained for the plotting functions we make log-log graphs of all the functions for the vertical axis based on the material balance pseudotime (\( \frac{q_{g}}{p_{pi} - p_{pwf}} \) and \( \frac{q_{g}}{p_{pi} - p_{pwf}} \) ) versus \( \tilde{t}_a \). Also, we make log-log plots of the same vertical functions but based on \( t_{cp} \) (as explained in Step 4) versus \( t_{cp} \). We then match the data plots based on \( \tilde{t}_a \) to the harmonic (\( b=1 \)) stems on the Fetkovich\(^8\) and McCray\(^{30}\) type curves. In addition, we match the plots based on \( t_{cp} \) to the exponential (\( b=0 \)) stems on the Fetkovich\(^8\) and McCray\(^{30}\) type curves. From any of these matches we read a match point and the appropriate \( r_{ed} \) value. Using the match point we compute the formation permeability and skin as well as the final
estimate of gas-in-place. If we assume a well centered in a circular reservoir geometry, we should use the following relations for these computations

\[ G = \frac{1}{c_{ti}} \left( \frac{t_a}{(t_M)_{M.P.}} \right) \left( \frac{q_g}{(p_{pi} - p_{pwf})_{fun}} \right) \left( \frac{\ll_n}{(q_{M.P.})_{M.P.}} \right) \]

\[ k = 141.2 \frac{B_g \mu_{gi}}{h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] \left( \frac{q_g}{(p_{pi} - p_{pwf})_{fun}} \right) \left( \frac{\ll_n}{(q_{M.P.})_{M.P.}} \right) \]

and

\[ r_w' = \frac{r_e}{r_{d0}} \quad s = - \ln \left( \frac{r_w}{r_w} \right) \]

We must state at this point that in the following examples, the different type curves and data plots are scaled identically so that the match point is the same for all the plots related to the same case. That is, once we select a match point with one of the type curves (i.e., Fetkovich or either of McCray's type curves), this match point should apply for all the other type curves involved in that particular case.

6.3 CONSTANT BOTTOMHOLE FLOWING PRESSURE EXAMPLE

The first illustration we present to show how the new techniques can be used for interpreting gas production data, is a simulated case of gas production at constant bottomhole flowing pressure. The critical data input to the simulator are provided in Table 6.1.

We first obtained the appropriate flow rate profile with a numerical simulator. Then, we followed the step procedure described in Section 6.2, using option B for the computation of gas-in-place. The resulting plot of gas-in-place, \( G \), versus time is shown here as Fig. 6.1.
Table 6.1
Data Input for First Simulated Example

<table>
<thead>
<tr>
<th>Reservoir Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir Area, $A$</td>
<td>40 acres ($r_e=745$ ft)</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$</td>
<td>0.2 ft</td>
</tr>
<tr>
<td>Net pay thickness, $h$</td>
<td>30 ft</td>
</tr>
<tr>
<td>Porosity, $\phi$ (fraction)</td>
<td>0.3</td>
</tr>
<tr>
<td>Permeability, $k$</td>
<td>100 md</td>
</tr>
<tr>
<td>Initial Reservoir Pressure, $p_i$</td>
<td>4800 psia</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fluid Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas gravity, $\gamma_g$</td>
<td>0.65 (air=1)</td>
</tr>
<tr>
<td>Gas volume factor at $p_i$, $B_{gi}$</td>
<td>0.6929 RB/Mscf</td>
</tr>
<tr>
<td>Gas viscosity at $p_i$, $\mu_{gi}$</td>
<td>$2.3485\times10^{-2}$ cp</td>
</tr>
<tr>
<td>Gas compressibility at $p_i$, $c_{gi}$</td>
<td>$1.4783\times10^{-4}$ psia$^{-1}$</td>
</tr>
<tr>
<td>Temperature, $T$</td>
<td>200 °F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Production Characteristics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottomhole pressure, $p_{wf}$ (constant)</td>
<td>500 psia</td>
</tr>
</tbody>
</table>

Fig. 6.1 - Calculated Gas-in-Place for a Simulated Example at Constant Bottomhole Flowing Pressure
The value of 4.15x10$^6$ Mscf that we read for gas-in-place at the horizontal region of Fig. 6.1 is close to the actual gas-in-place volume (4.04x10$^6$ Mscf). This verifies at least partially, the validity of the new algorithm.

After determining the gas-in-place value from Fig. 6.1, we used that value to compute the material balance pseudotime functions. Then, we calculated all the plotting functions described in Section 6.2 and plotted the results for type curve matching. The results of the type curve matching procedures are shown in Figs. 6.2-6.5. We note that a match to a harmonic decline ($b=1$) is always obtained for the data plots based on $\bar{t}_a$, and a match to an exponential ($b=0$) decline for those based on $t_{cp}$. This fact constitutes a partial proof of the theoretical developments presented in Chapter IV. The value of the match point that we obtained in this case with the different type curves is

| Match Point with Fetkovich and McCray Type Curves (Constant Pressure Case) |
|-----------------|-----------------|-----------------|-----------------|
| Type Curve Variables (Dimensionless) | Data Plot Variables (Dimensional) |
| Function | Value | Function | Value |
| $q_{Dd}, q_{Ddi}$ or $q_{Ddid}$ | 1 | $(q_g/\Delta P_p)|_{un}$ | 155 |
| $t_{Dd}$ | 1 | $\bar{t}_a$ or $t_{cp}$ | 3.9 |

Using this match point we calculate the corrected gas-in-place volume and the formation permeability values using. The value of gas-in-place is computed as

\[
G = \frac{1}{c_{ii}} \left( \frac{q_g}{(P_{pi} - P_{pwf}) f} \right)_{M.P.} \left( \frac{q_{Dd}}{q_{Ddid}} \right)_{M.P.}
\]

\[
G = \frac{1}{1} \left( \frac{3.9 \text{ days}}{155 \text{ scf/D/psi}} \right) = \frac{1}{1} \left( \frac{1}{1.4783 \times 10^{-4} \text{ psi}^{-1}} \right) = 4.089 \times 10^6 \text{ Mscf}
\]

Now, assuming a zero value for the skin factor as is considered in the data input to the simulator, we estimate the formation permeability as
Fig. 6.2 - Type Curve Match with a Harmonic Decline for Simulated Case at Constant Bottomhole Flowing Pressure. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
Fig. 6.3 - Type Curve Match with an Exponential Decline for Simulated Case at Constant Bottomhole Flowing Pressure. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve.
Fig. 6.4 - Type Curve Match with a Harmonic Decline for Simulated Case at Constant Bottomhole Flowing Pressure. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
Fig. 6.5 - Type Curve Match with an Exponential Decline for Simulated Case at Constant Bottomhole Flowing Pressure. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCrory's Type Curves.
\[ k_g = 141.2 \frac{B_g \mu_g i}{h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] \left( \frac{q_g}{(q_{pi} - p_{pwf})f} \right) M.P. \]

\[ k_g = 141.2 \left( \frac{0.6929 \text{ RB/Mscf}}{30 \text{ ft}} \right) \left( \frac{2.3485 \times 10^2 \text{ cp}}{0.2 \text{ ft}} \right) \left[ \ln \left( \frac{745 \text{ ft}}{0.2 \text{ ft}} \right) - \frac{1}{2} \right] \left( \frac{155 \text{ scf/D/psia}}{1} \right) = \]

\[ k_g = 91.7 \text{ md} \]

We see that the results obtained after performing the type curve match are in good agreement with the input data for both gas-in-place, \( G \), and permeability, \( k_g \). These results give additional verification of the techniques proposed in this thesis.

### 6.4 VARIABLE BOTTOMHOLE FLOWING PRESSURE EXAMPLE

The second example we present is also a simulated case. This case, however, represents a gas well producing at variable bottomhole flowing pressure. In this example, not only is \( p_{wf} \) assumed to change but also the permeability value is reduced from 100 md to 1 md. The rest of the input values are exactly the same as those seen in Table 6.1. The selected bottomhole flowing pressure profile is shown below

<table>
<thead>
<tr>
<th>( t ) [days]</th>
<th>( p_{wf} ) [psia]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-365</td>
<td>4000</td>
</tr>
<tr>
<td>366-730</td>
<td>2000</td>
</tr>
<tr>
<td>731-1095</td>
<td>800</td>
</tr>
<tr>
<td>1096-2191</td>
<td>500</td>
</tr>
<tr>
<td>2191-3287</td>
<td>350</td>
</tr>
</tbody>
</table>

We present the gas-in-place plots obtained in this case as Figs. 6.6 and 6.7. The first plot is a semilog plot of \( G \) versus time, while the second is the corresponding cartesian plot. We include the two forms of the gas-in-place plot to help to visualize the real effects of the variable pressure profile and also to facilitate the choice of the correct \( G \) value. We clearly see in these graphs that every sudden change in bottomhole flowing pressure introduces a transient in the reservoir which causes some spikes in the expected behavior of
Fig. 6.6 - Semilog Plot of Calculated Gas-in-Place versus Time for a Simulated Example of Variable Bottomhole Flowing Pressure

Legend:
- Step Wise Changing Bottomhole Pressure
- Actual Gas-in-Place = 4.04 bscf
- Computed Gas-in-Place = 4.16 bscf

Fig. 6.7 - Cartesian Plot of Calculated Gas-in-Place versus Time for a Simulated Example of Variable Bottomhole Flowing Pressure

Legend:
- Step Wise Changing Bottomhole Pressure
- Actual Gas-in-Place = 4.04 bscf
- Computed Gas-in-Place = 4.16 bscf
$G$ with time. Nevertheless, when the pseudosteady-state (or boundary dominated) flow is reached, the data points tend to lie on a horizontal straight line as expected. The gas-in-place volume we read at the horizontal region of Figs. 6.6 and 6.7 is $4.16 \times 10^6$ Mcf, which compares favorably to the actual value of $4.04 \times 10^6$ Mcf. We then use the value read for gas-in-place to compute the material balance pseudotimes and perform the type curve matching procedure. We show the results of these matching procedures in Figs. 6.8 and 6.9.

We note from Figs. 6.8 and 6.9 that the effect of the transients introduced have a significant effect in the results and must be commented on. We first recognize from Fig. 6.8 that the plot of normalized rates versus $\tilde{t}_a$ closely matches Fetkovich\(^8\) ($b=1$) harmonic decline. In addition, we see in Fig. 6.9 that the plot of normalized rates versus $t_{cp}$ matches Fetkovich\(^8\) ($b=0$) exponential decline. However, we observe no match of the flow rate integral in any of the two cases (versus $\tilde{t}_a$ or $t_{cp}$). This departure from the correct model is due to the sudden changes in bottomhole flowing pressures. In fact, the simulator response to a sudden reduction in $p_{wf}$ includes spike flow rates that are not realistic. When these "spike" rates are normalized for type curve matching, the boundary dominated flow equations are still capable of modeling the situation fairly accurately. Unfortunately, when the normalized rate data are integrated, the transient effects are magnified making it very difficult to model the situation with boundary dominated flow equations.

Since we know that the numerical computation of a derivative involves not only a procedure where a degree of accuracy is lost compared to that of the original function but also in this particular case, that the input data for differentiation would be unrealistic, we do not include the match with the integral derivative type curve of McCray.\(^{30}\)

We must highlight that the analysis techniques based on $\tilde{t}_a$ and $t_{cp}$ do work. The only limitation in this case is related to the integral type of analysis which is only a side consideration of this thesis. Therefore, using the normalized rate data that do yield a good match to the Fetkovich\(^8\) type curve we can compute the values of $G$ and $k_g$. The match point values obtained in this case with the Fetkovich type curve are shown below.
Fig. 6.8 - Type Curve Match with a Harmonic Decline for Simulated Case at Variable Bottomhole Flowing Pressure. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
Fig. 6.9 - Type Curve Match with an Exponential Decline for Simulated Case at Variable Bottomhole Flowing Pressure. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
### Match Point for Fetkovich and McCray Type Curves (Variable Pressure Case)

<table>
<thead>
<tr>
<th>Type Curve Variables (Dimensionless)</th>
<th>Data Plot Variables (Dimensional)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function: ( q_{Dd}, q_{Ddi} )</td>
<td>Function: ( q_g/\Delta P_p )_{fun}</td>
</tr>
<tr>
<td>Value: 1</td>
<td>Value: 1.9</td>
</tr>
<tr>
<td>Function: ( t_{Dd} )</td>
<td>Function: ( t_a ) or ( t_{cp} )</td>
</tr>
<tr>
<td>Value: 1</td>
<td>Value: 315</td>
</tr>
</tbody>
</table>

Using this match point we compute the gas-in-place as

\[
G = \frac{1}{c_{ii}} \left( \frac{t_a}{t_{Dd}} \right)_{M.P.} \left[ \frac{q_g}{(p_{pi} - p_{pwf})} \right]_{M.P.} \left[ \frac{f}{(q_{Dd})_{M.P.}} \right]
\]

\[
G = \frac{1}{315 \text{ days} \cdot 1.9 \text{ scf/D/psia}} = \frac{1}{1 (1.4783 \times 10^{-4} \text{ psia}^{-1})}
\]

\[
G = 4.049 \times 10^6 \text{ Mscf}
\]

and assuming a zero value for the skin factor (as we did in the data input to the simulator), the formation permeability is calculated as

\[
k_g = 141.2 \cdot \frac{B_g \cdot \mu_g \cdot i}{h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \left( \frac{q_g}{(p_{pi} - p_{pwf})} \right)_{M.P.} \left( \frac{f}{(q_{Dd})_{M.P.}} \right) \right]
\]

\[
k_g = 141.2 \left( \frac{0.6929 \text{ RB/Mscf} \cdot 2.3485 \times 10^{-2} \text{ cp}}{30 \text{ ft}} \right) \left[ \ln \left( \frac{745 \text{ ft}}{0.2 \text{ ft}} \right) - \frac{1}{2} \left( 1.9 \text{ scf/D/psia} \right) \right] = 1.12 \text{ md}
\]

The value of 4.049 \times 10^6 Mscf obtained for gas-in-place from type curve matching compares very well with both the value obtained from the algorithm (4.16 \times 10^6 Mscf), and the actual value of 4.04 \times 10^6 Mscf. The same agreement is seen for the calculated \( k_g \) value of 1.12 md when compared to the actual value of 1.0 md.
6.5 VARIABLE FLOW RATE EXAMPLE

The third example we illustrate is a case that considers simulated step wise changes in gas flow rate. The basic input data are the same shown in Table 6.1 except that now the flow rate varies in a step wise manner. The $q_g$ profile taken is arbitrary, and is given by a series of decreasing and increasing changes as shown below.

<table>
<thead>
<tr>
<th>$t$ [days]</th>
<th>$q_g$ [Mscf/D]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-365</td>
<td>700</td>
</tr>
<tr>
<td>366-1095</td>
<td>600</td>
</tr>
<tr>
<td>1096-2191</td>
<td>450</td>
</tr>
<tr>
<td>2192-2556</td>
<td>500</td>
</tr>
<tr>
<td>2557-2922</td>
<td>650</td>
</tr>
<tr>
<td>2923-3650</td>
<td>780</td>
</tr>
</tbody>
</table>

We present again the gas-in-place plot in both Cartesian and semilog form in Figs. 6.10 and 6.11. We note that these plots show an analogous situation to that seen in the preceding example, as once again sudden changes in gas flow rate introduce transients in the reservoir. These transients cause some "spikes" in the expected behavior of $G$ with time, but once pseudosteady-state (or boundary dominated) flow is reached, the data points tend to lie back on the expected horizontal straight line.

We also see in Figs. 6.10 and 6.11 that the data points corresponding to the last two sudden changes in flow rate do not reach stabilization where they level off to a horizontal straight line, as was the case for the first four rate changes. We can describe this behavior as the result of transients introduced whose effects did not disappear. These effects prevent the reservoir from achieving pseudosteady-state (or boundary dominated) flow. Since the flow equation involved in the algorithm that we propose to compute gas-in-place is a boundary dominated flow equation, the behavior observed in the gas-in-place plots is not unexpected.

In spite of this apparent discrepancy, we can still define the adequate horizontal straight line by the first four rate changes and estimate the gas-in-place with a sufficient degree of accuracy. The approximate value of gas-in-place was computed as $4.082 \times 10^6$ Mscf which compares well to the actual gas-in-place value of $4.04 \times 10^6$ Mscf.
Fig. 6.10 - Semilog Plot of Calculated Gas-in-Place versus Time for a Simulated Example of Variable Flow Rate

Fig. 6.11 - Cartesian Plot of Calculated Gas-in-Place versus Time for a Simulated Example of Variable Flow Rate
Using the computed gas-in-place value we calculate the pseudotime functions, plot the results, and match these results with the Fetkovich and McCray decline type curves. We present the results of the type curve matching procedures in Figs. 6.12-6.15. We note from these figures that although the variable rate profile also produces spikes in the bottomhole flowing pressure values, the order of magnitude of the pressure changes was much lower than that obtained for rates in the previous example. Therefore, in this case, the equations used accurately model the output data from the simulator without preventing the type curve matching to any of the Fetkovich or McCray decline type curves. Subsequently, we can perform the gas-in-place and permeability computations using the match point read with any of the type curves. The values read at the match point are seen below.

<table>
<thead>
<tr>
<th>Match Point for Fetkovich and McCray Type Curves (Variable Rate Case)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type Curve Variables (Dimensionless)</strong></td>
</tr>
<tr>
<td>Function</td>
</tr>
<tr>
<td>$q_{Dd}, q_{Ddi}$ or $q_{Ddid}$</td>
</tr>
<tr>
<td>$t_{Dd}$</td>
</tr>
</tbody>
</table>

We compute the gas-in-place value as

$$G = \frac{1}{c_{ti}} \left( \frac{t_{Dd}}{t_{Dd}} \right)_{M.P.} \left[ \frac{q_g}{(P_{pi} - P_{pwf})/f} \right]_{M.P.}$$

$$G = \frac{1}{(1.4783 \times 10^{-4} \text{ psia}^{-1})} \frac{340 \text{ days}}{1} \frac{1}{1} \frac{1.75 \text{ scf/D/psia}}{1} =$$

$$G = 4.025 \times 10^6 \text{ Mcf}$$

and assuming that the skin factor has a zero value, we obtain the formation permeability as

$$k_g = 141.2 \frac{B_g h \mu_{kg}}{h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] \left[ \frac{q_g}{(P_{pi} - P_{pwf})/f} \right]_{M.P.}$$

$$k_g = 141.2 \left( \frac{0.6929 \text{ RB/Mcf}}{30 \text{ ft}} \right) \left( \frac{2.3485 \times 10^{-2} \text{ cp}}{30 \text{ ft}} \right) \left[ \ln \left( \frac{745 \text{ ft}}{0.2 \text{ ft}} \right) \frac{1}{2} \right] \frac{1.75 \text{ scf/D/psia}}{1} =$$

$$k_g = 1.035 \text{ md}$$
Fig. 6.12 - Type Curve Match with a Harmonic Decline for Simulated Case at Step Wise Changing Flow Rate Profile. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve.
Fig. 6.13 - Type Curve Match with an Exponential Decline for Simulated Case at Step Wise Changing Flow Rate Profile. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve.
Fig. 6.14 - Type Curve Match with a Harmonic Decline for Simulated Case at Step Wise Changing Flow Rate Profile. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
Fig. 6.15 - Type Curve Match with an Exponential Decline for Simulated Case at Step Wise Changing Flow Rate Profile. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves.
We note that there is good agreement among the gas-in-place value from the new algorithm \((4.082 \times 10^6 \text{ Mscf})\), that obtained with type curve matching \((4.025 \times 10^6 \text{ Mscf})\), and the actual value of \(4.04 \times 10^6 \text{ Mscf}\). The same is true for the computed permeability value of 1.035 md when it is compared to the actual input value of 1.0 md.

6.6 FIELD CASE EXAMPLE - WELL A

We now use the proposed new techniques to analyze actual data sets obtained from gas fields. The first well we analyze is well A whose original data were taken from Ref. 14. The same well was studied previously by Fraim and Wattenbarger\(^{16}\) and we present a comparison to the results of these authors.

Well A is a hydraulically fractured gas well completed in the Onondaga chert in West Virginia, which had a problem of water loading up the wellbore. The well was initially produced for 200 days and then shut in for 106 days to determine average reservoir pressure from a build-up test. The estimated value for gas-in-place using the material balance equation was \(G = 3.36 \text{ bscf}\), whereas that computed by Fraim and Wattenbarger\(^{16}\) was \(G = 3.03 \text{ bscf}\).

We summarize the values of the pertinent formation and fluid properties in Table 6.2.

### Table 6.2
Data Input for Well A

<table>
<thead>
<tr>
<th>Reservoir Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Wellbore radius, (r_w)</td>
<td>0.354 ft</td>
</tr>
<tr>
<td>Net pay thickness, (h)</td>
<td>70 ft</td>
</tr>
<tr>
<td>Porosity, (\phi) (fraction)</td>
<td>0.06</td>
</tr>
<tr>
<td>Permeability, (k)</td>
<td>0.1 md</td>
</tr>
<tr>
<td>Initial Reservoir Pressure, (p_i)</td>
<td>4175 psia</td>
</tr>
<tr>
<td>Bottomhole Flowing Pressure, (p_{wf})</td>
<td>710 psia</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fluid Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas gravity, (\gamma_g)</td>
<td>0.57 (air = 1)</td>
</tr>
<tr>
<td>Gas volume factor at (p_i), (B_{gi})</td>
<td>0.7099 RB/Mscf</td>
</tr>
<tr>
<td>Gas viscosity at (p_i), (\mu_{gi})</td>
<td>2.1670\times10^{-2} \text{ cp}</td>
</tr>
<tr>
<td>Gas compressibility at (p_i), (c_{gi})</td>
<td>1.8227\times10^{-4} \text{ psia}^{-1}</td>
</tr>
<tr>
<td>Temperature, (T)</td>
<td>160 °F</td>
</tr>
</tbody>
</table>
We must note that this is a "constant bottomhole pressure case." Fraim and Wattenbarger\textsuperscript{16} mention that they had to raise the line pressure from the original $p_{wef}$ of 500 psia to 710 psia, "to represent the lifetime average bottomhole pressure." This latter value is therefore what we use for our analysis.

Following the proposed new procedure to analyze this kind of data we obtain the gas-in-place plots shown as Figs. 6.16 and 6.17. We recognize that these plots do not exhibit the horizontal straight line that should be obtained for the fixed value of gas-in-place. This behavior may be due to the fact that the theoretical developments that we present in this work assume an unfractured system and single phase flow. These two fundamental assumptions are being violated in this case to some extent, so that a high degree of data scatter is not unexpected. Therefore, we believe that the departure from the ideal solution is mostly due to two-phase flow in the fracture, and liquid loading up the wellbore.

Although the gas-in-place plots are not ideal, we select an approximate gas-in-place value of $2.75\times10^6$ Mcf for the material balance pseudotime computation from Figs. 6.16 and 6.17. We base our selection on what appears to be an appropriate value around which the gas-in-place plot would exhibit a horizontal straight line. After computing the pseudotime values, we calculate the data functions for the vertical axis and make the plots for type curve matching based on both $t_a$ and $t_{cp}$. Then, we match the resulting log-log plots with the Fetkovich\textsuperscript{8} and McCray\textsuperscript{30} type curves. We show the results of these matches in Figs. 6.18-6.21.

We find that the plots of the well production data based on $\tilde{t}_a$ lie on a $b=1$ stem on the decline type curves,\textsuperscript{8,30} indicating a harmonic type of decline as predicted from theory alone. In addition, when we use $t_{cp}$ as the plotting function for the horizontal axis, the production data lie on a $b=0$ stem on the decline type curves, as theory predicts. However, we recognize that in this case the use of the $\tilde{t}_a$ function not only produces much smoother plots than does $t_{cp}$, but also the plots made using $\tilde{t}_a$ extend farther into the depletion region of the type curves. These two facts indicate that the material balance pseudotime, $\tilde{t}_a$, may be much more favorable for analysis than the constant pressure analog time, $t_{cp}$.
Fig. 6.16 - Semilog Plot of Calculated Gas-in-Place versus Time for Well A

Fig. 6.17 - Cartesian Plot of Calculated Gas-in-Place versus Time for Well A
Fig. 6.18 - Type Curve Match with a Harmonic Decline for Well A. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
Fig. 6.19 - Type Curve Match with an Exponential Decline for Well A. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve.
Fig. 6.20 - Type Curve Match with a Harmonic Decline for Well A. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
Fig. 6.21 - Type Curve Match with an Exponential Decline for Well A. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
The match we obtain to the harmonic and/or exponential stems allow the computation of a corrected gas-in-place volume. The match point values for this case are seen below.

<table>
<thead>
<tr>
<th>Match Point for Fetkovich and McCray Type Curves (Well A)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type Curve Variables (Dimensionless)</strong></td>
</tr>
<tr>
<td>Function</td>
</tr>
<tr>
<td>$q_{Dd}, q_{Ddi}$ or $q_{Ddid}$</td>
</tr>
<tr>
<td>$t_{Dd}$</td>
</tr>
</tbody>
</table>

The corrected value of the gas-in-place volume is computed as

$$G = \frac{1}{(\hat{t}_{a})_{M.P.}} \frac{\left( \frac{q_{g}}{(P_{pi} - P_{pwf})_{f}} \right)_{M.P.}}{(q_{Dd})_{M.P.}} \frac{730 \text{ days} \ 0.75 \text{ scf/D/psia}}{(0.18227 \times 10^{-3} \text{ psia}^{-1})} = \frac{1}{1} \frac{1}{1} = \frac{1}{1}$$

$$G = 3.003 \times 10^6 \text{ Mcf}$$

This value of gas-in-place we obtain is in good agreement with that found previously by Fraim and Wattenbarger\textsuperscript{16} of 3.03 $\times$ 10$^6$ Mcf. Since both the drainage area and the formation damage are unknown, the permeability computation can not be performed with the results obtained from this analysis.

6.7 FIELD CASE EXAMPLE - WELL B

The second field example we present involves the study of a gas well for which only the basic information was available. These data are presented in Table 6.3.

