SEMI-ANALYTICAL METHODS FOR THE ANALYSIS AND INTERPRETATION OF WELL TEST DATA DISTORTED BY WELLBORE STORAGE AND SKIN EFFECTS

A Thesis

by

SOMPONG PRACHUMCHON

Submitted to the Office of Graduate Studies of Texas A&M University in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 1996

Major Subject: Petroleum Engineering
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Approved as to style and content by:

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December 1996

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ABSTRACT

Semi-Analytical Methods for the Analysis and Interpretation of Well Test Data Distorted by Wellbore Storage and Skin Effects. (December 1996)
Sompong Prachumchon, B.S., Texas A&M University
Chair of Advisory Committee: Dr. Thomas A. Blasingame

Our objective is to develop approximations of the pressure-time behavior for use in analyzing the pressure response of a well in an infinite-acting reservoir influenced by wellbore storage and skin effects. Our resulting approximate models are semi-analytical, closed-form solutions. We propose five approximate models by assuming the behavior of dimensionless constant rate pressure at the sandface, $p_{stD}(t_D)$, as follows:

<table>
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<th>$p_{stD}(t_D)$</th>
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<td>1</td>
<td>$a$ (Constant)</td>
<td>Constant Model</td>
</tr>
<tr>
<td>2</td>
<td>$at_D$</td>
<td>Linear Model-Zero Intercept</td>
</tr>
<tr>
<td>3</td>
<td>$a_0 + a_1 t_D$</td>
<td>Linear Model-General</td>
</tr>
<tr>
<td>4</td>
<td>$a_0 \exp(a_1 t_D)$</td>
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</tr>
<tr>
<td>5</td>
<td>$a_0 + a_1 t_D + a_2 t_D^2$</td>
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The advantage of our approach is that these solutions can be used to predict the pressure behavior of a well influenced by wellbore storage and skin effects in the real time domain. To our knowledge, this work is the first attempt to develop and apply real time domain solutions for all times of interest.

To verify our new $p_{wCD}(t_D)$ models, we used the numerical inversion solution as the "exact" solution for wellbore pressure behavior that has been influenced by wellbore
storage and skin effects. In particular, we have focused on the cylindrical source solution. As a comparison, we generated the “type curve” solutions for a well in an infinite-acting reservoir with wellbore storage and skin effects using the numerical inversion solution (as the “exact” solution) and our new relations as the “approximate” solutions. We also provided a graphical comparison of the residuals for each approximate solution (as compared to the numerical solution).

All of the approximate solutions exhibited excellent agreement and consistency with the numerical inversion solution (except for Model 2). As a practical consideration, we found that it is generally better to compute the pressure derivative functions numerically, rather than by analytical differentiation, because in most cases the analytical derivatives are too tedious for hand calculations.

For practical applications, we compare our new solutions to a number of field data cases. In order to perform history-matching of our new solutions, as well as the numerical inversion solution we developed and applied new modules for the history-matching program (PERANA). Again, for comparison, the results from the numerical inversion solution are used as the base results. In general, all of the models are as accurate as the Laplace transform numerical inversion solution, though Model 1 is clearly not appropriate in certain cases.
DEDICATION

To my loving parents, Poo and Samean Prachumchon. In dedication for your love, your thoughtfulness, your guidance, and your giving without expectation—this makes me who I am, and guides me to who I will become. To my grandmother Team Yusuksawat, my brothers Saman, Samart, Wasan, and Pimon Prachumchon, my sisters Yupin and Somsri Prachumchon, for your love and support.

I also dedicate this work to my teacher Keit Sithipong and my guardian Pra Somchai Sriratharo, who provided me with unselfish love and support. Both have given me the desire and strength to achieve my goals.
ACKNOWLEDGMENTS

I want to first thank the Royal Thai Army for providing the financial support for my seven years of study in the United States.

I also thank the chairman of my Graduate advisory committee, Dr. Thomas A. Blasingame, for his support, guidance, and encouragement during my graduate studies at Texas A&M University. Tom, it has been a great pleasure to work with you and to get to know you. Again, thanks for your faithful friendship, Tom.

I thank Drs. W. John Lee and Robert R. Berg for their advice and suggestions, and for their service on my advisory committee.

I would like to thank those individuals directly or indirectly involved in the completion of this work—in particular, Col. Buck Henderson, Mrs. Sithiraporn Prommar, Ms. Puangsuree Meta, and Ms. Mala Farmer.

In addition, I wish to thank the faculty and staff at the Department of Petroleum Engineering. The invaluable assistance of Ms. Gail Krueger is gratefully acknowledged.

I would also like to acknowledge the help of all my colleagues in the Well Testing Research Group, especially Javier Velarde, Joseph Ansah, and Taufan Marhaendrajana for their comments and suggestions regarding this research, as well as their encouragement throughout my graduate program.

Finally, I thank all of the members of the Thai Students Association at Texas A&M University, especially my friends who provided strength and encouragement throughout this work—in particular, my good friend Sarich Chotipanich.
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CHAPTER I

INTRODUCTION

The evaluation of formation properties using well tests has evolved into a significant component of the reservoir characterization activities performed in the petroleum industry. Because of the expense of extended pressure-drawdown or buildup tests, the trend in the petroleum industry has been to develop methods for the analysis and interpretation of “early-time” well test data. “Early-time” pressure data are defined as the pressure data obtained prior to the onset of the typical semilog straight-line portion of a well test \(i.e.,\) the radial flow data. These early data are influenced by wellbore storage, various “skin” effects, and possibly non-Darcy flow.

Pressure buildup tests and other types of pressure transient tests have been used for many years to evaluate reservoir fluid flow characteristics and well completion efficiency. The developments of the basic theory, as well as the modeling and analysis equations for wellbore storage distortion, are well documented. However, in order to analyze pressure data influenced by wellbore storage and skin effects, we must use solutions which account for wellbore storage and skin effects. As we know, the skin effect can be added directly into the constant rate pressure solution—but wellbore storage is not an “additive” condition like that of the skin factor. The current solution for transient radial flow influenced by wellbore storage and skin effects is given by Agarwal, \textit{et al.}\(^1\) An exact solution is derived by using the Laplace transformation. However, this “exact” solution requires tedious numerical integration.

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This thesis follows the style and format of the \textit{Journal of Petroleum Technology}.\(^1\)
The difficulty with the Laplace domain solutions is that these solutions typically cannot be inverted analytically, and are thus typically inverted numerically using a sampling algorithm such as the one proposed by Stehfest. While the numerical inversion solutions are accurate these methods, unfortunately, do not provide explicit, closed-form results. This lack of closed form solutions is our motivation to develop approximate solutions for the pressure behavior influenced by wellbore storage and skin effects. Our goal is to develop solutions which are easy enough for use in hand calculations and, at the same time, are as accurate as the numerical inversion solutions.

Approximate solutions may provide significant aid in well test analysis since these relations can be used to model the behavior of pressure test data for a well in an infinite-acting reservoir with wellbore storage and skin effects. The results of these approximate solutions may also provide new methods of analysis (although such developments were not considered in this work). In order to apply our new solutions we have developed additional modules for a well test history-matching program called PERANA (Reservoir Performance Analysis), which uses these approximate solutions for matching pressure test data (in addition to the option of using the numerical inversion solution). Obviously the new solutions are less complex and run times are considerably less than for the numerical inversion case (using the Stehfest algorithm). We also expect the development of several analysis techniques for wellbore storage influenced pressure data to arise from the computational formulae derived in this work. The relations derived in this work should be particularly useful for interpreting the "unit slope" line and "transition" flow regimes. We also expect the development of convolution and deconvolution analysis techniques to arise from the results of this work.

We have verified the new solutions developed in this work using the PERANA program. Field data used in this validation include cases for an unfractured well in an infinite-acting homogeneous reservoir, an unfractured well in an infinite-acting dual porosity reservoir, and a fractured well in an infinite-acting homogeneous reservoir.
1.1 OBJECTIVES OF THIS RESEARCH

The purpose of our research is to develop, verify, and apply simple and accurate closed-form approximations for the transient pressure behavior of a well influenced by wellbore storage and skin effects. This research is a significant contribution to well test analysis because the approximations we propose can be used to model the behavior of well test data. In addition, this work also offers the potential for developing new analysis relations for wellbore storage distorted well test data. Our goal is that this work will provide more options for the analysis of pressure transient test data.

1.2 ORGANIZATION OF THIS THESIS

Chapter II summarizes the historical development of the current methods used for analyzing pressure transient test data. The earliest study of the flow of fluids through porous media was given by Muskat.³ Muskat’s study was similar to previous work performed in study of the heat conduction in solids. Refs. 4-12 provide the original developments for the computation of wellbore storage solutions. van Everdingen¹³ and Hurst,¹⁴ and later Ramey,¹⁰ provided relations to compute the wellbore pressure response in the presence of wellbore storage and skin effects by using an exponential sandface rate model. Ramey¹⁰ also provided limited analysis techniques for wellbore storage influenced pressure data based on this exponential sandface rate model. Horne¹⁵ provided an alternate solution for such analysis by using the Fenske conservation method, and this effort did result in a closed form solution—much the same as our current efforts, but the Fenske-Horne solution will be shown to be considerably less accurate than our new solutions.

Blasingame, et al.¹⁶ provided an important new approach for developing approximate relations to model the behavior of pressure transient data influenced by wellbore storage and phase redistribution effects. However, these authors did not provide any methods to interpret well test data, only relations for modeling performance. Our goal is to extend
their work by developing analysis and interpretation methods, and verifying these developments using field data. In the following sections we summarize the process of these developments.

Chapter III presents the development of our new approximate solutions. The first section in this chapter summarizes the definitions of the dimensionless variables that we used in the development of our approximate solution models. We provide graphical plots of the behavior of each model as well as plots showing the deviation of the approximate solutions from the numerical solution for each model. Our observations indicate that during the transition from the wellbore storage dominated flow regime to the constant sandface flowrate regime (i.e., no wellbore storage effects), all of the approximate models show the highest value of deviation from the numerical inversion solution during this period. Recall that the wellbore storage dominated flow regime is only for the flow of fluids within the wellbore, but the constant sandface flowrate regime represents the flow of fluids through the porous medium—which is described by the diffusivity equation. Of our approximate models, the linear case with a non-zero intercept, as well as the exponential and quadratic models, appear to be as accurate as the numerical inversion solution for essentially every case we considered.

At the end of this chapter, we will discuss the Fenske-Horne method for modeling pressure data influenced by wellbore storage in an infinite-acting, homogeneous reservoir. As we noted earlier, we found that our approximate solutions can predict the pressure behavior during the wellbore storage domination and the transition flow regimes much more accurately and consistently than the Fenske-Horne method.

In Chapter IV we provide verification of our new approximate solutions. These example cases include data from fractured and unfractured wells producing from an infinite-acting homogeneous or dual porosity reservoir, influenced by wellbore storage and skin effects. We outline the general data requirements for history-matching pressure test data
and we provide 9 separate example cases. The primary data required are the bottomhole flowing (or shut-in) pressure, along with the basic rock and fluid properties. We also provide a general procedure for applying the PERANA program for history-matching pressure test data.

We summarize the fluid and reservoir properties for each of the field data case and we provide a discussion of the well history whenever such data were available. We also provide tables of the analysis results obtained using our new approximate $p_{sd}(t_D)$ solutions (the quadratic, the linear (non-zero intercept), and the constant approximate solutions), as well as the numerical inversion solution. We also provide the definitions of the statistical terms used in this work: the sum of squared errors, variance, standard deviation, root mean square error, and the average error. Finally, we discuss our history-matching results from the PERANA program using our new approximate solutions as well as the results we obtained using the numerical inversion solution.

Chapter V discusses our conclusions and our recommendations for applying the results of this research. The simple, semi-analytical approximate solutions that we have developed are highlighted. We encourage future investigators to attempt development of the explicit analysis relations using our new solutions.
CHAPTER II

LITERATURE REVIEW

In this chapter, we review the techniques that are used to analyze pressure test data influenced by wellbore storage and skin effects. This review ranges from the early developments in petroleum engineering (and other related fields) to the current methods being used in the industry. We also include discussion of other applications of the "wellbore storage" solutions.

Early studies regarding how fluids flow through porous media correspond to work performed for problems involving heat conduction. We will discuss and review both fields of study. However, we will concentrate on the petroleum industry more than other applications.

Jaeger\textsuperscript{17-18} provided complete results on the radial flow of heat in solids bounded internally or externally by a circular cylinder. Jaeger and Carslaw\textsuperscript{19} provided numerical solutions for these cases and Bullard\textsuperscript{20} applied the solutions given by refs. 17-19 to estimate the flow of heat through the floor of the Atlantic Ocean. Bullard used these solutions to develop a technique for measuring the temperature gradient and the thermal conductivity, where these results can be used to calculate the magnitude of heat flow. This process directly relates to the calculation of parameters used to describe the flow of fluids through porous media.

In 1934, Muskat\textsuperscript{3} presented partial differential equations for the flow of compressible fluids through porous media. Muskat derived and applied these solutions for various cases of pressure distributions and showed that these solutions corresponded to heat conduction problems.
The problem of a well produced at a constant "surface" flowrate, including wellbore storage and skin effects was first presented by van Everdingen and Hurst. This paper shows that the Laplace transformation, which was used to solve the diffusivity equation (as well as other partial differential equations), can be solved explicitly in terms of Fourier-Bessel series. As we know, the Laplace transformation simplifies a partial differential equation into an ordinary differential equation and is especially useful when the solution for the constant "surface" rate case is known. In addition, the Laplace transform solution for the constant bottomhole pressure case can be derived directly from the constant sandface case, or vice-versa. This interchange can be performed only by using the appropriate transformations. The final, real domain solution is then calculated by inversion (using analytical and/or numerical methods). van Everdingen and Hurst gave the essential theorems for applying the Laplace transformation to the diffusivity equation and they also introduced the inner boundary condition for the wellbore storage effect, without including skin effects.

In addition, van Everdingen and Hurst later extended the problem to include steady-state "skin" effects. They presented solutions for the radial flow diffusivity equation using the assumption of a line-source well. The solution for wellbore pressure affected by wellbore storage and skin effects was obtained using the real-time domain inversion integration of the Laplace transformation solution (i.e., the Mellin inversion integral), as well as a long-time approximation. This inner boundary condition (i.e., wellbore storage and skin effects) is a special case given by Jaeger, applied to an unsteady-state heat conduction problem.

In 1954, Blackwell presented a new theory for determining thermal constants, where this procedure has also been used to derive solutions for fluid flow problems. Blackwell provided large and small-time approximate solutions for the pressure in the wellbore, for the "infinite conductivity" cylinder (i.e., wellbore) case.
provided large and small-time approximate solutions for the pressure in the wellbore, for the "infinite conductivity" cylinder (i.e., wellbore) case.

Jaeger\(^6\) provided a solution for the heat-conduction problem analogous to that of the fluid-flow problem and presented an alternative method for measuring the thermal conductivities of the rock performed by observing the temperature in a borehole. Later, in 1956, Beck, et al.\(^7\) illustrated a "type-curve matching" for matching borehole temperature data for an unsteady-state process in order to determine the thermal constants of the well.

Lesem, et al.\(^8\) discussed a method for calculating the distribution of temperatures in flowing gas wells. These results are identical to the solutions for undistorted radial flow in an infinite-acting reservoir.

In 1958, Hurst\(^9\) provided an simple approach for predicting reservoir performance by using the Laplace transformation in material balance formulas. Using the Laplace transformation we can easily isolate the pressure term in the various material balance equations instead of using tedious iteration procedures.

Ramey\(^10-11\) developed a method for estimating the duration of wellbore storage effects by extending the Gladfelter, et al. method for correcting wellbore storage distorted pressure data in drawdown testing. The Gladfelter, et al. method is another approach for calculating wellbore storage effects without using superposition. However, this method requires extremely accurate sandface flowrate data that simply are not available. Ramey also investigated the effects of non-Darcy flow in gas wells and concluded that a combination of non-Darcy flow and wellbore storage effects can severely distort pressure data taken from short duration gas well tests.
Agarwal, et al.\textsuperscript{1} are among several authors who provide major contributions to the study of wellbore storage distorted transient pressure behavior. These authors presented the definitive study on the importance of wellbore storage and skin effects. Agarwal, et al. derived solutions for wellbore storage and skin effects by using analytical methods and the numerical integration of a Laplace transformation inversion integral. They also presented tabular and graphical results for various cases of the dimensionless wellbore storage coefficient and the dimensionless skin factor. The wellbore storage problem is given in the Laplace domain by:

\[
\bar{P}_{wCD}(u) = \frac{K_0(\sqrt{u}) + s\sqrt{u}K_1(\sqrt{u})}{u[\sqrt{u}K_1(\sqrt{u}) + C_Du\{K_0(\sqrt{u}) + s\sqrt{u}K_1(\sqrt{u})\}]}
\]  \hspace{1cm} (2-1)

This solution was originally studied by Blackwell\textsuperscript{5}, who applied the Residue Theorem to invert this Laplace domain solution back to the real domain. This result is

\[
P_{wCD}(t_{CD},C_D,s) = \frac{4}{\pi^2} \int_0^{\infty} \frac{(1-e^{-\tau^2t_D})}{\tau^2\left\{\tau C_D J_0(\tau) - (1-\tau^2 C_D) J_1(\tau)\right\}^2 + \left\{\tau C_D Y_0(\tau) - (1-\tau^2 C_D) Y_1(\tau)\right\}^2} \, d\tau \quad \ldots (2-2)
\]

Eq. 2-2 is the cylindrical source solution (specifically, the exact solution) for the case of a well in an infinite-acting homogeneous reservoir with wellbore storage and skin effects.

In 1969, Wattenbarger and Ramey,\textsuperscript{12} investigated the influence of wellbore storage and skin effects for unsteady-state liquid flow by using a finite-difference solution of the basic partial differential equation. These solutions showed excellent agreement with the analytical solutions of Agarwal, et al.\textsuperscript{1} The finite-difference solutions significantly differ from the analytical solutions only for the cases of a negative skin factor (although this is somewhat expected given the physical concept of a "negative" skin factor).

In 1970, McKinley\textsuperscript{21} presented another approach for analyzing pressure buildup data influenced by wellbore storage and skin effects where his solution estimated near well
permeability, and provided insight into the behavior of the “wellbore” transmissibility and the relationship between the near-wellbore and formation flow capacity. Fair\textsuperscript{22} presented a detailed development and analysis of the effects of the wellbore phase redistribution on pressure buildup tests (where this condition includes wellbore storage effects). Fair provided an array of type curve solutions illustrating the quantitative effects of wellbore phase redistribution.

In 1985, Joseph \textit{et al.}\textsuperscript{23-24} proposed an approximate approach for modeling early-time pressure data that were influenced by wellbore storage effects. These authors proposed several numerical models that included regression of the van Everdingen and Hurst model\textsuperscript{4} for determining the pressure and sandface flowrate profiles. This approach provided methods to analyze pressure-time data and to estimate the wellbore storage coefficient, $C_s$. Joseph \textit{et al.} also reviewed the Schapery numerical technique for the inversion of the Laplace transform, but we continue to use the Stehfest inversion algorithm due to its stability and ease of application.

In 1986, Correa and Ramey\textsuperscript{25} presented a new method for analyzing buildup pressure where various bottomhole pressure conditions are modeled by using a special inner boundary condition. This approach does not require direct superposition for modeling variations in wellbore storage, but does require a Gaussian quadrature integration for the “coupling” condition.

Bourdet and Gringarten\textsuperscript{26} presented a set of type curves for analyzing pressure drawdown and buildup in wells with wellbore storage and skin effects in dual porosity reservoirs. These type curves apply to damaged, acidized, and fractured wells in fissured formations. This work gives insight into the grouping $C_D e^{2s}$, for both homogeneous and dual-porosity systems.
Using a different formulation of the solution to the diffusivity equation, Horne\textsuperscript{15} also attempted to develop a direct method for analyzing well test pressure data that are influenced by wellbore storage and skin effects. Horne used the Fenske conservation method to include wellbore storage and skin effects in the pressure behavior, and as such, he replaced a complex mathematical procedure with a much more simple approximation, which eliminated both the use of the Laplace transformation and the need for the evaluation of tedious Bessel functions.

Ayestaran and Minhas\textsuperscript{27} used convolution and deconvolution methods to analyze transient pressure buildup and drawdown test data. By using the downhole pressure and the sandface flowrate, along with the Gladfelter deconvolution technique, these authors developed a procedure for system identification and parameter estimation for single layer and multilayer reservoirs.

Blasingame \textit{et al.}\textsuperscript{28} applied deconvolution methods to analyze gas well test data that are influenced by wellbore storage. Their approach is to use an explicit deconvolution formula which is much easier to use than previous methods, which are both computationally tedious and difficult to apply. This work also presented new methods to estimate the sandface flowrate using a wellbore material balance. However, this approach requires an extremely accurate measurement of the sandface flowrate and the wellbore pressure data. Blasingame \textit{et al.} also introduced Laplace domain methods for deconvolution. However, these methods are again limited by the accuracy of the measured pressure and sandface flowrate data.

In 1991 Blasingame \textit{et al.}\textsuperscript{16} introduced a similar approach for developing approximate solutions to represent the behavior of pressure data influenced by wellbore storage and skin effects. These approximations are explicit (and hence, easy to use), and may provide new methods for the interpretation of transient well test data.
CHAPTER III

DEVELOPMENT OF $p_{wCD}(t_D)$ MODELS

3.1 INTRODUCTION
In this chapter we first define several dimensionless variables; these include the dimensionless pressure and pressure derivative functions, the dimensionless time function, and the dimensionless wellbore storage coefficient. We then discuss the development of various approximate, semi-analytical solutions for the dimensionless pressure function, $p_{wCD}(t_D)$, where this solution includes wellbore storage and skin effects. We also discuss the Fenske and Horne solution\textsuperscript{15} which was developed to model wellbore pressure data influenced by wellbore storage and skin effects.

3.2 DEFINITIONS OF DIMENSIONLESS VARIABLES

*Dimensionless Pressure Functions*

The dimensionless wellbore pressure, $p_D$, for a constant flowrate is defined as:

$$p_D = \frac{1}{141.2 \frac{k h}{q B_H}} \Delta p \tag{3-1}$$

where the pressure drop, $\Delta p$, for a pressure drawdown test is given by:

$$\Delta p = p_i - p_{wf} \tag{3-2}$$

and for buildup tests we have:

$$\Delta p = p_{ws} - p_{wf} \tag{3-3}$$

The dimensionless wellbore pressure, $p_{sD}$, for a constant rate system with skin effects is defined as:

$$p_{sD} = p_D + s \tag{3-4}$$

The dimensionless wellbore pressure, $p_{wCD}$, for a system with wellbore storage and skin effects is defined by the “convolution” as:
\[ P_{wCD}(t_D) = \int_{0}^{t_D} \frac{d}{d\tau} [q_{wCD}(\tau)] P_{sD}(t_D - \tau) d\tau \] \hspace{1cm} (3-5)

**Dimensionless Time Function**

The dimensionless time, \( t_D \), based on the wellbore radius, \( r_w \), is given as

\[ t_D = 0.0002637 \frac{k}{\phi \mu c_t r_w^2} t \] \hspace{1cm} (3-6)

**Dimensionless Pressure Derivative Functions**

The dimensionless pressure derivative function, \( p_D' \), used in type curve analysis is defined by two forms which are mathematically identical. The first form is given in logarithmic form as

\[ p_D' = \frac{dp_D}{d(\ln t_D)} \] \hspace{1cm} (3-7)

and the second form is given as

\[ p_D' = t_D \frac{dp_D}{dt_D} \] \hspace{1cm} (3-8)

**Dimensionless Wellbore Storage Coefficients**

The dimensionless wellbore storage coefficient, \( C_D \), based on the wellbore radius, \( r_w \), is given as

\[ C_D = \frac{0.894 C_s}{\phi c_t h r_w^2} \] \hspace{1cm} (3-9)

and the dimensionless wellbore storage coefficient, \( C_{Df} \), based on the fracture half-length, \( L_f \), is defined as:

\[ C_{Df} = \frac{0.894 C_s}{\phi c_t h L_f^2} \] \hspace{1cm} (3-10)
3.3 DEVELOPMENT OF THE NEW APPROXIMATE $p_{wCD}(t_D)$ MODELS

In this section, we discuss the development of the semi-analytical approximate solutions that we derived for the dimensionless pressure function, $p_{wCD}(t_D)$, where this solution includes wellbore storage and skin effects. In Appendix A, we derive the Laplace transform identities (i.e., the $\bar{p}_{wCD}(u)$ formulations) which are required to develop the various $p_{wCD}(t_D)$ approximations that we propose. These identities are developed rigorously from the convolution integral for the case of a well with wellbore storage and skin effects. In Appendix B we develop approximations for $p_{wCD}(t_D)$ based on the various cases of the constant rate dimensionless pressure solution, $p_{sD}(t_D)$. To conclude in this chapter, we discuss the Fenske and Horne solution\textsuperscript{15} that was developed for modeling the pressure behavior in a well influenced by wellbore storage and skin effects.

3.3.1 $p_{wCD}(t_D)$ Based on $p_{sD} = a \ (a$ Is Constant)

In Appendix B we develop the case where we assume that the $p_{sD}(t_D)$ function is constant near some particular time of interest. For this case, the $p_{wCD}(t_D)$ result is given by

$$p_{wCD}(t_D) = p_{sD}(t_D) \left[ 1 - \exp\left( \frac{-t_D}{p_{sD}(t_D)C_D} \right) \right] \quad \text{(3-11)}$$

Eq. 3-11 states that the $p_{wCD}(t_D)$ function is an exponentially increasing function of the $p_{sD}(t_D)$ relation. This is a somewhat intuitive result since we know that the $p_{wCD}(t_D)$ function increases monotonically until such time where the $p_{wCD}(t_D)$ function becomes identical to the $p_{sD}(t_D)$ function. We verify Eq. 3-11 against the numerical inversion solution (computed using the Stehfest inversion algorithm) for the case of an infinite-acting homogeneous reservoir with wellbore storage and skin effects.
We note that fundamentally, the assumption of the constant $p_{sD}(t_D)$ model violates the theory of the Laplace transformation for a continuous function. We note our observations for this case in the sections below.

- Fig. 3.1 shows behavior of $p_{wCD}(t_D)$ for $0.1 \leq t_D/C_D \leq 1 \times 10^4$, for 18 separate values of $C_D e^{2s}$ ranging from 0.5 to $1 \times 10^{100}$. This plot clearly shows that for $t_D/C_D \leq 100$ this model (the constant $p_{sD}(t_D)$ case) fails to represent the behavior of pressure data influenced by wellbore storage and skin effects.

- Fig. 3.2 shows the behavior of the $p_{wCD}'(t_D)$ function which is obtained using analytical differentiation of $p_{wCD}(t_D)$ (Eq. B-13). As we would expect, the approximate (dimensionless) pressure derivative solution deviates over the same ranges of parameters as the $p_{wCD}(t_D)$ function. In fact, the deviation is slightly more pronounced because the derivative function is more sensitive to change than the pressure function.

- Fig. 3.3 shows the behavior of the $p_{wCD}'(t_D)$ function which is calculated by using numerical differentiation of the $p_{wCD}(t_D)$ function. The results shown in Fig. 3.3 suggests that Eq. 3-11 and its derivative function fail to represent the behavior of pressure data influenced by wellbore storage when $t_D/C_D \leq 100$, for $C_D e^{2s} \leq 1 \times 10^4$. We also note that the results shown in Figs. 3.2 and 3.3 are essentially identical.
Fig. 3.4 shows the distribution of the residual functions (computed using the new approximate solutions and the numerical inversion solutions). We note the residuals are almost perfectly uniform, except for $t_D/C_D$ values of $1$ to $1 \times 10^3$. The uniformity of these residuals suggests that this simple model (Eq. 3-11) may not be the most accurate approximation, but clearly, the deviation from the "true" solution (the numerical inversion results) are both uniform and reproducible.

Finally we consider a plot of the $P_{wCD}(t_D)$ residual function versus the $t_D/(C_DP_{sD}(t_D))$ function—where these results are shown in Fig. 3.5. We immediately note an excellent correlation of results (except for the cases of $0.5 \leq C_DE^{2s} \leq 10$). While we have not conclusively determined the reason for poor correlation (i.e., deviation) for these cases, we suspect that the $t_D/(C_DP_{sD}(t_D))$ "correlation" is only strictly valid for cases where the "log approximation" is valid for $P_{sD}(t_D)$.

In conclusion, we find that the $P_{wCD}(t_D)$ function based on $P_{sD}(t_D)$=constant gives a reasonably good performance, except for small values of $t_D/C_D$ and small values of $C_DE^{2s}$. We do not recommend using this solution for general applications.
Figure 3.1 - Type Curve Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir.

$p_{wCD}(t_D)$ Computed Using $p_{sd}(t_D) = \text{Constant}$.
Figure 3.2 - Type Curve Plot of $P_{wCD}'(t_D)$ for a Homogeneous Reservoir. $P_{wCD}'(t_D)$ Computed Using Analytical Differentiation of $P_{wCD}(t_D)$, $P_{wCD}(t_D)$ Computed Using $P_{3D}(t_D) = \text{Constant.}$
Figure 3.3 - Type Curve Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir. $P_{wCD}''(t_D)$ Computed Using Numerical Differentiation of $P_{wCD}(t_D)$ (3-Point Lagrange Formula), $P_{wCD}(t_D)$ Computed Using $P_{st}(t_D) = \text{Constant}$.
Figure 3.4 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/C_D$ Format).

$p_{wCD}(t_D)$ Computed Using $p_{sD}(t_D) = \text{Constant}$. 

Model: $p_{sD}(t_D) = \text{constant}$
Figure 3.5 - Residual Plot of $P_{wCD} (t)$ for a Homogeneous Reservoir ($t_p / (C_P d P_{wCD} (t))$ Format).

Model: $p_{wCD} (t) = \text{constant}$.
3.3.2. \( p_{wCD}(t_D) \) Based on \( p_{sD} = at_D \)

In Appendix B we provide the development of the case where we assume that \( p_{sD}(t_D) \) is a direct function of \( t_D \) only (zero intercept). The \( p_{wCD}(t_D) \) result for this case is given by

\[
p_{wCD}(t_D) = \frac{1}{\frac{1}{p_{sD}(t_D)} + \frac{1}{t_D/C_D}}
\]

(3-12)

Eq. 3-12 suggests that the “early-time” solution \( (t_D/C_D) \) and the “late-time” solution \( (p_{sD}(t_D)) \) can simply be coupled in the same manner as resistors in parallel. While this has considerable benefit from the standpoint of analysis, our results show that this concept does not properly model the “transition” from “early” to “late” time behavior.

- Fig. 3.6 confirms that this approximation does in fact model the correct behavior of \( p_{wCD}(t_D) \) at early and late times. However, we note considerable disparity between the results of the approximate and the numerical inversion solutions for the “transition” times. Obviously Eq.3-12 would not be a preferred result for general applications.

- Fig. 3.7 (analytical derivative functions) and Fig. 3.8 (numerical derivative functions) confirm and emphasize this disparity and force us to conclude that we should not use this model for modeling or analyzing pressure test data that are influenced by wellbore storage and skin effects.
Figure 3.6 - Type Curve Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir.

$P_{wCD}(t_D)$ computed using $P_{30}(t_D) = a_0 f_D$. 

Model: $P_{30}(t_D) = a_0 f_D$ 

- Numerical Inversion Solution 
- Approximate Solution 

$P_{30}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm.
Figure 3.7 - Type Curve Plot of $P_{wCD}'(t_D)$ for a Homogeneous Reservoir. $P_{wCD}'(t_D)$ Computed Using Analytical Differentiation of $P_{wCD}(t_D)$. $P_{wCD}(t_D)$ Computed Using $P_{sD}(t_D) = a_0 t_D$. 
Figure 3.8 - Type Curve Plot of $p_{wCD}'(t_D)$ for a Homogeneous Reservoir. $p_{wCD}'(t_D)$ Computed Using Numerical Differentiation of $p_{wCD}(t_D)$ (3-Point Lagrange Formula), $p_{wCD}(t_D)$ Computed Using $p_{sD}(t_D) = a_{0}t_D$. 
3.3.3 $p_{wCD}(t_D)$ Based on $p_{sD} = a_0 + a_1 t_D$

This model assumes that the $p_{sD}(t_D)$ function is linear (with a non-zero intercept) near some particular time of interest. In Appendix B we develop the following result for $p_{wCD}(t_D)$:

$$p_{wCD}(t_D) = \frac{\beta}{a} \left[ 1 - \exp(-at_D) \right] + \frac{\theta}{a^2} \left[ \exp(-at_D) + at_D - 1 \right]$$

(3-13)

where the generalized coefficients for this case; $\alpha, \beta, \text{and} \ \theta$ are defined as:

$$\alpha = \frac{1 + C_D a_1}{C_D a_0} \quad \beta = \frac{1}{C_D} \quad \theta = \frac{a_1}{a_0 C_D}$$

The coefficients $a_0$ and $a_1$ which appear in the $p_{sD}(t_D)$ model are given by

$$p_{sD}(t_D) = a_0 + a_1 t_D$$

To use Eq. 3-13 to compute $p_{wCD}(t_D)$ requires that we know the values for the coefficients $a_0$ and $a_1$. We can solve for the coefficient $a_1$ by differentiating the above relation for $p_{sD}(t_D)$ with respect to $t_D$. This gives

$$\frac{dp_{sD}(t_D)}{dt_D} = a_1$$

(3-14)

The coefficient $a_0$ can be determined directly from our $p_{sD}(t_D)$ model given above using the value of $a_1$ from Eq. 3-14. Rearranging our $p_{sD}(t_D)$ relation we obtain

$$a_0 = p_{sD}(t_D) - t_D \frac{dp_{sD}(t_D)}{dt_D}$$

(3-15)

Finally, we can substitute all of the coefficients terms in Eq. 3-13 (including Eqs. 3-14 and 3-15) to obtain
\[ p_{wCD}(t_D) = \left\{ \frac{p_{sD}(t_D) - p_{sD}'(t_D)}{\frac{t_D}{p_{sD}(t_D)C_D} + \frac{1}{p_{sD}(t_D) + \frac{1}{t_D/C_D} + \frac{1}{p_{sD}(t_D) - p_{sD}'(t_D)}}} \right\} \\
\left\{ 1 - \exp \left[ - \frac{1}{p_{sD}(t_D) - p_{sD}'(t_D)} \right] \right\} + \frac{1}{p_{sD}'(t_D)} + \frac{1}{t_D/C_D} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}

\text{where } p_{sD}'(t_D) = t_D \frac{d}{dt} \left[ p_{sD}(t_D) \right] \]

- Fig. 3.9 shows the dimensionless pressure behavior of \( p_{wCD}(t_D) \) for this case. We note that \( p_{wCD}(t_D) \) functions agree extremely well with the numerical inversion solutions for essentially all comparisons. The only variance of any significance occurs for \( C_D e^{2s} \leq 1 \times 10^4 \) during the time function \( t_D/C_D \leq 10 \). This suggests that Eq. 3-13 (or Eq. 3-16) should be an excellent approximation for the true \( p_{wCD}(t_D) \) behavior.

- Fig. 3.10 illustrates the behavior of the \( p_{wCD}'(t_D) \) function by using the \( p_{sD}(t_D) \) derivative computed using the Stehfest inversion algorithm. We note that for this work we consider the Stehfest inversion solution to be the “exact” solution so using \( p_{sD}'(t_D) \) to compute \( p_{wCD}(t_D) \) should, in theory, produce the most accurate results. Keeping this in mind, we observe excellent agreement between the approximate and numerical inversion solutions for \( p_{wCD}'(t_D) \). Note that \( p_{wCD}'(t_D) \) is computed using a 3-point Lagrange formula for the derivative.
• For comparison we also computed $p_{wCD}(t_D)$ (then $p_{wCD}'(t_D)$) using the line source solution for $p_{sd}(t_D)$. Fig. 3.11 shows the behavior of the $p_{wCD}'(t_D)$ function where the $p_{sd}(t_D)$ derivative is taken from the "line source" solution for $p_{sd}(t_D)$. The only obvious deviation for the $p_{wCD}'(t_D)$ functions are oscillations at $C_D e^{2s}=0.5$ for $t_D/C_D \leq 2$. However, this behavior does not preclude us from recommending this as a general approach.

It is important to note that because of the extremely tedious nature of developing the closed form derivatives for this case, we chose to compute the derivative functions numerically. We do not believe that the numerical differentiation process contributes any significant deviation to the approximate $p_{wCD}'(t_D)$ function.

• Fig. 3.12 shows the distribution of the residual functions computed using the approximate solutions and the numerical inversion solutions. As with the constant $p_{sd}(t_D)$ case, we note excellent correlation (uniformity) of the residual functions (except for $1 \leq t_D/C_D \leq 1 \times 10^3$).

• In Fig. 3.13 we again compare the residual functions by plotting these functions against $t_D/(C_D p_{sd}(t_D))$. We note an excellent correlation of the residual functions (except for $C_D e^{2s} \leq 10$). We also find that the entire residual trend begins to separate for all $C_D e^{2s}$ values, for $t_D/(C_D p_{sd}(t_D)) > 20$.
Figure 3.9 - Type Curve Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir.

$p_{wCD}(t_D)$ Computed Using $p_{sd}(t_D) = a_0 + a_1 t_D$. 

Model: $p_{sd}(t_D) = a_0 + a_1 t_D$

- Numerical Inversion Solution
- Approximate Solution

$P_{wCD}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm.
Figure 3.10 - Type Curve Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir. $P_{wCD}(t_D)$ Computed Using Numerical Differentiation of $P_{wCD}(t_D)$ (3-Point Lagrange Formula), $P_{wCD}(t_D)$ Computed Using $P_{sD}(t_D) = a_0 + a_1 t_D$ ($P_{sD}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm).
Figure 3.11 - Type Curve Plot of $p_{wCD}'(t_D)$ for a Homogeneous Reservoir. $p_{wCD}'(t_D)$ Computed Using Numerical Differentiation of $p_{wCD}(t_D)$ (3-Point Lagrange Formula), $p_{wCD}(t_D)$ Computed Using $p_{sD}(t_D) = a_0 + a_1 t_D$ ($p_{sD}(t_D)$ Derivative Computed Using the Line Source Solution).
Figure 3.12 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/C_D$ Format).

$p_{wCD}(t_D)$ Computed Using $p_{sD}(t_D) = a_0 + a_1 t_D$. 