In order to have an idea of the type of variation in bottomhole flowing pressures and flow rates exhibited by the well, we include Figs. 6.22 and 6.23. The first of these plots is a Cartesian plot of bottomhole flowing pressure versus flowing time. Even though the pressure profile does not exhibit a wide variation, we can not consider this a constant bottomhole pressure production scenario.
Table 6.3
Data Input for Well B

<table>
<thead>
<tr>
<th>Reservoir and Fluid Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Reservoir Pressure, $p_i$</td>
<td>6900 psia</td>
</tr>
<tr>
<td>Gas gravity, $\gamma_g$</td>
<td>0.67 (air = 1)</td>
</tr>
<tr>
<td>Gas volume factor at $p_i$, $B_{gi}$</td>
<td>0.5562 RB/Mscf</td>
</tr>
<tr>
<td>Gas viscosity at $p_i$, $\mu_{gi}$</td>
<td>2.8966x10^{-2} cp</td>
</tr>
<tr>
<td>Gas compressibility at $p_i$, $c_{gi}$</td>
<td>8.8116x10^{-5} psia^{-1}</td>
</tr>
<tr>
<td>Temperature, $T$</td>
<td>200 °F</td>
</tr>
</tbody>
</table>

Fig. 6.22 - Bottomhole Flowing Pressure Profile for Well B

Fig. 6.23 is a semilog graph of the rate profile in time. In this plot we see that even though the rate profile does not exhibit wide variation, a rigorous analysis can not be performed with a simple straight line model. Therefore, we set out to use the proposed new proposed procedure with option "B". We present the resulting gas-in-place plots in Figs. 6.24 and 6.25.
Fig. 6.23 - Gas Flow Rate Profile in Time for Well B

Fig. 6.24 - Semilog Plot of Calculated Gas-in-Place versus Time for Well B
Fig. 6.25 - Cartesian Plot of Calculated Gas-in-Place versus Time for Well B

We note that the gas-in-place plots do not exhibit the expected horizontal line, but the choice of an approximate value for gas-in-place is still possible. As shown in Figs. 6.24 and 6.25 we choose a gas-in-place value of 6x10^6 Mscf for the pseudotime computations. We then make the data plots to be matched to the different type curves. We illustrate the results of the matches in Figs. 6.26-6.29.

In this case the matches obtained are not acceptable in any case. Nevertheless, we recognize that the match of the plots based on \( \bar{t}_a \) are much better than those obtained for the plots based on \( t_{cp} \). We must also note that the plots made to match the integral derivative type curves\(^{30}\) (seen in Figs. 6.28 and 6.29) exhibit too much noise and they are not useful for analysis. In spite of this, we select a match point based on Fig. 6.26 alone, which yields the best approximation. The values for the selected match point are seen below.

<table>
<thead>
<tr>
<th>Match Point for Fetkovich and McCray Type Curves (Well B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type Curve Variables (Dimensionless)</td>
</tr>
<tr>
<td>-------------------------------------</td>
</tr>
<tr>
<td>( q_{Dd}, q_{Ddi} )</td>
</tr>
<tr>
<td>( t_{Dd} )</td>
</tr>
</tbody>
</table>
Fig. 6.26 - Type Curve Match with a Harmonic Decline for Well B. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
Fig. 6.27 - Type Curve Match with an Exponential Decline for Well B. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray’s Type Curve
Analysis for Gas Well B:
\( \frac{q_g \Delta P_p}{q D_d} = 2.6 \text{ Mscf/D/psia} \)
\( t_{a,\text{bar}} / t_{D_d} = 210 \text{ days} \)
\( G = 6.20 \text{ bscf} \)

Fig. 6.28 - Type Curve Match with a Harmonic Decline for Well B. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
Fig. 6.29 - Type Curve Match with an Exponential Decline for Well B. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves.
We can then calculate the gas-in-place volume as

\[
G = \frac{1}{c_t i (tDd)_{M.P.}} \left[ \frac{q_g}{(P_{pi} - P_{pwf})_{f}} \right]_{M.P.} \left[ \frac{(P_{pi} - P_{pwf})_{f}}{(qDd)_{M.P.}} \right]_{M.P.}
\]

\[
G = \frac{1}{(8.8116 \times 10^{-5} \text{ psia}^{-1})} \frac{210 \text{ days}}{1} \frac{2.60 \text{ scf/D/psia}}{1} =
\]

\[
G = 6.20 \times 10^6 \text{ Mscf}
\]

The actual value of gas-in-place for this well is unknown, but we notice that the gas-in-place computed from the type curve match is consistent with the value obtained by using the new algorithm (6.0 \times 10^6 \text{ Mscf}).

### 6.8 FIELD CASE EXAMPLE - WELL C

The final field example we illustrate, represents a production data set obtained from well C which belongs to a gas field where the production characteristics are extremely non-ideal. Well C is located in a Mid-Continent reservoir with permeability values around 0.1 md. Also, the pressures found in the area are moderate, with well C having an original reservoir pressure of 2625 psia. The information required for performing the analysis, i.e., bottomhole flowing pressures and flow rates, is recorded on a daily basis. In addition, a very important feature regarding the production from this well is that very significant liquid loading and seasonal effects have been detected. We present the values of the pertinent formation and fluid properties in Table 6.4.

<table>
<thead>
<tr>
<th>Table 6.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Input for Well C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reservoir Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir Area, ( A )</td>
</tr>
<tr>
<td>Wellbore radius, ( r_w )</td>
</tr>
<tr>
<td>Net pay thickness, ( h )</td>
</tr>
<tr>
<td>Porosity, ( \phi ) (fraction)</td>
</tr>
<tr>
<td>Permeability, ( k )</td>
</tr>
<tr>
<td>Initial Reservoir Pressure, ( p_i )</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>6.6 acres (( r_e = 303 \text{ ft} ))</td>
</tr>
<tr>
<td>0.3 ft</td>
</tr>
<tr>
<td>48 ft</td>
</tr>
<tr>
<td>0.09</td>
</tr>
<tr>
<td>0.1 md</td>
</tr>
<tr>
<td>2625 psia</td>
</tr>
</tbody>
</table>
Table 6.4 (Continued)

<table>
<thead>
<tr>
<th>Fluid Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas gravity, $\gamma_g$</td>
<td>0.72 (air = 1)</td>
</tr>
<tr>
<td>Gas volume factor at $p_i$, $B_{gi}$</td>
<td>1.166 RB/Mscf</td>
</tr>
<tr>
<td>Gas viscosity at $p_i$, $\mu_{gi}$</td>
<td>$1.785 \times 10^{-2}$ cp</td>
</tr>
<tr>
<td>Gas compressibility at $p_i$, $c_{gi}$</td>
<td>$3.649 \times 10^{-4}$ psia$^{-1}$</td>
</tr>
<tr>
<td>Temperature, $T$</td>
<td>275 °F</td>
</tr>
</tbody>
</table>

In order to show the type of variation in bottomhole flowing pressures and flow rates exhibited by the well, we present Figs 6.30 and 6.31. The first of these plots (Fig. 6.30) is a Cartesian plot of bottomhole flowing pressure versus flowing time. We note from this Fig. 6.30 that the bottomhole flowing pressure varies widely with no defined patterns. The second plot, Fig. 6.31, is a logarithmic plot of normalized instantaneous rates versus real time. The normalized rates are the result of dividing the instantaneous gas flow rates by the difference between the pseudopressure evaluated at initial pressure and the pseudopressure at the corresponding bottomhole flowing pressure. By looking at Fig. 6.31 we can state that our efforts to obtain a match of this data set to any decline type curve would be virtually useless at this point.

If we use the proposed new procedure with option "B" to analyze this data set, we obtain the gas-in-place plots shown in Figs. 6.32 and 6.33. We note that these plots do not exhibit the horizontal straight line that should be obtained for the fixed value of gas-in-place. We should not be surprised by this phenomenon given the production characteristics of this well. In fact, we must recall that all the theory presented in this thesis assumes single phase gas flow, and the significant liquid loading existing in this well attempts to invalidate this fundamental assumption. In addition, the analysis of this data set includes points that fall outside any possible physical range. There are normalized rates and bottomhole pressure values that do not appear in either Fig. 6.30 or 6.31 because they would distort the scales totally. However, we do include these values in the analysis. This means that we do not leave out any measurement regardless of its accuracy to ensure that our analysis really describes the actual situation.
Fig. 6.30 - Bottomhole Flowing Pressure Profile for Well C

Fig. 6.31 - Normalized Flow Rate Profile in Time for Well C
Although the gas-in-place plots (Figs. 6.32 and 6.33) are not ideal, we observe a horizontal tendency around a gas-in-place value of $0.2 \times 10^6$ Mscf. In order to determine if an interpretation of the data set from well C is possible at all, we use the computed gas-in-place value to calculate the material balance pseudotimes.

For purposes of illustration, we replot the normalized rate profile versus the calculated material balance pseudotimes rather than actual time as was done in Fig. 6.31. We show the new appearance of the normalized rate profile in Fig. 6.34. When we compare Figs. 6.31 and Fig. 6.34, we observe a significant improvement in smoothness in the latter one, so that we can now match the normalized rate profile as seen in Fig. 6.34 to a Fetkovich\textsuperscript{8} decline type curve.

*Fig. 6.32 - Semilog Plot of Calculated Gas-in-Place versus Time for Well C*
Fig. 6.33 - Cartesian Plot of Calculated Gas-in-Place versus Time for Well C

Fig. 6.34 - Normalized Flow Rate Profile versus Material Balance Pseudotimes for Well C
After following the rest of the procedure presented in Section 6.2, we obtain the type curve matches seen in Figs. 6.35-6.37. In these figures we can observe that the plots based on \( t_a \) yield a match to a \( b=1 \) stem in all cases, except for the flow rate integral derivative. In fact, we see a very good match of normalized flow rates and the integral of normalized flow rates to a harmonic type of decline in Fig. 6.35. In Fig 6.37 we again observe the good match of the flow rate integral but a poor match of the flow rate integral derivative. This poor match may be due to the degree of accuracy lost in a differentiation process, which in this case, appears to affect the result significantly.

From Fig. 6.36 we note that the plot of the production data of this well using \( t_{cp} \) is not a helpful tool since none of the data plots match a \( b=0 \) stem. We find the explanation to this discrepancy when we plot the variables (integrand versus integration variable) involved in the integration process to calculate \( t_{cp} \) against each other (see Section 6.2; step 4). The points on this plot exhibit a spike that resemble a straight angle. This abrupt variation does not allow us to make useful data plots for type curve matching based on \( t_{cp} \). This difficulty explains why we do not present the plot that includes the flow rate integral derivative based on \( t_{cp} \).

We must recall that the plots based on \( t_a \) do follow the theoretical principles, making this a significant result especially because of the unfavorable conditions involved in the analysis. The match to the harmonic decline stems allows the computation of a corrected gas-in-place volume. The match point we find in this case is shown below.

<table>
<thead>
<tr>
<th>Match Point for Fetkovich and McCray Type Curves (Well C)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Type Curve Variables (Dimensionless)</td>
<td>Data Plot Variables (Dimensional)</td>
</tr>
<tr>
<td>Function</td>
<td>Value</td>
</tr>
<tr>
<td>( q_{Dd}, q_{Ddi} )</td>
<td>1</td>
</tr>
<tr>
<td>( t_{Dd} )</td>
<td>1</td>
</tr>
</tbody>
</table>

The gas-in-place value is obtained as

\[
G = \frac{1}{c_{ii}} \left( \frac{(t_a)_{M.P.}}{(t_{Dd})_{M.P.}} \right) \left( \frac{q_{g}}{(P_{pi} - P_{pwf})} \right) \left( \frac{f_{M.P.}}{(q_{Dd})_{M.P.}} \right)
\]
Fig. 6.35 - Type Curve Match with a Harmonic Decline for Well C. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve
Fig. 6.36 - Type Curve Match with an Exponential Decline for Well C. Match of Flow Rate on Fetkovich Type Curve and Flow Rate Integral on McCray's Type Curve.
Analysis for Gas Well C:

\[ \left( \frac{q_g}{\Delta P_p} \right)_{jou} / qD_d = 0.78 \text{ Mscf/D/psia} \]

\[ t_{a,bar} / t_{Dd} = 105 \text{ days} \]

\[ G = 0.22 \text{ bscf} \]

**Fig. 6.37** - Type Curve Match with a Harmonic Decline for Well C. Match of Flow Rate Integral and Flow Rate Integral Derivative on McCray's Type Curves
\[
G = \frac{1}{(3.649 \times 10^{-4} \text{ psia}^{-1})} \left( \frac{105 \text{ days}}{1} \frac{0.78 \text{ scf/D/psia}}{1} \right) = \\
G = 0.224 \times 10^6 \text{ Mscf}
\]

and the formation permeability is calculated as

\[
k_g = 141.2 \frac{B_g \mu_{gi}}{h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \right] \left( \frac{q_g}{(P_{pi} - P_{pwf})} \right) f M.P. \\
k_g = 141.2 \frac{(1.166 \text{ RB/Mscf})(1.785 \times 10^{-2} \text{ cp})}{48 \text{ ft}} \left[ \ln \left( \frac{303 \text{ ft}}{0.3 \text{ ft}} \right) - \frac{1}{2} \right] \frac{0.78 \text{ scf/D/psia}}{(q_D) M.P.} = \\
k_g = 0.306 \text{ md}
\]

We note that the corrected gas-in-place of 0.224 \times 10^6 \text{ Mscf} is in good agreement with the value provided by the algorithm (0.2 \times 10^6 \text{ Mscf}). Also, any value of permeability in the area is expected to be within the tenths of millidarcies, so that our value of 0.306 md represents a fair estimate of this formation parameter. If we knew the value of the skin factor, our permeability estimate could surely be improved significantly.

From the outcome of all the analyses presented in this Chapter we can arrive at several conclusions: (1) even for highly non-ideal situations, we can obtain a sufficiently accurate approximation to the gas-in-place value with the new algorithm, (2) the value of gas-in-place and average reservoir pressure profile we obtain with the new algorithm, allows us to calculate the pseudotime functions that are required to rigorously analyze gas production under actual field conditions, and (3), in most instances, accurate results can be obtained from either an exponential or a harmonic decline when we use \( t_{cp} \) or \( \tilde{t}_a \), respectively, as the plotting function for the horizontal axis. However, we recognize that for gas flow, the material balance pseudotime, \( \tilde{t}_a \), appears to produce smoother plots that extend farther into the depletion region than those obtained by using the constant pressure analog time, \( t_{cp} \). This fact constitutes a significant advantage of \( \tilde{t}_a \) over \( t_{cp} \).
CHAPTER VII

SUMMARY AND CONCLUSIONS

We have discussed both existing and proposed new methods of production data analysis for the case of single phase flow of liquid or gas. The description that we make of the current existing technology includes sufficient detail as for any reservoir engineer to understand from it alone how any specific method or technique has been devised, and how it may assist in the analysis of actual production data.

We have presented the gas flow techniques in a "liquid equivalent form." Therefore, the liquid analysis techniques, or at least their form, can be also used for the analysis and interpretation of gas production data.

We consider the following to be the most important comprehensive ideas and conclusions of this work. These conclusions represent either new concepts or validations of statements previously made by other authors regarding the matter of interest.

7.1 We have proven analytically and by means of simulation techniques that the long term solution for the case of liquid production at constant bottomhole flowing pressure corresponds exactly to an exponential decline. In other words, the stem corresponding to \( b=0 \) on the Fetkovich\(^8\) type curve is suitable for the analysis of liquid production under these circumstances using actual pressure, and actual time.

7.2 The exponential stem \((b=0)\) on the Fetkovich\(^8\) type curve is also suitable for data analysis of single phase gas production under a constant bottomhole flowing pressure. However, in this case, we need to make the data plot for type curve matching using pseudopressure rather than pressure, and pseudotime rather than actual time.

7.3 In the case of single phase liquid flow, the use of a time function given as cumulative production divided by flow rate, allows us to obtain a flow equation that has the form of a harmonic decline. This equation permits the analysis of
variable pressure/variable rate production data. Therefore, virtually any liquid data set can be matched to the \( b=1 \) stem on Fetkovich type curve regardless of the production conditions under which the given data set is obtained.

7.4 There exist reasonable approximations which avoid the use of pseudotime to treat the gas problem. However, we have recognized that the rigorous analysis of this gas production data is not possible by the use of pseudopressure alone. The two pseudovariables are required in order to achieve an effective (although not total) linearization of the diffusivity equation.

7.5 We have provided the analytical proof for the material balance pseudotime. We have proven that such this function has the exact theoretical definition for the analysis of gas production data during boundary dominated flow.

7.6 The use of the material balance pseudotime allows us to obtain a gas flow equation which has the same form as the equation for liquid flow. This equation permits the analysis of gas production data obtained under variable pressure/variable rate conditions through a match to a harmonic \( (b=1) \) stem on a decline type curve.

7.7 We have introduced a new technique that allows for the independent explicit computation of the value of gas-in-place, facilitating the analysis of gas production data using the material balance pseudotime. This technique has proved useful in the decline curve analysis of field data.

7.8 In general, the harmonic type of decline \( (b=1) \) is useful for any sort of analysis regardless of the fluid type or the conditions under which the data are obtained. The only requirement for the validity of this statement is that we use the proper plotting variables according to the type of fluid flowing.

7.9 For the case of gas flow at constant flow rate, the plotting of the normalized flow rates versus actual time only allows for an approximate solution. This kind of plot departs from the analytical solution \( (b=1) \) more or less rapidly, depending on the magnitude of the pressure drop (that controls the change in the gas properties), and less markedly, on how large the flow rate is. The larger the pressure drop and the flow rate, the sooner our solution leaves the harmonic stem of \( b=1 \).
7.10 The flow rate integral and flow rate integral derivative functions may be helpful to obtain a more accurate match with decline type curves than would be obtained with flow rate data alone. This is true for production conditions actually found in the field.

7.11 The constant pressure analog time should allow a match to an exponential decline stem to be obtained under variable pressure/variable rate production conditions. We have devised a new form to compute the constant pressure analog time which does not require previous knowledge of the reservoir properties, but we recommend the use of the material balance pseudotime previous to the constant pressure analog time. We propose this because the material balance pseudotime yields decline plots that extend farther into the depletion region, and appears to provide a good match under a variety of circumstances.

7.12 We have introduced a very accurate approximation that avoids the necessity of working in the Laplace space, to obtain the constant pressure analytical solution. This approximation should find utility in the prediction, analysis, and interpretation of reservoir performance data.

7.13 In every attempt to analyze production data, we must identify completion and producing practices which may limit our ability to analyze and interpret production the given data.

7.14 The analysis and interpretation of production data is relatively easy given the right tools. Moreover, this analysis can provide us with just about the same information as conventional pressure transient analysis. Then, since production data are usually plentiful and fairly easy to acquire, their analysis should constitute a must in reservoir engineering because there is no cost in lost production.
NOMENCLATURE

\( A \) = drainage area, ft\(^2\)
\( b \) = decline curve exponent, dimensionless
\( B_g \) = gas formation volume factor, RB/Mscf
\( B_{gi} \) = gas formation volume factor at initial reservoir pressure, RB/Mscf
\( B_o \) = oil formation volume factor, RB/STB
\( C_A \) = reservoir shape factor, dimensionless
\( c_g \) = gas compressibility, psia\(^{-1}\)
\( \bar{c}_g \) = gas compressibility at average reservoir pressure, psia\(^{-1}\)
\( c_{gi} \) = gas compressibility at original reservoir pressure, psia\(^{-1}\)
\( c_i \) = total system compressibility, psia\(^{-1}\)
\( c_{ii} \) = total system compressibility at original reservoir pressure, psia\(^{-1}\)
\( D \) = decline rate, days\(^{-1}\)
\( D_i \) = initial decline rate, days\(^{-1}\)
\( G \) = original gas-in-place, Mscf
\( G_p \) = cumulative gas production, Mscf
\( h \) = formation thickness, ft
\( I_0(x) \) = modified Bessel function of the first kind, zero order
\( I_1(x) \) = modified Bessel function of the first kind, first order
\( k_o \) = effective permeability to oil, md
\( k_g \) = effective permeability to gas, md
\( K_0(x) \) = modified Bessel function of the second kind, zero order
\( K_1(x) \) = modified Bessel function of the second kind, first order
\( N \) = original oil-in-place, STB
\( N_p \) = cumulative oil production, STB
\( p \) = pressure, psia
\( \bar{p} \) = average reservoir pressure, psia
\( p_D \) = dimensionless pressure response
\( p_D \) = logarithmic derivative of the dimensionless wellbore pressure response
\( p_{D, \text{avg}} \) = dimensionless pressure response based on average reservoir pressure
\( \bar{p}_D(u) \) = Laplace transform of the dimensionless pressure response
\[ p_{fun} = \text{appropriate pressure-like type of function for the stems on the Fetkovich-Carter type curve, psia} \]

\[ p_i = \text{initial reservoir pressure, psia} \]

\[ p_p = \text{normalized pseudopressure, psia} \]

\[ \bar{p}_p = \text{normalized pseudopressure at average reservoir pressure, psia} \]

\[ p_{p,abn} = \text{normalized pseudopressure at abandonment pressure, psia} \]

\[ p_{pi} = \text{normalized pseudopressure at initial reservoir pressure, psia} \]

\[ p_{pwf} = \text{normalized pseudopressure at bottomhole flowing pressure, psia} \]

\[ \Delta p_{tof} = \text{appropriate pressure function difference defined in table 4.4, psia} \]

\[ p_{wD} = \text{dimensionless wellbore pressure response} \]

\[ \bar{p}_{wD}(u) = \text{Laplace transform of the dimensionless wellbore pressure response} \]

\[ p_{wf} = \text{bottomhole flowing pressure, psia} \]

\[ q = \text{surface flow rate, STB/days or Mscf/days} \]

\[ q' = \text{derivative of the surface flow rate, STB/days}^2 \text{ or Mscf/days}^2 \]

\[ q_D = \text{dimensionless rate solution} \]

\[ \bar{q}_D(u) = \text{Laplace transform of the dimensionless rate solution} \]

\[ q_{Dd} = \text{dimensionless decline rate} \]

\[ q_{iD} = \text{derivative of the dimensionless decline rate} \]

\[ q_{Ddc} = \text{dimensionless decline rate function used to generate Carter's type curves for constant rate} \]

\[ q_{Ddi} = \text{dimensionless decline rate integral} \]

\[ q_{Ddd} = \text{derivative of the dimensionless decline rate integral} \]

\[ q_i = \text{initial flow rate for any type of fluid (liquid or gas) STB/days or Mscf/days} \]

\[ q_g = \text{gas flow rate, Mscf/days} \]

\[ q_o = \text{oil flow rate, STB/days} \]

\[ q_{oi} = \text{initial oil flow rate, STB/days} \]

\[ q_{tof} = \text{flow rate according to the type of fluid as defined in table 4.4, STB/days} \]

\[ Q = \text{cumulative production for any type of fluid (liquid or gas), STB or Mscf} \]

\[ Q_{Dd} = \text{dimensionless decline cumulative production} \]

\[ Q_{tof} = \text{cumulative production-like function defined by Eq. 4.48 or 4.49, STB or Mscf} \]

\[ r_D = \text{dimensionless radius at any point in the reservoir} \]

\[ r_e = \text{drainage radius, ft} \]

\[ r_dD = \text{dimensionless drainage radius} \]
\( r_w \) = wellbore radius, ft
\( r'_w \) = effective wellbore radius (includes formation damage effects), ft
\( s \) = formation damage (skin) factor, dimensionless
\( t \) = time, days
\( t^* \) = dummy time variable (\( t \) or \( t_a \)), days
\( \tilde{t} \) = material balance time (\( N_p / q_o \) or \( G_p / q_g \)), days
\( t_a \) = conventional normalized pseudotime, days
\( \tilde{t}_a \) = normalized material balance pseudotime, days
\( t_{a,DA} \) = dimensionless time based on drainage area and conventional normalized pseudotime
\( \tilde{t}_{a,DA} \) = dimensionless time based on drainage area and normalized material balance pseudotime
\( t_{a, Dd} \) = dimensionless decline time based on conventional normalized pseudotime
\( \tilde{t}_{a, Dd} \) = dimensionless decline time based on normalized material balance pseudotime
\( t_{cp} \) = constant pressure analog time, days
\( t_D \) = dimensionless time based on wellbore radius
\( t_{DA} \) = dimensionless time based on drainage area
\( \tilde{t}_{DA} \) = dimensionless decline time based on drainage area and material balance time
\( t_{Dd} \) = dimensionless decline time
\( \tilde{t}_{Dd} \) = dimensionless decline time based on material balance time
\( t_{dum} \) = actual time, \( t \), for liquid or pseudotime, \( t_a \), for gas, days
\( t_{fun} \) = appropriate time-like function for every stem in the Fetkovich-Carter type curve, days
\( t_{pss} \) = time for the onset of pseudosteady state, days
\( \tilde{t}_{tof} \) = time function defined in table 4.4, days
\( T \) = temperature, °F
\( u \) = Laplace parameter, dimensionless
\( V \) = total fluid volume, STB or Mcf
\( V_p \) = reservoir pore volume, RB or Mcf
\( y \) = dummy pressure type of variable, psia
\( z \) = real gas deviation factor, dimensionless
\( \tilde{z} \) = real gas deviation factor at average reservoir pressure, dimensionless
\( z_i \) = real gas deviation factor at initial reservoir pressure, dimensionless
Greek Letter Variables

\( \phi \) = formation porosity, fraction
\( \gamma \) = Euler's constant = 0.5772156649
\( \gamma_g \) = gas specific gravity, dimensionless
\( \lambda \) = Carter's drawdown parameter, dimensionless
\( \mu \) = fluid viscosity, cp
\( \mu_g \) = gas viscosity, cp
\( \bar{\mu}_g \) = gas viscosity at average reservoir pressure, cp
\( \mu_{gi} \) = gas viscosity at initial reservoir pressure, cp
\( \mu_o \) = oil viscosity, cp
\( \rho \) = fluid density, lb/scf
\( \tau \) = Dummy Time-Type of Variable, days

Special Subscripts and Operators

\( a \) = "adjusted" variable for gas well test analysis (use of these variables in gas well test analysis yields an equivalent liquid system).
\( avg \) = variable in terms of average reservoir pressure
\( CP \) = constant pressure
\( CR \) = constant rate
\( D \) = dimensionless variable
\( L \) = Laplace space operator (from real space to Laplace space)
\( L^{-1} \) = inverse Laplace space operator (from Laplace space to real space)
\( M.P. \) = match point
\( pss \) = pseudosteady-state
\( tof \) = type of fluid (liquid or gas)
REFERENCES


APPENDIX A

DEVELOPMENT OF LIQUID AND GAS DIFFUSIVITY EQUATIONS AND
DEFINITION OF DIMENSIONLESS VARIABLES

DIFFUSIVITY EQUATION FOR SINGLE PHASE FLOW IN ANY FLOW
GEOMETRY

The objective of this section is to develop a mathematical expression which
describes the flow of a single fluid in a porous media with respect to time and distance.
This relation should make no assumptions about rates or pressures which may be imposed
on the system. The diffusivity equation should be derived for a general flow geometry and
adapted to a specific flow geometry (e.g., radial, linear, spherical, etc.) when the need
arises.

As is typical in mathematical physics, the diffusivity equation is expected to be a
partial differential equation relating the pressure (or some pressure-like function) to
differentials in both space and time (or some time-like function). This partial differential
equation, and the appropriate initial and boundary conditions will be used to develop
solutions later in this thesis.

Derivation of the Single Phase Diffusivity Equation in Terms of Density

In order to develop a partial differential equation for the flow of fluids in porous
media the following physical concepts must be included:

1. Mass continuity equation, mass balance on the system.
2. Equation of motion (force balance)
3. Equation of state which relates fluid density with the pressure, temperature, and
   composition of the fluid.
4. Constitutive equation which relates shear rate and shear stress of fluid.
5. Energy conservation equation, energy balance on the system.
The mass continuity equation is physically described as

\[
\begin{pmatrix}
\text{rate of mass flow into the system} \\
\text{rate of mass flow out of the system} \\
\text{accumulation}
\end{pmatrix} = \begin{pmatrix}
\text{rate of mass flow} \\
\text{rate of mass flow} \\
\text{rate of mass flow}
\end{pmatrix} \quad \text{.........}(A-1)
\]

The mathematical form of the mass continuity equation for a general flow geometry is

\[
\nabla (\rho \vec{v}) = -\frac{\partial (\phi \rho)}{\partial t} \quad \text{.........................}(A-2)
\]

where

\[
\begin{align*}
\vec{v} & = \text{fluid velocity vector} \\
\rho & = \text{fluid density} \\
\phi & = \text{porosity, fraction of bulk volume} \\
t & = \text{time}
\end{align*}
\]

The equation of motion used for fluid flow in porous media is known as Darcy's law. Although Darcy's law was empirically derived by flowing water through sand packs and was only partially verified in an analytic sense, it is considered accurate for laminar flow in porous media. Darcy's law is given as

\[
\vec{v} = -\frac{k}{\mu} \left( \nabla p + \rho \vec{g} \right) \quad \text{.........................}(A-3)
\]

where

\[
\begin{align*}
\nabla p & = \text{pressure gradient} \\
k & = \text{effective permeability} \\
\mu & = \text{fluid viscosity} \\
\vec{g} & = \text{gravity vector}
\end{align*}
\]

Combining Eqs. A-2 and A-3 gives us

\[
\nabla \cdot \left[ \frac{k}{\mu} \left( \nabla p + \rho \vec{g} \right) \right] = \frac{\partial (\phi \rho)}{\partial t} \quad \text{.........................}(A-4)
\]

If we assume horizontal flow, the gravity term can be dropped, which reduces Eq. A-4 to

\[
\nabla \cdot \left[ \frac{k}{\mu} \nabla p \right] = \frac{\partial (\phi \rho)}{\partial t} \quad \text{.........................}(A-5)
\]

The \nabla operators for radial flow are given as

\[
\nabla \cdot \alpha = \frac{\partial \alpha}{\partial y} \nabla y
\]

where pressure will be used as the "y" variable, and
\[ \nabla \cdot \nabla \alpha = \nabla^2 \alpha = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \alpha}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \alpha}{\partial \phi^2} + \frac{\partial^2 \alpha}{\partial z^2} \]

For developments in this section of the Appendix, all the results will be left in \( \nabla \) notation for a general flow geometry. In later developments these operators for radial flow will be incorporated.

Eq. A-5 is the fundamental form of the diffusivity equation in terms of fluid density. This relation represents the starting point for future developments in terms of pressure, pressure squared, and pseudopressure. At this point the equation of state has not been incorporated in the diffusivity equation because this substitution is not yet necessary. In fact, the diffusivity equation can be worked in terms of density although this form would not be practical for field applications.

However, Eq. A-5 does include the constitutive equation implicitly through the viscosity term in Darcy's law (Eq. A-3). Virtually all well testing conditions are considered isothermal, therefore the conservation of energy equation can be neglected.

One further simplification of Eq. A-5 assumes the permeability, \( k \), and fluid viscosity, \( \mu \), to be constant. This gives us

\[ \nabla \cdot [\rho \nabla p] = \frac{\mu}{k} \frac{\partial (\phi p)}{\partial t} \] ..............................................................(A-6)

**Derivation of the Single Phase Diffusivity Equation in Terms of Pressure - Slightly Compressible Liquid Case**

If we expand the terms on the left-hand-side of Eq. A-6 using the product rule, and we expand the term on the right-hand-side using the chain rule, then we obtain

\[ \left( \nabla \cdot \rho \right) \nabla p + \rho \nabla \cdot \nabla p = \frac{\mu}{k} \frac{\partial (\phi p)}{\partial \phi} \frac{\partial \phi}{\partial t} \] ..............................................................(A-7)

Using the chain rule for the grad terms, the first term on the left-hand-side of Eq. A-7 can be written as

\[ \left( \nabla \cdot \rho \right) \nabla p = \frac{\partial p}{\partial \phi} \nabla p \nabla p = \frac{\partial p}{\partial \phi} (\nabla p)^2 \] ..............................................................(A-8)

and the second term on the left-hand-side of Eq. A-7 can be reduced using the \( \nabla \) notation.
\[ \rho \nabla \cdot \nabla p = \rho \nabla^2 p \] 

(A-9)

Combining Eqs. A-7 to A-9 we obtain

\[ \frac{\partial p}{\partial t} (\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k} \frac{\partial (\phi p)}{\partial t} \frac{\partial p}{\partial t} \] 

(A-10)

By applying the product rule to the \( \frac{\partial (\phi p)}{\partial t} \) of Eq. A-10 we have

\[ \frac{\partial (\phi p)}{\partial t} = \phi \frac{\partial p}{\partial t} + \rho \frac{\partial \phi}{\partial t} \]

or

\[ \frac{\partial (\phi p)}{\partial t} = \phi p \left[ \frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{1}{\phi} \frac{\partial \phi}{\partial t} \right] \]

Recalling the definition of fluid compressibility, we have

\[ c = \frac{1}{\rho} \frac{\partial p}{\partial t} \] 

(A-11)

and the definition of pore-volume compressibility is given as

\[ c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial t} \] 

(A-12)

Combining the definition of compressibility with the expansion of \( \frac{\partial (\phi p)}{\partial t} \), we obtain

\[ \frac{\partial (\phi p)}{\partial t} = \phi p (c + c_f) \]

Since "total" compressibility is defined as \( c_t = c + c_f \), the last relation may be simply written

\[ \frac{\partial (\phi p)}{\partial t} = \phi p c_t \] 

(A-13)

Combining Eqs. A-10 and A-13 we obtain

\[ \frac{\partial p}{\partial t} (\nabla p)^2 + \rho \nabla^2 p = \rho \frac{\phi c_t}{k} \frac{\partial p}{\partial t} \]

Dividing through by the fluid density gives

\[ \frac{1}{\rho} \frac{\partial p}{\partial t} (\nabla p)^2 + \nabla^2 p = \frac{\rho c_t}{k} \frac{\partial p}{\partial t} \]

and using the definition of fluid compressibility, Eq. A-11, we obtain

\[ c (\nabla p)^2 + \nabla^2 p = \frac{\rho c_t}{k} \frac{\partial p}{\partial t} \] 

(A-14)
The $c(\nabla p)^2$ term in Eq. A-14 is clearly non-linear as this term provides a dependent variable ($p$) squared and a function of the dependent variable ($c$) multiplied by another function of the dependent variable ($\nabla p)^2$. Solving Eq. A-14 analytically is very difficult and the solution given by Finjord and Aadnoy, is not well suited for the analysis of well test data.

However, if we assume a small and constant compressibility, as suggested by the equation of state for a "slightly" compressible liquid, then the $c(\nabla p)^2$ term can be neglected. The equation of state for a "slightly" compressible liquid is given as

$$\rho = \rho_0 \exp [c(p - p_0)] \tag{A-15}$$

Thus, if the $c(\nabla p)^2$ in Eq. A-14 is neglected, this relation becomes

$$\nabla^2 p = \frac{\phi \mu c_i}{k} \frac{\partial p}{\partial t} \tag{A-16}$$

Eq. A-16 is familiar as a typical diffusion-type partial differential equation and can be solved analytically in a variety of geometries, assuming that the diffusivity term $\frac{\phi \mu c_i}{k}$ is constant.

**Derivation of the Single Phase Diffusivity Equation in terms of Normalized Pseudopressure and Pseudotime - General Case**

The developments shown in the last section focused on the derivation of a diffusivity equation for single phase liquid flow where the liquid was considered to be of small and constant compressibility. Generally, these conditions are only satisfied by black oils above the bubble point pressure. For black oils below the bubble point pressure and for natural gases Eq. A-16 will be an approximation at best.

Therefore, we must develop a form of the diffusivity equation where general behavior of $\rho$, $\mu$, and $k$ can be assumed. The $\frac{\partial k}{\mu}$ term in Eq. A-5 will be treated as general, with no assumptions on its behavior with pressure. Eq. A-5 was shown to be

$$\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi p)}{\partial t} \tag{A-5}$$
For a black oil, the oil density is given by
\[ \rho_o = \frac{\rho_{osc}}{B_o} \] .................................................. (A-17)

For gases, the density is given by
\[ \rho_g = \frac{pM}{zRT} \] .................................................. (A-18)
and the gas formation volume factor is given by
\[ B_g = \frac{p_{sc}}{z_{sc}T_{sc}} \frac{zT}{p} \] .................................................. (A-19)

Combining Eqs. A-18 and A-19 we obtain
\[ \rho_g = \frac{p_{sc}M}{z_{sc}RT_{sc}} \frac{1}{B_g} \]
or using the definition of gas density at standard conditions we have
\[ \rho_g = \frac{\rho_{gsc}}{B_g} \] .................................................. (A-20)
where
\[ \rho_{gsc} = \frac{p_{sc}M}{z_{sc}RT_{sc}} \]

Substituting Eq. A-17 into Eq. A-5 gives us the general diffusivity equation for a liquid
\[ \nabla \cdot \left( \frac{k_o}{\mu_o B_o} \nabla p \right) = \frac{\partial}{\partial t} \left( \frac{\phi}{B_o} \right) \] .................................................. (A-21)

and substitution of Eq. A-20 into Eq. A-5 gives us the general diffusivity equation for the flow of a single gas phase
\[ \nabla \cdot \left( \frac{k_g}{\mu_g B_g} \nabla p \right) = \frac{\partial}{\partial t} \left( \frac{\phi}{B_g} \right) \] .................................................. (A-22)

We note that the forms of Eqs. A-21 and A-22 are identical. By using a set of general parameters \( \mu, k, \) and \( B, \) a single form of diffusivity equation for both oil and gas flow can be developed. We must also note that the \( \frac{k}{\mu B} \) terms in Eqs. A-21 and A-22 represent non-linearities which must be resolved. In particular, we must define a function that combines these variables but retains the \( \frac{k}{\mu B} \nabla p \) form. Such a function is called "pseudopressure."35

The normalized pseudopressure function, \( p_p, \) is defined as
\[ p_p = \left( \frac{\mu B}{k} \right)_n \int_{p_{base}}^{p} \frac{k}{\mu B} \, dp \]  \hspace{1cm} \text{(A-23)}

where \( n \) indicates the normalizing condition. Throughout this work, the initial pressure, \( p_i \), will be used as the normalizing pressure. In addition, the pseudopressure gradient, \( \nabla p_p \), can be written as

\[ \nabla p_p = \frac{\partial p_p}{\partial p} \, \nabla p \]  \hspace{1cm} \text{(A-24)}

Combining Eqs. A-23 and A-24 gives us

\[ \frac{\partial p_p}{\partial p} = \left( \frac{\mu B}{k} \right)_n \frac{k}{\mu B} \]  \hspace{1cm} \text{(A-25)}

The partial derivative of the pseudopressure function with respect to time is obtained using the chain rule. This result is given as

\[ \frac{\partial p_p}{\partial t} = \frac{\partial p_p}{\partial p} \frac{\partial p}{\partial t} \]  \hspace{1cm} \text{(A-26)}

Combining these developments using a generic form given by Eqs. A-21 and A-22, we have as our starting point

\[ \nabla \cdot \left[ \frac{k}{\mu B} \, \nabla p \right] = \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) \]  \hspace{1cm} \text{(A-27)}

Expanding the right-hand-side of Eq. A-27 using the chain rule gives us

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) = \frac{\partial}{\partial p} \left( \frac{\phi}{B} \right) \frac{\partial p}{\partial t} \]

Using the product rule we have

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) = \left[ \frac{1}{B} \, \frac{\partial \phi}{\partial p} - \frac{\phi}{B^2} \, \frac{\partial B}{\partial p} \right] \frac{\partial p}{\partial t} \]

which after some rearranging into a more familiar form gives us

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) = \frac{\phi}{B} \left[ \frac{1}{\phi} \, \frac{\partial \phi}{\partial p} - \frac{1}{B} \, \frac{\partial B}{\partial p} \right] \frac{\partial p}{\partial t} \]  \hspace{1cm} \text{(A-28)}

Recalling the definition of fluid compressibility, we have

\[ c = - \frac{1}{B} \frac{\partial B}{\partial p} \]  \hspace{1cm} \text{(A-29)}
Recalling the definition of pore volume compressibility we have

\[ c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial p} \]  \hspace{1cm} (A-12)

Substituting Eqs. A-12 and A-29 into Eq. A-28 and using the definition \( c_t = c + c_f \) into Eq. A-28 gives us

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) = \frac{\phi c_t}{B} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A-30)

Substituting Eq. A-30 back into Eq. A-27 we obtain

\[ \nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \frac{\phi c_t}{B} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A-31)

Combining Eqs. A-24 and A-25 and solving for \( \frac{k}{\mu B} \nabla \nabla p \) we have

\[ \frac{k}{\mu B} \nabla p = \left( \frac{k}{\mu B} \right)_n \nabla \nabla p \]  \hspace{1cm} (A-32)

Also, if we combine Eqs. A-25 and A-26 and solve for \( \frac{\partial p}{\partial t} \) we obtain

\[ \frac{\partial p}{\partial t} = \left( \frac{k}{\mu B} \right)_n \frac{\mu B}{k} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A-33)

Substituting Eqs. A-32 and A-33 into Eq. A-31 yields

\[ \nabla \cdot \left[ \frac{k}{\mu B} \right] \nabla \nabla p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \]

Canceling the \( \left( \frac{k}{\mu B} \right)_n \) terms we obtain

\[ \nabla \cdot \nabla \nabla p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \]

or if we use the \( \nabla \) notation that \( \nabla \cdot \nabla = \nabla^2 \), we have as our final result

\[ \nabla^2 \nabla p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A-34)

While the left-hand-side of Eq. A-34 has no non-linear terms, the \( \mu, c_t, \) and \( k \) terms on the right-hand-side are not assumed constant and represent non-linearities due to the multiplication of these terms with \( \frac{\partial p}{\partial t} \) as well as with each other. In order to develop analytical solutions using Eq. A-34, we must either assume \( \frac{\mu c_t}{k} \) to be constant, or we must
develop a linearizing pseudotime that accounts for changes in $\frac{\mu c_i}{k}$ with respect to pressure.
For gas reservoirs, we typically assume that the effective permeability, $k$, is constant in
general, and that $\mu c_i$ is constant during transient pressure drawdown flow conditions.

Because we know that the $\frac{\mu c_i}{k}$ term is not really constant, we must develop a
linearizing function (in this case a pseudotime) to account for the variation in the $\frac{\mu c_i}{k}$ term.
Agarwal\textsuperscript{37} proposed the following "intuitive" pseudotime function

$$t_a = (\mu c_i)_n \int_0^t \frac{1}{\mu(p)c_i(p)} \, dt \quad \cdots \quad (A-35)$$

Lee and Holditch\textsuperscript{38} verified this function for the analysis of pressure buildup tests
and gave analytic criteria for the application of Eq. A-35 using wellbore pressures to
evaluate $\mu$ and $c_i$, for the radial flow case.

Fraim and Wattenbarger\textsuperscript{16} proposed a slightly different form of the Agarwal\textsuperscript{37}
pseudotime function for boundary dominated (post-transient) flow. This pseudotime is
given by

$$t_a = (\mu c_i)_n \int_0^t \frac{1}{\mu(p)c_i(p)} \, dt \quad \cdots \quad (A-36)$$

where the average reservoir pressure, $\bar{p}$, is used in place of the wellbore pressure, for the
evaluation of $\mu c_i$. While Eq. A-36 is rigorously correct for boundary dominated flow,
simulation suggests that Eq. A-36 is also correct for gas wells during transient flow.

Our present objective is to combine the definition of the pseudotime with the
general form of the diffusivity equation, Eq. A-34. Taking the partial derivative of Eq.
A-35 with respect to time gives

$$\frac{\partial t}{\partial t} = (\mu c_i)_n \quad \cdots \quad (A-37)$$

Applying the chain rule to the $\frac{\partial p}{\partial t}$ in Eq. A-34 we have
\[ \frac{\partial \rho_p}{\partial t} = \partial_{\alpha} \frac{\partial \rho_p}{\partial \alpha} \]

Combining this result with Eq. A-37 gives us

\[ \mu c \frac{\partial \rho_p}{\partial t} = \frac{(\mu c)_n}{\partial_{\alpha}} \frac{\partial \rho_p}{\partial \alpha} \]

Substituting Eq. A-34 into Eq. A-38, we obtain

\[ \nabla \cdot \rho_p = \phi \frac{\mu c}{k} \frac{\partial \rho_p}{\partial \alpha} \]

(A-39)

where the normalized pseudopressure function, \( \rho_p \), is given by

\[ \rho_p = \frac{\mu B}{k} \int_{p_{base}}^{p} \frac{k}{\mu B} dp \]

(A-23)

and the normalized pseudotime function, \( t_\alpha \), is

\[ t_\alpha = \frac{1}{\mu c} \int_{0}^{t} \frac{1}{\mu(p)c(p)} dp \]

(A-35)

If we assume that the effective permeability, \( k \), is constant, as is usually done, Eq. A-23 reduces to

\[ \rho_p = \frac{\mu B}{n} \int_{p_{base}}^{p} \frac{1}{\mu B} dp \]

(A-40)

We note that the definitions of pseudopressure and pseudotime are valid for the single phase flow of both gases and liquids. Because the diffusivity equation is also written in a general form, we can use the solutions derived for the "slightly compressible liquid" case to model the single phase flow of gas and compressible liquids, assuming that the appropriate pseudopressure and pseudotime functions are used for each case.
THE RADIAL FLOW DIFFUSIVITY EQUATION FOR SINGLE PHASE FLOW

In the previous section we developed diffusivity equations for general flow geometries in terms of pressure, pseudopressure and pseudotime. In this section the $\nabla^2 p$ and $\nabla^2 p_p$ terms will be expanded for radial flow geometries for development of specific solutions. Using a general variable, $\alpha$, it is known\textsuperscript{42} that $\nabla^2 \alpha$ is

$$\nabla \cdot \nabla \alpha = \nabla^2 \alpha = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \alpha}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \alpha}{\partial \theta^2} + \frac{\partial^2 \alpha}{\partial z^2} \tag{A-41}$$

For horizontal, radial flow with neither angular flow nor vertical flow effects, Eq. A-41 is reduced to

$$\nabla^2 \alpha = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \alpha}{\partial r} \right) \tag{A-42}$$

Slightly Compressible Liquid Case

The development of a diffusivity equation for the horizontal, radial flow of a slightly compressible liquid is the most fundamental case in reservoir engineering and well testing. The approach is to consider a liquid of small and constant compressibility ("slightly compressible") and constant fluid viscosity. Although these conditions are not always met in practice, these conditions are usually valid for black oil reservoirs producing at pressures above the bubble point.

Our goal, is to develop a form of the radial flow diffusivity equation, which is a partial differential equation, that can be solved analytically. Therefore, the resulting partial differential equation must be linear. That is, none of the pressure or pressure gradient terms can be raised to a power or multiplied by another pressure or pressure gradient term. This non-linear behavior will become apparent as we discuss relations for gas flow in terms of pressure or pressure-squared rather than pseudopressure, and in terms of time rather than pseudotime

The diffusivity equation for the flow of a "slightly compressible liquid" of small and constant compressibility, in a general flow geometry is
\[ \nabla^2 p = \frac{\phi \mu C_l}{k} \frac{\partial p}{\partial t} \] ........................................ (A-16)

We recall that the \( \frac{\phi \mu C_l}{k} \) term is usually assumed constant for liquid flow, but this term represents a potential non-linearity that must be accounted for. If \( \frac{\phi \mu C_l}{k} \) is constant then we can consider Eq. A-16 as a linear partial differential equation. By contrast, if the \( \frac{\phi \mu C_l}{k} \) term is not constant with respect to pressure, then it is unlikely that Eq. A-16 can be solved analytically in terms of known solutions, except using the perturbation approach, which is only an approximation.

We will start by assuming that \( \frac{\phi \mu C_l}{k} \) is approximately constant. Combining Eq. A-16 with the horizontal, radial flow operator, Eq. A-42, we have

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{\phi \mu C_l}{k} \frac{\partial p}{\partial t} \] ........................................ (A-43)

We can expand the left-hand-side of Eq. A-43 using the product rule which gives us

\[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi \mu C_l}{k} \frac{\partial p}{\partial t} \] ........................................ (A-44)

Eqs. A-43 and A-44 are the most basic forms of the horizontal, radial flow diffusivity equation for the flow of a slightly compressible liquid given in terms of pressure. While this result is the most widely applied form of the diffusivity equation, in terms of both solutions as well as analysis and interpretation methodologies, the assumption of constant viscosity and small and constant fluid compressibility are not universally applicable. This non-constant viscosity-compressibility behavior is especially true for the flow of gas, but is also true for the flow of "compressible" liquids (i.e., oils where the pressure is below the bubble point pressure--solution gas drive systems).

**Concept of the "Equivalent" Liquid Case:**

The widespread application of analysis and interpretation methods based on the fluid model for a slightly compressible liquid makes the development of an "equivalent" liquid model quite desirable. Such an "equivalent" liquid model would be used when the
assumptions for a slightly compressible liquid are not met as is the case of gas flow or the flow of a compressible liquid.

Again, our goal is to develop a form of the radial flow diffusivity equation that can be solved analytically. As we showed previously the pseudopressure-pseudotime form of the diffusivity equation is given by

$$\nabla^2 p_p = \frac{\phi}{k} (\mu c)_n \frac{\partial p_p}{\partial t} \hspace{1cm} (A-39)$$

Eq. A-39 appears to be completely linearized with no non-linear terms present anywhere in it. However, Eq. A-39 includes in an implicit manner the variation of gas properties with distance on the right-hand-side. The interested reader is referred to Lee and Holditch\textsuperscript{38} for a complete discussion of pseudotime for radial flow. Assuming that Eq. A-39 is sufficient, we substitute the geometry term for horizontal, radial flow, based on the relation given by Eq. A-42 to yield

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p_p}{\partial r} \right) = \frac{\phi}{k} (\mu c)_n \frac{\partial p_p}{\partial t} \hspace{1cm} (A-45)$$

Expanding the left-hand-side of Eq. A-45 using the product rule, we obtain

$$\frac{\partial^2 p_p}{\partial r^2} + \frac{1}{r} \frac{\partial p_p}{\partial r} = \frac{\phi}{k} (\mu c)_n \frac{\partial p_p}{\partial t} \hspace{1cm} (A-46)$$

where Eq. A-46 represents a linearized form of the diffusivity equation for horizontal, radial flow that should be more than sufficient for modeling and analysis. The porosity, $\phi$, is usually evaluated at the initial pressure of the reservoir, $p_i$, as is also the $(\mu c)_n$ product.

These relations for pressure-time (Eq. A-44), pseudopressure-time (Eq. A-34), and pseudopressure-pseudotime (Eq. A-46) will be discussed when we develop analytical solutions that can be used for the analysis and prediction of reservoir performance.
DEVELOPMENT OF A GENERALIZED GAS DIFFUSIVITY EQUATION

Our objective is to develop the pressure, pressure-squared, and pseudopressure forms of the diffusivity equation from the general form given by Eq. A-5.