Model: $p_{sD}(t_D) = a_0 + a_1 t_D$
Figure 3.13 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/(C_D P_{sl}(t_D))$ Format).

$p_{wCD}(t_D)$ Computed Using $P_{sl}(t_D) = a_0 + a_1 t_D$. 

Model: $p_{sl}(t_D) = a_0 + a_1 t_D$
3.3.4 $P_{wCD}(t_D)$ Based on $P_{sD} = a_2 \exp(a_1 t_D)$

This model assumes that the $P_{sD}(t_D)$ function acts like a single-term exponential function near a particular time of interest. The development of this $P_{wCD}(t_D)$ result is given in Appendix B as

$$
P_{wCD}(t_D) = C_0 \frac{1}{2\sqrt{\beta - \frac{\alpha^2}{4}}} \left[ \exp \left[ -\left( \frac{\alpha}{2} - \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right] \right] - \exp \left[ -\left( \frac{\alpha}{2} + \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right]
$$

where the generalized coefficients for this case, $\alpha$, $\beta$, and $C_0$ are defined as

$$
\alpha = \frac{1}{C_D a_0} \quad \beta = -\frac{a_1}{C_D a_0} \quad C_0 = \frac{1}{C_D}
$$

- In Fig. 3.14 we show the $P_{wCD}(t_D)$ function given by Eq. 3-17 and we again note that $P_{wCD}(t_D)$ functions compare favorably with the numerical inversion solutions. In fact, the results for this case appear to be virtually identical to those for the linear non-zero intercept case (i.e., Fig. 3.9). The only significant deviations in the $P_{wCD}(t_D)$ solutions occur for $C_D e^{2s} < 10$, for $t_D/C_D < 10$. This performance suggests that Eq. 3-17 should be an excellent approximation for the $P_{wCD}(t_D)$ function, and by comparison, these results should be quite similar to the results for the linear, non-zero intercept model for $P_{sD}(t_D)$.

- Fig. 3.15 illustrates the behavior of $P_{wCD}'(t_D)$ function using the $P_{sD}(t_D)$ derivative ($P_{sD}'(t_D)$) computed using the Stehfest inversion algorithm. Again, as with the linear, non-zero intercept case for $P_{sD}(t_D)$, we observe excellent agreement of the semi-analytical approximate solutions and the numerical inversion solutions. Numerical differentiation of Eq. 3-17 is performed using a 3-point Lagrange formula.
- Fig. 3.16 shows the behavior of the $p_{wCD}'(t_D)$ function where the $p_{sD}(t_D)$ derivative (i.e., $p_{sD}'(t_D)$) which is obtained using the line source solution. As with the linear, non-zero intercept case, we again see that the approximate $p_{wCD}'(t_D)$ function oscillates for the $C_D e^{2s} = 0.5$ case when $t_D/C_D < 2$.

We note that both the $p_{wCD}(t_D)$ and $p_{wCD}'(t_D)$ compare quite well for this case and we further note that this case (exponential model for $p_{sD}(t_D)$) and the previous case (linear, non-zero intercept model for $p_{sD}(t_D)$) compare extremely well—suggesting that these cases may have a common mathematical basis—though that issue is not addressed in this thesis.

- Fig. 3.17 shows the distribution of the residual functions computed using the approximate and numerical solutions. The performance of these residual functions is, as expected, very similar to those for the linear, non-zero intercept case.

- In Fig. 3.18 we plot the residual functions versus $t_D/(C_D p_{sD}(t_D))$ to establish or dispute a correlation of these functions. We note good correlation of these functions for $C_D e^{2s} \geq 10$, and $t_D/(C_D p_{sD}(t_D)) > 10$. As with the linear, non-zero intercept case, we also note a "fan" effect where the residual functions diverge away from a unified trend for $t_D/(C_D p_{sD}(t_D)) > 20$.

The similarities between this case (which uses an exponential for $p_{sD}(t_D)$) and linear, non-zero intercept $p_{sD}(t_D)$ case are truly striking. For continuity, as well as ease of application, we recommend using the linear, non-zero intercept $p_{sD}(t_D)$ case for general applications.
Figure 3.14 - Type Curve Plot of $P_{\text{wCD}}(t_D)$ for a Homogeneous Reservoir.

$P_{\text{wCD}}(t_D)$ Computed Using $P_{\text{SB}}(t_D) = a_0 \exp(a_1 t_D)$. 

Model: $P_{\text{SB}}(t_D) = a_0 \exp(a_1 t_D)$
Numerical Inversion Solution
Approximate Solution

$P_{\text{SB}}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm.
Figure 3.15 - Type Curve Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir. $p_{wCD}(t_D)$ Computed Using Numerical Differentiation of $p_{wCD}(t_D)$ (3-Point Lagrange Formula), $p_{wCD}(t_D)$ Computed Using $p_{sd}(t_D) = a_0 \exp(a_1 t_D)$ ($p_{sd}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm).
Figure 3.16 - Type Curve Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir. $P_{wCD}(t_D)$ Computed Using Numerical Differentiation of $P_{wCD}(t_D)$ (3-Point Lagrange Formula), $P_{wCD}(t_D)$ Computed Using $p_{sD}(t_D) = a_0 \exp(a_1 t_D)$ ($p_{sD}(t_D)$ Derivative Computed Using the Line Source Solution).
Figure 3.17 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/C_D$ Format).

$p_{wCD}(t_D)$ Computed Using $p_{stD}(t_D) = a_0\exp(a_1 t_D)$. 

Model: $p_{stD}(t_D) = a_0\exp(a_1 t_D)$ 

$p_{stD}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm.
Figure 3.18 - Residual Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/(C_D sD(t_D))$ Format).

$P_{wCD}(t_D)$ Computed Using $P_{sD}(t_D) = a_0 \exp(a_1 t_D)$.
3.3.5 $p_{wCD}(t_D)$ Based on $p_{sD} = a_0 + a_1 t_D + a_2^* t_D^2$

This model assumes that the $p_{sD}(t_D)$ function varies as a quadratic function of time near a particular time of interest. The $p_{wCD}(t_D)$ function for this case is developed in Appendix B and this result is given by

$$p_{wCD}(t_D) = c_0 f_0(t_D) + c_1 f_1(t_D) + c_2 f_2(t_D)$$  \hspace{1cm} (3-18)

The coefficients $a_0$, $a_1$, and $a_2^*$ that appear in our quadratic $p_{sD}(t_D)$ model are given by

$$p_{sD}(t_D) = a_0 + a_1 t_D + a_2^* t_D^2, \quad (a_2^* = a_2/2)$$  \hspace{1cm} (3-19)

To use Eq. 3-18 to compute $p_{wCD}(t_D)$ we must know the values of the coefficients $a_0$, $a_1$, and $a_2^*$. We can solve for coefficient $a_2^*$ by differentiating Eq. 3-19 with respect to $t_D$.

This gives:

$$a_2^* = \frac{1}{2} \frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right]$$  \hspace{1cm} (3-20)

The coefficient $a_1$ can be obtained by differentiating Eq. 3-19 with respect to $t_D$ and rearranging. This gives

$$a_1 = \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] - t_D \left[ \frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right] \right]$$  \hspace{1cm} (3-21)

The coefficient $a_0$ can be determined directly from Eq. 3-19 and using the values of $a_1$ and $a_2^*$ from Eq. 3-20 and Eq. 3-21, respectively. Making these substitutions and solving for $a_0$ gives us

$$a_0 = p_{sD}(t_D) - t_D \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] + \frac{t_D^2}{2} \frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right]$$  \hspace{1cm} (3-22)

- Fig. 3.19 shows exceptional performance of the $p_{wCD}(t_D)$ function for this case. This behavior suggests that Eq. 3-18 should be an excellent approximation for the $p_{wCD}(t_D)$ function and, while not directly verified, we believe by induction that this approach should work for any case regardless of the true $p_{sD}(t_D)$ model.
• In Fig. 3.20 we show the behavior of the \( p_{wCD}'(t_D) \) function where \( p_{wCD}(t_D) \) is computed using \( p_{sd}(t_D) \) (and its derivatives) using the Stehfest inversion algorithm. For this case we observe virtually perfect agreement of the semi-analytical and numerical inversion solutions.

• In Fig. 3.21 we show \( p_{wCD}'(t_D) \) computed from \( p_{wCD}(t_D) \) where \( p_{sd}(t_D) \) and its derivatives are computed using the “line source” solution. We note excellent agreement in the \( p_{wCD}(t_D) \) functions, but we can also clearly see that the \( p_{wCD}'(t_D) \) approximate function again oscillates for the \( C_D e^{2s}=0.5 \), for \( t_D/C_D < 2 \). Recall that we noted similar behavior for the linear, non-zero intercept \( p_{sd}(t_D) \) case and the exponential \( p_{sd}(t_D) \) case.

• As also with our previous cases, we have tried to “correlate” the residuals computed by comparing our semi-analytical and numerical inversion solutions. Fig. 3.22 shows the residual functions for this case, and we note the residuals for \( C_D e^{2s} \leq 10 \) do not “correlate” for \( t_D/C_D \leq 30 \), nor for \( C_D e^{2s} > 1 \times 10^{30} \) for \( t_D/C_D \geq 10 \).

• In Fig. 3.23 we show the residual functions plotted against \( t_D/(C_D p_{sd}(t_D)) \) and we confirm the observations noted above for small values of \( C_D e^{2s} \) and large values of \( t_D/C_D \).
Figure 3.19 - Type Curve Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir.

$p_{wCD}(t_D)$ Computing Using $p_{sp}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$. 

Model: $p_{sp}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$

- Numerical Inversion Solution
- Approximate Solution
Figure 3.20 - Type Curve Plot of $p_{wCD}'(t_D)$ for a Homogeneous Reservoir. $p_{wCD}'(t_D)$ Computing Using Numerical Differentiation of $p_{wCD}(t_D)$ (3-Point Lagrange Formula), $p_{wCD}(t_D)$ Computed Using $P_{sD}(t_D) = a_0 + a_1t_D + a_2t_D^2$. ($P_{sD}(t_D)$ Derivative Computed Using the Stehfest Inversion Algorithm).
Figure 3.21 - Type Curve Plot of $p_{wCD}'(t_D)$ for a Homogeneous Reservoir. $p_{wCD}'(t_D)$ Computing Using Numerical Differentiation of $p_{wCD}(t_D)$ (3-Point Lagrange Formula), $p_{wCD}(t_D)$ Computed Using $p_{stD}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$ ($p_{stD}(t_D)$ Derivative Computed Using the Line Source Solution).
Figure 3.22 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/C_D$ Format).

$p_{wCD}(t_D)$ Computing Using $p_{sD}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$.
Figure 3.23 - Residual Plot of $p_{wCD}(t_D)$ for a Homogeneous Reservoir ($t_D/(C_Dp_{sd}(t_D))$ Format).

$p_{wCD}(t_D)$ Computing Using $p_{sd}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$. 

Model: $p_{sd}(t_D) = a_0 + a_1 t_D + a_2 t_D^2$
3.3.6 $p_{wCD}(t_D)$ Based on the Fenske-Horne Model

The final discussion in this chapter focuses on the Fenske-Horne$^{15}$ model, where the derivation for this model is well documented in the ref. 15. Fig. 3.24 shows the behavior of $p_{wCD}(t_D)$ as predicted by the Fenske-Horne solution. We clearly see deviation of $p_{wCD}(t_D)$ function from the numerical inversion solutions for small values of both $C_D e^{2s}$ and $t_D/C_D$, as well as for large values of $C_D e^{2s}$ for $10 \leq t_D/C_D \leq 1 \times 10^3$. Fig. 3.25 illustrates the behavior of $p_{wCD}'(t_D)$ function where $p_{wCD}'(t_D)$ is computed from $p_{wCD}(t_D)$ using a 3-point Lagrange formula. This model provides acceptable results only for $3 \leq C_D e^{2s} \leq 10^3$. This comparison demonstrates the relative advantage of our approximate solutions.
Figure 3.24 - Type Curve Plot of $P_{wCD}(D)$ for a Homogeneous Reservoir.

$P_{wCD}(D)$ Computed Using the Fenske-Horne Model.
Figure 3.25 - Type Curve Plot of $P_{wCD}(t_D)$ for a Homogeneous Reservoir. $P_{wCD}(t_D)$ Computed Using Numerical Differentiation of $P_{wCD}(t_D)$ (3-Point Lagrange Formula), $P_{wCD}(t_D)$ Computed Using the Fenske-Horne Model.
CHAPTER IV

VALIDATION OF THE NEW APPROXIMATE SOLUTIONS

4.1 INTRODUCTION
This chapter summarizes the application of our new approximate solutions that were discussed in Chapter III. The use of computers for well test analysis plays an important role in the application of our new approximate models and as part of this research, we modified an existing computer program to include our new solutions. The PERANA (Reservoir Performance Analysis) program (written in Fortran), is used to "match" our approximate and numerical inversion solutions to the field pressure test data using non-linear regression methods.

We used the PERANA program to analyze field pressure data, where this analysis included the estimation of parameters such as permeability, skin factor, dimensionless wellbore storage coefficient, etc. The following approximate $P_{wCD}(t_D)$ models were used: the constant, the linear (non-zero intercept), and the quadratic $P_{sD}(t_D)$ models—as well as the numerical inversion solution. Later in this chapter we provide definitions of the statistical variables and we discuss and summarize the results of the analysis and interpretation for each case.

4.2 DATA REQUIRED FOR ANALYSIS
We require the following data in order to perform history-matching (using our approximate solutions as well as the numerical inversion solutions): first, and most vital requirement is the record of bottomhole flowing (or shut-in) pressures. The second data requirement is that of having representative fluid and reservoir properties. These rock and fluid properties are usually obtained from either a direct laboratory report of the fluid properties or from the computation of all pertinent fluid parameters by means of
published correlations. We also require knowledge of the total producing time prior to the shut-in a well (in the case of a pressure buildup test) in order to use Agarwal effective time (a superposition technique). If the producing time is unknown, then shut-in time must be used instead of Agarwal effective time.

4.3 WELL TEST ANALYSIS USING THE PERANA PROGRAM
A general procedure for well test analysis is provided below and includes four separate components:

1. Compilation of reservoir properties,
2. Identification of the reservoir model,
3. Conventional well test analysis (graphical techniques), and

The development of components 1-3 is required before the PERANA program can be used to match pressure test data.

4.3.1 Reservoir Properties
This section describes the basic data that are required for the PERANA program. These data consist of:

1. Table of time-pressure data.
2. The reservoir rock and fluid properties. The required properties are:
   a. Reference pressure (initial pressure for a drawdown test, and initial shut-in pressure for a buildup test).
   b. Flowrate during the test (pressure drawdown) or at shut-in (pressure buildup).
   c. Fluid properties—formation volume factor, viscosity, and total compressibility.
   d. Formation properties—reservoir thickness, wellbore radius, and porosity.

4.3.2 Reservoir System
This component requires knowledge of the reservoir system—specifically such information as whether or not the reservoir system is producing at infinite-acting or
pseudosteady-state flow conditions, whether or not the well is fractured, whether, the reservoir is homogeneous or not, etc. For the purposes of this work, we use only the solutions for a well in an infinite-acting reservoir.

4.3.3 History-Matching Process

This section describes the data preparation and initialization required for the PERANA program.

1. Select the relevant solution (i.e., the constant, the linear (non-zero intercept), the quadratic $p_{1D}(t_D)$ solutions, or the numerical inversion solution).

2. Select the Stehfest parameter as follows
   a. Use 12 for an unfractured well
   b. Use 8 for a fractured well

3. Select the method for the pressure integral function (i.e., trapezoidal or power rule).

4. Select the Bourdet derivative parameter (start with $L=0.15$).

5. Select the appropriate pressure function for history-matching (i.e., pressure drop, pressure drop derivative, pressure drop integral, and/or pressure drop integral derivative function—or a combination of any of these).

6. Select the initial parameters for history-matching. This is the most important step because it requires knowledge of the reservoir system (i.e., a homogeneous or dual porosity reservoir) where the system dictates specific reservoir parameters. Example cases include:
   a. An unfractured well in a homogeneous reservoir. Solution requires: permeability, skin factor, and the wellbore storage constant.
   b. A fractured well in a homogeneous reservoir. Solution requires: permeability, the wellbore storage constant, fracture conductivity, and the fracture half-length.
   c. An unfractured well in a dual porosity reservoir. Solution requires: permeability, skin factor, the wellbore storage constant, fracture storativity ratio, and the interporosity flow coefficient.
4.4 DISCUSSION OF STATISTICAL PROPERTIES

In this section we discuss the definition and use of the statistical variables which are used in the non-linear least-squares regression analysis (*i.e.*, the history-matching sequence).

A. Sum of Squared Errors, *SSR*

B. Variance

C. Standard Deviation

D. Root Mean Square Error, *RMS*

E. Average (Relative) Error

Though all of these statistical parameters are defined in ref. 37, we summarize the definitions of these variables to enhance comprehension in later discussions.

4.4.1 Sum of Squared Errors

The sum of squared errors (*SSR*) shows how the predicted trend deviates from the original data. *SSR* is defined as

\[
SSR = \sum_{i=1}^{n} (y_{obs,i} - y_{cal,i})^2
\]

(4-1)

where

\( y_{obs} \) is an "observed" value

\( y_{cal} \) is a calculated or predicted value

\( n \) is the total number of data points

4.4.2 Variance

Variance indicates the variability of the data from the mean (or average) value. Data distributions with a large variance indicate that many of the data points lie far from the average value of that data. On the other hand, small values of variance indicate that most of the data points lie fairly close to the average value. Variance is defined as:

\[
Variance = \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{y_{obs,i} - y_{cal,i}}{y_{cal,i}} \right)^2
\]

(4-2)

where
\( y_{obs} \) is an “observed” value

\( y_{cal} \) is a calculated or predicted value

\( n \) is the total number of data points

### 4.4.3 Standard Deviation

The standard deviation, \( \sigma \), also indicates the variability of the data from the mean or average value. The standard deviation has the advantage of having with the same units as the original data.

\[
\sigma = \sqrt{\text{Variance}} \tag{4-3}
\]

where

\( \sigma \) is the standard deviation of a set of variables

### 4.4.4 Root Mean Square Error

\( RMS \) is defined as:

\[
RMS = \left[ \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{y_{obs,i} - y_{cal,i}}{y_{cal,i}} \right)^2 \right]^{1/2} \tag{4-4}
\]

where

\( y_{obs} \) is an “observed” value

\( y_{cal} \) is a calculated or predicted value

\( n \) is the total number of data points

### 4.4.5 Average Error

\( AVE \) is defined as:

\[
AVE = \frac{100}{n} \sum_{i=1}^{n} \left| \frac{y_{obs,i} - y_{cal,i}}{y_{cal,i}} \right| \tag{4-5}
\]

where

\( y_{obs} \) is an “observed” value
\( y_{\text{cal}} \) is a calculated or predicted value

\( n \) is the total number of data points

4.5 FIELD EXAMPLES

In this section, we analyze several field cases using the PERANA program and we closely follow the procedures shown in Section 4.3.3. These field data were taken from both the petroleum literature and from various corporate sources—and include data for unfractured and fractured wells, and for homogeneous and dual porosity reservoirs. Our approach is as follows:

1. We first summarize the reservoir and fluid property data required to perform an analysis of pressure test data using the PERANA program.
2. Next we plot the pressure drop and pressure drop derivative functions against the appropriate time function on a log-log scale to identify the reservoir model. Using pressure drop and/or pressure drop derivative functions—we use the PERANA program to obtain an optional set of solution parameters.
3. We also provide tables that summarize the results from history-matching. These tables contain values the matching parameters and the statistical results.
4. Finally, we provide discussions of the results for each example. These discussions include observations and commentary on the data matches—as well as discussions as to the validity of a particular model.