Gas Pseudopressure Relations

Before proceeding into the following derivations, we would like to establish the proper nomenclature and formulations for applying pseudopressure to gas systems. The definition of the formation gas volume factor, $B_g$, is given by Eq. A-19 as

$$B_g = \frac{p_{sc}}{z_{sc} T_{sc}} \frac{z T}{p} \quad \text{(A-19)}$$

The general definition of pseudopressure is given as

$$p_p = \left(\frac{\mu B}{k}\right)_n \int_{p_{base}}^p \frac{k}{\mu B} dp \quad \text{(A-23)}$$

Because our efforts are intended for gas flow we can assume that the effective permeability, $k$, remains constant at its initial value. This assumption reduces Eq. A-23 to

$$p_p = (\mu B)_n \int_{p_{base}}^p \frac{1}{\mu B} dp \quad \text{(A-40)}$$

Combining Eqs. A-19 and A-45 gives us

$$p_{pg} = \left(\frac{\mu z T}{p}\right)_n \int_{p_{base}}^p \frac{p}{\mu z T} dp$$

If we assume isothermal flow conditions ($T=T_n$), which is usually the case as cooling or heating is considered insignificant, then we can write

$$p_{pg} = \left(\frac{\mu z}{p}\right)_n \int_{p_{base}}^p \frac{p}{\mu z} dp \quad \text{(A-47)}$$

Eq. A-47 is equivalent in detail and application to Eq. A-40, but Eq. A-47 allows us to make comments regarding the behavior of $\mu$, $z$, and $p$, as a function of pressure.
Starting from the generalized density form of the diffusivity equation, Eq. A-5, we have
\[
\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi p)}{\partial t} \tag{A-5}
\]

Recalling the definition of gas density, Eq. A-18, gives us
\[
\rho_g = \frac{pM}{zRT} \tag{A-18}
\]
Combining Eqs. A-5 and A-18 and eliminating the \(\frac{M}{RT}\) factor, we obtain
\[
\nabla \cdot \left[ \frac{k}{\mu} \frac{p}{z} \nabla p \right] = \frac{\partial}{\partial t} \left( \phi \frac{p}{z} \right) \tag{A-48}
\]

If we assume that the effective permeability, \(k\), is constant and expand the right-hand-side using the product rule, then Eq. A-48 becomes
\[
\nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{1}{k} \frac{\partial}{\partial t} \left( \phi \frac{p}{z} \right) = \frac{1}{k} \left[ \frac{p}{z} \frac{\partial \phi}{\partial t} + \phi \frac{\partial}{\partial t} \left( \frac{p}{z} \right) \right]
\]
Expanding the time derivatives using the chain rule gives us
\[
\nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{1}{k} \left[ \frac{p}{z} \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial t} + \phi \frac{\partial}{\partial p} \left( \frac{p}{z} \right) \frac{\partial p}{\partial t} \right]
\]
Factoring out the porosity term we have
\[
\nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{\phi}{k} \left[ \frac{p}{z} \frac{1}{\phi} \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial t} + \phi \frac{\partial}{\partial p} \left( \frac{p}{z} \right) \frac{\partial p}{\partial t} \right] \tag{A-49}
\]
Recalling the definition of pore-volume compressibility, \(c_f\), (Eq. A-12) we have
\[
c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial p} \tag{A-12}
\]
The combination of Eq. A-12 with Eq. A-49 yields
\[
\nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{\phi}{k} \left[ \frac{p}{z} c_f \frac{\partial p}{\partial t} + \phi \frac{\partial}{\partial p} \left( \frac{p}{z} \right) \frac{\partial p}{\partial t} \right] \tag{A-50}
\]
Given the definition of isothermal gas compressibility, \(c_g\), we have
\[
c_g = \left( \frac{1}{p} - \frac{1}{z} \frac{\partial z}{\partial p} \right)_{T} \tag{A-51}
\]
The alternative form of the definition of gas compressibility is (again for isothermal conditions, but dropping the \(T\) subscript)
\[ c_g = \frac{z}{p} \frac{\partial}{\partial p} \left( \frac{P}{z} \right) \]  \hspace{1cm} (A-52)

Rearranging this form we have
\[ \frac{\partial}{\partial p} \left( \frac{P}{z} \right) = \frac{P}{z} c_g \]  \hspace{1cm} (A-53)

Combining Eqs. A-50 and A-53 gives us
\[ \nabla \cdot \left[ \left( \frac{P}{\mu z} \right) \nabla p \right] = \frac{\phi}{k} \left( \frac{P}{z} c_f \frac{\partial p}{\partial t} + \frac{P}{z} c_g \frac{\partial p}{\partial t} \right) \]  \hspace{1cm} (A-54)

Using \( c_t = c_g + c_f \), Eq. A-54 reduces to
\[ \nabla \cdot \left[ \left( \frac{P}{\mu z} \right) \nabla p \right] = \frac{\phi c_t}{k} \frac{P}{z} \frac{\partial p}{\partial t} \]  \hspace{1cm} (A-55)

Eq. A-55 is the generalized diffusivity equation for gas flow. We have already mentioned that the pseudopressure and pseudotime concepts can be applied to linearize Eq. A-55, as we showed in the previous section. The pseudopressure gradient in Eq. A-24, \( \nabla p_p \), converted to the gas nomenclature is
\[ \nabla p_p = \frac{\partial p_p}{\partial p} \nabla p \]

or in terms of the pressure gradient we have
\[ \nabla p = \frac{\partial p}{\partial p_p} \nabla p_p \]  \hspace{1cm} (A-56)

The pseudopressure derivative with respect to time of Eq. A-55 can be expanded using the chain rule to yield
\[ \frac{\partial p}{\partial t} = \frac{\partial p}{\partial p_p} \frac{\partial p_p}{\partial t} \]  \hspace{1cm} (A-57)

Combining Eqs. A-55, A-56, and A-57 we obtain
\[ \nabla \cdot \left[ \left( \frac{P}{\mu z} \right) \frac{\partial p}{\partial p_p} \nabla p_p \right] = \frac{\phi c_t}{k} \frac{P}{z} \frac{\partial p}{\partial p_p} \frac{\partial p_p}{\partial t} \]  \hspace{1cm} (A-58)

Taking the derivative of the gas pseudopressure function (Eq. A-47) with respect to pressure gives us
\[ \frac{\partial p_p}{\partial p} = \left( \frac{\mu z}{P} \right) \frac{P}{n \mu z} \]  \hspace{1cm} (A-59)
Combining Eqs. A-58 and A-59 we have

\[ \nabla \cdot \left[ \frac{P}{\mu_z} \left( \frac{1}{\mu_z} \frac{P}{k} \nabla p_{pg} \right) \right] = \phi c_t \frac{P}{k} \left( \frac{1}{\mu_z} \frac{P}{k} \right) \frac{\partial p_{pg}}{\partial t} \]

Factoring out the constant terms we have

\[ \nabla \cdot \nabla p_{pg} = \frac{\phi c_t}{k} \frac{\partial p_{pg}}{\partial t} \]

or

\[ \nabla^2 p_{pg} = \frac{\phi c_t}{k} \frac{\partial p_{pg}}{\partial t} \]  \hspace{1cm} \text{(A-60)}

The important point to note is that Eqs. A-34 and A-60 are exactly equivalent. These relations differ only in the definition of pseudopressure used in each relation. The pseudopressure definitions illustrate that even though those relations are different, the pseudopressure concept is general and should produce accurate modeling and analysis -- so long as the pseudopressure definition used includes the pertinent variables that would present non-linearities if left untransformed on the left-hand-side of the diffusivity equation.

We note that the $\mu c_t$ product in Eqs. A-34 and A-60 is not constant, but can be a strong function of pressure. Fortunately, the effect of a changing $\mu c_t$ product is usually small or even negligible for the analysis and interpretation of transient flow data from pressure drawdown tests in gas wells. However, for data from boundary-dominated flow tests (as well as production data) and for data from pressure buildup tests, the effect of the changing $\mu c_t$ product is very significant and can cause analysis based on liquid solutions to be in considerable error. Therefore, we need to use another "equivalent liquid" variable (as we have done with pseudopressure to account for $\frac{P}{\mu_z}$) that can be used to resolve the changing $\mu c_t$ product. This "equivalent liquid" variable is the pseudotime function defined in the previous section by Eqs. A-35 (general form) and A-36 (form for boundary dominated flow). The combination of either definition (Eq. A-35 or A-36) with Eq. A-60 gives

\[ \nabla^2 p_{pg} = \frac{\phi c_t}{k} \frac{\partial p_{pg}}{\partial \eta} \]  \hspace{1cm} \text{(A-61)}
For unsteady-state or transient flow in gas wells, Lee and Holditch\textsuperscript{38} verified the use of Eq. A-35 by simulation and they gave an approximate analytical proof for the analysis of pressure buildup tests. Lee and Holditch\textsuperscript{38} showed that for the analysis of pressure buildup tests, wellbore pressures can be used to evaluate $\mu$ and $c_t$, for the radial flow case.

Although intuitive, the use of wellbore pressures to compute pseudotime for post-transient and boundary-dominated flow tests is not correct because pseudotime is a function of both time and space. Towards this end, Fraim and Wattenbarger\textsuperscript{16} proposed that the average reservoir pressure, $\bar{p}$, be used in the pseudotime integral. They showed that the definition given by Eq. A-36 is exact for boundary-dominated (or pseudosteady-state) flow for gas wells produced at a constant bottomhole flowing pressure.

Eq. A-36 is also exactly correct for gas wells produced a constant flowrate during boundary-dominated flow, and additional research (semi-analytic proofs and numerical simulation) suggests that Eq. A-36 is also correct during transient flow. Unfortunately, the application of Eq. A-36 for transient flow in gas wells is somewhat difficult to apply because the average reservoir pressure profile must be known. Eq. A-36 is fairly easy to apply for boundary-dominated flow data because the average reservoir pressure profile can be estimated and refined using the gas material balance equation and the boundary dominated flow identity that relates average reservoir pressures and wellbore pressures.

Development of the Pressure-Squared form of the Gas Diffusivity Equation

The application of the pressure-squared form of the gas diffusivity equation has been one of the most argued topics in gas reservoir engineering. We recognize that while the pressure-squared form provides a significant reduction in complexity from the rigorous pseudopressure form, the validity of applying the pressure-squared form of the diffusivity equation is often debated. This controversy is primarily due to two considerations. First, the pressure-squared formulation of the diffusivity equation is an approximation that has definable limits of applicability. Second, the application of the pressure-squared form has been extremely broad, from company policies to regulatory requirements.

The argument often arises as to when is the pressure-squared formulation valid. We will show that the pressure-squared formulation of the diffusivity equation can be
derived by two methods. First, by explicitly assuming that $\mu z=$ constant, and second, from a completely general development that results in a condition where $\mu z$ must be constant to obtain the pressure-squared form. Most references claim that $\mu z=$ constant for gas reservoirs where $p \leq 2000$ psia. We will show that this condition is somewhat arbitrary, and the assumption of $\mu z=$ constant must be coupled to the behavior of the pressure, temperature, and the composition of the gas in the system.

Specifically, we must both quantify as well as qualify the assumption of $\mu z=$ constant. This condition is specifically valid for an ideal gas, which has the following properties

- constant gas viscosity
- $z = 1$, (i.e., no volume correction)

For real gases the conditions where $\mu z=$ constant must be carefully investigated. The verification of this hypothesis requires plots of $\mu z$ versus pressure for various cases of temperature and gas composition.

**Derivation of the Pressure-Squared form of the Gas Diffusivity Equation**

**Pseudopressure Approach**

In this approach we consider the behavior of the gas pseudopressure function, Eq. A-47, by assuming that $\mu z$ is constant with pressure, temperature, and gas composition. Recalling Eq. A-47 we have

$$p_{pg} = \left(\frac{\mu z}{p}\right) \int_{p_{base}}^{p} \frac{p}{\mu z} \, dp \quad \text{.......................................................... (A-47)}$$

If we assume that $\mu z = (\mu z)_n =$ constant, Eq. A-47 becomes

$$p_{pg} = \frac{1}{p_n} \int_{p_{base}}^{p} p \, dp \quad \text{.......................................................... (A-62)}$$

Completing the integration of Eq. A-62 we obtain

$$p_{pg} = \frac{1}{2p_n} \left(p^2 - p_{base}^2\right)$$

or by normalizing with respect to $2p_n$ we have
\[ p_{pg} = \frac{p^2}{2p_n} - \frac{p_{base}^2}{2p_n} \] ................................................................. (A-63)

Recalling the pseudopressure form of the gas diffusivity equation, (Eq. A-60), we have
\[ \nabla^2 p_{pg} = \frac{\phi \mu c_l}{k} \frac{\partial p_{pg}}{\partial t} \] ................................................................. (A-60)

The combination of Eqs. A-60 and A-63 yields
\[ \nabla^2 \left( \frac{p^2}{2p_n} - \frac{p_{base}^2}{2p_n} \right) = \frac{\phi \mu c_l}{k} \frac{\partial}{\partial t} \left( \frac{p^2}{2p_n} - \frac{p_{base}^2}{2p_n} \right) \]

which after eliminating the constant \( \frac{p_{base}^2}{2p_n} \) terms from these derivatives gives us
\[ \nabla^2 \left( \frac{p^2}{2p_n} \right) = \frac{\phi \mu c_l}{k} \frac{\partial}{\partial t} \left( \frac{p^2}{2p_n} \right) \]

and, finally eliminating the constant \( \frac{1}{2p_n} \) factors, we are left with
\[ \nabla^2 \left( p^2 \right) = \frac{\phi \mu c_l}{k} \frac{\partial}{\partial t} \left( p^2 \right) \] ................................................................. (A-64)

where Eq. A-64 is strictly valid for a constant \( \mu z \) term in Eq. A-47. It must be noted that no assumptions have been made regarding the \( \mu c_l \) product on the right-hand-side of Eq. A-64. Unfortunately, the development of Eq. A-64 starts with the assumption of a constant \( \mu z \) term and using this assumption it was possible for us not to develop the residual term or terms which must explicitly account for the \( \mu z \) product, without regard as to whether this term is constant or not.

**Derivation of the Pressure-Squared form of the Gas Diffusivity Equation**

**General Approach**

This approach is taken from the 1975 Energy Resources Conservation Board Manual.\(^\text{43}\) We start with the general form of the gas diffusivity equation, Eq. A-55, which is given as
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \frac{\phi c_l}{k} \frac{P}{z} \frac{\partial p}{\partial t} \] ................................................................. (A-55)
Using the chain rule on the pressure-time derivative term in Eq. A-55 we have
\[ \nabla \cdot \left( \frac{p}{\mu z} \nabla p \right) = \frac{\phi c_t p}{k} \frac{1}{z} \frac{\partial}{\partial p} \left( p^2 \right) \frac{\partial}{\partial t} \left( p^2 \right) \] .................................................. (A-65)

In order to "convert" Eq. A-65 into pressure-squared form, we must use the following identities
\[ \frac{\partial}{\partial p} \left( p^2 \right) = 2p \] ................................................................. (A-66)
and
\[ \nabla (p^2) = 2p \nabla p \]

where this result can be rearranged to give us
\[ p \nabla p = \frac{1}{2} \nabla (p^2) \] ................................................................. (A-67)

\[ \nabla \cdot \left[ \frac{1}{\mu z} \frac{1}{2} \nabla (p^2) \right] = \frac{\phi c_t p}{k} \frac{1}{2z} \frac{\partial}{\partial t} \left( p^2 \right) \]

or, upon canceling p terms on the right-hand-side, we have
\[ \nabla \cdot \left[ \frac{1}{\mu z} \frac{1}{2} \nabla (p^2) \right] = \frac{\phi c_t}{k} \frac{1}{2z} \frac{\partial}{\partial t} \left( p^2 \right) \] .................................................. (A-68)

We can expand the left-hand-side of Eq. A-68 using the product rule to give
\[ \frac{1}{2} \mu z \nabla \cdot \nabla (p^2) + \frac{1}{2} \nabla \cdot \left[ 1 \right] \nabla (p^2) = \frac{\phi c_t}{k} \frac{1}{2z} \frac{\partial}{\partial t} \left( p^2 \right) \]

Multiplying through by 2\mu z and reducing the \( \nabla \cdot \nabla (p^2) \) term to \( \nabla^2 (p^2) \) gives
\[ \nabla^2 (p^2) + \mu z \nabla \cdot \left[ \frac{1}{\mu z} \nabla (p^2) \right] = \frac{\phi c_t}{k} \frac{1}{2z} \frac{\partial}{\partial t} \left( p^2 \right) \] .................................................. (A-69)

Expanding the
\[ \nabla \cdot \left[ \frac{1}{\mu z} \nabla (p^2) \right] \] term using \( \nabla \cdot \alpha = \frac{\partial}{\partial y} \nabla y \) with \( \alpha = \frac{1}{\mu z} \) and \( y = p^2 \) we have
\[ \nabla \cdot \left[ \frac{1}{\mu z} \nabla (p^2) \right] = \frac{\partial}{\partial p} \left[ \frac{1}{\mu z} \nabla (p^2) \right] \nabla (p^2) \]

or
\[ \nabla \cdot \left[ \frac{1}{\mu z} \nabla (p^2) \right] = \frac{\partial}{\partial p} \left[ \frac{1}{\mu z} \nabla (p^2) \right] \nabla (p^2)^2 \] .................................................. (A-70)
Combining Eqs. A-69 and A-70 we obtain

\[ \nabla^2 (p^2) + \mu_z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu_z} \right] \nabla (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial (p^2)}{\partial t} \]

Expanding the \( \mu_z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu_z} \right] \) term as an attempt to simplify

\[ \mu_z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu_z} \right] = \frac{\partial}{\partial p^2} \left[ \ln \left( \frac{1}{\mu_z} \right) \right] = -\frac{\partial}{\partial p^2} \left[ \ln (\mu z) \right] \]

(A-71)

Combining Eqs. A-70 and A-71 we have

\[ \nabla^2 (p^2) + \frac{\partial}{\partial p^2} \left[ \ln \left( \frac{1}{\mu z} \right) \right] \nabla (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial (p^2)}{\partial t} \]

(A-72)

or we can have the alternative form which is given as

\[ \nabla^2 (p^2) - \frac{\partial}{\partial p^2} \left[ \ln (\mu z) \right] \nabla (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial (p^2)}{\partial t} \]

(A-73)

We see here that if \( \mu z = \text{constant} \) then Eqs. A-72 and A-73 reduce to

\[ \nabla^2 (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial (p^2)}{\partial t} \]

(A-74)

We also note that the \( \mu c_t \) product is not assumed constant on the right-hand-side of Eq. A-74. Also, Eqs. A-72 and A-73 were developed without assuming any particular behavior of the \( \mu z \) term. On the other hand Eq. A-74 results from assuming that the \( \mu z \) term is constant. While this is plausible and perhaps even probable given the varied behavior of real gas systems, this assumption of the \( \mu c_t \) product being constant must be verified by inspection of plots of \( \mu z \) versus pressure.

**Validation of the Pressure-Squared Form of the Gas Diffusivity Equation Using \( \mu z \) Versus Pressure Plots**

The first thing that we must be remember regarding the condition of \( \mu z \) being constant is that it is only rigorously valid for an ideal gas, where by definition viscosity is constant and \( z=1 \).
We give three plots of $\mu z$ versus pressure for various temperatures and gas compositions to help us to recognize the applicability of the $\mu z$ product being approximately constant. These graphs are shown as Figs. A-1, A-2 and A-3. Each plot is based on a particular temperature (100, 200 and 300°F), and contains four curves, each curve representing a different gas gravity ($\gamma_g = 0.6, 0.8, 1.0, \text{ and } 1.2$, where air = 1.0). Log-log scales are used to emphasize the duration of the regions where $\mu z$ is approximately constant. While only three cases of temperature and four cases of gas composition may seem like a small sample, we believe that these plots illustrate the expected behavior in practice.

Wattenbarger and Ramey\textsuperscript{44} suggest that in general $\mu z$ is constant for pressure values less than 2000 psia. It is clear from Figs. A-1 through A-3 that the assumption of $\mu z = \text{constant}$ for that pressure range is quite reasonable for all temperatures and gas compositions. This verifies, in principle, the use of the "pressure-squared" diffusivity equation (Eq. A-64).

![Graph showing $\mu z$ versus pressure for various gas compositions at 100°F.](image)

**Fig. A-1**- Comparison of $\mu z$ versus Pressure for a Temperature of 100°F and Various Gas Compositions.
Fig. A-2 - Comparison of $\mu z$ versus Pressure for a Temperature of 200°F and Various Gas Compositions.

Fig. A-3 - Comparison of $\mu z$ versus Pressure for a Temperature of 300°F and Various Gas Compositions.
Derivation of the Pressure Form of the Gas Diffusivity Equation

In this case we assume that the $\frac{p}{\mu_z}$ product remains approximately constant with pressure. This condition has questionable origins, but yields a very convenient result, which is a gas diffusivity equation in terms of pressure. The validity of assuming $\frac{p}{\mu_z}$ constant will also have to be determined by observation of plots of $\frac{p}{\mu_z}$ versus pressure for various cases of temperature and gas composition.

Derivation of the Pressure Form of the Gas Diffusivity Equation

Pseudopressure Approach

In this approach we will start with the definition of gas pseudopressure, Eq. A-47. Recalling Eq. A-47 we have

$$p_{pg} = \left(\frac{\mu_z}{p}\right)_n \int_{p_{base}}^p \frac{p}{\mu_z} dp \quad \cdots \quad (A-47)$$

Assuming that $\frac{p}{\mu_z} = \left(\frac{p}{\mu_z}\right)_n = \text{constant}$, Eq. A-47 becomes

$$p_{pg} = \int_{p_{base}}^p dp \quad \cdots \quad (A-75)$$

Completing the integration gives us

$$p_{pg} = p - p_{base} \quad \cdots \quad (A-76)$$

Recalling the pseudopressure form of the gas diffusivity equation (Eq. A-60) we have

$$\nabla^2 p_{pg} = \frac{\phi \mu c_i}{k} \frac{\partial p_{pg}}{\partial t} \quad \cdots \quad (A-60)$$

Combining Eqs. A-60 and A-76 produces

$$\nabla^2 (p - p_{base}) = \frac{\phi \mu c_i}{k} \frac{\partial}{\partial t} (p - p_{base})$$

Eliminating the constant $p_{base}$ terms from these derivatives we obtain
\[ \nabla^2 p = \frac{\phi u c_L}{k} \frac{\partial p}{\partial t} \] ................................. (A-77)

where Eq. A-77 is strictly valid for a constant \( \frac{p}{\mu z} \) term. We note that we have not assumed that the \( \mu c_L \) product is constant on the right-hand-side of Eq. A-77. Unfortunately, the development of Eq. A-77 starts with the assumption of a constant \( \frac{p}{\mu z} \) term and using this assumption we are able not to develop the residual term or terms which must explicitly account for the \( \frac{p}{\mu z} \) group.

**Derivation of the Pressure Form of the Gas Diffusivity Equation**

**General Approach**

We have borrowed this approach from the 1975 ERCB Manual for Gas Well Testing.\(^{43}\) This development starts with the general form of the gas diffusivity equation, Eq. A-55.

\[ \nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{\phi c_L}{k} \frac{p}{z} \frac{\partial p}{\partial t} \] ................................. (A-55)

Expanding the left-hand-side of Eq. A-55 gives us

\[ \nabla \cdot \left[ \frac{p}{\mu z} \right] \nabla p + \frac{p}{\mu z} \nabla \cdot \nabla p = \frac{\phi c_L}{k} \frac{p}{z} \frac{\partial p}{\partial t} \]

or in \( \nabla^2 \) notation we have

\[ \nabla \cdot \left[ \frac{p}{\mu z} \right] \nabla p + \frac{p}{\mu z} \nabla^2 p = \frac{\phi c_L}{k} \frac{p}{z} \frac{\partial p}{\partial t} \]

Multiplying through by \( \frac{\mu z}{p} \) we have

\[ \nabla^2 p + \frac{\mu z}{p} \nabla \cdot \left[ \frac{p}{\mu z} \right] \nabla p = \frac{\phi c_L}{k} \frac{\partial p}{\partial t} \] ................................. (A-78)

We can write the \( \nabla \cdot \left[ \frac{p}{\mu z} \right] \nabla p \) term as

\[ \nabla \cdot \left[ \frac{p}{\mu z} \right] \nabla p = \frac{\partial}{\partial p} \left[ \frac{p}{\mu z} \right] \nabla p \nabla p = \frac{\partial}{\partial p} \left[ \frac{p}{\mu z} \right] \nabla p^2 \] ................................. (A-79)

Combining Eqs. A-78 and A-79 gives

\[ \nabla^2 p + \frac{\mu z}{p} \frac{\partial}{\partial p} \left[ \frac{p}{\mu z} \right] (\nabla p)^2 = \frac{\phi c_L}{k} \frac{\partial p}{\partial t} \] ................................. (A-80)
Also, the \( \frac{\mu_z}{p} \frac{\partial}{\partial p} \left[ p \right] \mu_z \) term may be written as
\[
\frac{\mu_z}{p} \frac{\partial}{\partial p} \left[ p \right] \mu_z = \frac{\partial}{\partial p} \left[ \ln \left( \frac{p}{\mu_z} \right) \right] = \frac{\partial}{\partial p} \left[ \ln \left( \frac{p}{\mu_z} \right) \right]
\] .................................................... (A-81)

Combining Eqs. A-80 and A-81 we obtain
\[
\nabla^2 p + \frac{\partial}{\partial p} \left[ \ln \left( \frac{p}{\mu_z} \right) \right] \left( \nabla p \right)^2 = \frac{\phi \mu_c I}{k} \frac{\partial p}{\partial t}
\] .................................................... (A-82)

The alternative form of Eq. A-82 is
\[
\nabla^2 p - \frac{\partial}{\partial p} \left[ \ln \left( \frac{p}{\mu_z} \right) \right] \left( \nabla p \right)^2 = \frac{\phi \mu_c I}{k} \frac{\partial p}{\partial t}
\] .................................................... (A-83)

From Eqs. A-82 and A-83 it is clear that if \( \frac{p}{\mu_z} \) is constant then we have
\[
\nabla^2 p = \frac{\phi \mu_c I}{k} \frac{\partial p}{\partial t}
\] .................................................... (A-84)

where Eq. A-84 is only rigorously valid for a constant \( \frac{p}{\mu_z} \) term. We note that we did not assume the product \( \mu c_i \) product to be constant on the right-hand-side of Eq. A-84.

**Validation of the Pressure Form of the Gas Diffusivity Equation Using \( p/\mu_z \) Versus Pressure Plots**

We give three plots of \( p/\mu_z \) versus pressure using the same temperatures and gas compositions that we considered in the validation of the pressure-squared (\( \mu z \)=constant) case. Wattenbarger and Ramey\(^{44}\) suggest that in general, \( p/\mu_z \) is constant for pressure values greater than 4000 psia. However, we immediately note from looking at Figs. A-4-A-6 that \( p/\mu_z \) is never truly constant. The \( \gamma_g \approx 0.6 \) cases appear to be constant for pressures greater than 5000 psia, which would seem to agree with Wattenbarger and Ramey's criterion.
Fig. A-4 - Comparison of $\frac{p}{\mu z}$ versus Pressure for a Temperature of 100°F and Various Gas Compositions.

Fig. A-5 - Comparison of $\frac{p}{\mu z}$ versus Pressure for a Temperature of 200°F and Various Gas Compositions.
Fig. A-6 - Comparison of $\frac{P}{\mu z}$ versus Pressure for a Temperature of 300°F and Various Gas Compositions.

On the other hand, the cases for $\gamma_g = 0.8, 1.0, \text{and } 1.2$ indicate only a short "constant" region near the point where the $\frac{P}{\mu z}$ versus pressure trend has a maximum and begins to decrease with increasing pressures. In light of this behavior, it would be ill-advised to recommend the use of the pressure diffusivity equation for gas (Eq. A-84), even at high pressures (>5000 psia).

We note that although Eq. A-84 is exactly the same in form as the slightly compressible liquid relation, Eq. A-16, there are some critical differences. These differences are illustrated by the governing assumptions for each relation, which are cited below.

Assumptions common to Eq. A-84 and A-16:
- laminar flow
- homogeneous and isotropic porous media
- isothermal flow
- negligible gravity forces

Assumptions exclusive to Eq. A-16:
- slightly compressible liquid (liquid of small and constant compressibility)
- constant fluid viscosity
Assumptions exclusive to Eq. A-84:

- $p/\mu z$ = constant
- $\mu c_f$ is assumed to vary with pressure

Given these assumptions, the following conclusions are made:

- For general applications in gas flow the pseudopressure form of the diffusivity equation, Eq. A-60, must be used.

- The "pressure-squared" form of the diffusivity equation (Eq. A-64) may be used for cases where all pressures are less than 2000 psia. However, it is advised that the pseudopressure form, Eq. A-60, be always used.

- The "pressure" form of the gas diffusivity equation (Eq. A-84) should not be used under any circumstances for modeling of gas flow, or the analysis of gas data.

DEFINITION OF DIMENSIONLESS VARIABLES IN TERMS OF NORMALIZED PSEUDOFUNCTIONS FOR THE ANALYSIS OF GAS FLOW

In this section we will use the general form of the diffusivity equation and its initial and inner boundary conditions to develop a unique set of dimensionless variables. It must be acknowledged that this treatment is similar to the one developed by Jennifer Johnston in a new well testing text by Lee et al. (unpublished).

This work is particularly interested in showing the relation of the dimensionless variables to the diffusivity equation written in terms of normalized pseudopressure and normalized pseudotime. The normalized pseudotime was discussed earlier in this Appendix (Eq. A-35). Recalling the gas diffusivity equation in terms of normalized pseudopressure and normalized pseudotime (Eq. A-39) which is given as

$$\nabla^2 p_p = \frac{\phi}{k} \left( \mu c_i v_n \right) \frac{\partial p_p}{\partial t_{\sigma}} \hspace{1cm} \text{(A-39)}$$
and using the initial reservoir pressure as the normalizing condition, we have

\[ \nabla^2 p_p = \frac{\phi \mu_i c_{ii}}{k_g} \frac{\partial p_p}{\partial t_a} \]  

(A-85)

However, in order to keep a general form of the diffusivity equation for any type of fluid, we will use two dummy variables, "y" and "\( \tau \)". The "y" variable represents a pressure type of function. This can be assumed to be pressure, pressure-squared or pseudopressure according to the needs. Also, the "\( \tau \)" variable represents a time function that can be taken as actual time or pseudotime time depending again on our needs.

Writing the generalized diffusivity equation, we have

\[ \frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} = \frac{\phi \mu c_i}{k} \frac{\partial y}{\partial \tau} \]  

(A-86)

where \( y \) is a general pressure function (e.g., \( p \), \( p^2 \), or \( p_p \)) and \( \tau \) is a general time function (\( t \) or \( t_i \)). The definition of the dimensionless radius is well known to be given by

\[ r_D = \frac{r}{r_w} \]  

(A-87)

or

\[ r = r_w r_D \]  

(A-88)

Substituting Eq. A-88 into Eq. A-86 we have

\[ \frac{\partial}{\partial (r_w r_D)} \frac{\partial y}{\partial (r_w r_D)} + \frac{1}{r_w r_D} \frac{\partial y}{\partial r_w r_D} = \frac{\phi \mu c_i}{k} \frac{\partial y}{\partial \tau} \]

Factoring out the \( r_w \) terms from inside the derivatives we have

\[ \frac{1}{r_w^2} \frac{\partial^2 y}{\partial r_D^2} + \frac{1}{r_w^2} \frac{1}{r_D} \frac{\partial y}{\partial r_D} = \frac{\phi \mu c_i}{k} \frac{\partial y}{\partial \tau} \]

Multiplying through by \( r_w^{-2} \) we obtain

\[ \frac{\partial^2 y}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial y}{\partial r_D} = \frac{\mu c_i r_w^{-2}}{k} \frac{\partial y}{\partial \tau} \]  

(A-90)

Our objective is to develop a dimensionless pressure function, \( p_D \), that accounts for both the initial and inner boundary conditions. The initial condition is

\[ y (r, \tau \leq 0) = y_i \]  

(A-91)

We would like to have a \( p_D \) definition such that the following initial condition exists

\[ p_D (r, \tau \leq 0) = 0 \]  

(A-92)
Eq. A-92 yields a mathematical convenience for developing analytical solutions of the dimensionless diffusivity equation. Eq. A-92 then suggests the following form of the dimensionless pressure function, \( p_D \)

\[
p_D = \frac{y_i - y}{y_{ch}} \tag{A-93}
\]

where \( y_{ch} \) is a "characteristic" value of the pressure function, \( y \). This \( y_{ch} \) will be determined using the inner boundary condition. Rearranging Eq. A-93 and solving for \( y \) it is seen that

\[
y = y_i - y_{ch} p_D \tag{A-94}
\]

Substituting Eq. A-94 into Eq. A-90 we have

\[
\frac{\partial}{\partial r_D} \left( \frac{\partial}{\partial r_D} (y_i - y_{ch} p_D) \right) + \frac{1}{r_D} \frac{\partial}{\partial r_D} (y_i - y_{ch} p_D) = \frac{\phi \mu c r_w^2}{k} \frac{\partial}{\partial \tau} (y_i - y_{ch} p_D) \tag{A-95}
\]

or

\[
\frac{\partial^2}{\partial r_D^2} (y_i) - y_{ch} \frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial}{\partial r_D} (y_i) - \frac{1}{r_D} y_{ch} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu c r_w^2}{k} y_{ch} \frac{\partial p_D}{\partial \tau} \tag{A-96}
\]

Expanding the constant terms from Eq. A-96 we have

\[
- y_{ch} \frac{\partial^2 p_D}{\partial r_D^2} \cdot \frac{y_{ch}}{r_D} \frac{\partial p_D}{\partial r_D} = - \frac{\phi \mu c r_w^2}{k} y_{ch} \frac{\partial p_D}{\partial \tau} \tag{A-97}
\]

Canceling the \( y_{ch} \) terms in Eq. A-97 gives

\[
\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu c r_w^2}{k} \frac{\partial p_D}{\partial \tau} \tag{A-98}
\]

From Darcy's law the inner boundary condition is

\[
\left[ r \frac{\partial p}{\partial r} \right]_{r_w} = \frac{q B \mu}{2 \pi k h} \tag{A-99}
\]

Using the chain rule we can express Eq. A-99 in terms of \( y \) as

\[
\left[ r \frac{\partial y}{\partial r} \right]_{r_w} = \frac{q B \mu}{2 \pi k h} \frac{\partial y}{\partial p} \tag{A-100}
\]

Substituting the definitions of dimensionless radius, \( r_D \), and dimensionless pressure, \( p_D \) gives us
\[
\left[ (r_w r_D) \frac{\partial}{\partial (r_w r_D)} (y_i - y_{ch} p_D) \right]_{r_D = \frac{r_x}{r_w}} = \frac{qB\mu}{2\pi kh} \frac{\partial y}{\partial \rho}
\]

or canceling like terms and eliminating \( \frac{\partial y_i}{\partial r_D} \) we have

\[
\left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D = 1} = - \frac{1}{y_{ch}} \frac{qB\mu}{2\pi kh} \frac{\partial y}{\partial \rho} \quad \text{(A-101)}
\]

Again as a mathematical convenience, we would like to have the inner boundary condition expressed as

\[
\left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D = 1} = - 1 \quad \text{(A-102)}
\]

Comparing Eqs. A-101 and A-102 we obtain

\[
y_{ch} = \frac{qB\mu}{2\pi kh} \frac{\partial y}{\partial \rho} \quad \text{(A-103)}
\]

It must be recalled that \( y \) can be \( p, p^2 \), or \( p_p \), so the \( \frac{\partial y}{\partial \rho} \) is readily obtained once \( y \) is specified. Combining Eqs. A-93 and A-103 the final definition of \( p_D \) definition is given by

\[
p_D = \frac{2\pi kh}{qB\mu} (y_i - y) \quad \text{(A-104)}
\]

where \( \frac{\partial y}{\partial \rho} \) is one if \( y=p, 2p \) if \( y=p^2 \), and finally \( \frac{(\mu B)n}{\mu B} \) if \( p=p_p \). It must be noted that the "\( n \)" subscript means normalizing conditions, and we recommend that the initial conditions be used.

We also note at this time that \( \tau \) usually represents actual time if the working fluid is liquid, or pseudotime if the fluid is gas. Now, an "intuitive" definition of the appropriate dimensionless time-like function \( t_D \) should be

\[
t_D = \frac{\tau}{\tau_{ch}} \quad \text{(A-105)}
\]

where \( \tau_{ch} \) is a characteristic time-like function to be determined from Eq. A-98. Rearranging Eq. A-105 we have

\[
\tau = \tau_{ch} t_D \quad \text{(A-106)}
\]

Combining Eq. A-98 and Eq. A-106 we obtain
\[ \frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu_c r_w^2}{k} \left( \frac{\partial p_D}{\partial (\tau r_D)} \right) \frac{1}{\tau c h} \frac{\partial p_D}{\partial t_D} \]

If we define \( \tau_{ch} \) as
\[ \tau_{ch} = \frac{\phi \mu_c r_w^2}{k} \] ...........................(A-107)

then the final form of the dimensionless diffusivity equation is given by
\[ \frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \] ...........................................(A-108)

where,
\[ r_D = \frac{r_w}{r} \] ...........................................(A-87)
\[ p_D = \frac{y_i - y}{y_{ch}} \] ...........................................(A-93)
\[ y_{ch} = \frac{q B u \partial y}{2 \pi k h \partial p} \] ...........................................(A-103)
\[ p_D = \frac{2 \pi k h}{q B u \partial y} (y_i - y) \] ...........................................(A-104)
\[ t_D = \frac{\tau}{\tau_{ch}} \] ...........................................(A-105)
\[ \tau_{ch} = \frac{\phi \mu_c r_w^2}{k} \] ...........................................(A-107)
\[ t_D = - \frac{k \tau}{\phi \mu_c r_w^2} \] ...........................................(A-109)

and, the initial and boundary conditions are given by

Initial Condition:
\[ p_D (r_D, t_D \leq 0) = 0 \] ...........................................(A-110)

Inner Boundary Condition: (constant rate at the well)
\[ \left[ r_D \frac{\partial p_D}{\partial r_D} \right] r_D = -1 \] ...........................................(A-111)

Outer Boundary Condition:
Case 1: infinite outer boundary
\[ p_D (r_D \to 0, t_D) = 0 \] ...........................................(A-112)
Case 2: no-flow outer boundary
\[ \left[ r_D \frac{\partial p_D}{\partial r} \right]_{r_D} = 0 \]...........................(A-113)

Case 3: constant pressure outer boundary (constant at initial pressure)
\[ p_D (r_{eD}, t_D) = 0 \]...................................................(A-114)

**Dimensionless Variables for the Radial Flow Diffusivity Equation in Terms of Field and SI Units**

Unfortunately several unit systems are currently being used within the petroleum industry. The original definition of Darcy's law lead to the "Darcy" unit system where the fundamental definition of permeability is given as

\[ k \left[ \text{1 darcy} \right] = \frac{q \left[ 1 \text{ cm}^3 \text{ sec}^{-1} \right] B \left[ \text{cm}^3/\text{cm}^3 \right] \mu \left[ \text{cp} \right]}{A \left[ 1 \text{ cm}^2 \right] \frac{\partial p}{\partial r} \left[ 1 \text{ atm}/1 \text{ cm} \right]} \]

or in words as stated by Amyx *et al.*45:

"A porous medium has permeability of one Darcy when a single phase fluid of one centipoise viscosity that completely fills the voids of the medium will flow through it under conditions of viscous (laminar) flow at a rate of one cubic centimeter per second per square centimeter cross-sectional area under a pressure or potential gradient of one atmosphere per centimeter".

In order to work in a particular unit system the \( t_D \) and \( p_D \) identities must be converted from the Darcy unit system to the chosen field or SI unit system. To facilitate this task, the units associated with each variable in each of the three unit systems have been summarized below. First however, some useful conversions are provided. These conversions are

\[
\begin{align*}
1 \text{ ft} & = 30.48 \text{ cm} \\
1 \text{ atm} & = 14.696 \text{ psia} = 101.325 \text{ kPa} \\
1 \text{ bbl} & = 5.615 \text{ ft}^3 \\
1 \text{ cp} & = 1 \text{ mPa\cdotsec}
\end{align*}
\]

We have summarized the important units in the Darcy, field and SI units in Table A-1.
Table A-1
Summary of Variables Associated with the Darcy, Field, and SI Unit Systems

<table>
<thead>
<tr>
<th>Variable</th>
<th>Darcy Unit</th>
<th>Field Unit</th>
<th>SI Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$</td>
<td>d</td>
<td>md</td>
<td>md</td>
</tr>
<tr>
<td>Net pay thickness, $h$</td>
<td>cm</td>
<td>ft</td>
<td>m</td>
</tr>
<tr>
<td>$y$ ($p$ or $p_p$)</td>
<td>atm</td>
<td>psia</td>
<td>kPa</td>
</tr>
<tr>
<td>$y$ ($p^2$)</td>
<td>atm$^2$</td>
<td>psia$^2$</td>
<td>(kPa)$^2$</td>
</tr>
<tr>
<td>Flow rate, $q$</td>
<td>cm$^3$/sec</td>
<td>STB/D (MSCF/D)</td>
<td>m$^3$/D</td>
</tr>
<tr>
<td>Form, volume factor, $B$</td>
<td>rcm$^3$/std cm$^3$</td>
<td>RB/STB(or RB/MSCF)</td>
<td>rm$^3$/std m$^3$</td>
</tr>
<tr>
<td>Fluid viscosity, $\mu$</td>
<td>cp</td>
<td>cp</td>
<td>mPa$\cdot$s</td>
</tr>
<tr>
<td>$\frac{\partial y}{\partial p}$ ($y = p$ or $p_p$)</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$\frac{\partial y}{\partial p}$ ($y = p^2$)</td>
<td>atm</td>
<td>psia</td>
<td>kPa</td>
</tr>
<tr>
<td>Time function $\tau$ ($t$ or $t_a$)</td>
<td>sec</td>
<td>hr</td>
<td>hr</td>
</tr>
<tr>
<td>Total compressibility, $c_t$</td>
<td>atm$^{-1}$</td>
<td>psia$^{-1}$</td>
<td>kPa$^{-1}$</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$</td>
<td>cm</td>
<td>ft</td>
<td>m</td>
</tr>
</tbody>
</table>

The conversion of the $p_D$ function requires one minor adjustment to set the $p^2$ relation as equivalent in form to the $p$ and $p_p$ cases. This "adjustment" comes as a lumping of the $B \frac{\partial y}{\partial p}$ product into a new variable, $B_{psq}$, given by

$$B_{psq} = B \frac{\partial (p^2)}{\partial p}$$

which we can write equivalently as

$$B_{psq} = 2p B$$  \hspace{1cm} \text{(A-115)}

The use of $B_{psq}$ in place of the $B \frac{\partial y}{\partial p}$ product will make the $p^2$ form exactly the same as the $p$ and $p_p$ forms, and result in a single unit conversion for all three cases ($p$, $p^2$, $p_p$).
Conversion to Field Units

Using the appropriate conversions from Darcy to field units in Eq. A-104 gives us

\[ p_D = \frac{2\pi k [md][\frac{1d}{1000md}][h[ft][\frac{30.48cm}{1ft}][y_i - y][\text{psia}][\frac{1\text{atm}}{14.696 \text{ psia}}]}{q \left[ \text{STB/D} \right] \left[ \frac{5.615 \text{ ft}^3}{\text{bbl}} \right] \left[ \frac{30.48 \text{ cm}}{1 \text{ ft}} \right]^3 \left[ \frac{1 \text{D}}{24 \text{hr}} \right] \left[ \frac{1 \text{ hr}}{3600 \text{ sec}} \right] B \left[ \text{RB} \right] \left[ \mu \left[ \text{cp} \right] \right] \frac{\partial y}{\partial p}} \]

\[ p_D = 2\pi \left[ 1.127 \times 10^{-3} \right] \left[ \frac{d \text{ cm atm cm}^3}{\text{md ft psia bbl sec}} \right] \frac{kh(y_i - y)}{qB\mu \frac{\partial y}{\partial p}} \]

Finally we have

\[ p_D = 7.081\times 10^{-3} \frac{kh(y_i - y)}{qB\mu \frac{\partial y}{\partial p}} \]

or in terms of the common form for well testing, we have

\[ p_D = \frac{kh(y_i - y)}{141.2 qB\mu \frac{\partial y}{\partial p}} \] .................................(A-116)

Using the appropriate conversions from Darcy to field units in Eq. A-116, we obtain

\[ t_D = \frac{k [md][\frac{1d}{1000md}][\tau [hr][\frac{3600 \text{ sec}}{1hr}]}{\phi \left[ \text{fraction} \right] \mu [\text{cp}] c_i \left[ \frac{1}{\text{psia}} \right] \left[ \frac{14.696 \text{ psia}}{1 \text{atm}} \right] r_w^2 \left[ \frac{30.48 \text{ cm}^2}{1 \text{ft}^2} \right]} \]

and after reducing terms we have

\[ t_D = 2.637\times 10^{-3} \left[ \frac{d \text{ sec atm ft}^2}{\text{md hr psia cm}^2} \right] \frac{k\tau}{\phi\mu c_r r_w^2} \]

Dropping the units on the conversion constant it must be clear that

\[ t_D = 2.637\times 10^{-3} \frac{k\tau}{\phi\mu c_r r_w^2} \] .................................(A-117)

We must recall that the "y" variable in Eq. A-116 represents any pressure-like type of function. That is, pressure, pressure squared or pseudopressure. Similarly, in Eq. A-117 the variable "\( \tau \)" represents any time-like type of function. That is, actual time or pseudotime.
Conversion to SI Units

Using the appropriate conversions from Darcy to SI in Eq. A-104 gives us

\[ p_D = \frac{2\pi k [\text{md}] \left[ \frac{1 \text{d}}{1000 \text{md}} \right] h [\text{m}] \left[ \frac{100 \text{cm}}{\text{m}} \right] (y_i - y) [\text{kPa}] \left[ \frac{1 \text{atm}}{101,33 \text{ kPa}} \right]}{q [\text{std m}^3/\text{D}] \left[ \frac{100 \text{cm}}{\text{m}} \right]^3 \left[ \frac{1 \text{D}}{24 \text{hr}} \right] \left[ \frac{1 \text{hr}}{3600 \text{ sec}} \right] B \left[ \frac{\text{rm}^3}{\text{std m}^3} \right] \mu [\text{mPa\cdotsec}] \left[ \frac{1 \text{ cp}}{1 \text{ mPa\cdotsec}} \right] \frac{\partial y}{\partial p}}\]

\[ p_D = 2\pi \left[ 8.527 \times 10^{-5} \right] \left[ \frac{d \text{ cm atm cm}^3 \text{ sec}}{\text{md m kPa std m D mPa sec}} \right] \frac{kh(y_i - y)}{qB\mu \frac{\partial y}{\partial p}} \]

Finally we have

\[ p_D = 5.3574 \times 10^{-4} \frac{kh(y_i - y)}{qB\mu \frac{\partial y}{\partial p}} \] \hspace{1cm} (A-118)

Using the appropriate conversions from Darcy to SI units in Eq A-109 it follows that

\[ t_D = \frac{k [\text{md}] \left[ \frac{1 \text{d}}{1000 \text{md}} \right] \tau [\text{hr}] \left[ \frac{3600 \text{sec}}{\text{hr}} \right]}{\phi [\text{fraction}] \mu [\text{mPa \cdot sec}] \left[ \frac{1 \text{cp}}{1 \text{mPa \cdot sec}} \right] c_i \left[ \frac{1}{\text{kPa}} \right] \left[ \frac{101.33 \text{ kPa}}{1 \text{atm}} \right] r_w^2 [\text{m}^2] \left[ \frac{100 \text{cm}}{1 \text{m}} \right]^2} \]

or upon reduction we have

\[ t_D = 3.557 \times 10^{-6} \left[ \frac{d \text{ sec mPa sec atm m}^2}{\text{md hr cp kPa cm}^2} \right] \frac{k\tau}{\phi \mu c_i r_w^2} \]

which after dropping the units on the conversion constant gives us

\[ t_D = 3.557 \times 10^{-6} \frac{k\tau}{\phi \mu c_i r_w^2} \] \hspace{1cm} (A-119)

Summary of Unit Conversion Developments

A summary of the definitions presented in this section can be given by writing \( t_D \) and \( p_D \) in the following general form

\[ t_D = t_{De} \frac{k\tau}{\phi \mu c_i r_w^2} \] \hspace{1cm} (A-120)
and
\[ p_D = p_{Dc} \frac{kh(y_i - y)}{qB\mu \frac{\partial y}{\partial p}} \]  \hspace{2cm} (A-121)

The following tables summarize unit conversion constants for the dimensionless time constant, \( t_{Dc} \), dimensionless pressure constant, \( p_{Dc} \), as well as the assignments for the pressure dummy variable "y", and the time dummy variable "\( \tau \)" in each system of units.

**Table A-2**

Conversion Constants for the \( t_D \) and \( p_D \) Functions

<table>
<thead>
<tr>
<th>Constant</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{Dc} )</td>
<td>1</td>
<td>2.637x10^{-4}</td>
<td>3.557x10^{-6}</td>
</tr>
<tr>
<td>( p_{Dc} )</td>
<td>2p</td>
<td>7.081x10^{-3}</td>
<td>5.356x10^{-4}</td>
</tr>
<tr>
<td>( p_{Dc} = \frac{1}{p_{Dc}} )</td>
<td>1/(2p)</td>
<td>141.2</td>
<td>1867.1</td>
</tr>
</tbody>
</table>

**Table A-3**

Possible Assignments for the Dummy Pressure Variable "y".

<table>
<thead>
<tr>
<th>Function</th>
<th>Pressure</th>
<th>Press. Squared</th>
<th>Normalized Pseudopressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>p</td>
<td>( p^2 )</td>
<td>( p_{pn} = \mu_nB_n \int_{p \text{ base}}^{p} \frac{1}{\mu_B} , dp )</td>
</tr>
<tr>
<td>( \frac{\partial y}{\partial p} )</td>
<td>1</td>
<td>2p</td>
<td>( \frac{\mu_n B_n}{\mu B} )</td>
</tr>
</tbody>
</table>

**Table A-4**

Possible Assignments for the Dummy Time Variable "\( \tau \)".

<table>
<thead>
<tr>
<th>Function</th>
<th>Time</th>
<th>Normalized Pseudotime</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>( t )</td>
<td>( t_a = \mu_n c_{tn} \int_{0}^{t} \frac{1}{\mu c_t} , dt )</td>
</tr>
</tbody>
</table>
APPENDIX B

SOLUTION OF THE RADIAL DIFFUSIVITY EQUATION USING LAPLACE TRANSFORM METHODS

We begin this development by recalling the general form of the dimensionless diffusivity equation that we derived in Appendix A of this thesis. This relation is

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \quad \text{.................................................. (B-1)}$$

where

$$r_D \equiv \frac{r}{r_w}$$

and for liquid we have the following definitions in terms of field units

$$p_D \equiv \frac{k_{ph}}{141.2 q_b B_o \mu_o} (p_i - p_{wf})$$

$$t_D \equiv \frac{0.00633 k_{ph}}{\phi \mu_o C_{it} r_w^2} \quad (t \text{ in days})$$

whereas for gas we have the following definitions in terms of field units

$$p_D \equiv \frac{k_{ph}}{141.2 q_b B_i \mu_i} (p_{pi} - p_{pwf})$$

$$t_D \equiv \frac{0.00633 k_{ph}}{\phi \mu_i C_{it} r_w^2} \quad (t \text{ in days})$$

CONSTANT RATE FLOW IN A BOUNDED CIRCULAR RESERVOIR WITH NO FLOW OUTER BOUNDARY

In this section we will solve the dimensionless diffusivity equation and its initial boundary conditions for the case of a well centered in a circular reservoir with a no-flow outer boundary. Then, the initial and boundary conditions are given by the following relations
Initial Condition

\[ p_D(r_D, t_D \leq 0) = 0 \quad \text{(uniform pressure distribution)} \] ................. (B-2)

Constant Rate Inner Boundary Condition

\[ \left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D = 1} = -1 \] ................. (B-3)

No Flow Outer Boundary Condition

\[ \left[ r_D \frac{d \tilde{p}_D}{d r_D} \right]_{r_D = r_e} = 0 \] ................. (B-4)

Taking the Laplace transform of Eq. B-1 we have

\[ \frac{d^2 \tilde{p}_D}{d r_D^2} + \frac{1}{r_D} \frac{d \tilde{p}_D}{d r_D} = u \tilde{p}_D - p_D(t_D = 0) \]

where

\[ \tilde{p}_D = L\{p_D(t_D, r_D)\} \]

and \( u \) is the Laplace parameter. When the initial condition, Eq. B-2, is combined with the last relation we have

\[ \frac{d^2 \tilde{p}_D}{d r_D^2} + \frac{1}{r_D} \frac{d \tilde{p}_D}{d r_D} = u \tilde{p}_D \]

Using the variable of substitution, \( x = \sqrt{u} r_D \), we obtain

\[ x^2 \frac{d^2 \tilde{p}_D}{d x^2} + x \frac{d \tilde{p}_D}{d x} = x^2 \tilde{p}_D \] ................. (B-5)

This type of equation has a general solution\(^{46}\) given by

\[ \tilde{p}_D(r_D, u) = AI_0 (\sqrt{u} r_D) + BK_0 (\sqrt{u} r_D) \] ................. (B-6)

and the derivative of the general solution with respect to \( r_D \) is given as

\[ \left[ r_D \frac{d \tilde{p}_D(r_D, u)}{d r_D} \right] = A\sqrt{u}r_D I_1(\sqrt{u}r_D) - B\sqrt{u}r_D K_1(\sqrt{u}r_D) \] ................. (B-7)

Taking the Laplace transform of the boundary conditions (Eqs. B-3 and B-4) we have

\[ \left[ r_D \frac{d \tilde{p}_D}{d r_D} \right]_{r_D = 1} = -\frac{1}{u} \] ................. (B-8)
and
\[
\left. r_D \frac{d\bar{P}_D}{d\bar{r}_D} \right|_{\bar{r}_D = r_{eD}} = 0 \quad \mbox{................................................................. (B-9)}
\]

We note that we used the initial condition (Eq. B-2) in the development of the general Laplace transform relation, Eq. B-5.