4.5.1 Example 1 (Unfractured Well in a Homogeneous Reservoir)

These buildup data\(^ {29} \) are for an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects. These pressure test data have no discussion of the historical or geological development.
Reservoir, Fluid Property, and Production Data:

*Reservoir Properties:*
- Wellbore radius, $r_w$ = 0.29 ft
- Net pay thickness, $h$ = 107 ft
- Porosity, $\phi$ (fraction) = 0.25

*Fluid Properties:*
- Oil formation volume factor, $B$ = 1.06 RB/STB
- Oil viscosity, $\mu$ = 2.5 cp
- Total compressibility, $c_i$ = $4.20 \times 10^{-6}$ psia$^{-1}$

*Production Parameters:*
- Production rate, $q$ = 174.0 STB/D
- Production time, $t_p$ = 15.33 hr
- Pressure at shut-in, $P_{wf, \Delta t=0}$ = 3086.33 psia

Discussion of Analysis Results: (Example 1)

The first case we examine is for an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects—as illustrated in Figs. 4.1-4.4. Figs. 4.1-4.4 show the pressure drop functions plotted against the shut-in effective time, $\Delta t_e$, on a log-log scale. We clearly see the transition from the wellbore storage regime to the transient flow regime. The wellbore storage regime ends at approximately $\Delta t_e=0.2$ hours and the effective time for the start of transient flow is at about 9 hours. The optimization parameters are: permeability, $k$, skin factor, $s$, and the dimensionless wellbore storage coefficient, $C_D$. We optimize first with the numerical inversion solution—followed by the quadratic $p_{SD}(t_D)$ approximate solution, the linear (non-zero intercept) $p_{SD}(t_D)$ approximate solution, and finally with the constant $p_{SD}(t_D)$ approximate solution.
• Fig. 4.1 provides the history-matching results from the optimization which used the numerical inversion solution—we note excellent agreement for both the pressure drop and pressure drop derivative functions. The quality of this match implied that these results are both accurate and reproducible.

• Fig. 4.2 shows the history-matching results using the quadratic $p_{sD}(t_D)$ approximate solution for $p_{wCD}(t_D)$ —we again note an excellent match that is comparable to the Stehfest numerical inversion solution.

• Fig. 4.3 gives the history-matching results obtained using the linear (non-zero intercept) $p_{sD}(t_D)$ approximate solution—and yet again we obtain an excellent match of the data and the computed functions. We also note that this model gives results comparable to the results obtained from the Stehfest numerical inversion solution.

• Fig. 4.4 shows the history-matching results obtained using the constant $p_{sD}(t_D)$ approximate solution. In this case we note a good match although the abrupt turn of the pressure derivative function is not evident in the data—this feature is an artifact of this particular model. However, given the quality of the match, the results should be comparable to other cases.

• Table 4.1 summarizes the results of history-matching from these solutions, including the values of the optimized parameters and the statistical variables.
In general, this example shows excellent agreement in the history-matching results, except for the constant $p_{SD}(t_D)$ approximate solution case, which shifts abruptly from transition to undistorted radial flow. In general, these results should be considered accurate and reproducible. All of the approximate solutions show lower permeability and skin factor values compared to the numerical inversion solution (but this difference is only slight and should be considered insignificant for practical purposes). The dimensionless wellbore storage coefficient varies slightly in each solution (again such differences should be considered negligible).

The statistical results provide us with another tool for comparing the consistency and accuracy of these solutions. We focus on the average error of the pressure drop and pressure drop derivative functions and note that the average error in pressure drop is less than 3 percent, and the average error in the pressure drop derivative is on the order of 5 percent. Except for the constant $p_{SD}(t_D)$ approximate model, where the average error in the pressure drop derivative is slightly less than 8 percent. Recall that this is the most simple of our approximate models—so such behavior should be the rule, rather than the exception.
Figure 4.1 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 1, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Figure 4.2 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 1, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Quadratic $p_{sd}(t_D)$ Model.
Figure 4.3 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 1, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Linear $P_{dD}(t_D)$ Model.
Figure 4.4 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 1, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Constant $p_{sD}(t_D)$ Model.
### TABLE 4.1

Optimization Results for Example 1 (Pressure Buildup Test Analysis for an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wcD}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wcD}(t_D)$ Using Quadratic $P_{sd}(t_D)$</th>
<th>$P_{wcD}(t_D)$ Using Linear $P_{sd}(t_D)$</th>
<th>$P_{wcD}(t_D)$ Using Constant $P_{sd}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>12.494</td>
<td>10.92</td>
<td>10.89</td>
<td>10.85</td>
</tr>
<tr>
<td>Skin factor, $s$</td>
<td>9.708</td>
<td>7.725</td>
<td>7.712</td>
<td>7.606</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., $C_D$</td>
<td>$8.517 \times 10^2$</td>
<td>$8.492 \times 10^2$</td>
<td>$8.583 \times 10^2$</td>
<td>$8.838 \times 10^2$</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi²</td>
<td>$1.964 \times 10^4$</td>
<td>$2.043 \times 10^4$</td>
<td>$2.011 \times 10^4$</td>
<td>$2.749 \times 10^4$</td>
</tr>
<tr>
<td>Variance, psi²</td>
<td>$1.964 \times 10^2$</td>
<td>$2.043 \times 10^2$</td>
<td>$2.011 \times 10^2$</td>
<td>$2.749 \times 10^2$</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>14.01</td>
<td>14.29</td>
<td>14.19</td>
<td>16.58</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>13.81</td>
<td>14.08</td>
<td>13.97</td>
<td>16.34</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>2.491</td>
<td>2.687</td>
<td>2.319</td>
<td>1.677</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>4.094</td>
<td>5.136</td>
<td>4.443</td>
<td>7.773</td>
</tr>
</tbody>
</table>
4.5.2 Example 2 (Unfractured Well in a Homogeneous Reservoir)

The pressure buildup test data for this case are taken from the literature\(^\text{30}\) for the case of an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects. As with Example 1, no historical or geological development was given for these data.

**Reservoir, Fluid Property, and Production Data:**

*Reservoir Properties:*
- Wellbore radius, \(r_w\) = 0.401 ft
- Net pay thickness, \(h\) = 73.0 ft
- Porosity, \(\phi\) (fraction) = 0.20

*Fluid Properties:*
- Oil formation volume factor, \(B\) = 1.3 RB/STB
- Oil viscosity, \(\mu\) = 0.5 cp
- Total compressibility, \(c_i\) = \(1 \times 10^{-5}\) psia\(^{-1}\)

*Production Parameters:*
- Production rate, \(q\) = 1500.0 STB/D
- Production time, \(t_p\) = 18.04 hr
- Pressure at shut-in, \(P_{wf,\Delta t=0}\) = 2235.28 psia

**Discussion of Analysis Results:** (Example 2)

The second case we examine is an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects. Our results are shown in Figs. 4.5-4.8. Figs. 4.5-4.8 show the pressure drop functions plotted against shut-in effective time, \(\Delta t_e\), on a log-log scale. From these data we clearly see the transition from wellbore storage regime to transient flow regime, and we note that the shape of this plot clearly indicates a low (possibly negative) skin factor value for this case.
The results obtained from each $P_{wCD}(t_D)$ solution confirm our observation (see Table 4.2), where $-4 \leq s \leq -3.52$. The wellbore storage domination regime ends at approximately $\Delta t_e=3\times10^2$ hours and the time for the start of transient flow occurs about $\Delta t_e=5$ hours. The optimization parameters for this case are: permeability, $k$, skin factor, $s$, and the dimensionless wellbore storage coefficient, $C_D$. Again, we optimize first using the numerical inversion solution—followed by the quadratic $P_{sd}(t_D)$ approximate solution, the linear (non-zero intercept) $P_{sd}(t_D)$ approximate solution, and the constant $P_{sd}(t_D)$ approximate solution.

- Fig. 4.5 shows the history-matching results obtained from the optimization which used the numerical inversion solution—we note excellent agreement for both the pressure drop and pressure drop derivative functions, for the entire data range.

- Fig. 4.6 provides the history-matching results obtained using the quadratic $P_{sd}(t_D)$ approximate solution—we note an excellent match that is clearly comparable to the Stehfest numerical inversion solution.

- Fig. 4.7 gives the history-matching results obtained using the linear (non-zero intercept) $P_{sd}(t_D)$ approximate solution—we again obtain an excellent match that is also comparable to results from the Stehfest numerical inversion solution and the quadratic $P_{sd}(t_D)$ approximate solution.
• Fig. 4.8 shows the history-matching results that we obtained using the constant $p_{SD}(t_D)$ approximate solution. This model produces an anomalous solution that should be considered a failure, even though the optimized results are comparable to the other cases. This solution produces an artifact where the wellbore storage transition region abruptly changes to undistorted radial flow. We noted this in our previous example as well, but we will continue to use this model as an attempt to verify its usefulness in pressure test analysis.

• Table 4.2 summarizes the results of our history-matching efforts using these solutions, and we include both estimated parameters as well as a summary of the statistical variables for each case.

This example gave average errors for the pressure drop and pressure drop derivative function of 2 percent or less for pressure drop, and less than 5 percent for the pressure drop derivative—except for the constant $p_{SD}(t_D)$ approximate solution. Again, all of the approximate solutions show slightly lower permeability and skin factor values, and slightly higher values of the dimensionless wellbore storage coefficient as compared to the results obtained using the numerical inversion solution. Again, as with Example 1, we believe the slightly differences in the estimated parameters are insignificant. We also believe that, with the notable exception of the constant $p_{SD}(t_D)$ approximation case, all of the solutions provide representative and reproducible results.
Figure 4.5 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 2, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_p)$ from the Stehfest Inversion Algorithm.
Figure 4.6 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 2, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Quadratic $p_{sD}(t_D)$ Model.
Figure 4.7 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 2, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_wCD(t_D)$ from the Linear $p_{sd}(t_D)$ Model.
Figure 4.8 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 2, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Constant $p_{ID}(t_D)$ Model.
TABLE 4.2
Optimization Results for Example 2 (Pressure Buildup Test Analysis for an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$p_{weD}(t_D)$ Using Stehfest Solution</th>
<th>$p_{weD}(t_D)$ Using Quadratic $p_{sd}(t_D)$</th>
<th>$p_{weD}(t_D)$ Using Linear $p_{sd}(t_D)$</th>
<th>$p_{weD}(t_D)$ Using Constant $p_{sd}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>22.28</td>
<td>21.77</td>
<td>21.19</td>
<td>18.55</td>
</tr>
<tr>
<td>Skin factor, $s$</td>
<td>-3.518</td>
<td>-3.574</td>
<td>-3.645</td>
<td>-4.001</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., $C_D$</td>
<td>$3.011 \times 10^3$</td>
<td>$3.143 \times 10^3$</td>
<td>$3.194 \times 10^3$</td>
<td>$3.297 \times 10^3$</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi$^2$</td>
<td>$5.076 \times 10^2$</td>
<td>$6.171 \times 10^2$</td>
<td>$1.065 \times 10^3$</td>
<td>$3.154 \times 10^3$</td>
</tr>
<tr>
<td>Variance, psi$^2$</td>
<td>5.127</td>
<td>6.234</td>
<td>10.758</td>
<td>31.86</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>2.264</td>
<td>2.497</td>
<td>3.279</td>
<td>5.645</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>2.231</td>
<td>2.459</td>
<td>3.231</td>
<td>5.561</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>1.809</td>
<td>1.912</td>
<td>2.011</td>
<td>2.473</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>3.509</td>
<td>3.762</td>
<td>4.781</td>
<td>7.439</td>
</tr>
</tbody>
</table>
4.5.3 Example 3 (Unfractured Well in a Homogeneous Reservoir)

The pressure buildup test data for this case are taken from the literature\textsuperscript{31} for the case of an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects. As with Examples 2 and 3, no historical or geological development were provided for these buildup data.

Reservoir, Fluid Property, and Production Data:

\textit{Reservoir Properties:}
- Wellbore radius, \( r_w \) = 0.198 ft
- Net pay thickness, \( h \) = 69.0 ft
- Porosity, \( \phi \) (fraction) = 0.039

\textit{Fluid Properties:}
- Oil formation volume factor, \( B \) = 1.136 RB/STB
- Oil viscosity, \( \mu \) = 0.8 cp
- Total compressibility, \( c_t \) = \( 17 \times 10^{-6} \) psia\(^{-1} \)

\textit{Production Parameters:}
- Production rate, \( q \) = 250.0 STB/D
- Production time, \( t_p \) = 13,630 hr
- Pressure at shut-in, \( P_{w,\Delta t=0} \) = 3534.0 psia

Discussion of Analysis Results: (Example 3)

The third example case we consider is for an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects, where these data are shown in Figs. 4.9-4.12. Our typical approach is to use Agarwal effective time as the time-axis plotting function, but in this particular case \( t_p \gg \Delta t_{max} \), so the use of effective time is not necessary, and we can simply use shut-in time. Figs. 4.9-4.12 show the pressure drop functions plotted against the shut-in time, \( \Delta t \), on a log-log scale.

We clearly see three flow regimes on these plots: wellbore storage distortion, transition (the backside of the "hump" in the pressure drop derivative function), and the transition
to boundary-dominated flow. This does not appear to be a clear development of the undistorted radial flow regime (i.e., a horizontal pressure drop derivative), but the region $7 \leq \Delta t \leq 20$ hours could be considered undistorted radial flow.

As we currently only consider transient flow, we matched only the portion of the data which does not exhibit boundary effects because these boundary effects would bias the regression procedure if included. The wellbore storage domination regime (i.e., the "unit slope" line) appears to end approximately at $\Delta t = 0.2$ hours and all wellbore storage effects appear to end by $\Delta t = 7$ hours, at which time we assume that transient flow begins. We also assume that boundary effects begin at approximately $\Delta t = 20$ hours. The optimization parameters used for this case are: permeability, $k$, skin factor, $s$, and the dimensionless wellbore storage coefficient, $C_D$. As with the previous examples we will use the Stehfest numerical inversion solution, as well as the quadratic $p_{sd} (t_D)$ approximate solution, the linear (non-zero intercept) $p_{sd} (t_D)$ approximate solution, and the constant $p_{sd} (t_D)$ approximate solution.

- Fig. 4.9 presents the history-matching results obtained using the numerical inversion solution. We note excellent agreement for both the pressure drop and the pressure drop derivative functions.
- Fig. 4.10 provides the history-matching results that we obtained using the quadratic $p_{sd} (t_D)$ approximate solution—we obtain an excellent match that is comparable to the Stehfest numerical inversion solution.
- Fig. 4.11 shows the history-matching results obtained using the linear (non-zero intercept) $p_{sd} (t_D)$ approximate solution—the match is comparable to results from the Stehfest numerical inversion solution and the quadratic $p_{sd} (t_D)$ approximate solution.
- Fig. 4.12 shows our history-matching results obtained using the constant $p_{sd} (t_D)$ approximate solution—we note a good match of the data and the computed
functions although the abrupt drop and turn exhibited by the pressure drop derivative function (from the wellbore storage transition period to the undistorted radial flow regime) again illustrates the approximate nature of this particular model.

- Table 4.3 summarizes the results of our history-matching effort using these solutions and includes the parameter estimates as well as the statistical results.

This example gave average errors for the pressure drop matches of less than 2 percent for all cases, including the constant \( p_{SD}(t_D) \) approximate solution. However, this example also showed relatively high errors in the pressure drop derivative matches (range of 6.2 to 22.6 percent), but this can be mostly attributed to the sparse nature of the data set which causes significant variations in the pressure drop derivative data function. The important conclusion to be noted is that consistent estimates of the individual parameters were obtained, despite some significant differences in the statistics for a given case.

In summary, the quadratic \( p_{SD}(t_D) \) approximate solution appears to be the most accurate of the approximate solutions for use in analyzing well test data for an unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects. In general, the linear (non-zero intercept) \( p_{SD}(t_D) \) approximation is also very accurate and should be considered valid for universal application. On the other hand, the constant \( p_{SD}(t_D) \) approximate solution has been shown to give inconsistent matches (the computed results do not match the original data trend), while still yielding reasonable parameter estimates. This appears to be an artifact of this particular solution, and we do not recommend the constant \( p_{SD}(t_D) \) approximate solution for general applications.
Figure 4.9 - Log-log Plot (In Terms of Shut-In Time, \( \Delta t \)) for Example 3, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using \( p_{wCD}(t_D) \) from the Stehfest Inversion Algorithm.
Figure 4.10 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 3, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Quadratic $p_{sD}(t_D)$ Model.
Figure 4.11 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 3, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Linear $P_{sD}(t_D)$ Model.
Figure 4.12 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 3, Case of an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_{D})$ from the Constant $p_{D}(t_{D})$ Model.
### TABLE 4.3

Optimization Results for Example 3 (Pressure Buildup Test Analysis for an Unfractured Well in an Infinite-Acting, Homogeneous Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>( P_{wCD}(t_D) ) Using Stehfest Solution</th>
<th>( P_{wCD}(t_D) ) Using Quadratic ( p_{sD}(t_D) )</th>
<th>( P_{wCD}(t_D) ) Using Linear ( p_{sD}(t_D) )</th>
<th>( P_{wCD}(t_D) ) Using Constant ( p_{sD}(t_D) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, ( k ), md</td>
<td>7.657</td>
<td>7.829</td>
<td>8.131</td>
<td>8.258</td>
</tr>
<tr>
<td>Skin factor, ( s )</td>
<td>6.609</td>
<td>6.145</td>
<td>6.677</td>
<td>6.759</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., ( C_D )</td>
<td>( 5.815 \times 10^3 )</td>
<td>( 5.854 \times 10^3 )</td>
<td>( 5.994 \times 10^3 )</td>
<td>( 6.521 \times 10^3 )</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi(^2)</td>
<td>( 9.254 \times 10^3 )</td>
<td>( 8.747 \times 10^3 )</td>
<td>( 8.455 \times 10^3 )</td>
<td>( 1.458 \times 10^4 )</td>
</tr>
<tr>
<td>Variance, psi(^2)</td>
<td>( 6.609 \times 10^2 )</td>
<td>( 6.053 \times 10^2 )</td>
<td>( 6.039 \times 10^2 )</td>
<td>( 8.582 \times 10^2 )</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>25.71</td>
<td>24.61</td>
<td>24.57</td>
<td>29.29</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>23.33</td>
<td>22.33</td>
<td>22.30</td>
<td>27.01</td>
</tr>
<tr>
<td>( \Delta p ) Average Error, percent</td>
<td>1.799</td>
<td>1.619</td>
<td>1.40</td>
<td>1.726</td>
</tr>
<tr>
<td>( \Delta p' ) Average Error, percent</td>
<td>9.747</td>
<td>6.157</td>
<td>6.63</td>
<td>22.57</td>
</tr>
</tbody>
</table>
4.5.4 Example 4 (Fractured Well in a Homogeneous Reservoir)
This example presents pressure falloff test data taken from Well No. 5408 in the Southwest Garza Unit (SWG) (ref. 32). These buildup data are analyzed using the model for a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects.