**Development of Laplace Transform Solution**:

Our objective is to determine the particular solution of Eq. B-6 using the given boundary conditions. In other words, we must determine \( A \) and \( B \) using Eqs. B-7 to B-9. Eq. B-7 is the fundamental relation and Eqs. B-8 and B-9 are the boundary conditions that we will use to determine the constants \( A \) and \( B \).

Combining Eqs. B-7 and B-8 we have
\[
\left. r_D \frac{d\bar{P}_D}{d\bar{r}_D} \right|_{\bar{r}_D = 1} = - \frac{1}{\bar{u}} = A \bar{v} \bar{u}_1(\bar{v}\bar{u}) \right. - B \bar{v} \bar{u} K_1(\bar{v}\bar{u})
\]
or
\[
A \bar{v} \bar{u}_1(\bar{v}\bar{u}) - B \bar{v} \bar{u} K_1(\bar{v}\bar{u}) = - \frac{1}{\bar{u}} \quad \mbox{................................................................. (B-10)}
\]

Combining Eqs. B-7 and B-9 we obtain
\[
\left. r_D \frac{d\bar{P}_D}{d\bar{r}_D} \right|_{\bar{r}_D = r_{eD}} = 0 = A \bar{v} \bar{u}(r_{eD}) I_1(\bar{v}\bar{u} r_{eD}) - B \bar{v} \bar{u}(r_{eD}) K_1(\bar{v}\bar{u} r_{eD})
\]
or
\[
A \bar{v} \bar{u} r_{eD} I_1(\bar{v}\bar{u} r_{eD}) - B \bar{v} \bar{u} r_{eD} K_1(\bar{v}\bar{u} r_{eD}) = 0
\]

which after eliminating the \( r_{eD} \) terms gives us
\[
A \bar{v} \bar{u} I_1(\bar{v}\bar{u} r_{eD}) - B \bar{v} \bar{u} K_1(\bar{v}\bar{u} r_{eD}) = 0 \quad \mbox{................................................................. (B-11)}
\]

Solving Eq. B-11 for the constant \( B \) we have
\[
B = A \frac{I_1(\bar{v}\bar{u} r_{eD})}{K_1(\bar{v}\bar{u} r_{eD})} \quad \mbox{................................................................. (B-12)}
\]

Substituting Eq. B-12 into Eq. B-10 gives us
\[
A \bar{v} \bar{u} I_1(\bar{v}\bar{u}) - A \frac{I_1(\bar{v}\bar{u} r_{eD})}{K_1(\bar{v}\bar{u} r_{eD})} \bar{v} \bar{u} K_1(\bar{v}\bar{u}) = - \frac{1}{\bar{u}}
\]

Multiplying through by \( K_1(\bar{v}\bar{u} r_{eD}) \) we have
\begin{align*}
A\sqrt{u}I_1(\sqrt{u})K_1(\sqrt{v}r_d) - A\sqrt{u}r_d K_1(\sqrt{v}u) &= -\frac{K_1(\sqrt{v}r_d)}{u} \\
\text{Solving this expression for the constant } A \text{ we obtain} \\
A &= \frac{K_1(\sqrt{v}r_d)}{u[I_1(\sqrt{v}r_d)\sqrt{u} K_1(\sqrt{v}u) - \sqrt{u} I_1(\sqrt{v}u) K_1(\sqrt{v}r_d)]} \quad \text{(B-13)}
\end{align*}

Substitution of Eq. B-13 into Eq. B-12 gives the B constant as
\begin{align*}
B &= \frac{I_1(\sqrt{v}r_d)}{u[I_1(\sqrt{v}r_d)\sqrt{u} K_1(\sqrt{v}u) - \sqrt{u} I_1(\sqrt{v}u) K_1(\sqrt{v}r_d)]} \quad \text{(B-14)}
\end{align*}

Finally, the substitution of Eqs. B-13 and B-14 into Eq. B-6 gives us the particular solution for this case
\begin{align*}
\bar{p}_D(r_D, u) &= \frac{I_0(\sqrt{v}r_d)K_1(\sqrt{v}r_d) + K_0(\sqrt{v}r_d)I_1(\sqrt{v}r_d)}{u[I_1(\sqrt{v}r_d)\sqrt{u} K_1(\sqrt{v}u) - \sqrt{u} I_1(\sqrt{v}u) K_1(\sqrt{v}r_d)]} \quad \text{(B-15)}
\end{align*}

We admit that the direct inversion of Eq. B-15 into known functions appears very difficult at best, and impossible at worst. However, we can use the behavior of the Bessel functions $I_1(z)$ and $K_1(z)$, to help simplify Eq. B-15 into a directly invertible expression. The following approach will consist of assuming that $u$ tends toward zero, that is, a “long” time solution. Then, the result can be tested against the numerical inversion of Eq. B-15 using the Stehfest algorithm.

Abramowitz and Stegun\textsuperscript{46} present expressions for $I_1(z)$ and $K_1(z)$ as $z$ approaches zero (their Eqs. 9.6.7 and 9.6.9, respectively). These are
\begin{align*}
I_1(z) &= \frac{1}{2}z \quad \text{as } z \rightarrow 0 \\
K_1(z) &= \frac{1}{z} \quad \text{as } z \rightarrow 0 \\
\text{Multiplying by } z \text{ gives us} \\
z I_1(z) &= \frac{1}{2}z^2 \quad \text{as } z \rightarrow 0 \\
z K_1(z) &= 1 \quad \text{as } z \rightarrow 0
\end{align*}

These approximations imply that the $\sqrt{u} K_1(\sqrt{v}u)$ and $\sqrt{u} I_1(\sqrt{v}u)$ terms will show the following behavior for $u \rightarrow 0$,
\begin{align*}
\sqrt{u} K_1(\sqrt{v}u) &= 1 \quad \text{as } u \rightarrow 0 \quad \text{..........................................................(B-16)} \\
\sqrt{u} I_1(\sqrt{v}u) &= 0 \quad \text{as } u \rightarrow 0 \quad \text{..........................................................(B-17)}
\end{align*}
Combining Eqs. B-15, B-16, and B-17 it follows that

\[ \bar{p}_D(r_D,u) = \frac{I_0(\sqrt{\alpha} r_{ed}) K_1(\sqrt{\alpha} r_{ed}) + K_0(\sqrt{\alpha} r_{ed}) I_1(\sqrt{\alpha} r_{ed})}{u I_1(\sqrt{\alpha} r_{ed})} \]  ..............(B-18)

which we can rearrange to give

\[ \bar{p}_D(r_D,u) = \frac{1}{u} K_0(\sqrt{\alpha} r_{ed}) + \frac{1}{u} \frac{K_1(\sqrt{\alpha} r_{ed})}{I_1(\sqrt{\alpha} r_{ed})} I_0(\sqrt{\alpha} r_{ed}) \]  ..............(B-19)

By inspection of Eq. B-19 we note that the \( \frac{1}{u} K_0(\sqrt{\alpha} r_{ed}) \) term is the constant rate solution for an infinite-acting reservoir. This observation suggests that the solution is composed of two parts. These are, an infinite-acting part \( \frac{1}{u} K_0(\sqrt{\alpha} r_{ed}) \), and a finite-acting part \( \frac{1}{u} \frac{K_1(\sqrt{\alpha} r_{ed})}{I_1(\sqrt{\alpha} r_{ed})} I_0(\sqrt{\alpha} r_{ed}) \). The inversion of the infinite-acting part is the well known exponential integral solution which leaves us only with the inversion of the finite-acting of this relation.

Eq. B-19 may be written in a more compact form as

\[ \bar{p}_D(r_D,u) = \bar{p}_{D,\text{inf}}(r_D,u) + \frac{1}{u} \frac{K_1(a)}{I_1(a)} I_0(b) \]  ..............(B-20)

where

\[ a = \sqrt{\alpha} r_{ed} \]  ..............(B-21)

\[ b = \sqrt{\alpha} r_D \]  ..............(B-22)

\[ \bar{p}_{D,\text{inf}}(u,r_D) = \frac{1}{u} K_0(\sqrt{\alpha} r_D) \]  ..............(B-23)

Looking at the \( \frac{K_1(a)}{I_1(a)} \) term, we see the need of a product or quotient expression involving \( I_n(z) \) and \( K_n(z) \) to resolve this term into a form that can be inverted into simple known functions. Using the Wronskian (cross-product) relation for modified Bessel functions given in Abramowitz and Stegun\(^46\) (Eq. 9.6.15) we have

\[ I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = \frac{1}{z} \]  ..............(B-24)

We see that for \( n = 0 \) and \( n = 1 \) we obtain the following identities

\[ I_0(z)K_1(z) + I_1(z)K_0(z) = \frac{1}{z} \]

and

\[ I_1(z)K_2(z) + I_2(z)K_1(z) = \frac{1}{z} \]
Equate and isolating the $I_1(z)$ and $K_1(z)$ terms it may be stated that

$$I_0(z)K_1(z) + I_1(z)K_0(z) = I_1(z)K_2(z) + I_2(z)K_1(z)$$

and

$$I_1(z)[K_0(z) - K_2(z)] = K_1(z)[I_2(z) - I_0(z)]$$

Solving the last relation for $\frac{K_1(a)}{I_1(a)}$, we obtain

$$\frac{K_1(z)}{I_1(z)} = \frac{K_2(z) - K_0(z)}{I_0(z) - I_2(z)}$$

(B-25)

The $I_0(z)$-$I_2(z)$ term is not helpful in our attempt to develop an invertible form. However, another Bessel function identity can be used. In this case, we use the recursion relation in $I_n(z)$ to help reduce the $I_0(z)$-$I_2(z)$ term. From Abramowitz and Stegun\(^46\) (Eq. 9.6.26), we have

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z}I_n(z)$$

For the case of $n = 1$

$$I_0(z) - I_2(z) = \frac{2}{z}I_1(z)$$

(B-26)

Combining Eqs. B-25 and B-26 gives us

$$\frac{K_1(z)}{I_1(z)} = \frac{z}{2I_1(z)}[K_2(z) - K_0(z)]$$

(B-27)

and combining Eqs. B-20 and B-27 gives

$$\overline{p}_D(u,r_D) = \overline{p}_{D,ref}(u,r_D) + \frac{1}{2}[K_2(a) - K_0(a)]\frac{a I_0(b)}{2 I_1(a)}$$

(B-28)

We suffice it to say that the $K_0(a)$ and $K_2(a)$ terms are explicitly invertible, but we must reduce the $\left[a I_0(b)\right]$ term into a less complicated expression. In fact, as time grows large we assume that $a$ and $b$ approach zero. The “ascending” series form of $I_v(z)$ given in Abramowitz and Stegun\(^46\) (Eq. 9.6.10) is

$$I_v(z) = \left(\frac{1}{2}z\right)^v \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(v+k+1)}$$

or if $v = n$, an integer, then we have

$$I_n(z) = \left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k! (n+k)!}$$
or

\[ I_n(z) = \left( \frac{1}{2} z \right)^n \left[ \frac{1}{n!} + \frac{z^2}{4(n+1)!} + \frac{z^4}{(16 \cdot 2)! (n+2)!} + \frac{z^6}{(64 \cdot 3)! (n+3)!} + \ldots \right] \]

For \( n=0 \) the last relation becomes

\[ I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \ldots \]

whereas for \( n=1 \) we have

\[ I_1(z) = \frac{1}{2} z \left[ 1 + \frac{z^2}{8} + \frac{z^4}{192} + \ldots \right] \]

Now, as \( z \to 0 \), we will be assume that \( z^4 \) and greater terms are negligible so that

\[ I_0(z) = 1 + \frac{z^2}{4} \quad \text{as } z \to 0 \] \hspace{1cm} \text{(B-29)}

\[ I_1(z) = \frac{z}{2} + \frac{z^3}{16} \quad \text{as } z \to 0 \] \hspace{1cm} \text{(B-30)}

We can expand the \( \left[ \frac{a I_0(b)}{2 I_1(a)} \right] \) term in Eq. B-28 using Eqs. B-29 and B-30 to yield

\[ \frac{a I_0(b)}{2 I_1(a)} = \frac{a}{2} \left( \frac{1 + \frac{b^2}{4}}{a + \frac{a^3}{16}} \right) = \frac{a}{2} \left( \frac{1 + \frac{b^2}{4}}{1 + \frac{a^2}{8}} \right) \]

or more compactly we have

\[ \frac{a I_0(b)}{2 I_1(a)} = \left( \frac{1 + \frac{b^2}{4}}{1 + \frac{a^2}{8}} \right)^{-1} \] \hspace{1cm} \text{(B-31)}

Since it was assumed that \( a \to 0 \), then \( \frac{a^2}{8} \ll 1 \). This means that the term \( \left( 1 + \frac{a^2}{8} \right)^{-1} \) can be expressed as a binomial series of the form \((1+x)^{-1}\). Abramowitz and Stegun 46 give the binomial series as (Eq. 3.6.10)

\[ (1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \quad (|x| < 1) \]

Applying this expansion to the expression \( \left( 1 + \frac{a^2}{8} \right)^{-1} \) and neglecting the terms \( a^2 \) and greater, we are left with

\[ \left( 1 + \frac{a^2}{8} \right)^{-1} = 1 - \frac{a^2}{8} \] \hspace{1cm} \text{(B-32)}
Combining Eqs. B-31 and B-32 we obtain

\[
\frac{a \cdot I_0(b)}{I_1(a)} = \left(1 + \frac{b^2}{4}\right) \left(1 - \frac{a^2}{8}\right)
\]

or

\[
\frac{a \cdot I_0(b)}{I_1(a)} = 1 + \frac{b^2}{4} - \frac{a^2}{8} - \frac{a^2 b^2}{32}
\]

If we neglect the \( \frac{a^2 b^2}{32} \) term, then we have

\[
\frac{a \cdot I_0(b)}{I_1(a)} = 1 + \frac{b^2}{4} - \frac{a^2}{8}
\]

(B-33)

Combining Eqs. B-28 and B-33 gives us

\[
\bar{p}_D (r_D, u) = \bar{p}_{D, \inf} (r_D, u) + \frac{1}{u} \left[ K_2(a) - K_0(a) \right] \left[ 1 + \frac{b^2}{4} - \frac{a^2}{8} \right]
\]

where we can rearrange this expression to yield

\[
\bar{p}_D (r_D, u) = \bar{p}_{D, \inf} (r_D, u) + \frac{1}{u} \left[ K_2(a) - K_0(a) \right] + \frac{\left( \frac{b^2}{4} - \frac{a^2}{8} \right)}{u} \left[ K_2(a) - K_0(a) \right].
\]

(B-34)

At this point the definitions of \( a, b, \) and \( \bar{p}_{D, \inf} (u, r_D) \) (Eqs. B-21, B-22, and B-23, respectively) must be recalled

\[
a = \sqrt{u} \, r_{eD}
\]

(B-21)

\[
b = \sqrt{u} \, r_D
\]

(B-22)

\[
\bar{p}_{D, \inf} (u, r_D) = \frac{1}{u} \, K_0(\sqrt{u} r_D)
\]

(B-23)

The combination of these three relations with Eq. B-34 produces

\[
\bar{p}_D (r_D, u) = \frac{1}{u} \, K_0(\sqrt{u} r_D) + \frac{1}{u} \left[ K_2(\sqrt{u} r_{eD}) - K_0(\sqrt{u} r_{eD}) \right]
\]

\[
+ \frac{1}{4} \left( \frac{r_D^2 - r_{eD}^2}{2} \right) \left[ K_2(\sqrt{u} r_{eD}) - K_0(\sqrt{u} r_{eD}) \right]
\]

(B-35)

For convenience, we define the constant \( c \) as

\[
c = \frac{1}{4} \left( \frac{r_D^2 - r_{eD}^2}{2} \right)
\]

(B-36)

so that it can now be written that

\[
\bar{p}_D (r_D, u) = \frac{1}{u} \, K_0(\sqrt{u} r_D) + \frac{1}{u} \left[ K_2(\sqrt{u} r_{eD}) - K_0(\sqrt{u} r_{eD}) \right]
\]

\[
+ c \left[ K_2(\sqrt{u} r_{eD}) - K_0(\sqrt{u} r_{eD}) \right]
\]

(B-37)
The inversion of Eq. B-37 is best accomplished in a term by term fashion. The following collection of transforms, real space expressions, and sources summarizes the inversion effort.

<table>
<thead>
<tr>
<th>$\tilde{f}(u)$</th>
<th>$f(t)$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{u} K_0(\alpha \sqrt{u})$</td>
<td>$\frac{1}{2} E_1 \left( \frac{\alpha^2}{4t} \right)$</td>
<td>Carslaw and Jaeger⁴⁷, Eq. 26</td>
</tr>
<tr>
<td>$\frac{1}{u} K_2(\alpha \sqrt{u})$</td>
<td>$\frac{2t}{\alpha^2} \exp \left( \frac{-\alpha^2}{4t} \right)$</td>
<td>Roberts and Kaufman⁴⁸, Eq. 13.2.13</td>
</tr>
<tr>
<td>$K_0(\alpha \sqrt{u})$</td>
<td>$\frac{1}{2t} \exp \left( \frac{-\alpha^2}{4t} \right)$</td>
<td>Roberts and Kaufman⁴⁸, Eq. 13.2.1</td>
</tr>
<tr>
<td>$K_2(\alpha \sqrt{u})$</td>
<td>$\frac{2t}{\alpha^2} \left( 2, \frac{\alpha^2}{4t} \right)$</td>
<td>Roberts and Kaufman⁴⁸, Eq. 13.2.14</td>
</tr>
</tbody>
</table>

From Spanier and Oldham⁴⁹ we also note that $\Gamma(2,x) = (1+x) \exp(-x)$. Therefore,

$$\frac{2}{\alpha^2} \Gamma \left( 2, \frac{\alpha^2}{4t} \right) = \frac{2}{\alpha^2} \left( 1 + \frac{\alpha^2}{4t} \right) \exp \left( \frac{-\alpha^2}{4t} \right)$$

$$\frac{2}{\alpha^2} \Gamma \left( 2, \frac{\alpha^2}{4t} \right) = \frac{1}{2t} \exp \left( \frac{-\alpha^2}{4t} \right) + \frac{2}{\alpha^2} \exp \left( \frac{-\alpha^2}{4t} \right)$$

These relations can be used to obtain the inverse Laplace transform of Eq. B-37 term-by-term. Inverting term-by-term gives us,

$$p_D(r_D,t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) \cdot \frac{1}{2} E_1 \left( \frac{r_{D_0}^2}{4t_D} \right) + \frac{2t_D}{r_{D_0}^2} \exp \left( \frac{-r_{D_0}^2}{4t_D} \right)$$

$$+ c \left[ \frac{1}{2t_D} \exp \left( \frac{-r_{D_0}^2}{4t_D} \right) + \frac{2}{r_D^2} \exp \left( \frac{-r_D^2}{4t_D} \right) \cdot \frac{1}{2t_D} \exp \left( \frac{-r_{D_0}^2}{4t_D} \right) \right]$$

Now, if the constant $c$ is substituted with its definition, Eq. B-36, we have

$$p_D(r_D,t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) \cdot \frac{1}{2} E_1 \left( \frac{r_{D_0}^2}{4t_D} \right) + \frac{2t_D}{r_{D_0}^2} \exp \left( \frac{-r_{D_0}^2}{4t_D} \right)$$

$$+ \frac{1}{4} \left( \frac{r_D^2 - r_{D_0}^2}{2} \right) \frac{2}{r_D^2} \exp \left( \frac{-r_D^2}{4t_D} \right)$$
which reduces to
\[
p_{D}(r_D,t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) - \frac{1}{2} E_1 \left( \frac{r_{eD}^2}{4t_D} \right) + \frac{2t_D}{r_{eD}^2} \exp \left( \frac{-r_{eD}^2}{4t_D} \right) + \left( \frac{r_D^2}{2r_{eD}^2} \cdot \frac{1}{4} \right) \exp \left( \frac{-r_{eD}^2}{4t_D} \right) \tag{B-38}
\]

The following result has been derived and presented in the literature for this case
\[
p_{D}(r_D,t_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) - \frac{1}{2} E_1 \left( \frac{r_{eD}^2}{4t_D} \right) + \frac{2t_D}{r_{eD}^2} \exp \left( \frac{-r_{eD}^2}{4t_D} \right) \tag{B-39}
\]

Eqs. B-38 and B-39 are identical except for the last term in Eq. B-38. This at least provides a partial verification of our result, Eq. B-38. In order for us to verify the validity of Eq. B-38, we can compute results using Eq. B-15 via a numerical inversion algorithm.

The advantage of having a closed form, real space solution such as Eq. B-38 or Eq. B-39 is that we can use such a relation to develop theoretical results, analysis relations, and for non-inversion computation of the solution.

From the standpoint of analysis and interpretation of well test data, it is also valuable to have the time derivative of Eq. B-38. We can develop this function using the following identities
\[
\frac{dE_1(z)}{dt_D} = \frac{dz}{dt_D} \frac{dE_1(z)}{dz} \tag{B-40}
\]
and
\[
\frac{d[\exp(z)]}{dt_D} = \frac{dz}{dt_D} \frac{d[\exp(z)]}{dz} \tag{B-41}
\]

The derivative of \( E_1(z) \) is also needed, and it is given by Abramowitz and Stegun\textsuperscript{46} (Eq. 5.1.26) as
\[
\frac{dE_n(z)}{dz} = -E_{n-1}(z)
\]

We also need an expression for \( E_{n-1}(z) \), in particular for \( n=1 \). This expression is taken from Abramowitz and Stegun\textsuperscript{46} (Eq. 5.1.24).
\[
E_0(z) = \frac{\exp(-z)}{z}
\]

Combining the last two relations we obtain
\[
\frac{dE_1(z)}{dz} = \frac{-\exp(-z)}{z}
\]

\[\text{Equation (B-42)}\]

We note that the derivative of \(\exp(z)\) is

\[
\frac{d[\exp(z)]}{dz} = \exp(z)
\]

\[\text{Equation (B-43)}\]

Combining Eqs. B-40 and B-42 produces

\[
\frac{dE_1(z)}{d\tau_D} = \frac{dz}{d\tau_D} \left[ \frac{-\exp[-z]}{z} \right]
\]

\[\text{Equation (B-44)}\]

and combining Eqs. B-41 and B-43 gives us

\[
\frac{d[\exp(z)]}{d\tau_D} = \frac{dz}{d\tau_D} \exp(z)
\]

\[\text{Equation (B-45)}\]

Eqs. B-44 and B-45 are used to complete the differentiation of Eq. B-38. Thus, we obtain

\[
\frac{d}{d\tau_D} [p_D (r_D, \tau_D)] = -\frac{1}{2} \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D^2} \right] \frac{d}{d\tau_D} \left[ \frac{4\tau_D}{r_D^2} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]

\[
+ \frac{1}{2} \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D^2} \right] \frac{d}{d\tau_D} \left[ \frac{4\tau_D}{r_D^2} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]

\[
+ \frac{2}{r_D^2} \left[ \exp \left[ -\frac{r_D^2}{4\tau_D} \right] + \tau_D \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right] \right]
\]

\[
+ \left( \frac{r_D^2}{2r_D^2} - \frac{1}{4} \right) \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]

where we can reduce this relation to

\[
\frac{d}{d\tau_D} [p_D (r_D, \tau_D)] = -\frac{1}{2} \left[ \frac{r_D^2}{4\tau_D^2} \right] \frac{d}{d\tau_D} \left[ \frac{4\tau_D}{r_D^2} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]

\[
+ \frac{1}{2} \left[ \frac{r_D^2}{4\tau_D^2} \right] \frac{d}{d\tau_D} \left[ \frac{4\tau_D}{r_D^2} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]

\[
+ \frac{2}{r_D^2} \left[ \exp \left[ -\frac{r_D^2}{4\tau_D} \right] + \tau_D \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right] \right]
\]

\[
+ \left( \frac{r_D^2}{2r_D^2} - \frac{1}{4} \right) \frac{d}{d\tau_D} \left[ \frac{r_D^2}{4\tau_D} \right] \exp \left[ -\frac{r_D^2}{4\tau_D} \right]
\]
Canceling equal terms with opposite signs we have

\[
\frac{d}{dt_D} \left[ p_D \left( r_D, t_D \right) \right] = \frac{1}{2t_D} \exp \left( -\frac{r_D^2}{4t_D} \right) + \frac{2}{r_{e_D}^2} \exp \left( -\frac{r_{e_D}^2}{4t_D} \right)
\]

\[
+ \frac{1}{2t_D^2} \left( \frac{r_D^2}{4} - \frac{r_{e_D}^2}{8} \right) \exp \left( -\frac{r_{e_D}^2}{4t_D} \right)
\]...............(B-46)

A convenient plotting function in reservoir engineering, and particularly in well testing is the pressure derivative function " or logarithmic derivative", which is defined as

\[
p_D' \left( r_D, t_D \right) = \frac{d}{d \ln(t_D)} \left[ p_D \left( r_D, t_D \right) \right] = t_D \frac{d}{dt_D} \left[ p_D \left( r_D, t_D \right) \right]
\]............................(B-47)

Multiplying through Eq. B-46 by \( t_D \) we have

\[
p_D' \left( r_D, t_D \right) = \frac{1}{2} \exp \left( -\frac{r_D^2}{4t_D} \right) + \frac{2t_D}{r_{e_D}^2} \exp \left( -\frac{r_{e_D}^2}{4t_D} \right)
\]

\[
+ \frac{1}{2t_D^2} \left( \frac{r_D^2}{4} - \frac{r_{e_D}^2}{8} \right) \exp \left( -\frac{r_{e_D}^2}{4t_D} \right)
\]..........................(B-48)

**SOLUTION FOR A WELL PRODUCED AT A CONSTANT FLOW RATE IN A BOUNDED CIRCULAR RESERVOIR WITH A CONSTANT PRESSURE OUTER BOUNDARY**

The only modification that we must make in this case, compared to the previous case, is in the outer boundary condition. The equation we must solve is Eq. B-1, given by

\[
\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D}
\].................................(B-1)

subject to the following initial and boundary conditions.

**Initial Condition**

\[
p_D \left( r_D, t_D \leq 0 \right) = 0 \quad \text{(uniform pressure distribution)} \]............(B-49)
Constant Rate Inner Boundary Condition

$$[r_D \frac{\partial p_D}{\partial r_D}]_{r_D=1} = -1 \hspace{1cm} \text{(B-50)}$$

Constant Pressure Outer Boundary Condition

$$p_D(r_e, t_D) = 0 \hspace{1cm} \text{(B-51)}$$

Following exactly the same procedure as in the previous case, we conclude that the general solution for Eq. B-1 given by Eq. B-6. That equation is repeated here for clarity as Eq. B-52

$$\bar{p}_D(r_D,u) = A l_0(\bar{v}u) + BK_0(\bar{v}u) \hspace{1cm} \text{(B-52)}$$

Also, the derivative of the general solution is again Eq. B-7, repeated here as Eq. B-53

$$\left[ r_D \frac{d\bar{p}_D}{dr_D}(r_D,u) \right] = A \bar{v}u l_1(\bar{v}u) - B \bar{v}u K_1(\bar{v}u) \hspace{1cm} \text{(B-53)}$$

If we take the Laplace transform of the boundary conditions, Eq. B-50 and Eq. B-51, we obtain Eqs. B-54, and B-55, respectively

$$\left[ r_D \frac{d\bar{p}_D}{dr_D} \right]_{r_D=1} = -\frac{1}{u} \hspace{1cm} \text{(B-54)}$$

and

$$\bar{p}_D(r_e,u) = 0 \hspace{1cm} \text{(B-55)}$$

Combining Eqs. B-53 and B-54, and recalling that $$r_D$$ is unity, we obtain

$$A \bar{v}u l_1(\bar{v}u) - B \bar{v}u K_1(\bar{v}u) = -\frac{1}{u}$$

which, after solving for $$A$$, results in

$$A = \frac{Bu \bar{v}u K_1(\bar{v}u) - 1}{u \bar{v}u l_1(\bar{v}u)} \hspace{1cm} \text{(B-56)}$$

Combining Eqs. B-52 and B-55 yields

$$A l_0(\bar{v}u r_e) + BK_0(\bar{v}u r_e) = 0$$

which, when solved for $$B$$ produces

$$B = -\frac{A l_0(\bar{v}u r_e)}{K_0(\bar{v}u r_e)} \hspace{1cm} \text{(B-57)}$$

Substituting Eq. B-57 into Eq. B-56, we have
\[ A \nu \bar{u} l_1(\bar{\nu}u) = - A \nu \bar{u} K_1(\bar{\nu}u) \frac{l_0(\bar{\nu}u e^D)}{K_0(\bar{\nu}u e^D)} - 1 \]

Multiplying this last relation through by \( K_0(\bar{\nu}u e^D) \) and solving for \( A \), we obtain

\[ A = - \frac{K_0(\bar{\nu}u e^D)}{u \bar{u} [l_1(\bar{\nu}u) K_0(\bar{\nu}u e^D) + K_1(\bar{\nu}u) l_0(\bar{\nu}u e^D)]} \] .................................(B-58)

Substituting Eq. B- 57 into Eq. B-58, we have

\[ B = \frac{l_0(\bar{\nu}u e^D)}{u \bar{u} [l_1(\bar{\nu}u) K_0(\bar{\nu}u e^D) + K_1(\bar{\nu}u) l_0(\bar{\nu}u e^D)]} \] .................................(B-59)

We can now substitute the expressions obtained for the constants \( A \) and \( B \) (Eqs. B-58 and B-59) into Eq. B-52, to obtain the expression that governs the constant rate pressure solution. This solution is

\[ -\bar{\rho}_D = - \frac{K_0(\bar{\nu}u e^D) l_0(\bar{\nu}u e^D) + l_0(\bar{\nu}u e^D) K_0(\bar{\nu}u e^D)}{u \bar{u} [l_1(\bar{\nu}u) K_0(\bar{\nu}u e^D) + K_1(\bar{\nu}u) l_0(\bar{\nu}u e^D)]} \] .................................(B-60)

The Laplace transform inversion of Eq. B- 60 is difficult to obtain directly. Thus, we use the behavior of the Bessel functions to simplify the inversion process. The relations we need are taken from Abramowitz and Stegun\(^4^6\) (Eq. 9.6.7 and 9.6.9), and they are

\[ l_1(z) = \frac{1}{2} z \quad \text{as} \quad z \to 0 \]
\[ K_1(z) = \frac{1}{z} \quad \text{as} \quad z \to 0 \]

Multiplying by \( z \), these expressions result in

\[ z l_1(z) = \frac{1}{2} z^2 \quad \text{as} \quad z \to 0 \]

or

\[ z l_1(z) = 0 \quad \text{as} \quad z \to 0 \]

and

\[ z K_1(z) = 1 \quad \text{as} \quad z \to 0 \]

Therefore we conclude that

\[ \sqrt{\nu} l_1(\sqrt{\nu}u) \equiv 0 \quad \text{as} \quad z \to 0 \]

and

\[ \sqrt{\nu} K_1(\sqrt{\nu}u) \equiv 1 \quad \text{as} \quad z \to 0 \]
We use these identities in Eq. B-60 to obtain a "long term" approximation (as \( z \to 0, \ t \to \infty \)). Thus,

\[
\bar{p}_D = \frac{K_0(\sqrt{\alpha} r_D)}{u} \cdot \frac{1}{u} \ K_0(\sqrt{\alpha} r_{ed}) \ I_0(\sqrt{\alpha} r_{ed}) \ I_0(\sqrt{\alpha} r_D) \]  \tag{B-61}

The first term on the right hand side of Eq. B-61 represents the infinite acting part, whereas the second one represents the finite acting part. If we use the same relations in this case as in the previous case, we can write

\[
\bar{p}_D = \frac{K_0(b)}{u} - \frac{1}{u} \ K_0(a) \ I_0(b) \ I_0(a) \]  \tag{B-62}

where

\[
a = \sqrt{\alpha} r_{ed} \tag{B-63}
\]

and

\[
b = \sqrt{\alpha} r_D \tag{B-64}
\]

Since we are deriving a long term approximation, we assume that both \( a \) and \( b \) approach zero. This is because they are direct functions of the Laplace variable which is in turn function of the inverse of time. Thus, as time grows bigger, \( u \) must be smaller.

By analogy with the previous case, the difficulty in inverting Eq. B-62 lies in the quotient \( \frac{I_0(b)}{I_0(a)} \), because the other terms can be inverted directly. Thus, we make use of Eq. B-29 whose derivation was presented earlier. Eq. B-29 was given as

\[
I_0(z) = 1 + \frac{z^2}{4} \quad \text{as } z \to 0 \]  \tag{B-29}

Therefore \( \frac{I_0(b)}{I_0(a)} \) is given by the following expression

\[
\frac{I_0(b)}{I_0(a)} = \left(1 + \frac{b^2}{4}\right) \left(1 + \frac{a^2}{4}\right)^{-1} \]  \tag{B-65}

The second term on the right hand side may be expanded using the binomial series given by

\[
(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \quad (|x| < 1)
\]

Thus, using this expansion and neglecting the terms \( a^4 \) and greater, it follows that

\[
\left(1 + \frac{a^2}{4}\right)^{-1} = 1 - \frac{a^2}{4}
\]

which when we substitute back into Eq. B-65, produces
\[
\frac{l_0(b)}{l_0(a)} = \left(1 + \frac{b^2}{4}\right)\left(1 - \frac{a^2}{4}\right) = 1 - \frac{a^2 b^2}{16} + \frac{b^2}{4} - \frac{a^2}{4}
\]

Again, since we assume that both \(a\) and \(b\) approach zero, for a long time solution, the term \(\frac{a^2 b^2}{16}\) should be small enough to be ignored, and the term \(\frac{l_0(b)}{l_0(a)}\) can be represented by
\[
\frac{l_0(b)}{l_0(a)} = 1 + \frac{b^2}{4} - \frac{a^2}{4}
\]

The combination of the last relation with Eqs. B-62, B-63 and B-64 produces
\[
\bar{p}_D(u, r_D) = \frac{K_0(\bar{\nu}r_D)}{u} - \frac{1}{u} K_0(\bar{\nu}r_{D'}) \left[ 1 + \frac{b^2}{4} - \frac{a^2}{4} \right]
\]

which can be rearranged as
\[
\bar{p}_D(u, r_D) = \frac{K_0(\bar{\nu}r_D)}{u} \cdot K_0(\bar{\nu}r_{D'}) \frac{(r_D^2 - r_{D'}^2)}{4} K_0(\bar{\nu}r_{D'}) \tag{B-66}
\]

At this point we must use the transforms as in the previous case. Specifically, we require the following two functions

<table>
<thead>
<tr>
<th>(\bar{f}(u))</th>
<th>(f(t))</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{u} K_0(\alpha \bar{v}u))</td>
<td>(\frac{1}{2} E_1\left(\frac{\alpha^2}{4t}\right))</td>
<td>Carslaw and Jaeger\textsuperscript{47}, Eq. 26</td>
</tr>
<tr>
<td>(K_0(\alpha \bar{v}u))</td>
<td>(\frac{1}{2t} \exp\left(-\frac{\alpha^2}{4t}\right))</td>
<td>Roberts and Kaufman\textsuperscript{48}, Eq. 13.2.1</td>
</tr>
</tbody>
</table>

Combining these inversions with Eq. B-66, we obtain
\[
p_D(r_{D', D}) = \frac{1}{2} E_1\left(\frac{r_D^2}{4t_D}\right) \cdot \frac{1}{2} E_1\left(\frac{r_{D'}^2}{4t_{D'}}\right) \cdot \frac{(r_D^2 - r_{D'}^2)}{4t_D} \exp\left(-\frac{r_{D'}^2}{4t_{D'}}\right)
\]

which we rewrite as
\[
p_D(r_{D', D}) = \frac{1}{2} E_1\left(\frac{r_D^2}{4t_D}\right) \cdot \frac{1}{2} E_1\left(\frac{r_{D'}^2}{4t_{D'}}\right) \frac{(r_D^2 - r_{D'}^2)}{8t_D} \exp\left(-\frac{r_{D'}^2}{4t_{D'}}\right) \tag{B-67}
\]

The first term on the right-hand-side corresponds to the infinite-acting part and the other terms to the finite acting part.
In Chapter IV of this thesis, we introduce an approximation to calculate the constant pressure solution from the constant rate solution that requires the use of Eq. B-67 and its derivative. To fulfill this requirement we present the calculation of the derivative of Eq. B-67 below. In order to perform this task, we will use several of the relations previously presented for function derivatives. These relations are

\[
\frac{dE_1(z)}{dt_D} = \frac{dz}{dt_D} \left[ -\exp\left(-\frac{z}{2}\right) \right] \tag{B-44}
\]

and

\[
\frac{d[\exp\{z\}]}{dt_D} = \frac{dz}{dt_D} \exp(z) \tag{B-45}
\]

Using these two results to differentiate Eq. B-67 it follows that

\[
\frac{d}{dt_D} [p_D(r_D,t_D)] = -\frac{1}{2} \frac{d}{dt_D} \left[ \frac{r_D^2}{4t_D} \right] \exp\left[ -\frac{r_D^2}{4t_D} \right]
+ \frac{1}{2} \frac{d}{dt_D} \left[ \frac{r_{eD}^2}{4t_D} \right] \exp\left[ -\frac{r_{eD}^2}{4t_D} \right]
+ \frac{r_D^2 - r_{eD}^2}{8t_D} \exp\left[ -\frac{r_{eD}^2}{4t_D} \right]
- \exp\left[ -\frac{r_{eD}^2}{4t_D} \right] \frac{d}{dt_D} \left[ \frac{r_D^2 - r_{eD}^2}{8t_D} \right]
\]

which can be reduced to

\[
\frac{d}{dt_D} [p_D(r_D,t_D)] = -\frac{1}{2} \frac{d}{dt_D} \left[ \frac{r_D^2}{4t_D} \right] \exp\left[ -\frac{r_D^2}{4t_D} \right]
+ \frac{1}{2} \frac{d}{dt_D} \left[ \frac{r_{eD}^2}{4t_D} \right] \exp\left[ -\frac{r_{eD}^2}{4t_D} \right]
+ \frac{r_D^2 - r_{eD}^2}{8t_D} \exp\left[ -\frac{r_{eD}^2}{4t_D} \right]
- \exp\left[ -\frac{r_{eD}^2}{4t_D} \right] \left( \frac{r_D^2 - r_{eD}^2}{8t_D^2} \right)
\]

and finally to the following expression

\[
\frac{d}{dt_D} [p_D(r_D,t_D)] = -\frac{1}{2t_D} \exp\left[ -\frac{r_D^2}{4t_D} \right] - \frac{1}{2t_D} \exp\left[ -\frac{r_{eD}^2}{4t_D} \right]
+ \left( \frac{r_{eD}^2 - r_D^2}{8t_D^2} \right) \left( \frac{r_{eD}^2}{4t_D^2} - 1 \right) \exp\left[ -\frac{r_{eD}^2}{4t_D} \right] \tag{B-68}
\]
We have already mentioned that in reservoir engineering, and in particular in well testing, the pressure derivative function is defined as

\[ p_D'(r_D,t_D) = t_D \cdot \frac{d}{dt_D} [p_D(r_D,t_D)] \] ..........................(B-47)

Thus, if we multiply Eq. B-68 through by \( t_D \), we have the following relationship for the pressure derivative function

\[
p_D'(r_D,t_D) = \frac{1}{2} \left[ \exp \left( -\frac{r_D^2}{4t_D} \right) - \exp \left( -\frac{r_{eD}^2}{4t_D} \right) \right]
+ \left( \frac{r_{eD}^2 - r_D^2}{8t_D} \right) \left( \frac{r_D^2}{4t_D} - 1 \right) \exp \left( -\frac{r_{eD}^2}{4t_D} \right) \] ..........................(B-69)

REAL SPACE APPROXIMATION TO THE CONSTANT PRESSURE RATE SOLUTION

In Appendix C of this work, we show how van Everdingen and Hurst\(^{33}\) obtained the relation between the constant pressure rate and the constant rate pressure solution in Laplace space. In Chapter III, we show how the same authors used this result to obtain the real space constant pressure solution which was the exponential decline equation for the case of single phase liquid.

The general solution for the case of constant rate flow in a bounded circular reservoir with a no flow outer boundary is given by Eq. B-15

\[
\bar{p}_{D, CR}(r_D,u) = \frac{l_0(\bar{v}\bar{u} \ r_D)K_1(\bar{v}\bar{u} \ r_{eD}) + K_0(\bar{v}\bar{u} \ r_D)I_1(\bar{v}\bar{u} \ r_{eD})}{\bar{u}[I_1(\bar{v}\bar{u} \ r_{eD})\bar{v}\bar{u} K_1(\bar{v}\bar{u}) - \bar{v}\bar{u} I_1(\bar{v}\bar{u}) K_1(\bar{v}\bar{u} r_{eD})]} \] ..........................(B-15)

The long time approximation to this solution is given by Eq. B-19

\[
\bar{p}_{D, CR}(r_D,u) = \frac{1}{\bar{u}} K_0(\bar{v}\bar{u} \ r_D) + \frac{1}{\bar{u}} \frac{K_1(\bar{v}\bar{u} \ r_{eD})I_0(\bar{v}\bar{u} \ r_D)}{[I_1(\bar{v}\bar{u} \ r_{eD})]} \] ..........................(B-19)

where the first term on the right hand side represents the constant rate solution for an infinite acting reservoir and the second term corresponds to the finite acting part.

Also, the general solution for the case of constant rate flow in a bounded circular reservoir with a constant pressure outer boundary (at initial pressure) is given by Eq. B-60
\[ \bar{p}_{D,CR}(r_D,u) = \frac{-K_0(\sqrt{u} \ r_{ed}) I_0(\sqrt{u} \ r_D) + I_0(\sqrt{u} \ r_{ed}) K_0(\sqrt{u} \ r_D)}{u \sqrt{u} \ [I_1(\sqrt{u}) K_0(\sqrt{u} \ r_{ed}) + K_1(\sqrt{u}) I_0(\sqrt{u} \ r_{ed})]} \]  \hspace{1cm} (B-60)

The long term approximation to this solution is given by Eq. B-61

\[ \bar{p}_{D,CR} \ (r_D,u) = \frac{1}{u} K_0(\sqrt{u} \ r_D) - \frac{1}{u} \frac{K_0(\sqrt{u} \ r_{ed}) I_0(\sqrt{u} \ r_D)}{I_0(\sqrt{u} \ r_{ed})} \]  \hspace{1cm} (B-61)

The relation between the constant rate and constant pressure solution in Laplace space at the wellbore is derived as Eq. C-4 in Appendix C. That relation is repeated here as Eq. B-70.

\[ \bar{q}_{D,CP} \ (r_D=1,u) = \frac{1}{u^2} \frac{1}{\bar{p}_{D,CR} \ (r_D=1,u)} \]  \hspace{1cm} (B-70)

If we couple the long-term solution, Eq. B-19, with Eq. B-70 we obtain the following result, that is valid only at the wellbore (where \( r_D = 1 \)).

\[ \bar{q}_{D,CP} \ (r_D=1,u) = \frac{I_1(\sqrt{u} \ r_{ed})}{u [I_0(\sqrt{u}) K_1(\sqrt{u} \ r_{ed}) + K_0(\sqrt{u}) I_1(\sqrt{u} \ r_{ed})]} \]  \hspace{1cm} (B-71)

If we couple the long-term solution, Eq. B-61, with Eq. B-70 we obtain the following result, that is also valid only at the wellbore (where \( r_D = 1 \))

\[ \bar{q}_{D,CP} \ (r_D=1,u) = \frac{I_0(\sqrt{u} \ r_{ed})}{u [K_0(\sqrt{u}) I_0(\sqrt{u} \ r_{ed}) - K_0(\sqrt{u} \ r_{ed}) I_0(\sqrt{u})]} \]  \hspace{1cm} (B-72)

It is very unlikely that we can find an analytical direct inversion into real space of these \( \bar{q}_{D,CP} \) solutions in terms of known functions. That is the reason we prove in this section that a simple assumed approximation to the constant rate pressure solution yields a helpful relation from which we can compute the constant pressure rate solution.

Rather than transforming the actual equation for the constant rate pressure solution into Laplace space, we assume that it can be approximated in real space by a simple linear function. That function depends on dimensionless time and is expressed as follows

\[ p_{D,CR} = a_1 + a_2 t_D \]  \hspace{1cm} (B-73)

This approximation is strictly valid only for pseudosteady state flow with the appropriate definitions of \( a_1 \) and \( a_2 \). The parameters \( a_1 \) and \( a_2 \) are not constants but functions of \( r_D \) and \( t_D \). However, we will assume that they are not functions of \( t_D \) for the
purposes of the following Laplace transformation operations. Thus, the Laplace transform of Eq. B-73 is given by

\[ \bar{p}_{D, CR} = \frac{a_1}{u} + \frac{a_2}{u^2} \]

We combine this relation with Eq. B-70, to obtain

\[ \bar{q}_{D, CP}(r, D=1, u) = \frac{1}{u^2} \left[ \frac{a_1}{u} + \frac{1}{u^2} \right] \]

which, after rearranging, is written as

\[ \bar{q}_{D, CP}(r, D=1, u) = \frac{1}{(a_1 u + a_2)} \]

We further rearrange this last equation to obtain the following transform

\[ \bar{q}_{D, CP}(r, D=1, u) = \frac{1}{a_1} \frac{1}{a_2} + u \]

We can invert Eq. B-74 into real space by using a table of Laplace transforms such as that presented by Abramowitz and Stegun. In this case the inversion produces the following result

\[ q_{D, CP}(r, D=1, t_D) = \frac{1}{a_1} \exp \left( -\frac{a_2}{a_1} t_D \right) \] (B-75)

We must note that Eq. B-75 is valid only at the wellbore. The question remains as to how to find \( a_1 \) and \( a_2 \). We recall that these parameters are not really constants but we assumed them to be time independent for purposes of Laplace transformations. We will present an approach to determine these parameters, which, although not rigorous (because \( a_1 \) and \( a_2 \) are not time independent), has proven accurate enough for general applications in reservoir engineering. This method involves the derivative of Eq. B-73 which is given by

\[ \frac{dP_{D, CR}}{dt_D} = \frac{d}{dt_D} (a_1 + a_2 t_D) = a_2 \] (B-76)

Combining Eq. B-76 with Eq. B-73 we obtain

\[ a_1 = p_{D, CR} \cdot \frac{dP_{D, CR}}{dt_D} t_D \] (B-77)
We can write Eq. B-77 in a more compact form by noting that the second term on the right corresponds exactly to the pressure derivative function used in well testing which is symbolized by \( p_D \). Thus,

\[
a_1 = p_{D, CR} - p_D \tag{B-78}
\]

Now, combining Eqs. B-75, B-76, and B-77, we can see that

\[
q_{D, CP} = \frac{1}{p_{D, CR}(t_D) - t_D \frac{dp_{D, CR}}{dt_D}} \exp \left[ - \frac{t_D}{p_{D, CR}(t_D) - t_D \frac{dp_{D, CR}}{dt_D}} \right]
\]

or

\[
q_{D, CP} = \frac{1}{p_{D, CR}(t_D) - t_D \frac{dp_{D, CR}}{dt_D}} \exp \left[ - \frac{1}{t_D \frac{dp_{D, CR}}{dt_D} - 1} \right] \tag{B-79}
\]

which we can write more compactly as

\[
q_{D, CP} = \left[ \frac{1}{p_{D, CR}(t_D) - p_{D, CR}(t_D)} \right] \exp \left[ - \frac{1}{1 - \frac{p_{D, CR}(t_D)}{t_D \frac{dp_{D, CR}}{dt_D}}} \right] \tag{B-80}
\]

Eq. B-80 provides a straightforward method to obtain the constant pressure rate solution from the constant rate pressure solution directly in real space. In other words, the solution of Eq. B-80 is much easier than that of the rigorous relations given by Eq. B-71 or B-72. Thus, if the model \( p_D \) is known, it can be input along with its derivative into Eq. B-80 to get the desired constant pressure rate solution. This represents the real advantage of having derived Eq. B-80 because it avoids the need to perform the complex process involved in the use of van Everdingen and Hurst solution. That process is described as follows. First, the constant rate solution \( p_D \), must be transformed into Laplace space using some numerical method. Then, using the relation in Laplace space between \( \bar{p}_{D, CR} \) and \( q_{D, CP} \), we would obtain a Laplace transform for \( q_{D, CP} \). Finally, making use of an inversion algorithm from Laplace to real space, such as Stehfest, we would calculate the actual \( q_{D, CP} \) solution.
APPENDIX C

DERIVATION OF THE RELATION BETWEEN THE CONSTANT PRESSURE AND CONSTANT RATE SOLUTIONS IN LAPLACE SPACE

In order to establish the link between $q_{D,CP}$ and $p_{wD,CR}$ we use the principle of superposition. Therefore, we present a clear explanation of this basic principle at the beginning of this Appendix.

Because of the linearity of the diffusivity equation we can apply the superposition theorem as a sequence of constant terminal pressures or constant rates. In the constant rate case, we can separate a variable rate profile into pieces, each one of which has a constant rate period as is illustrated by Fig. C-1.

![Diagram showing flow rate over time](image)

Fig. C-1 - Actual Approach for the Constant Rate Case

Our goal is to obtain the total pressure drop $(p_i - p_{wf})_t$, since the initial time ($t=0$) until $t > t_3$. The subscript "t" in this case stands for total.
In order to obtain the desired result we assume at first that we have produced at a constant rate, \(q_1\), during the whole time interval \((0 < t < t)\) as shown in Fig. C-2. In this case the pressure drop from \(p_i\) to \(p_{wf1}\) is

\[
(p_i - p_{wf1}) = \frac{(q_1 - q_0)}{q_r} \Delta p_{scr} (t - t_0)
\]

where

- \(q_0\) = initial rate = 0
- \(t_0\) = initial time = 0
- \(q_r\) = arbitrary reference flow rate
- \(\Delta p_{scr} = \Delta p_{formation} + \Delta p_{skin}\) (for the constant rate case)
- \(\Delta p_{skin}\) = additional pressure drop due to a damaged near wellbore zone

![Graph showing flow rate over time](image)

**Fig. C-2 - Initial Approximation to Compute the Total Pressure Drop**

In the case shown in Fig. C-2, the pressure drop would be higher than the actual pressure drop, because the flow rate is declining \((q_1 > q_2 > q_3)\). In order to account for this we add a correction term. Such a correction term should be negative in order to reduce the pressure drop. The first part of this correction term is
\[
(p_i - p_{wf})_2 = \frac{(q_2 - q_1)}{q_r} \Delta p_{scr} (t - t_1)
\]

If we add this result to the previous pressure drop \((p_i - p_{wf})_1\), it is easy to see that since \(q_2 < q_1\), the correction will be negative and \((p_i - p_{wf})_1 + (p_i - p_{wf})_2\) will be less than \((p_i - p_{wf})_1\). This result is closer to the actual response \((p_i - p_{wf})_1\) at \(t\). The net result of adding these two "fake" pressure drops is shown below in Fig. C-3.