SWG Well 5408 was drilled and completed as a water injection well in the San Andres Formation of West Texas in December 1984. At completion the well was stimulated with 16,000 gal of foamed acid. The initial injection rate for this well was 100 BWPD at 425 psig wellhead pressure. In July 1990 the well was given a hydraulic fracture stimulation treatment consisting of 42,000 lbs of sand. After the treatment, the injection rate was 500 BWPD at 510 psig wellhead pressure. As of September 1992, the injection rate was 350 BWPD with a cumulative water injection of 360,000 bbls. This particular pressure falloff test was run from September 14-16, 1992, and the pressure data were recorded at the surface using an electronic gauge.

Reservoir, Fluid Property, and Injection Data:

*Reservoir Properties:*
- Wellbore radius, \( r_w \) = 0.32 ft
- Net pay thickness, \( h \) = 196 ft
- Porosity, \( \phi \) (fraction) = 0.18
- Nominal well spacing = 10 acres

*Fluid Properties:*
- Water formation volume factor, \( B_w \) = 1.002 RB/STB
- Water viscosity, \( \mu \) = 0.9344 cp
- Total compressibility, \( c_t \) = \( 6.53 \times 10^{-6} \) psia\(^{-1}\)

*Injection Parameters:*
- Injection rate, \( q \) = 350 STB/D
- Pressure at shut-in, \( P_{wf, t=0} \) = 2518.1 psia
Discussion of Analysis Results: (Example 4)

Our fourth example case is for a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects, as shown in Figs. 4.13-4.16. As this well had almost eight (8) years of injection prior to performing this pressure falloff test, we chose to use the shut-in time, $\Delta t$, as the time-axis function. Figs. 4.13-4.16 show the pressure drop functions plotted against the shut-in time, $\Delta t$, on a log-log scale.

The most obvious of feature shown on these figures is the declining pressure drop derivative function, where this decline begins at about $\Delta t=7$ hours. We believe that this is most likely an interference feature due to pressure communication with a nearby injection well, and as such, we optimized only the data in the transient flow regime—ignoring the apparent interference effects. The parameters we chose to optimize are: permeability, $k$, fracture half-length, $x_f$, dimensionless fracture conductivity, $C_{fD}$, and the dimensionless fracture wellbore storage coefficient, $C_{Df}$.

As with the previous examples, we begin by optimizing with the numerical inversion solution, followed by the quadratic $p_{sD}(t_D)$ approximate solution, linear (non-zero intercept) $p_{sD}(t_D)$ approximate solution, and finally with the constant $p_{sD}(t_D)$ approximate solution.

- Fig. 4.13 shows our history-matching results obtained using the numerical inversion solution—we note excellent agreement for both the pressure drop and pressure drop derivative functions during the transient flow period (recall that we omitted the boundary affected data from our regression analysis).
- Fig. 4.14 shows the history-matching results obtained using the quadratic $p_{sD}(t_D)$ approximate solution—we obtained an excellent match that is comparable to the Stehfest numerical inversion solution.
• Fig. 4.15 presents our history-matching results which used the linear (non-zero intercept) \( p_{SD}(t_D) \) approximate solution—this model also gave an excellent match that gave results that are comparable to results obtained from the Stehfest numerical inversion solution and the quadratic \( p_{SD}(t_D) \) approximate solution.

• Fig. 4.16 gives the history-matching results obtained using the constant \( p_{SD}(t_D) \) approximate solution. This model again produces an anomalous solution (note that the computed trends do not have the same shape as the data trends) and as such, this case should be considered a failure. We suggest that this case is a “failure” even though the results are comparable to the other cases because the anomalous behavior may be non-unique and may yield different results given a different starting point.

• Table 4.4 summarizes our history-matching results using these solutions and in this table we include our parameter estimates as well as the statistical results.

This example has average errors for the pressure drop matches of less than 1.5 percent for all of the cases, except for the constant \( p_{SD}(t_D) \) approximate solution (where the error was 5.6 percent). Similarly, the average errors for the pressure drop derivative functions are less than 2.5 percent, except for the \( p_{SD}(t_D) \) approximation case where the average error was approximately 10 percent. Recall that we matched only the portion of the data which did not exhibit boundary effects.

Obviously our approach should be extended to include the effects of reservoir boundaries—currently we only consider transient undistorted flow conditions. This would be a relatively easy enhancement, but not a goal of this thesis. Our goal in this thesis is to validate and apply our approximate solutions for modeling pressure behavior which includes wellbore storage and skin effects.
Figure 4.13 - Log-log Plot (In Terms of Shut-In Time, Δt) for Example 4, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Figure 4.14 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 4, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Quadratic $p_{sD}(t_D)$ Model.
Figure 4.15 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 4, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Linear $p_{ID}(t_D)$ Model.
Figure 4.16 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 4, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Constant $P_{st}(t_D)$ Model.
<table>
<thead>
<tr>
<th>Variable</th>
<th>$p_{\text{wet}}(t_D)$ Using Stehfest Solution</th>
<th>$p_{\text{wet}}(t_D)$ Using Quadratic $P_{SD}(t_D)$</th>
<th>$p_{\text{wet}}(t_D)$ Using Linear $P_{SD}(t_D)$</th>
<th>$p_{\text{wet}}(t_D)$ Using Constant $P_{SD}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>1.01</td>
<td>1.009</td>
<td>1.01</td>
<td>1.01</td>
</tr>
<tr>
<td>Fracture Half-length, $x_f$, ft</td>
<td>29.29</td>
<td>29.29</td>
<td>29.30</td>
<td>29.28</td>
</tr>
<tr>
<td>Dim.-less Frac. Conductivity, $C_{fD}$</td>
<td>$1.0 \times 10^3$</td>
<td>$1.0 \times 10^3$</td>
<td>$1.0 \times 10^3$</td>
<td>$9.986 \times 10^2$</td>
</tr>
<tr>
<td>Dim.-less Frac. Wellbore Storage Coef., $C_{Df}$</td>
<td>$9.408 \times 10^{-2}$</td>
<td>$9.708 \times 10^{-2}$</td>
<td>$1.028 \times 10^{-1}$</td>
<td>$1.507 \times 10^{-1}$</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi$^2$</td>
<td>$5.512 \times 10^3$</td>
<td>$3.465 \times 10^3$</td>
<td>$3.396 \times 10^3$</td>
<td>$6.089 \times 10^4$</td>
</tr>
<tr>
<td>Variance, psi$^2$</td>
<td>$3.579 \times 10^1$</td>
<td>$2.250 \times 10^1$</td>
<td>$2.205 \times 10^1$</td>
<td>$3.954 \times 10^2$</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>5.983</td>
<td>4.743</td>
<td>4.696</td>
<td>19.88</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>5.906</td>
<td>4.683</td>
<td>4.636</td>
<td>19.63</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>1.505</td>
<td>1.204</td>
<td>1.07</td>
<td>5.627</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>2.498</td>
<td>2.076</td>
<td>2.189</td>
<td>10.08</td>
</tr>
</tbody>
</table>
4.5.5 Example 5 (Fractured Well in a Homogeneous Reservoir)

This example presents pressure falloff test data taken from Well No. 203 in the Southwest Garza Unit (SWG) (ref. 32). These data are analyzed using the model for a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects.

SWG Well 203 was originally drilled as an oil well in the San Andres Formation (in West Texas) in February 1946. The initial production was 160 STB/D of oil with no water production. The well was deepened and given a hydraulic fracture stimulation treatment with 12,000 lbs of sand in the 1950’s. The well was converted to injection in September 1979. In November of 1990, SWG Well 203 was given another fracture treatment, this time consisting of 18,000 lbs of sand. As of September 1992, the injection rate was 350 BWPD with a cumulative water injection of 1.1 million bbls. This particular pressure falloff test was conducted from September 17-21, 1992 and the pressure data were recorded at the surface using an electronic gauge.

Reservoir, Fluid Property, and Injection Data:

Reservoir Properties:
- Wellbore radius, $r_w$ = 0.198 ft
- Net pay thickness, $h$ = 235 ft
- Porosity, $\phi$ (fraction) = 0.18
- Nominal well spacing = 10 acres

Fluid Properties:
- Water formation volume factor, $B$ = 1.002 RB/STB
- Water viscosity, $\mu$ = 0.934 cp
- Total compressibility, $c_t$ = $6.53 \times 10^{-6}$ psia$^{-1}$

Injection Parameters:
- Injection rate, $q$ = 334.0 STB/D
- Pressure at shut-in, $p_{wf, \Delta t=0}$ = 2334.1 psia
Discussion of Analysis Results: (Example 5)

The fifth example case we consider is that of a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects, as illustrated in Figs. 4.17-4.20. As with our previous example, we do not know the total injection time, so we must use the shut-in time as our time-axis plotting function. Figs. 4.17-4.20 show the pressure drop functions plotted against the shut-in time, $\Delta t$, on a log-log scale. This well exhibits some wellbore storage distortion at the start of the test, but no interference effects are seen in this test. The undistorted radial flow period appears to begin at about $\Delta t=70$ hours. The parameters that we chose to optimize are: permeability, $k$, fracture half-length, $x_f$, dimensionless fracture conductivity, $C_{fD}$, and the dimensionless fracture wellbore storage coefficient, $C_{DF}$.

- Fig. 4.17 shows the history-matching results that we obtained using the numerical inversion solution—we note excellent agreement of the data and computed functions. The slight deviations at late times are most likely an attempt by the regression algorithm to “balance” errors, whereas an analyst might emphasize this region more than a statistical algorithm.

- Fig. 4.18 gives the history-matching results obtained using the quadratic $P_{SD}(t_D)$ approximate solution—we again obtain an excellent match and the results are comparable to those obtained from the Stehfest numerical inversion solution, as is also the slight deviation in the computed and data functions at late times.

- Fig. 4.19 presents the history-matching results that we obtained using the linear (non-zero intercept) $P_{SD}(t_D)$ approximate solution—again we obtain an excellent match and our results are comparable to those obtained from the Stehfest numerical inversion solution as well as the quadratic $P_{SD}(t_D)$ approximate solution.
- Fig. 4.20 shows our history-matching results obtained using the constant $p_{sD}(t_D)$ approximate solution—we obtain a reasonable match of the data and the computed functions, although this case does show anomalous behavior at early times. However, our results are comparable to those obtained from the other cases.

- Table 4.5 summarizes the results of our history-matching efforts, including the estimated parameter values and the statistical results.

This example shows good agreement of the history-matching results, including the constant $p_{sD}(t_D)$ approximate solution case. In general, these results should be considered both accurate and reproducible. All of the average errors for the pressure drop functions are less than 5 percent, and all of the average errors for the pressure drop derivative functions less than 7 percent, with relatively tight groupings on all of the errors.
Figure 4.17 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 5, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Figure 4.18 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 5, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wcD}(t_D)$ from the Quadratic $p_{sd}(t_D)$ Model.
Figure 4.19 - Log-log Plot (In Terms of Shut-In Time, \( \Delta t \)) for Example 5, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using \( P_{wCD}(t_D) \) from the Linear \( p_{sD}(t_D) \) Model.
Figure 4.20 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 5, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Constant $p_{ID}(t_D)$ Model.
### TABLE 4.5

Optimization Results for Example 5 (Pressure Falloff Test Analysis for a Fractured Well in an Infinite-Acting, Homogeneous Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wcd}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wcd}(t_D)$ Using Quadratic $p_{sp}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Linear $p_{sp}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Constant $p_{sp}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>0.773</td>
<td>0.773</td>
<td>0.773</td>
<td>0.773</td>
</tr>
<tr>
<td>Fracture Half-length, $x_f$, ft</td>
<td>41.78</td>
<td>41.78</td>
<td>41.78</td>
<td>41.78</td>
</tr>
<tr>
<td>Dim.-less Frac. Conductivity, $C_{fp}$</td>
<td>4.999</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Dim.-less Frac. Wellbore Storage Coef., $C_{Df}$</td>
<td>$8.258 \times 10^{-3}$</td>
<td>$1.0 \times 10^{-2}$</td>
<td>$8.993 \times 10^{-3}$</td>
<td>$1.169 \times 10^{-2}$</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi$^2$</td>
<td>$4.575 \times 10^5$</td>
<td>$3.858 \times 10^5$</td>
<td>$3.814 \times 10^5$</td>
<td>$3.636 \times 10^5$</td>
</tr>
<tr>
<td>Variance, psi$^2$</td>
<td>$9.673 \times 10^2$</td>
<td>$8.157 \times 10^2$</td>
<td>$8.063 \times 10^2$</td>
<td>$7.687 \times 10^2$</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>31.10</td>
<td>28.56</td>
<td>28.39</td>
<td>27.73</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>30.97</td>
<td>28.44</td>
<td>28.28</td>
<td>27.61</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>4.769</td>
<td>4.951</td>
<td>4.742</td>
<td>4.034</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>6.885</td>
<td>5.216</td>
<td>5.389</td>
<td>6.079</td>
</tr>
</tbody>
</table>
4.5.6 Example 6 (Fractured Well in a Homogeneous Reservoir)

This example is for a pressure falloff test from Emmons Well No. 207 (West Texas). The pressure falloff data show the behavior of a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects.

Emmons Well 207 was drilled and completed as an injection well in the San Andres Formation (of West Texas) in September 1974. The completion interval is 4524 to 4627 ft and the well was stimulated at completion with 3200 gals of 15% HCL acid. The initial injection rate was 408 BWPD at 0 psig wellhead pressure. In September 1985 the well was treated with a polymer profile modification process that included 2000 bbls of 2350 ppm biopolymer. In May 1992 the well was acidized with 2500 gal of 15 percent NEFE acid. As of June 1992, the injection rate was 1050 BWPD and the cumulative water injected was approximately 6.2 million bbls. This pressure falloff test was conducted using an electronic gauge at the surface from October 16-19, 1992.

Reservoir, Fluid Property, and Injection Data:

Reservoir Properties:
- Wellbore radius, \( r_w \) = 0.3 ft
- Net pay thickness, \( h \) = 103 ft
- Porosity, \( \phi \) (fraction) = 0.11
- Nominal Well Spacing = 10 acres

Fluid Properties:
- Water formation volume factor, \( B \) = 1.000 RB/STB
- Water viscosity, \( \mu \) = 0.92 cp
- Total compressibility, \( c_t \) = 7.70×10^{-6} psia^{-1}

Injection Parameters:
- Injection rate, \( q \) = 1053.0 STB/D
- Pressure at shut-in, \( p_{wf, \Delta t=0} \) = 3119.41 psia

Discussion of Analysis Results: (Example 6)

Our sixth example is for a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects, as illustrated in Figs. 4.21-4.24. As
with the previous examples in this section we do not know the total injection time, so we again use the shut-in time, $\Delta t$. Figs. 4.21-4.24 show pressure drop functions plotted against the shut-in time, $\Delta t$, on a log-log scale. This well exhibits wellbore storage effects followed by an apparent transient regime, with no obvious interference effects. These data have some minor fluctuations as shown in the pressure drop derivative data trend, but this should not affect our analyses. The parameters that were chosen for optimization are: permeability, $k$, fracture half-length, $x_f$, dimensionless fracture conductivity, $C_{fD}$, and the dimensionless fracture wellbore storage coefficient, $C_{Df}$.

- Fig. 4.21 illustrates the history-matching results that we obtained using the numerical inversion solution—we note an excellent agreement of the data and computed functions for the entire data range.

- Fig. 4.22 presents the history-matching results obtained using the quadratic $p_{sD}(t_D)$ approximate solution—we again obtained an excellent match and we note that these results are comparable to those we obtained from the Stehfest numerical inversion solution.

- Fig. 4.23 shows the history-matching results obtained using the linear (non-zero intercept) $p_{sD}(t_D)$ approximate solution. We again obtain an excellent match and the results are comparable to those obtained from the Stehfest numerical inversion solution and the quadratic $p_{sD}(t_D)$ approximate solution.

- Fig. 4.24 illustrates the history-matching results that we obtained using the constant $p_{sD}(t_D)$ approximate solution. We obtain a reasonable match of the data and the computed functions, but the computed pressure drop derivative function appears to show an artifact at early times, though this feature is not clear.

- Table 4.6 summarizes the history-matching results including the parameter estimates and the statistical results.
This example shows good agreement of the history-matching results for all cases. The average errors for the pressure drop matches are less than 1.5 percent for all cases, except for the constant $p_{SD}(t_D)$ approximate solution, where the average error is 2.2 percent. Similarly, the average errors for the pressure drop derivative functions were less than 5 percent for all cases, where these errors were fairly tightly grouped.

In summary, the PERANA program has been successfully verified for the analysis of pressure test data for a fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects. The quadratic and the linear (non-zero intercept) $p_{SD}(t_D)$ approximate solutions provide an excellent mechanism for the estimation of parameters and give the same order of accuracy as the numerical inversion solution. We recognize the inconsistencies of the constant $p_{SD}(t_D)$ approximate solution and we recommend against using the constant $p_{SD}(t_D)$ approximate solution for analyzing well test data from a fractured well.
Figure 4.21 - Log-log Plot (In Terms of Shut-In Time, Δt) for Example 6, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Data for Example 6:

- $B = 1.00$ RB/STB
- $c_r = 7.7 \times 10^{-6}$ psi$^{-1}$
- $\phi = 0.11$ (fraction)
- $\mu = 0.92$ cp
- $r_w = 0.3$ ft
- $h = 103$ ft
- $q = 1053$ STB/D
- $P_{wf,\Delta t=0} = 3119.4$ psia

Legend: Fractured Well in an Infinite-Acting, Homogeneous Reservoir
(Emmons Well No. 207)

- $k = 14.62$ md
- $x_f = 307.7$ ft
- $C_{pD} = 1.001 \times 10^3$
- $C_{\delta p} = 1.413 \times 10^2$

Optimized Simultaneously on $\Delta p$ and $\Delta p'$

Statistics:
- $SSR = 3.436 \times 10^3$ psi$^2$
- Variance = 11.61 psi$^2$
- Std. Dev. = 3.407 psi
- RMS = 3.385 psi
- $\Delta p$ Avg. Error = 1.33 percent
- $\Delta p'$ Avg. Error = 4.608 percent

Legend: Example 6
- $\Delta p$ Raw Data
- $\Delta p'$ Raw Data ($L=0.15$)
- Optimization Results Using $P_{wCD}(t_d)$ from the Quadratic $p_{sd}(t_d)$ Model

Figure 4.22 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 6, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_d)$ from the Quadratic $p_{sd}(t_d)$ Model.
Data for Example 6:
\( B = 1.00 \text{ RB/STB} \)
\( c_r = 7.7 \times 10^{-6} \text{ psi}^{-1} \)
\( \phi = 0.11 \text{ (fraction)} \)
\( \mu = 0.92 \text{ cp} \)
\( r_w = 0.3 \text{ ft} \)
\( h = 103 \text{ ft} \)
\( q = 1053 \text{ STB/D} \)
\( P_{wD,\Delta t=0} = 3119.4 \text{ psia} \)

Legend: Fractured Well in an Infinite-Acting, Homogeneous Reservoir
(Emmons Well No. 207)

Analysis Results:
\( k = 14.62 \text{ md} \)
\( x_f = 307.7 \text{ ft} \)
\( C_{pD} = 1.001 \times 10^3 \)
\( C_{pD'} = 1.489 \times 10^2 \)

Optimized Simultaneously on \( \Delta p \) and \( \Delta p' \)
Statistics:
\( SSR = 3.414 \times 10^3 \text{ psi}^2 \)
Variance = 11.53 psi^2
Std. Dev. = 3.396 psi
RMS = 3.373 psi
\( \Delta p \) Avg. Error = 1.32 percent
\( \Delta p' \) Avg. Error = 4.596 percent

Legend: Example 6
- \( \Delta p \) Raw Data
- \( \Delta p' \) Raw Data \((L=0.15)\)
- Optimization Results
- Using \( p_{wCD}(t_D) \) from the Linear \( p_{CD}(t_D) \) Model

Figure 4.23 - Log-log Plot (In Terms of Shut-In Time, \( \Delta t \)) for Example 6, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using \( p_{wCD}(t_D) \) from the Linear \( p_{CD}(t_D) \) Model.
Figure 4.24 - Log-log Plot (In Terms of Shut-In time, $\Delta t$) for Example 6, Case of a Fractured Well in an Infinite-Acting, Homogeneous Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Constant $p_sD(t_D)$ Model.
TABLE 4.6

Optimization Results for Example 6 (Pressure Falloff Test Analysis for a Fractured Well in an Infinite-Acting, Homogeneous Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wCD}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wCD}(t_D)$ Using Quadratic $p_{sp}(t_D)$</th>
<th>$P_{wCD}(t_D)$ Using Linear $p_{sp}(t_D)$</th>
<th>$P_{wCD}(t_D)$ Using Constant $p_{sp}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fracture Half-length, $x_h$, ft</td>
<td>307.7</td>
<td>307.7</td>
<td>307.7</td>
<td>307.7</td>
</tr>
<tr>
<td>Dim.-less Frac. Conductivity, $C_{FD}$</td>
<td>$9.999 \times 10^2$</td>
<td>$1.0 \times 10^3$</td>
<td>$1.001 \times 10^3$</td>
<td>$1.001 \times 10^3$</td>
</tr>
<tr>
<td>Dim.-less Frac. Wellbore Storage Coef., $C_{DF}$</td>
<td>$1.364 \times 10^{-2}$</td>
<td>$1.413 \times 10^{-2}$</td>
<td>$1.364 \times 10^{-2}$</td>
<td>$2.098 \times 10^{-2}$</td>
</tr>
<tr>
<td>Sum of Squared Errors, psi$^2$</td>
<td>$3.707 \times 10^3$</td>
<td>$3.436 \times 10^3$</td>
<td>$3.414 \times 10^3$</td>
<td>$5.257 \times 10^3$</td>
</tr>
<tr>
<td>Variance, psi$^2$</td>
<td>12.52</td>
<td>11.61</td>
<td>11.53</td>
<td>17.76</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>3.538</td>
<td>3.407</td>
<td>3.396</td>
<td>4.214</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>3.515</td>
<td>3.385</td>
<td>3.373</td>
<td>4.186</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>1.322</td>
<td>1.33</td>
<td>1.320</td>
<td>2.209</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>4.819</td>
<td>4.608</td>
<td>4.596</td>
<td>4.959</td>
</tr>
</tbody>
</table>
4.5.7 Example 7 (Unfractured Well in a Dual Porosity Reservoir)

This example presents the case of an unfractured well producing from an infinite-acting dual porosity (or fissured) reservoir influenced by wellbore storage and skin effects (ref. 29). As with some of our other literature examples, this case has no historical or geological development.

**Reservoir, Fluid Property, and Production Data:**

*Reservoir Properties:*
- Wellbore radius, \( r_w \) = 0.29 ft
- Net pay thickness, \( h \) = 20 ft
- Porosity, \( \phi \) (fraction) = 0.08

*Fluid Properties:*
- Oil formation volume factor, \( B \) = 1.3 RB/STB
- Oil viscosity, \( \mu \) = 1.3 cp
- Total compressibility, \( c_t \) = \( 5.0 \times 10^{-6} \) psia\(^{-1}\)

*Production Parameters:*
- Production rate, \( q \) = 880 STB/D
- Producing time, \( t_p \) = 6.75 hr
- Pressure at shut-in, \( p_{ref,\Delta t=0} \) = 7248 psia

**Discussion of Analysis Results: (Example 7)**

This seventh example case is for a pressure buildup test in an unfractured well producing from an infinite-acting dual porosity reservoir influenced by wellbore storage and skin effects, as illustrated in Figs. 4.25-4.28. Figs. 4.25-4.28 show the pressure drop functions plotted against the shut-in time, \( \Delta t \), on a log-log scale. The plot indicates transition from wellbore storage domination to transient fracture flow, and then on to transient radial flow for the entire system (*i.e.*, the final horizontal pressure drop derivative function).
During the transition from wellbore storage domination to transient radial flow in the fracture system, the well clearly shows the behavior of a fissured (or dual porosity) reservoir. The dual porosity character is seen on the log-log plot as the "dip" in the pressure drop derivative function. Wellbore storage distortion ends at approximately $\Delta t=8\times10^{-2}$ hours. The starting time for undistorted transient flow in the "total system" is at about $\Delta t=10$ hours. The parameters chosen for optimization are: permeability, $k$, skin factor, $s$, dimensionless wellbore storage coefficient, $C_D$, interporosity flow coefficient, $\lambda$, and the fracture storativity ratio, $\omega$.

- Fig. 4.25 shows the history-matching results that we obtained using the numerical inversion solution—we note excellent agreement of the data and the computed functions for the entire time range.

- Fig. 4.26 shows the history-matching results that we obtained using the quadratic $p_{sD}(t_D)$ approximate solution—we again note an excellent match, and the results for this case are comparable to those obtained from the Stehfest numerical inversion solution.

- Fig. 4.27 presents the history-matching results obtained using the linear (non-zero intercept) $p_{sD}(t_D)$ approximate solution—we again obtain an excellent match and our results are comparable to the results obtained using both the Stehfest numerical inversion solution and the quadratic $p_{sD}(t_D)$ approximate solution.
• Fig. 4.28 gives our the history-matching results obtained using the constant \( p_{sD}(t_D) \) approximate solution—we obtain only a fair match, but our results are comparable to the results obtained from the other techniques. We again note an anomalous trend as the computed solution emerges from wellbore storage distortion—which suggests that this model may not be appropriate for this particular case.

• Table 4.7 summarizes the history-matching results, including the estimated parameters and the statistical results.

All of the approximate solutions compare very well with pressure drop and pressure drop derivative functions predicted by the Stehfest numerical inversion solution. The average errors in the pressure drop function range from 0.6 to 0.97 percent and for the pressure drop derivative function the average errors range from 6.3 to 9.7 percent, with the worst performance being the numerical inversion solution. However, given the anomalous behavior shown in the constant \( p_{sD}(t_D) \) approximate solution—we do not recommend this case for general application in dual porosity reservoirs.
Figure 4.25 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 7, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Figure 4.26 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 7, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Quadratic $p_{sD}(t_D)$ Model.
Figure 4.27 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 7, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Linear $P_{sd}(t_D)$ Model.
Figure 4.28 - Log-log Plot (In Terms of Shut-In Time, $\Delta t$) for Example 7, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_d)$ from the Constant $p_{sd}(t_d)$ Model.
<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wcd}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wcd}(t_D)$ Using Quadratic $p_{sp}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Linear $p_{sp}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Constant $p_{sp}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>149.09</td>
<td>147.15</td>
<td>150.16</td>
<td>148.97</td>
</tr>
<tr>
<td>Skin factor, $s$</td>
<td>-1.028</td>
<td>-1.117</td>
<td>-0.995</td>
<td>-1.057</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., $C_D$</td>
<td>$1.58 \times 10^3$</td>
<td>$1.638 \times 10^3$</td>
<td>$1.666 \times 10^3$</td>
<td>$1.751 \times 10^3$</td>
</tr>
<tr>
<td>Interporosity Flow Coef., $\lambda$</td>
<td>$2.928 \times 10^{-6}$</td>
<td>$3.069 \times 10^{-6}$</td>
<td>$2.305 \times 10^{-6}$</td>
<td>$2.537 \times 10^{-6}$</td>
</tr>
<tr>
<td>Frac. Storativity Ratio, $\omega$</td>
<td>$9.894 \times 10^{-2}$</td>
<td>$8.114 \times 10^{-2}$</td>
<td>$7.619 \times 10^{-2}$</td>
<td>$8.989 \times 10^{-2}$</td>
</tr>
<tr>
<td>Sum of Squared Error, psi$^2$</td>
<td>$1.333 \times 10^4$</td>
<td>$7.039 \times 10^3$</td>
<td>$8.108 \times 10^3$</td>
<td>$1.243 \times 10^4$</td>
</tr>
<tr>
<td>Variance, psi$^2$</td>
<td>$9.007 \times 10^1$</td>
<td>$4.756 \times 10^1$</td>
<td>$5.478 \times 10^1$</td>
<td>$8.4 \times 10^1$</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>9.334</td>
<td>6.783</td>
<td>7.279</td>
<td>9.014</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>0.969</td>
<td>0.596</td>
<td>0.629</td>
<td>0.865</td>
</tr>
<tr>
<td>$\Delta p’$ Average Error, percent</td>
<td>9.716</td>
<td>7.563</td>
<td>6.326</td>
<td>6.673</td>
</tr>
</tbody>
</table>
4.5.8 Example 8 (Unfractured Well in a Dual Porosity Reservoir)

This example is for a pressure buildup test in an unfractured well producing from an infinite-acting dual porosity reservoir influenced by wellbore storage and skin effects. These pressure data are taken from ref. 33, and the authors gave no historical or geological development for this case.

Reservoir, Fluid Property, and Production Data:

Reservoir Properties:
- Wellbore radius, \( r_w \) = 0.29 ft
- Net pay thickness, \( h \) = 7 ft
- Porosity, \( \phi \) (fraction) = 0.05

Fluid Properties:
- Oil formation volume factor, \( B \) = 1.5 RB/STB
- Oil viscosity, \( \mu \) = 0.3 cp
- Total compressibility, \( c_t \) = 2.0\times10^{-5} \text{ psia}^{-1}

Production Parameters:
- Production rate, \( q \) = 830 STB/D
- Producing time, \( t_p \) = 16.45 hr
- Pressure at shut-in, \( P_{wf,At=0} \) = 3816.99 psia

Discussion of Analysis Results: (Example 8)

Our eighth example case is for an unfractured well producing from an infinite-acting dual porosity reservoir influenced by wellbore storage and skin effects, as illustrated in Figs. 4.29-4.32. We use the effective shut-in time as our time-axis plotted function and Figs. 4.29-4.32 show the pressure drop functions plotted against the shut-in effective time, \( \Delta t_e \), on a log-log scale. These plots indicate transition from wellbore storage domination to the "fracture," then "total system" transient flow regime, without any apparent boundary-dominated flow effects.
During the transition from wellbore storage domination to transient radial flow in the fracture system, the pressure drop derivative function shows the effects of a fissured (or dual porosity) formation by the "dip" feature in the pressure drop derivative function. The wellbore storage regime ends at approximately $\Delta t_e = 3 \times 10^{-2}$ hours. The beginning of the "total system" transient flow occurs at about $\Delta t_e = 7$ hours. The parameters chosen for optimization are: permeability, $k$, skin factor, $s$, dimensionless wellbore storage coefficient, $C_D$, interporosity flow coefficient, $\lambda$, and the fracture storativity ratio, $\omega$.

- Fig. 4.29 shows the history-matching results that we obtained using the numerical inversion solution—we note excellent agreement of the data and the computed functions for all times.

- Fig. 4.30 presents the history-matching results obtained using the quadratic $p_{sD}(t_D)$ approximate solution—we again note an excellent match and the results for this case are comparable to the results obtained from the Stehfest numerical inversion solution.

- Fig. 4.31 illustrates the history-matching results that we obtained using the linear (non-zero intercept) $p_{sD}(t_D)$ approximate solution—we again note an excellent match and the results are comparable to those obtained from the Stehfest numerical inversion solution as well as the quadratic $p_{sD}(t_D)$ approximate solution.
Fig. 4.32 shows our history-matching results obtained using the constant $p_{SD}(t_D)$ approximate solution—we obtain an acceptable match overall, but a poor match in the transition region. We also obtain an acceptable (i.e., good) match in the "total system" radial flow region. The poor match in the transition region suggests that this solution should not be used for dual porosity reservoir cases.

In Table 4.8 we summarize the results of our history-matching efforts. Included in this table are the parameter estimates and the statistical results.

All of the approximate solutions compare very well with the pressure drop and pressure drop derivative functions predicted by the Stehfest numerical inversion solution—except for the constant $p_{SD}(t_D)$ approximate solution. The average errors in the pressure drop function are less than 4.2 percent for all cases and the average errors in the pressure drop derivative function are less than 10 percent for each case, except for the constant $p_{SD}(t_D)$ approximate solution which gave an average error of approximately 13 percent. The obvious conclusion is that the constant $p_{SD}(t_D)$ approximate solution should not be used for general practice, particularly for history matching.
Figure 4.29 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 8, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $P_{wCD}(t_d)$ from the Stehfest Inversion Algorithm.
Figure 4.30 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 8, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Quadratic $p_{sD}(t_D)$ Model.
Figure 4.31 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 8, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Linear $p_{sD}(t_D)$ Model.
Figure 4.32 - Log-log Plot (In Terms of Shut-In Effective Time, $\Delta t_e$) for Example 8, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Constant $p_{sd}(t_D)$ Model.
<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wcd}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wcd}(t_D)$ Using Quadratic $P_{sd}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Linear $P_{sd}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Constant $P_{sd}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>478.25</td>
<td>467.39</td>
<td>451.42</td>
<td>444.40</td>
</tr>
<tr>
<td>Skin factor, $s$</td>
<td>-3.597</td>
<td>-3.732</td>
<td>-3.882</td>
<td>-3.963</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., $C_D$</td>
<td>2.929×10^4</td>
<td>2.998×10^4</td>
<td>3.105×10^4</td>
<td>3.481×10^4</td>
</tr>
<tr>
<td>Interporosity Flow Coef., $\lambda$</td>
<td>4.873×10^{-5}</td>
<td>6.489×10^{-5}</td>
<td>8.82×10^{-5}</td>
<td>9.875×10^{-5}</td>
</tr>
<tr>
<td>Frac. Storativity Ratio, $\omega$</td>
<td>2.09×10^{-1}</td>
<td>2.014×10^{-1}</td>
<td>1.818×10^{-1}</td>
<td>2.089×10^{-1}</td>
</tr>
<tr>
<td>Sum of Squared Error, psi^2</td>
<td>2.388×10^2</td>
<td>2.747×10^2</td>
<td>3.556×10^2</td>
<td>6.478×10^2</td>
</tr>
<tr>
<td>Variance, psi^2</td>
<td>1.349</td>
<td>1.552</td>
<td>2.009</td>
<td>3.66</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>1.161</td>
<td>1.246</td>
<td>1.417</td>
<td>1.913</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>1.145</td>
<td>1.228</td>
<td>1.398</td>
<td>1.887</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>2.689</td>
<td>2.623</td>
<td>3.297</td>
<td>4.196</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>8.159</td>
<td>9.33</td>
<td>9.646</td>
<td>12.96</td>
</tr>
</tbody>
</table>
4.5.9 Example 9 (Unfractured Well in a Dual Porosity Reservoir)

This example presents pressure test data from the MACH 3X Well in Venezuela. These drawdown test data\textsuperscript{34} indicate the model of an unfractured well producing from an infinite-acting, dual porosity reservoir influenced by wellbore storage and skin effects.

The MACH 3X Well was completed in a Cretaceous formation in a field located in western Venezuela (southwest of Lake Maracaibo). The formation permeability is less than 1 md and the primary porosity ranges from 3 to 6 percent. The fractured nature of the reservoir can be inferred from the lithological and petrophysical properties of the available core data. Buildup and drawdown tests (not sequentially) were performed on this well.

Reservoir, Fluid Property, and Production Data:

Reservoir Properties:
- Wellbore radius, $r_w$ = 0.2917 ft
- Net pay thickness, $h$ = 65 ft
- Porosity, $\phi$ (fraction) = 0.048
- Temperature, $T$ = 270 °F

Fluid Properties:
- Oil formation volume factor, $B$ = 1.8235 RB/STB
- Oil viscosity, $\mu$ = 0.362 cp
- Total compressibility, $c_t$ = $24.5 \times 10^{-6}$ psia\textsuperscript{-1}

Production Parameters:
- Production rate, $q$ = 2700 STB/D
- Initial reservoir pressure, $p_i$ = 11,600.0 psia

Discussion of Analysis Results: (Example 9)

Our last example case is for an unfractured well producing from an infinite-acting dual porosity reservoir influenced by wellbore storage and skin effects, as shown in Figs. 4.33-4.36. Figs. 4.33-4.36 show the pressure drop functions plotted against the flowing time, $t$, on a log-log scale. The data plots show transition from wellbore storage
domination to the “fracture” and “total system” transient radial flow regimes—without boundary-dominated effects. During the transition from wellbore storage domination to transient flow, the pressure drop derivative function clearly shows the “dip” feature associated with a dual porosity (or fissured) formation. The wellbore storage domination regime ends at approximately \( t=2 \) hours. The beginning of “total system” transient radial flow occurs at about \( t=100 \) hours. The parameters we chose to optimize are: permeability, \( k \), skin factor, \( s \), dimensionless wellbore storage coefficient, \( C_D \), interporosity flow coefficient, \( \lambda \), and the fracture storativity ratio, \( \omega \).

- Fig. 4.33 shows our history-matching results obtained using the numerical inversion solution—we find excellent agreement of the data and the computed functions for all times, and the characteristic “dip” feature of the pressure drop derivative function is matched as well.

- Fig. 4.34 presents the history-matching results that we obtained using the quadratic \( p_{sD}(t_D) \) approximate solution—we note an excellent match and our results are comparable to those results obtained from the Stehfest numerical inversion solution. This solution also matches the “dip” feature in the pressure drop derivative function.

- Fig. 4.35 shows the history-matching results obtained using the linear (non-zero intercept) \( p_{sD}(t_D) \) approximate solution—we again obtain an excellent match and the results are comparable to the results obtained from the Stehfest numerical inversion solution and the quadratic \( p_{sD}(t_D) \) approximate solution.

- Fig. 4.36 presents the history-matching results that we obtained using the constant \( p_{sD}(t_D) \) approximate solution. We also obtained an excellent match for this case and our results are comparable to the results obtained from the other cases. We also matched the characteristic “dip” feature using this solution.

- Table 4.9 summarizes our history-matching results for this case, including the parameter estimates and the statistical results.
domination to the “fracture” and “total system” transient radial flow regimes—without boundary-dominated effects. During the transition from wellbore storage domination to transient flow, the pressure drop derivative function clearly shows the “dip” feature associated with a dual porosity (or fissured) formation. The wellbore storage domination regime ends at approximately $t=2$ hours. The beginning of “total system” transient radial flow occurs at about $t=100$ hours. The parameters we chose to optimize are: permeability, $k$, skin factor, $s$, dimensionless wellbore storage coefficient, $C_D$, interporosity flow coefficient, $\lambda$, and the fracture storativity ratio, $\omega$.

- Fig. 4.33 shows our history-matching results obtained using the numerical inversion solution—we find excellent agreement of the data and the computed functions for all times, and the characteristic “dip” feature of the pressure drop derivative function is matched as well.