Fig. C-3 - First Correction Made to the Total Pressure Drop

The total area including both the dashed and the dotted regions, represents the result that would be obtained for the pressure drop had we considered only the first discretized rate value. The dotted area represents the correction that is being added to the total area to account for the first change in the discretized rate, i.e., from \(q_1\) to \(q_2\). Again, we note that since this correction is negative (because \(q_2 < q_1\)) the net effect is actually a subtraction. Therefore, rather than approximating \(\Delta p_i = (p_i - p_{wf})_1\) as \((p_i - p_{wf})_1\), we improve the approximation by making the correction

\[
\Delta p_i = (p_i - p_{wf})_1 + (p_i - p_{wf})_2
\]

and the net result is shown graphically as the dashed area in Figs. C-3 and C-4.
Fig. C-4 - Net Pressure Drop After Correcting for First Rate Change

It is important to note that had the flow rate increased, then the net effect of adding the correction would have been to increase the area (i.e. the pressure drop \((p_i - p_{wf})_t\). This is true simply because \(q_2\) would have been greater than \(q_1\) and \(\Delta p_i\) would have had a larger value after adding \((p_i - p_{wf})_1\) to \((p_i - p_{wf})_2\). This agrees with the physical principle that a higher flow rate would also imply a greater pressure drop.

We realize the result is still a poor approximation to the correct response (area). Using the same principle explained above, we make an additional correction to the total pressure drop (area). Thus, we add a third term

\[
(p_i - p_{wf})_3 = \frac{(q_3 - q_2)}{q_r} \Delta p_{Scr} (t - t_2)
\]

Again, this will be a negative term because \(q_3 < q_2\), but had \(q_3\) been greater than \(q_2\), the correction term would be positive and the effect of adding it to \(\Delta p_i\) would be reversed. Now, the approximation is written as

\[
\Delta p_i = (p_i - p_{wf})_1 + (p_i - p_{wf})_2 + (p_i - p_{wf})_3
\]

or
\[ \Delta p_i = \frac{(q_1 - q_0)}{q_r} \Delta p_{scr} (t - t_0) + \frac{(q_2 - q_1)}{q_r} \Delta p_{scr} (t - t_1) + \frac{(q_3 - q_2)}{q_r} \Delta p_{scr} (t - t_2) \]

This new correction is illustrated by Fig. C-5. The total area, both the dashed and dotted regions, represents the result that would be obtained for the pressure drop had the first two flow rate values been the only rates considered. The dotted region represents the correction that is being added to the total area to account for the second change in the discretized flow rate, i.e., from \( q_2 \) to \( q_3 \). This correction is negative given the decreasing rate profile that is chosen.

![Diagram showing flow rate and time intervals](image)

**Fig. C-5** - Second Correction Made to the Total Pressure Drop

We note that the corrections we have added are referred to different points in time. The first correction term was expressed using \( t_1 \) as the starting point. We see that the first correction term, \( (p_i - p_{wf})_2 \), was considered for a period of time from \( t_1 \) to \( t \), and not for the whole time interval. Likewise, the second correction term \( (p_i - p_{wf})_3 \) was allowed to act only for the period of time from \( t_2 \) to \( t \).

This mathematical manipulation of variables corresponds to the actual physical situation or to the situation represented by the discretized pattern. In other words, a
correction is not needed from the beginning to $t_1$ because the flow rate was $q_1$ during this period. Thus, the first assumption made that $q_1$ was the flow rate during the whole process does not conflict with the real situation up to the time $t_1$.

For the other time regions however, $q_1$ is not the flow rate. In fact from $t_1$ to $t_2$ the flow rate is $q_2$. Thus, when we added the first correction to $(p_i - p_{wf})_1$, we modeled a situation where the flowrate was $q_1$ from 0 to $t_1$, and $q_2$ from $t_1$ to $t$. However $q_2$ is not the actual rate from $t_1$ to $t$. Therefore, we required a third correction to model a flow rate of $q_1$ from 0 to $t_1$, $q_2$ from $t_1$ to $t_2$, and $q_3$ from $t_2$ to $t_3$.

Graphically, the final result is seen in Fig. C-6.

![Diagram](image)

Fig. C-6 - Final Result for Pressure Drop

By induction, we can state that had there been $n$ number of discretized flow rates, then the following term

$$(p_i - p_{wf})_n = \frac{(q_n - q_{n-1})}{q_r} \Delta p_{scr} (t - t_{n-1})$$

would be the last correction term and the actual pressure drop would be
\[ \Delta p_t = \frac{(q_1 - q_0)}{q_r} \Delta p_{scr} (t - t_0) + \frac{(q_2 - q_1)}{q_r} \Delta p_{scr} (t - t_1) + \frac{(q_3 - q_2)}{q_r} \Delta p_{scr} (t - t_2) + \ldots \]

\[ \ldots + \frac{(q_n - q_{n-1})}{q_r} \Delta p_{scr} (t - t_n) \]

Using the summation notation, this last relation is given by

\[ \Delta p_t = \sum_{j=1}^{n} (p_i - p_{wf})_j \]

or, in terms of rate

\[ \Delta p_t = \frac{1}{q_r} \sum_{j=1}^{n} (q_j - q_{j-1}) \Delta p_{scr} (t - t_{j-1}) \]

\[ (\text{C-1}) \]

If we multiply and divide by \( \Delta \tau = t_j - t_{j-1} \) then we rewrite Eq. C-1 as

\[ \Delta p_t = \frac{1}{q_r} \sum_{j=1}^{n} \left( \frac{q_j - q_{j-1}}{t_j - t_{j-1}} \right) \Delta p_{scr} (t - t_{j-1}) \Delta \tau \]

or if we use the following definition

\[ \Delta q = q_j - q_{j-1} \]

then Eq. C-1 becomes

\[ \Delta p_t = \frac{1}{q_r} \sum_{j=1}^{n} \left( \frac{\Delta q}{\Delta \tau} \right) \Delta p_{scr} (t - t_{j-1}) \Delta \tau \]

If we consider a number of small constant rate periods, we will represent the actual situation more closely. Thus, if we consider an infinite number \( (n \to \infty) \) of these periods, we can state that

\[ \Delta p_t = \frac{1}{q_r} \lim_{n \to \infty} \left[ \sum_{j=1}^{n} \frac{\Delta q}{\Delta \tau} \Delta p_{scr} (t - t_{j-1}) \Delta \tau \right] \]

or, in continuous form,

\[ \Delta p_t(t) = \frac{1}{q_r} \int_{0}^{t} q'(\tau) \Delta p_{scr}(t - \tau) d\tau \]

\[ (\text{C-2}) \]

where \( q'(\tau) \) is the derivative of the flow rate. Eq. C-2 is the rate convolution formula. Using integration by parts Eq. C-2 can also be written as
\[ \Delta p_t(t) = \frac{1}{q_r} \int_0^t q(\tau) \frac{d}{d\tau} [\Delta p_{scr}(t - \tau)] d\tau \]

Taking the Laplace transform (\( \mathcal{L} \)) of this relation we obtain

\[ \mathcal{L}[\Delta p_t(t)] = \frac{1}{q_r} \mathcal{L} \left[ \int_0^t q(\tau) \frac{d}{d\tau} [\Delta p_{scr}(t - \tau)] d\tau \right] \]

which we rewrite using Laplace transform properties,

\[ \mathcal{L}[\Delta p_t(t)] = \frac{1}{q_r} \left[ \mathcal{L}[q(t)] \mathcal{L}[\frac{d}{d\tau} \Delta p_{scr}(t - \tau)] \right] \]

since

\[ \mathcal{L} \left[ \frac{d}{d\tau} \Delta p_{scr}(t - \tau) \right] = u \Delta \bar{p}_{scr}(u) - \Delta p_{scr}(0) \]

where

\( u = \) Laplace transform parameter
\( \Delta \bar{p}_{scr} = \) Laplace transform of \( \Delta p_{scr} \)
\( \Delta p_{scr}(0) = 0 \)

Then,

\[ \mathcal{L}[\Delta p_t(t)] = \frac{1}{q_r} u \Delta \bar{p}_{scr}(u) \bar{q}(u) \]

where \( \bar{q}(u) \) is the Laplace transform of \( q(u) \). We can assume that \( \Delta p_t \) is constant, which is the same as assuming that we are producing against a fixed backpressure, \( p_{wf} \). This assumption does not violate any of the previous considerations, because the rate profile would still be variable for a fixed \( p_{wf} \). Thus, the discretization made previously to take small constant rate steps would still require a \( \Delta p_{scr} \) model at constant rate. The model would still be valid and the only consequence is that the Laplace transform of \( \Delta p_t(t) \) could be written as \( \frac{\Delta p_t}{u} \). Therefore, for constant \( \Delta p_t \), we have

\[ \frac{\Delta p_t}{u} = \frac{u}{q_r} \bar{q}_{CP}(u) \Delta \bar{p}_{scr}(u) \]

where \( \bar{q}_{CP}(u) \) is taken at constant pressure. The last relation may be rearranged as

\[ \frac{\bar{q}_{CP}(u) \Delta \bar{p}_{scr}(u)}{\Delta p_t} \frac{1}{q_r} = \frac{1}{u^2}. \]
Eq. C-3 represents the fundamental dimensional relation in Laplace space between the constant pressure (CP) and the constant rate (CR) solution. If Eq. C-3 is multiplied and divided by \( \frac{kh}{141.2 B \mu} \) we can write

\[
\bar{q}_{CP}(u) = \frac{kh}{141.2 B \mu} \frac{\Delta p_t}{q_r(141.2)B \mu} \bar{p}_{SCR}(u) = \frac{1}{u^2}
\]

The Laplace transform of the dimensionless rate response for constant pressure is

\[
\bar{q}_{D,CP} = \frac{141.2 B \mu}{kh} \bar{q}_{CP}
\]

and the Laplace transform of the dimensionless pressure response at constant rate is given by

\[
\bar{p}_{D,CR} = \frac{k_0 h}{141.2 q_r B \mu} \Delta \bar{p}_{SCR}
\]

Therefore, the combination of the last three equations leads to the following identity

\[
\bar{q}_{D,CP}(u) \bar{p}_{D,CR}(u) = \frac{1}{u^2} \ldots
\]

(C-4)

Eq. C-4 is the fundamental dimensionless relation in Laplace space between the constant rate and the constant pressure solution for any system regardless of the reservoir geometry (horizontal, fracture, etc.) or model. It is important to mention that Eq. C-4 is valid only at the wellbore where both rates and pressures are measured. van Everdingen and Hurst\(^{33}\) gave the mathematical proof of this by noting that

\[
\bar{p}_{D,CR} = \frac{1}{u} \frac{K_0(r_s \sqrt{u})}{\sqrt{u} K_1(\sqrt{u})}
\]

and

\[
\bar{q}_{D,CP} = \frac{1}{\sqrt{u}} \frac{K_1(\sqrt{u})}{K_0(\sqrt{u})}
\]

where \( K_0 \) and \( K_1 \) are modified Bessel functions. These authors combined the last three equations and obtained the relation shown below

\[
\bar{q}_{D,CP}(u) \bar{p}_{D,CR}(u) = \frac{1}{u^2} \frac{K_0(r_s \sqrt{u})}{K_0(\sqrt{u})}
\]

By comparing the last equation to Eq. C-4, we conclude that Eq. C-4 is valid only if \( r_D \) equals unity. In other words, since \( r_D = \frac{r}{r_w} \), \( r \) must be the wellbore radius \( r_w \), and the statement made above regarding the validity of Eq. C-4 at the wellbore, has a sound theoretical basis.
APPENDIX D

DERIVATION OF DECLINE CURVE RELATIONS

In this Appendix, we include all the algebraic manipulations of Arps' empirical equations, so that they may be solved not only for rate, but also for cumulative production, flow rate derivative, flow rate integral, and flow rate integral derivative. We present these results without identifying any particular type of fluid so that the variables may be related to either oil or gas. This is done in both dimensional and dimensionless forms.

We recall the relations that describe the exponential, hyperbolic and harmonic type of decline behaviors.

Dimensional Relations in Terms of Flow Rate

Exponential

\[ q = q_i \exp \left[-D_i t\right]. \] \hspace{1cm} (D-1)

Hyperbolic

\[ q = \frac{q_i}{\left[1 + bD_i t\right]^{1/b}}. \] \hspace{1cm} (D-2)

Harmonic

\[ q = \frac{q_i}{\left[1 + D_i t\right]}. \] \hspace{1cm} (D-3)

Relations in Terms of Dimensionless Flow Rate

If we define \( q_{Dd} = \frac{q}{q_i} \) and \( t_{Dd} = D_i t \), we obtain the equivalent dimensionless forms of Eqs. D-1, D-2 and D-3

Exponential

\[ q_{Dd} = \exp \left[-t_{Dd}\right]. \] \hspace{1cm} (D-4)
Hyperbolic
\[ q_{Dd} = \frac{1}{[1 + bt_{Dd}]^{1/b}} \]  \hspace{1cm} \text{(D-5)}

Harmonic
\[ q_{Dd} = \frac{1}{[1 + t_{Dd}]} \]  \hspace{1cm} \text{(D-6)}

Dimensional Relations in Terms of Cumulative Production

To write these relations in terms of cumulative production, \( Q \), we use the following definition
\[ Q = \int_0^t q(\tau)d\tau \]  \hspace{1cm} \text{(D-7)}

where \( \tau \) is a dummy variable of integration. Thus, substituting \( q \) from Eqs. D-1, D-2 or D-3 accordingly, the following relations for each situation are obtained

Exponential
\[ Q = \int_0^t q_i \exp [-D_i \tau] d\tau \]
\[ Q = \left[ \frac{q_i}{D_i} \exp(-D_i \tau) \right]_0^t = \frac{q_i}{D_i} (\exp(-D_i \tau) - 1) = \frac{q_i}{D_i} (1 - \exp(-D_i \tau)) \]  \hspace{1cm} \text{(D-8)}

Hyperbolic
\[ Q = \int_0^t \frac{q_i}{[1 + bD_i \tau]^{1/b}} d\tau \]

Letting \( w = 1 + bD_i \tau \), then \( dw = bD_i d\tau \), and the substitution of these two relations into the previous expression leads to
\[ Q = \frac{q_i}{bD_i} \int_0^t \frac{dw}{w^{1/(1/b)}} = \frac{q_i}{bD_i} \left[ \frac{w^{1/(1/b) + 1}}{1 - \frac{1}{b}} \right]_0^t \]
\[ Q = \frac{q_i}{bD_i (1 - \frac{1}{b})} \left( 1 + bD_i \tau \right)^{(1 - 1/b)} \int_0^t \]
\[ Q = \frac{q_i}{bD_i (1 - \frac{1}{b})} \left[ (1 + bD_i \tau)^{(1-1/b)} - 1 \right] \]

\[ Q = \frac{q_i}{D_i(1-b)} \left[ 1 - (1 + bD_i \tau)^{(1-1/b)} \right] \].................................(D-9)

**Harmonic**

\[ Q = \int_{0}^{\tau} \frac{q_i}{[1 + D_i \tau]} \, d\tau \]

Letting \( w = 1 + D_i \tau \), it follows that \( dw = D_i d\tau \). Substituting these two expressions into the previous equation we obtain

\[ Q = \frac{q_i}{D_i} \int_{0}^{\tau} \frac{d(w)}{w} \]

\[ Q = \frac{q_i}{D_i} \ln[w]\left|_{0}^{\tau} = \frac{q_i}{D_i} \ln[1 + D_i \tau]\right|_{0}^{\tau} \]

\[ Q = \frac{q_i}{D_i} \ln[1 + D_i \tau]..............................................(D-10) \]

**Relations in Terms of Dimensionless Cumulative Production**

If we define \( Q_{Da} = \frac{QD_i}{q_i} \), and \( t_{Da} = D_i \tau \), Eqs. D-8, D-9 and D-10 can be rewritten as follows

**Exponential**

\[ Q_{Da} = 1 - \exp[-t_{Da}]..............................................(D-11) \]

**Hyperbolic**

\[ Q_{Da} = \frac{1}{(1 - b)} \left[ 1 - (1 + bD_i \tau)^{(1/(1/b))} \right]..............................................(D-12) \]

**Harmonic**

\[ Q_{Da} = \ln(1 + t_{Da})..............................................(D-13) \]
**Dimensional Relations in Terms of Flow Rate Derivatives**

We now obtain the derivative relations for each case. In this case the derivative is not multiplied by time as in well testing practice, but we use the derivative as it is defined rigorously from the mathematical theory. However, the well testing relations can be obtained from any of the equations presented below by simply multiplying by time, \( t \). In the following relations we use the symbol \( q' \), to represent the derivative \( q' = \frac{dq}{dt} \).

**Exponential**

\[
q' = - q_i D_i \exp \left[ - D_i t \right] \tag{D-14}
\]

or since \( q = q_i \exp [- D_i t] \), we write Eq. D-14 in terms of the flow rate as

\[
q' = - q D_i \tag{D-15}
\]

**Hyperbolic**

\[
q' = \frac{d}{dt} \left[ q_i (1 + bD_i t)^{-1/b} \right] = \frac{- q_i}{b} (1 + bD_i t)^{-1/b - 1} (bD_i) \tag{D-16}
\]

For hyperbolic decline the flow rate is given by

\[ q = \frac{q_i}{(1 + bD_i t)^{1/b}} \]

therefore Eq. D-16 becomes

\[
q' = - q \frac{D_i}{(1 + bD_i t)} \tag{D-17}
\]

**Harmonic**

\[
q' = \frac{d}{dt} \left[ \frac{q_i}{1 + D_i t} \right] \tag{D-18}
\]

\[
q' = - q_i (1 + D_i t)^{-2} (D_i) = - \frac{q_i D_i}{[1 + D_i t]^2} \tag{D-18}
\]
The rate relation for harmonic decline is given by

\[ q = \frac{q_i}{[1 + D_i t]} \]

Substituting the rate into Eq. D-18, the rate derivative for harmonic decline is given by

\[ q' = - q \frac{D_i}{[1 + D_i t]} \]  \hspace{1cm} (D-19)

Relations in terms of Dimensionless Flow Rate Derivatives

We can now obtain expressions for the dimensionless flow rate derivatives, noting that \( q'_Dd = \frac{d q_{Dd}}{d t_{Dd}} \). Once more, the derivative is not multiplied by time as in well testing practice. However, the well testing dimensionless relations can be obtained from any of the equations presented in this section by simply multiplying by dimensionless time, \( t_{Dd} \).

Exponential

From Eq. D-4, we obtain

\[ q'_Dd = - \exp [- t_{Dd}] \]  \hspace{1cm} (D-20)

Combining Eqs. D-4 and D-20, we have

\[ q_{Dd}' = - q_{Dd} \]  \hspace{1cm} (D-21)

Hyperbolic

From Eq. D-5, we obtain

\[ q'_Dd = \frac{d}{d t} [1 + b t_{Dd}]^{-(1/b)} \]

\[ q'_Dd = \left( - \frac{1}{b} \right) [1 + b t_{Dd}]^{-(1/b) - 1} (b) \]

\[ q_{Dd}' = - \frac{1}{[1 + b t_{Dd}]^{1 + 1/b}} \]  \hspace{1cm} (D-22)

Combining Eqs. D-5 and D-22, we obtain

\[ q_{Dd}' = - \frac{q_{Dd}}{[1 + b t_{Dd}]} \]  \hspace{1cm} (D-23)
Harmonic

From Eq. D-6, we obtain

\[ q_D t = - \left[ 1 + t_D d \right] \frac{1}{\left[ 1 + t_D d \right]^2} \] ......................................................... (D-24)

Combining Eqs. D-6 and D-24, we obtain

\[ q_D t = - \frac{q_D t}{1 + t_D d} \] ......................................................... (D-25)

The collection of all the equations we have derived so far in this Appendix is shown in Table D-1. This table helps visualize all the variations of the decline relations when they are given in terms of different variables. Before presenting the table itself, we summarize the relations between the fundamental parameters involved in that table.

\[ q_D t = \frac{q}{q_i} \quad t_D d = D t \quad q' = \frac{dq}{dt} \]

\[ q_D t = \frac{dq_D t}{dt_D d} \quad Q = \int_{0}^{t} q(t) dt \quad Q_D d = \frac{QD_i}{q_i} \]

Specific Variables Used for Type Curve Matching

In the final section of this Appendix we include the definitions of the specific functions on which the different type curves used in Chapter VI of this work are based. These functions are simple combinations of those seen in Table D-1. These variables are always plotted versus \( t_D d \). Therefore, the following comments focus on the y-axis plotting functions only.

The first of these variables is \( q_D t \) whose different forms were already presented in Table D-1. The second variable is \( q_D t \). This plotting function is the "flow rate integral" and it is defined as

\[ q_D t = \frac{Q_D d}{t_D d} \] ......................................................... (D-26)
### Table D-1
Decline Curve Relations in Dimensional and Dimensionless Variables

<table>
<thead>
<tr>
<th>Dimensional</th>
<th>Exponential</th>
<th>Hyperbolic</th>
<th>Harmonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow Rate, ( q )</td>
<td>( q = q_i \exp[-D_i t] )</td>
<td>( q = \frac{q_i}{[1+bD_i t]^{1/b}} )</td>
<td>( q = \frac{q_i}{[1+D_i t]} )</td>
</tr>
<tr>
<td>Cumulative Production, ( Q )</td>
<td>( Q = \frac{q_i}{D_i} [1-\exp(-D_i t)] )</td>
<td>( Q = \frac{q_i}{D_i (1-b)} x ) ( \frac{1}{1-[1+bD_i t]^{1-(1/b)}} )</td>
<td>( Q = \frac{q_i}{D_i} \ln[1+D_i t] )</td>
</tr>
<tr>
<td>Flow Rate Derivative, ( q' )</td>
<td>( q' = -q_i D_i \exp[-D_i t] )</td>
<td>( q' = -\frac{q_i D_i}{[1+bD_i t]^{(1/b)+1}} )</td>
<td>( q' = -\frac{q_i D_i}{[1+D_i t]^2} ) or ( q' = -q_i D_i \frac{D_i}{[1+D_i t]} )</td>
</tr>
<tr>
<td>Dimensionless</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dimensionless Flow Rate, ( q_{Dd} )</td>
<td>( q_{Dd} = \exp[-t_{Dd}] )</td>
<td>( q_{Dd} = \frac{1}{[1+b t_{Dd}]^{1/b}} )</td>
<td>( q_{Dd} = \frac{1}{[1+t_{Dd}]} )</td>
</tr>
<tr>
<td>Dimensionless Cumulative Production, ( Q_{Dd} )</td>
<td>( Q_{Dd} = 1-\exp[-t_{Dd}] )</td>
<td>( Q_{Dd} = \frac{1}{(1-b)} x ) ( \frac{1}{1-[1+b t_{Dd}]^{1-(1/b)}} )</td>
<td>( Q_{Dd} = \ln(1+t_{Dd}) )</td>
</tr>
<tr>
<td>Dimensionless Flow Rate Derivative, ( q'_{Dd} )</td>
<td>( q'<em>{Dd} = - \exp[-t</em>{Dd}] )</td>
<td>( q'<em>{Dd} = -\frac{1}{[1+b t</em>{Dd}]^{(1+1/b)}} )</td>
<td>( q'<em>{Dd} = -\frac{1}{[1+t</em>{Dd}]^2} ) or ( q'<em>{Dd} = -\frac{q</em>{Dd}}{[1+b t_{Dd}]} )</td>
</tr>
<tr>
<td>or ( q'<em>{Dd} = -q</em>{Dd} )</td>
<td>or ( q'<em>{Dd} = -\frac{q</em>{Dd}}{[1+b t_{Dd}]} )</td>
<td>or ( q'<em>{Dd} = -\frac{q</em>{Dd}}{[1+t_{Dd}]} )</td>
<td></td>
</tr>
</tbody>
</table>
where we can use the results for the \(Q_{Dd}\) relations shown in Table D-1, to obtain the desired expression for either exponential, hyperbolic or harmonic decline.

The last variable of interest is the "flow rate integral derivative". The definition of this variable is given by

\[
q_{Ddid} = -t_{Dd} \frac{dq_{Ddi}}{dt_{Dd}} \tag{D-27}
\]

Substituting Eq. D-26 into Eq. D-27, we can simplify Eq. D-27 to obtain

\[
q_{Ddid} = -t_{Dd} \frac{d}{dt_{Dd}} \left[ \frac{Q_{Dd}}{t_{Dd}} \right]
\]

which is equivalent to

\[
q_{Ddid} = -t_{Dd} \frac{d}{dt_{Dd}} \left[ \int_{0}^{t_{Dd}} q_{Dd}(\tau) d\tau \right]
\]

If we use the quotient rule for differentiation of the previous formula, we have

\[
q_{Ddid} = -t_{Dd} \left[ \frac{t_{Dd} q_{Dd} \int_{0}^{t_{Dd}} q_{Dd}(\tau) d\tau - \int_{0}^{t_{Dd}} q_{Dd}(\tau) d\tau}{t_{Dd}^2} \right]
\]

After simplification, we can rearrange this expression to obtain

\[
q_{Ddid} = \frac{Q_{Dd}}{t_{Dd}} - q_{Dd}
\]

or

\[
q_{Ddid} = q_{Ddi} - q_{Dd} \tag{D-28}
\]

Eq. D-28 is the relation that McCray\(^{30}\) used for developing his "flow rate integral derivative" type curve.
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