- Fig. 4.34 presents the history-matching results that we obtained using the quadratic $p_{SD}(t_D)$ approximate solution—we note an excellent match and our results are comparable to those results obtained from the Stehfest numerical inversion solution. This solution also matches the “dip” feature in the pressure drop derivative function.

- Fig. 4.35 shows the history-matching results obtained using the linear (non-zero intercept) $p_{SD}(t_D)$ approximate solution—we again obtain an excellent match and the results are comparable to the results obtained from the Stehfest numerical inversion solution and the quadratic $p_{SD}(t_D)$ approximate solution.

- Fig. 4.36 presents the history-matching results that we obtained using the constant $p_{SD}(t_D)$ approximate solution. We also obtained an excellent match for this case and our results are comparable to the results obtained from the other cases. We also matched the characteristic “dip” feature using this solution.

- Table 4.9 summarizes our history-matching results for this case, including the parameter estimates and the statistical results.
All of the approximate solutions compare very well with the pressure drop and pressure drop derivative functions predicted by the Stehfest numerical inversion solution. The average errors for pressure drop and pressure drop derivative functions are less than 7 percent for all cases. Specifically the average error range for the pressure drop function is from 1.1 to 2.1 percent, with the Stehfest numerical inversion solution having the highest average error. Similarly the average error range for the pressure drop derivative function is from 3.9 to 6.8 percent, again with the numerical inversion solution giving the highest error. Although the constant $p_{sD}(t_D)$ approximate model gives excellent results for this case we still recommend against the general application of this solution.

In summary, we have verified and successfully applied each of our approximate solutions for unfractured and fractured wells, in homogenous and dual porosity reservoirs. For history-matching applications each of these solutions gives generally acceptable results. However, we strongly recommend against using the constant $p_{sD}(t_D)$ approximate solution.

In contrast, we do strongly recommend the linear (non-zero intercept) case for general applications as this model has proven accurate enough for general applications and the model is simple enough that other analysis techniques may be derived from this model.
Figure 4.33 - Log-log Plot (In Terms of Flowing Time, t) for Example 9, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Stehfest Inversion Algorithm.
Figure 4.34 - Log-log Plot (In Terms of Flowing Time, $t$) for Example 9, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $P_{wCD}(t_D)$ from the Quadratic $P_{sD}(t_D)$ Model.
Figure 4.35 - Log-log Plot (In Terms of Flowing Time, \( t \)) for Example 9, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using \( P_{wCD}(t_D) \) from the Linear \( p_d(t_D) \) Model.

Data for Example 9:
- \( B = 1.8235 \text{ RB/STB} \)
- \( c_i = 24.5 \times 10^{-6} \text{ psi}^{-1} \)
- \( \phi = 0.048 \text{ (fraction)} \)
- \( \mu = 0.362 \text{ cp} \)
- \( r_w = 0.2917 \text{ ft} \)
- \( h = 65.0 \text{ ft} \)
- \( p_r = 11,600.0 \text{ psia} \)
- \( q = 2700 \text{ STB/D} \)

Optimized Simultaneously on \( \Delta p \) and \( \Delta p' \)

Analysis Results:
- \( k = 3.383 \text{ md} \)
- \( s = -5.835 \)
- \( C_D = 4.334 \times 10^3 \)
- \( \lambda = 1.274 \times 10^{-1} \)
- \( \omega = 5.506 \times 10^{-2} \)

Legend: Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir
(Da Prat, et al., SPE 13054 (Sep. 1984))
Figure 4.36 - Log-log Plot (In Terms of Flowing Time, t) for Example 9, Case of an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir. History-Matching Results Computed Using $p_{wCD}(t_D)$ from the Constant $p_{sd}(t_D)$ Model.
TABLE 4.9

Optimization Results for Example 9 (Pressure Drawdown Test Analysis for an Unfractured Well in an Infinite-Acting, Dual Porosity Reservoir)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$P_{wcd}(t_D)$ Using Stehfest Solution</th>
<th>$P_{wcd}(t_D)$ Using Quadratic $p_{sd}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Linear $p_{sd}(t_D)$</th>
<th>$P_{wcd}(t_D)$ Using Constant $p_{sd}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$, md</td>
<td>3.735</td>
<td>3.476</td>
<td>3.383</td>
<td>3.169</td>
</tr>
<tr>
<td>Skin factor, $s$</td>
<td>-5.617</td>
<td>-5.783</td>
<td>-5.835</td>
<td>-5.962</td>
</tr>
<tr>
<td>Dim.-less Wellbore Storage Coef., $C_D$</td>
<td>2.436×10^3</td>
<td>3.692×10^3</td>
<td>4.334×10^3</td>
<td>5.099×10^3</td>
</tr>
<tr>
<td>Interporosity Flow Coef., $\lambda$</td>
<td>8.233×10^-2</td>
<td>1.16×10^-1</td>
<td>1.274×10^-1</td>
<td>1.578×10^-1</td>
</tr>
<tr>
<td>Sum of Squared Error, psi^2</td>
<td>2.878×10^5</td>
<td>1.225×10^5</td>
<td>1.068×10^5</td>
<td>1.048×10^5</td>
</tr>
<tr>
<td>Variance, psi^2</td>
<td>4.961×10^3</td>
<td>2.111×10^3</td>
<td>1.841×10^3</td>
<td>1.807×10^3</td>
</tr>
<tr>
<td>Standard Deviation, psi</td>
<td>7.044×10^1</td>
<td>4.596×10^1</td>
<td>4.291×10^1</td>
<td>4.251×10^1</td>
</tr>
<tr>
<td>Root Mean Square Error, psi</td>
<td>6.758×10^1</td>
<td>4.409×10^1</td>
<td>4.117×10^1</td>
<td>4.078×10^1</td>
</tr>
<tr>
<td>$\Delta p$ Average Error, percent</td>
<td>2.076</td>
<td>1.238</td>
<td>1.163</td>
<td>1.084</td>
</tr>
<tr>
<td>$\Delta p'$ Average Error, percent</td>
<td>6.794</td>
<td>5.078</td>
<td>4.708</td>
<td>3.898</td>
</tr>
</tbody>
</table>
CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 SUMMARY
In this work we have developed accurate, real-time approximate solutions for the dimensionless pressure function, \( p_{wCD}(t_D) \), where our solutions include the effects of wellbore storage and the near-well skin effect. We verified our approximate solutions against numerical inversion solutions, and we applied these new solutions to field cases. The quadratic and the linear (non-zero intercept) \( p_{hD}(t_D) \) approximate solutions appear to be as accurate as the numerical inversion solution for virtually all applications. Both of these approximate solutions (the quadratic and linear cases) provided accurate and consistent history-matching results using a computer program developed as part of this work.

5.2 CONCLUSIONS
We consider the following to be the most important conclusions:

1. We have successfully developed and verified five explicit relations for the computation of the dimensionless wellbore pressure with wellbore storage and skin effects (i.e., \( p_{wCD}(t_D) \)). These approximate solutions have been shown to have the same order of accuracy as the results obtained using numerical inversion (ref. 2).

2. We have successfully developed additional modules for a computer program used for history-matching well test data for the case of a well in an infinite-acting reservoir. This program uses our new approximate solutions as well as the Stehfest numerical inversion solution\(^2\).

3. We have successfully demonstrated the use of these new approximate solutions (via the PERANA program) for the following field data cases:
• An unfractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage and skin effects.

• A fractured well producing from an infinite-acting homogeneous reservoir influenced by wellbore storage effects.

• An unfractured well producing from an infinite-acting dual porosity reservoir influenced by wellbore storage and skin effects.

5.3 RECOMMENDATIONS FOR FUTURE WORK

Clearly we have not proven our relations for the case of a horizontal well. Future work can proceed along the same approach using the PERANA program, where the program would be modified to include the solution for a horizontal well. In addition, our approach may provide new methods for the deconvolution of well test data, where deconvolution removes the effects of wellbore storage. Finally, it may also be possible to develop explicit analysis relations using our new solutions; and future investigators are encouraged to make such attempts.
NOMENCLATURE

Dimensionless Variables

\( C_D \) = dimensionless wellbore storage coefficient
\( C_{Df} \) = dimensionless fracture wellbore storage coefficient
\( C_{fD} \) = dimensionless fracture conductivity
\( p_D \) = dimensionless pressure
\( p_{sD} \) = dimensionless pressure with skin effects
\( \bar{P}_{sD}(u) \) = Laplace transform of dimensionless pressure with skin effects
\( p_{wCD} \) = dimensionless pressure with wellbore storage and skin effects
\( \bar{P}_{wCD}(u) \) = Laplace transform of dimensionless pressure with wellbore storage and skin effects
\( q_{wCD} \) = dimensionless sandface flowrate with wellbore storage and skin effects
\( \bar{q}_{wCD}(u) \) = Laplace transform of dimensionless sandface flowrate with wellbore storage and skin effects
\( s \) = dimensionless skin factor
\( t_D \) = dimensionless time
\( u \) = Laplace transformation variable

Field Variables

\( A_{wb} \) = cross-section area of wellbore, \( \text{ft}^2 \)
\( B \) = formation volume factor, \( \text{RB/STB} \)
\( C_s \) = wellbore storage coefficient, \( \text{bbl/psi} \)
\( c_{wb} \) = compressibility of fluids in wellbore, \( \text{psia}^{-1} \)
\( c_t \) = total system compressibility, \( \text{psia}^{-1} \)
\( h \) = total formation thickness, ft
\( k \) = permeability, md
\( p_i \) = initial reservoir pressure, psia
\( p_{wf} \) = flowing pressure in wellhead, psia
\( p_{ws} \) = shut-in bottomhole pressure, psia
\( q \) = constant surface flowrate, STB/D
\( q_{sf} \) = sandface rate, STB/D
\( r_w \) = wellbore radius, ft
\( q \) = constant surface flowrate, STB/D
\( t \) = time, days
\( t_p \) = producing time, hours
\( T \) = Temperature, °F
\( V_{wb} \) = volume of wellbore, ft\(^3\)
\( x_f \) = fracture half-length, ft
\( J_0 \) = Bessel function of the first kind, zero order
\( J_1 \) = Bessel function of the first kind, first order
\( K_0 \) = Modified Bessel function of the second kind, zero order
\( K_1 \) = Modified Bessel function of the second kind, first order
\( Y_0 \) = Bessel function of the second kind, zero order
\( Y_1 \) = Bessel function of the second kind, first order

**Greek Variables**

\( \phi \) = reservoir porosity, fraction
\( \rho \) = fluid density, lb/ft\(^3\)
\( \tau \) = dummy time-type of variable
\( \lambda \) = interporosity flow coefficient

\( \omega \) = fracture storativity ratio

\( \mu \) = viscosity, cp
REFERENCES


APPENDIX A

DERIVATION OF WELLBORE STORAGE MODEL
(THE PHYSICAL PROBLEM)

Wellbore storage occurs when a well is turned on or off production and exists as the condition where fluids unload from the wellbore—with no flow from the formation to the wellbore in the case of production, and as continued flow into the well for the pressure buildup case. As time passes the sandface flowrate becomes equal to the surface flowrate and the amount of fluid “stored” in the wellbore becomes constant. In order to resolve the issues related to wellbore storage distortion, we must develop and verify a mathematical model as a means to observe the various aspects of wellbore storage distorted pressure behavior.

The mass balance in the wellbore is given as

\[
\left( \text{Rate of mass flow into wellbore at the sandface} \right) - \left( \text{Rate of mass flow out of wellbore at the surface} \right) = \left( \text{Rate of accumulation of mass in wellbore} \right)
\]

This balance can expressed mathematically as

\[
(q_{sf} - q)B = 24 C_s \left[ \frac{dp_{wf}}{dt} - \frac{dp_y}{dt} \right]
\]

where the wellbore storage coefficient, \( C_s \), is defined separately for the following two cases,

**Case 1:** Fluid-filled wellbore (Fig. A-1)

\[
C_s = c_{wb} V_{wb}
\]

**Case 2:** Rising or falling liquid level (Fig. A-2)

\[
C_s = \frac{144}{5.615} \frac{A_{wb}}{\rho (g/g_c)}
\]
Figure A-1 - Wellbore Schematic
Fluid Filled Wellbore Case

Figure A-2 - Wellbore Schematic
Rising or Falling Liquid Level
We can express Eq. A-1 in terms of dimensionless variables as\(^6\)

\[
q_{wCD} = 1 - C_D \left[ \frac{dp_{wD}}{dt_D} - \frac{dp_{tD}}{dt_D} \right] \quad \text{.................................................. (A-3)}
\]

where the dimensionless wellbore storage coefficient, \(C_D\), is given as

\[
C_D = \frac{0.894 C_s}{\phi c_T h r_w^2} \quad \text{.................................................. (A-4)}
\]

and dimensionless sandface flow rate, \(q_D\), is given by

\[
q_D = \frac{q_{sf}}{q} \quad \text{.................................................. (A-5)}
\]

In order to apply the wellbore storage concept using the rate condition given by Eq. A-3, we require the “convolution” integral, which is given as

\[
P_{wCD}(t_D) = \int_0^{t_D} \frac{d}{d\tau} [q_{wCD}(\tau)] p_{sD}(t_D - \tau) d\tau \quad \text{.................................................. (A-6)}
\]

where

\[
p_{sD} = p_D + s \quad \text{.................................................. (A-7)}
\]

\[
q_{wCD} = 1 - C_D \left[ \frac{dp_{wCD}(tD)}{dt_D} \right], (dp_{tD}/dt_D = 0 --\text{constant surface pressure}) \quad \text{.................................................. (A-8)}
\]

\(s\) is the dimensionless “skin factor,” which represents near-well damage or stimulation.

Taking the Laplace transform of Eq. A-6 gives us

\[
\tilde{P}_{wCD}(u) = u \tilde{q}_{wCD}(u) \tilde{p}_{sD}(u) \quad \text{.................................................. (A-9)}
\]

Taking the Laplace transform of Eq. A-8 gives

\[
\tilde{q}_{wCD}(u) = \frac{1}{u} - C_D u \tilde{p}_{wCD}(u) \quad \text{.................................................. (A-10)}
\]

Combining Eqs. A-9 and A-10 and solving for \(\tilde{P}_{wCD}(u)\) gives

\[
\tilde{P}_{wCD}(u) = \frac{1}{\left[ \frac{1}{\tilde{p}_{sD}(u)} + C_D u^2 \right]} \quad \text{.................................................. (A-11)}
\]

An alternative form of Eq. A-11 is obtained by multiplying through the right-hand-side (RHS) of Eq. A-11 by \(\tilde{p}_{sD}(u) / \tilde{p}_{sD}(u)\) and simplifying, this gives
\[
\bar{P}_{wCD}(u) = \frac{\bar{P}_{sD}(u)}{1 + C_D u^2 \bar{P}_{sD}(u)} \tag{A-12}
\]

Eqs. A-11 and A-12 are our fundamental identities for relating the constant rate solution (for a particular case) and the corresponding wellbore storage solutions. These formulations are valid for any reservoir condition (unfractured and fractured wells, horizontal wells, wells in dual porosity reservoirs, etc.)—provided that the constant \(C_s\) and \(p_{ef}\) conditions are met. If \(C_s\) and/or \(p_{ef}\) are not constant, then either Eq. A-1 (or its dimensionless form, Eq. A-3) must be used to model the “variable” wellbore storage condition.
APPENDIX B

DERIVATION OF NEW $p_{wCD}(t_D)$ MODELS

In this section we develop five explicit models for the prediction of pressure behavior distorted by wellbore storage. We begin our developments of the $p_{wCD}(t_D)$ models using the following approximate models for $p_{sD}(t_D)$ (recall that $p_{sD}(t_D) = p_D(t_D) + s$).

<table>
<thead>
<tr>
<th>Model</th>
<th>$p_{sD}(t_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a$ (Constant)</td>
</tr>
<tr>
<td>2</td>
<td>$at_D$</td>
</tr>
<tr>
<td>3</td>
<td>$a_0 + a_1t_D$</td>
</tr>
<tr>
<td>4</td>
<td>$a_0\exp(a_1t_D)$</td>
</tr>
<tr>
<td>5</td>
<td>$a_0 + a_1t_D + a_2't_D^2$</td>
</tr>
</tbody>
</table>

We will develop each model separately in the following sections.

**Model 1: $p_{sD} = a$ ($a$ is constant)**

This model is our most fundamental case and assumes that the $p_{sD}(t_D)$ function is constant for a particular time of interest. Clearly this is not physically constant but our hope is that this assumption will yield a useful approximation for the $p_{wCD}(t_D)$ function.

We begin with

$$p_{sD}(t_D) = a$$  \hspace{1cm} \text{.................................(B-1)}

Taking the Laplace transform of Eq. B-1 gives us

$$\bar{p}_{sD}(u) = \frac{a}{u}$$  \hspace{1cm} \text{.................................(B-2)}

Substituting Eq. B-2 into Eq. A-11 gives

$$\bar{p}_{wCD}(u) = \frac{1}{\left[\frac{u}{a} + C_Du^2\right]}$$ \hspace{1cm} \text{.................................(B-3)}
Factoring the denominator of Eq. B-3 and rearranging gives

\[ \bar{P}_{wCD}(u) = \frac{1}{C_D} \frac{1}{u \left[ u + \frac{1}{aC_D} \right]} \]  \hspace{1cm} \text{(B-4)}

or

\[ \bar{P}_{wCD}(u) = \frac{\alpha}{u \left[ u + \beta \right]} \]  \hspace{1cm} \text{(B-5)}

where the following terms are defined for convenience

\[ \alpha = \frac{1}{C_D} \]

and

\[ \beta = \frac{1}{aC_D} \]

The inverse Laplace transform of Eq. B-5 is obtained from ref. 35

\[ p_{wCD}(t_D) = \frac{\alpha}{\beta} \left[ 1 - \exp(-\beta t_D) \right] \]

Substituting the definitions of \( \alpha \) and \( \beta \) for this problem we obtain

\[ p_{wCD}(t_D) = a \left[ 1 - \exp\left(\frac{-t_D}{aC_D}\right) \right] \]  \hspace{1cm} \text{(B-6)}

From Eq. B-1 we assume that

\[ a \approx p_{sD}(t_D) \]  \hspace{1cm} \text{(B-7)}

Combining Eqs. B-6 and B-7 gives us

\[ p_{wCD}(t_D) = p_{sD}(t_D) \left[ 1 - \exp\left(\frac{-t_D}{p_{sD}(t_D)C_D}\right) \right] \]  \hspace{1cm} \text{(B-8)}

Taking the derivative of Eq. B-8 with respect to \( t_D \) we have two options. First, we can “blindly” differentiate Eq. B-8 assuming that \( p_{sD}(t_D) \) is constant. This approach gives

\[ \frac{d}{dt_D} \left[ p_{wCD}(t_D) \right] = \frac{1}{C_D} \exp\left(\frac{-t_D}{p_{sD}(t_D)C_D}\right) \]  \hspace{1cm} \text{(B-9)}
Second, we can differentiate Eq. B-8 assuming \( p_{sd}(t_D) = f(t_D) \). This gives
\[
\frac{d}{dt_D} \left[ p_{wCD}(t_D) \right] = \left[ 1 - \exp \left( \frac{-t_D}{p_{sd}(t_D) C_D} \right) \right] \frac{d}{dt_D} \left[ p_{sd}(t_D) \right] \\
+ \frac{1}{p_{sd}(t_D) C_D} \left[ p_{sd}(t_D) - t_D \frac{d}{dt_D} \left[ p_{sd}(t_D) \right] \right] \\
\exp \left( \frac{-t_D}{p_{sd}(t_D) C_D} \right)
\]
.............................................. (B-10)

Rearranging Eq. B-8 and solving for the exponential term gives
\[
\exp \left( \frac{-t_D}{p_{sd}(t_D) C_D} \right) = 1 - \frac{p_{wCD}(t_D)}{p_{sd}(t_D)} 
\]
.............................................. (B-11)

Combining Eqs. B-9 and B-11 gives us the following
\[
\frac{d}{dt_D} \left[ p_{wCD}(t_D) \right] = \frac{1}{C_D} \left[ 1 - \frac{p_{wCD}(t_D)}{p_{sd}(t_D)} \right] 
\]
.............................................. (B-12)

Combining Eqs. B-9, B-10 and B-11 gives us a result that is equivalent to Eq. B-12
\[
\frac{d}{dt_D} \left[ p_{wCD}(t_D) \right] = \frac{p_{wCD}(t_D)}{p_{sd}(t_D)} \frac{d}{dt_D} \left[ p_{sd}(t_D) \right] \\
+ \frac{1}{C_D} \left[ 1 - \frac{t_D}{p_{sd}(t_D)} \frac{d}{dt_D} \left[ p_{sd}(t_D) \right] \right] \left[ 1 - \frac{p_{wCD}(t_D)}{p_{sd}(t_D)} \right] 
\]
.............................................. (B-13)

Our purpose for obtaining these derivative functions is to generate plotting functions for type curve analysis. In practice we may use either an analytical result of a numerical derivative obtained from Eq. B-8.

**Model 2: \( p_{sd} = at_D \) (linear case, zero intercept)**

This model assumes that \( p_{sd}(t_D) \) is a function of \( t_D \) only (zero intercept). The appropriate model is
\[
p_{sd}(t_D) \equiv at_D 
\]
.............................................. (B-14)
Taking the Laplace transform of Eq. B-14 gives

$$\bar{p}_{sD}(u) = \frac{a}{u^2}$$ .................................................. (B-15)

Substituting Eq. B-15 into Eq. A-11 gives us

$$\bar{p}_{wCD}(u) = \frac{1}{\left[ \frac{u^2}{a} + C_D u^2 \right]}$$

or, factoring out the $u^2$ term in the denominator, we obtain

$$\bar{p}_{wCD}(u) = \frac{1}{C_D u^2} \left[ \frac{1}{1 + \frac{1}{a C_D}} \right]$$ .................................................. (B-16)

Since $a$ and $C_D$ are constants, the inverse Laplace transform of Eq. B-16 is

$$p_{wCD}(t_D) = \frac{t_D}{C_D} \frac{1}{1 + \frac{1}{a C_D}}$$ .................................................. (B-17)

From Eq. B-14 we have

$$a \approx \frac{p_{sD}(t_D)}{t_D}$$ .................................................. (B-18)

Substituting Eqs. B-18 into B-17 gives

$$p_{wCD}(t_D) = \frac{t_D}{C_D} \frac{1}{1 + \frac{t_D}{p_{sD}(t_D) C_D}}$$ .................................................. (B-19a)

or, upon rearranging, we obtain

$$p_{wCD}(t_D) = \frac{1}{\frac{1}{p_{sD}(t_D)} + \frac{1}{t_D / C_D}}$$ .................................................. (B-19b)
Taking the derivative Eq. B-19a with respect to $t_D$ gives us

$$\frac{d}{dt_D}[p_{wCD}(t_D)] = \frac{1}{C_D} \left[ \frac{1}{1 + \frac{t_D}{P_{sD}(t_D)C_D}} \right]$$

$$- \frac{t_D}{[P_{sD}(t_D)C_D]^2} \left[ P_{sD}(t_D) - t_D \frac{d}{dt_D}[P_{sD}(t_D)] \right] \left[ \frac{t_D}{1 + \frac{t_D}{P_{sD}(t_D)C_D}} \right]^2 \quad \text{(B-19c)}$$

**Model 3:** $p_{sD} = a_0 + a_1 t_D$ (linear case, non-zero intercept)

This model assumes that the $P_{sD}(t_D)$ function is linear near a particular time of interest.

We begin with the following model (non-zero intercept)

$$P_{sD}(t_D) = a_0 + a_1 t_D \quad \text{.......................................................... (B-20)}$$

Taking the Laplace transform of Eq. B-20 gives

$$\bar{P}_{sD}(u) = \frac{a_0}{u} + \frac{a_1}{u^2} \quad \text{.......................................................... (B-21)}$$

Substituting Eq. B-20 into Eq. A-12 gives us

$$\bar{P}_{wCD}(u) = \frac{\left( \frac{a_0}{u} + \frac{a_1}{u^2} \right)}{1 + C_D u^2 \left( \frac{a_0}{u} + \frac{a_1}{u^2} \right)} \quad \text{.......................................................... (B-22)}$$

Separating terms in the numerator, then expanding and combining terms in the denominator gives us
\[ \bar{p}_{wCD}(u) = \frac{a_0}{u + C_D a_0 u^2 + C_D a_1 u} + \frac{a_1}{u^2 + C_D a_0 u^3 + C_D a_1 u^2} \]
\[ = \frac{a_0}{u(1 + C_D a_0 u + C_D a_1)} + \frac{a_1}{u^2(1 + C_D a_0 u + C_D a_1)} \]
\[ = \frac{1}{C_D u} \left( \frac{1}{u + \frac{1 + C_D a_1}{C_D a_0}} \right) + \frac{a_1}{a_0 C_D u^2} \left( \frac{1}{u + \frac{1 + C_D a_1}{C_D a_0}} \right) \]
\[ = \beta \frac{1}{u(u + \alpha)} + \theta \frac{1}{u^2(u + \alpha)} \] .......................... (B-23)

where the following terms have been defined for convenience
\[ \alpha = \frac{1 + C_D a_1}{C_D a_0} \]
\[ \beta = \frac{1}{C_D} \]
\[ \theta = \frac{a_1}{a_0 C_D} \]

The inverse Laplace transform of Eq. B-23 is obtained from ref. 35
\[ p_{wCD}(t_D) = \frac{\beta}{a} \left[ 1 - \exp(-at_D) \right] + \frac{\theta}{\alpha^2} \left[ \exp(-at_D) + at_D - 1 \right] \] .......................... (B-24)

At this point we must consider a scheme for determining the coefficients \( a_0 \) and \( a_1 \) in the \( p_{sD}(t_D) \) model (Eq. B-20). Recalling Eq. B-20, we have
\[ p_{sD}(t_D) = a_0 + a_1 t_D \] .......................... (B-20)

Differentiating Eq. B-20 with respect to \( t_D \) gives
\[ a_1 = \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] \] .......................... (B-25)

Combining Eqs. B-20 and B-25 and solving for the \( a_0 \) coefficient gives
\[ a_0 = p_{sD}(t_D) - t_D \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] \]
or

\[ a_0 = p_{sD}(t_D) - p_{sD}'(t_D) \] ......................................................... (B-26)

where

\[ p_{sD}'(t_D) = t_D \frac{d}{dt_D}[p_{sD}(t_D)] \]

Taking the derivative of Eq. B-24 with respect to \( t_D \) by assuming \( \alpha, \beta, \text{ and } \theta \) constant gives

\[ \frac{d}{dt_D}[p_{wCD}(t_D)] = \frac{\theta}{\alpha^2} \left[ -\alpha \exp(-\alpha t_D) + \alpha \right] + \beta \left[ \exp(-\alpha t_D) \right] \] ......................................................... (B-27)

We can also develop an alternative form of Eq. B-24 by substituting the definitions of \( \alpha, \beta, \theta, a, \text{ and } b \) into Eq. B-24. Recalling Eq. B-24 we have

\[ p_{wCD}(t_D) = \frac{\beta}{\alpha} \left[ 1 - \exp(-\alpha t_D) \right] + \frac{\theta}{\alpha^2} \left[ \exp(-\alpha t_D) + at_D - 1 \right] \] ......................................................... (B-24)

Rearranging Eq. B-24 we obtain

\[ p_{wCD}(t_D) = \frac{\beta}{\alpha} \left[ 1 - \exp(-\alpha t_D) \right] - \frac{\theta}{\alpha^2} \left[ 1 - \exp(-\alpha t_D) - at_D \right] \]

or

\[ p_{wCD}(t_D) = \left( \frac{\beta}{\alpha} - \frac{\theta}{\alpha^2} \right) \left[ 1 - \exp(-\alpha t_D) \right] + \frac{\theta}{\alpha} t_D \] ......................................................... (B-28)

Taking "well testing" derivative of Eq. B-28 gives us

\[ p_{wCD}'(t_D) = t_D \left( \frac{d}{dt_D}[p_{wCD}(t_D)] \right) \]

\[ = \left( \frac{\beta}{\alpha} - \frac{\theta}{\alpha^2} \right) \alpha t_D \exp(-\alpha t_D) + \frac{\theta}{\alpha} t_D \]

\[ = \beta t_D \exp(-\alpha t_D) + \frac{\theta}{\alpha} t_D (1 - \exp(-\alpha t_D)) \] ......................................................... (B-29)

Substituting \( a_0 \) (Eq. B-26) and \( a_1 \) (Eq. B-25) into the definition of \( \theta \) and \( \alpha \) gives us

\[ \theta = \frac{t_D}{C_D} \frac{p_{sD}'(t_D)}{p_{sD}(t_D) - p_{sD}'(t_D)} \] ......................................................... (B-30)
\[\alpha = \frac{\frac{t_D}{C_D} + p_{sD}'(t_D)}{t_D p_{sD}(t_D) - p_{sD}'(t_D)} \]  \hspace{1cm} \text{(B-31)}

Dividing \( \beta \) by \( \alpha \) (Eq. B-31), gives us

\[\frac{\beta}{\alpha} = \frac{p_{sD}(t_D) - p_{sD}'(t_D)}{1 + \frac{p_{sD}'(t_D)C_D}{t_D}} \]  \hspace{1cm} \text{(B-32)}

Dividing \( \theta \) (Eq. B-30) by \( \alpha^2 \) gives us

\[\frac{\theta}{\alpha^2} = \frac{p_{sD}(t_D) - p_{sD}'(t_D)}{t_D + 2 + \frac{p_{sD}'(t_D)C_D}{t_D}} \]  \hspace{1cm} \text{(B-33)}

In addition, the \( \theta/\alpha \) term is given by

\[\frac{\theta}{\alpha} = \frac{p_{sD}'(t_D)}{t_D} \frac{1}{1 + \frac{p_{sD}'(t_D)C_D}{t_D}} \]  \hspace{1cm} \text{(B-34)}

Recalling Eq. B-28 we have

\[p_{wCD}(t_D) = \left( \frac{\beta}{\alpha} - \frac{\theta}{\alpha^2} \right) \left[ 1 - \exp(-\alpha t_D) \right] + \frac{\theta}{\alpha} t_D \]  \hspace{1cm} \text{(B-28)}

Solving for the \( (\beta/\alpha - \theta/\alpha^2) \) term in Eq. B-28 and substituting term-by-term, we have

\[\frac{\beta}{\alpha} - \frac{\theta}{\alpha^2} = \left[ p_{sD}(t_D) - p_{sD}'(t_D) \right] \frac{t_D}{p_{sD}'(t_D)C_D} \left( \frac{1}{p_{sD}'(t_D)} + \frac{1}{t_D/C_D} \frac{1}{p_{sD}'(t_D) + \frac{t_D}{C_D}} \right) \]  \hspace{1cm} \text{(B-35)}
Substituting terms \((\beta' \alpha - \theta' \alpha^2)\), \(a_D\), and \((\theta' \alpha)D\) into Eq. B-28 gives us

\[
p_{wCD}(t_D) = \left\{ p_{sD}(t_D) - p_{sD}'(t_D) \right\} \\
\quad \left\{ \frac{t_D}{p_{sD}(t_D)C_D} + \frac{1}{p_{sD}(t_D)/C_D} + \frac{1}{p_{sD}'(t_D) + t_D/C_D} \right\} \\
\quad \left\{ 1 - \exp\left[ -\frac{1}{p_{sD}(t_D)/C_D + p_{sD}'(t_D)} \right] \right\} \
\quad + \frac{1}{p_{sD}(t_D) + t_D/C_D} 
\]

......................................................... (B-36)

**Model 4:** \( p_{sD} = a_0 \exp(a_1 t_D) \) (exponential case)

This model assumes that the \( p_{sD}(t_D) \) function behaves like a single-term exponential function about a particular time of interest. Starting with the exponential model for \( p_{sD}(t_D) \) we have

\[
p_{sD}(t_D) = a_0 \exp(a_1 t_D) \quad \text{................................................................. (B-37)}
\]

Taking the Laplace transform of Eq. B-37 gives

\[
\bar{p}_{sD}(u) = \frac{a_0}{u - a_1} \quad \text{................................................................. (B-38)}
\]

Substituting Eq. B-38 into Eq. A-12 gives

\[
\bar{p}_{wCD}(u) = \frac{\left( \frac{a_0}{u - a_1} \right)}{1 + C_D u^2 \left( \frac{a_0}{u - a_1} \right)} \quad \text{................................................................. (B-39)}
\]
Rearranging terms in both the numerator and denominator, and collecting constant terms gives us

\[ \tilde{p}_{\text{wCD}}(u) = \frac{C_0}{u^2 + au + \beta} \]  \hspace{1cm} (B-40)

where the following terms have been defined for convenience

\[ \alpha = \frac{1}{C_D a_0} \]
\[ \beta = -\frac{a_1}{C_D a_0} \]
\[ C_0 = \frac{1}{C_D} \]

The inverse Laplace transform of Eq. B-40 is obtained from ref. 35

\[ p_{\text{wCD}}(t_D) = C_0 \frac{1}{2\sqrt{\beta - \frac{\alpha^2}{4}}} \left\{ \exp \left[ -\left( \frac{\alpha}{2} - \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right] \right\} \]

\[ -\exp \left[ -\left( \frac{\alpha}{2} + \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right] \]  \hspace{1cm} (B-41a)

Taking the derivative of Eq. B-41 with respect to \( t_D \), by assuming that \( \alpha \) and \( \beta \) are constant gives us

\[ \frac{d}{dt_D} \left[ p_{\text{wCD}}(t_D) \right] = C_0 \frac{1}{2\sqrt{\beta - \frac{\alpha^2}{4}}} \left\{ \left[ -\left( \frac{\alpha}{2} - \sqrt{\beta - \frac{\alpha^2}{4}} \right) \exp \left[ -\left( \frac{\alpha}{2} - \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right] \right\} \right\} \]

\[ + \left( \frac{\alpha}{2} + \sqrt{\beta - \frac{\alpha^2}{4}} \right) \exp \left[ -\left( \frac{\alpha}{2} + \sqrt{\beta - \frac{\alpha^2}{4}} \right) t_D \right] \]  \hspace{1cm} (B-41b)

**Model 5:** \( p_{sD}(t_D) = a_0 + a_1 t_D + a_2 t_D^2 \) (quadratic case)

This model assumes that the \( p_{sD}(t_D) \) function varies as a quadratic function of time near a particular time of interest. We start with a 3-term quadratic model which is given as
\[ p_{sD}(t_D) = a_0 + a_1 t_D + a_2 t_D^2, \quad (a_2' = a_2/2) \] .............................................. (B-42)

Taking the Laplace transform of Eq. B-42 gives

\[ \tilde{p}_{sD}(u) = \frac{a_0}{u} + \frac{a_1}{u^2} + \frac{a_2}{u^3} \] .............................................. (B-43)

Substituting Eq. B-43 into Eq. A-12 gives

\[ \tilde{p}_{wCD}(u) = \frac{\left( \frac{a_0}{u} + \frac{a_1}{u^2} + \frac{a_2}{u^3} \right)}{1 + C_D u^2 \left( \frac{a_0}{u} + \frac{a_1}{u^2} + \frac{a_2}{u^3} \right)} \] .............................................. (B-44)

Rearranging Eq. B-44 and collecting like terms we obtain

\[ \tilde{p}_{wCD}(u) = \frac{c_0}{u^2 + au + \beta} + \frac{c_1}{u(u^2 + au + \beta)} + \frac{c_2}{u^2 (u^2 + au + \beta)} \] .............................................. (B-45)

where the following definitions are made for convenience

\[ a_2 = 2a_2' \]

\[ b_0 = 1 + C_D a_1 \quad c_0 = a_0/b_1 \quad \alpha = b_0/b_1 \]

\[ b_1 = C_D a_0 \quad c_1 = a_1/b_1 \quad \beta = b_2/b_1 \]

\[ b_2 = C_D a_2 \quad c_2 = a_2/b_1 \]

And we also use the following definitions for convenience

\[ \theta = \sqrt{\beta - \frac{\alpha^2}{4}} \]

\[ \xi = \alpha/2 \]

And finally, we also use the following definitions to reduce the solution still further

\[ A = 1/(2\theta) \]

\[ B = \xi - \theta \]

\[ C = \xi + \theta \]

The inverse Laplace transform of Eq. B-45 is

\[ p_{wCD}(t_D) = c_0 f_0(t_D) + c_1 f_1(t_D) + c_2 f_2(t_D) \] .............................................. (B-46)
where
\[ f_0(t_D) = A \left[ \exp(-Bt_D) - \exp(-Ct_D) \right] \]
\[ f_1(t_D) = A \left[ \frac{1}{B} \left( 1 - \exp(-Bt_D) \right) - \frac{1}{C} \left( 1 - \exp(-Ct_D) \right) \right] \]
\[ f_2(t_D) = A \left[ \left( \frac{1}{B} - \frac{1}{C} \right) t_D - \left( \frac{1}{B^2} - \frac{1}{C^2} \right) + \frac{1}{B^2} \exp(-Bt_D) - \frac{1}{C^2} \exp(-Ct_D) \right] \]

We use following procedure to determine the coefficients in Eq. B-42. Recalling Eq. B-42, we have
\[ p_{sD}(t_D) = a_0 + a_1 t_D + a_2 t_D^2 \] ............................................................................ (B-42)

Taking the derivative of Eq. B-42 with respect to \( t_D \) gives
\[ \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] = a_1 + 2a_2 t_D \] ............................................................................ (B-47)

Taking the second derivative of Eq. B-42 with respect to \( t_D \) and solving for \( a_2' \) gives
\[ a_2' = \frac{\frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right]}{2} \] ............................................................................ (B-48)

Substituting Eq. B-48 into Eq. B-47 and solving for \( a_1 \) gives
\[ a_1 = \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] - t_D \left[ \frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right] \right] \] ............................................................................ (B-49)

Substituting Eqs. B-48 and B-49 into Eq. B-42 and solving for \( a_0 \) gives
\[ a_0 = p_{sD}(t_D) - t_D \frac{d}{dt_D} \left[ p_{sD}(t_D) \right] + \frac{t_D^2}{2} \frac{d^2}{dt_D^2} \left[ p_{sD}(t_D) \right] \] ............................................................................ (B-50)

Again, we use numerical differentiation to determine \( \frac{d}{dt_D} \left[ p_{wCD}(t_D) \right] \) because the analytical, closed form differentiation of Eq. B-46 is too complex for practical purposes.
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