The Laplace Transform Finite Difference (LTFD) Numerical Method for the Simulation of Compressible Liquid Flow in Reservoirs

SPE 22888 © 1994 Society of Petroleum Engineers

This paper presents a new numerical method which eliminates the need for time discretization in the simulation of flow of reservoir liquids.


SUMMARY
A new numerical method, the Laplace Transform Finite Difference (LTFD) method, was developed for the simulation of single-phase compressible liquid flow through porous media. The major advantage of LTFD is that it eliminates time dependency, the need for time discretization, and the problems stemming from the treatment of the time derivative in the flow equation by employing a Laplace transform formulation. The LTFD method yields a solution semi-analytical in time and numerical in space, and renders the effects of the time derivative on accuracy and stability irrelevant because time is no longer considered. The method was tested against results from three test cases obtained from a standard Finite Difference (FD) simulator, as well as from analytical models. For a single timestep, LTFD requires an execution time 6 to 10 times longer than the analogous FD requirement without an increase in storage. However, this disadvantage is outweighed by the fact that LTFD allows an unlimited timestep size. Execution times may be reduced by orders of magnitude because calculations are necessary only at the desired observation times, while in a standard FD method calculations are needed at all the intermediate times of the discretized time domain. Thus, FD may require several hundred timesteps and matrix inversions to reach the desired solution time, but LTFD requires only one timestep and no more than 6 to 10 matrix inversions. Moreover, LTFD yields a stable, non-increasing material balance error in addition to a more accurate solution than FD.

INTRODUCTION
In transient flow through porous media, the general Partial Differential Equation (PDE) to be solved is obtained by combining appropriate forms of Darcy's Law and the equation of mass conservation, yielding:

\[
\nabla \cdot \left[ k \frac{p}{\mu} (\nabla p - \rho g \nabla z) \right] = \frac{\partial}{\partial t}(\rho \phi) + w. \quad (1)
\]

Eq. 1 is generally nonlinear and contains the time derivative \(\partial p/\partial t\), of which the numerical approximation is consistently the most important source of instability and error in numerical models. The treatment of \(\partial p/\partial t\) in a traditional Finite Difference (FD) approximation scheme involves the discretization of the continuous time coordinate into a large number of small timesteps \(\Delta t\). Numerical solutions at a number of points \(\kappa\) in the domain are then sought at the discrete times

\[
t_0 = 0, \quad t_1 = \Delta t_1, \quad t_2 = \Delta t_1 + \Delta t_2, \ldots, \quad t_n = \sum_{\ell=1}^{n} \Delta t_{\ell}, \quad (2)
\]

where the dependent variable \(p_{\kappa}(t)\) is approximated by a set of values \(\tilde{p}_{\kappa}^{\ell}, \; \ell = 1, 2, \ldots, n\).

The PDE problem with a continuous smooth solution surface is thus reduced to a set of algebraic equations relating the discrete approximate values \(\tilde{p}_{\kappa}\) to each point \(\kappa\). A Taylor series approximation of the time derivative yields

\[
\left( \frac{\partial p_{\kappa}}{\partial t} \right)^{\nu} = \frac{1}{\Delta t_{n+1}} (\tilde{p}_{\kappa}^{n+1} - \tilde{p}_{\kappa}^{n}) + E_{\kappa,n}, \quad (3)
\]

where

\[
E_{\kappa,n} = p_{\kappa}^{n} \frac{\Delta t_{n+1}}{2} - p_{\kappa}^{n} \frac{\Delta t_{n+1}^2}{6} + \ldots \quad (4)
\]

is the truncation error, and \(n + 1\) and \(n\) denote the current time and previous times of the discretized time domain. For \(\nu = n\), Eq. 3 represents a forward difference approximation and results in the explicit formulation of the FD, which is not unconditionally stable; for \(\nu = n + 1\), Eq. 3 represents a backward difference approximation and results in the unconditionally stable implicit formulation of the FD. The above approximation introduces an error of order \(O(\Delta t)\). Accuracy (and, in the case of the explicit formulation, stability) considerations preclude the use of a large \(\Delta t\). Minimization of \(E_{\kappa,n}\) often dictates the use of a large number of small timesteps \(\Delta t\) between desired observation times, requiring longer execution times and resulting in potentially larger roundoff errors. The problem of restriction on the size of \(\Delta t\) is further exacerbated by the nonlinearity of the PDE, which is caused by the pressure dependence of the liquid-density and the formation porosity. This necessitates even shorter timesteps, dictates internal iterations within each timestep until a convergence criterion is met, and adds significantly to the computational load.

The Laplace Transform Finite Difference (LTFD) method belongs to a family of new, Laplace transform-based numerical methods recently introduced by Moridis and Reddell1-5. It was first applied to the solution of the diffusion-type (parabolic) PDE of incompressible flow through porous media1-2, and was extended to the solution of the advection-diffusion (hyperbolic) PDE of solute transport (miscible displacement) in porous media3. The major advantage of LTFD is that it eliminates the time dependency of the problem because of the Laplace transform formulation employed, and thus the need for time discretization for an accurate, stable solution. In essence, LTFD yields a solution semi-analytical in time and numerical in space. An unlimited \(\Delta t\) size is possible without loss of accuracy or stability, and the need for a large number of intermediate steps between desired observation times is eliminated.

In the present paper, the LTFD method was formulated to address the problem of slightly compressible, single-phase liquid flow through
porous media. The mathematical basis of the method is developed, and its performance is evaluated against analytical solutions and standard FD models.

The Nonlinear Liquid Flow Equation
A general equation for liquid density is

$$\rho = \rho_{STC} \beta, \quad \beta = \beta_0 R, \quad R = \exp[c_f(p - p_0)] ,$$

(5)

where $\beta$ is the inverse of the formation volume factor. Then

$$p = p_0 + \frac{1}{c_f} \ln \left( \frac{\rho}{\rho_0} \right) = p_0 + \frac{1}{c_f} \ln \left( \frac{\beta}{\beta_0} \right) = p_0 + \frac{1}{c_f} \ln R ,$$

(6)

where $\rho_0 = \rho_{STC} \beta_0$. Differentiation of Eq. 6 yields

$$\frac{\partial p}{\partial d^2} = \frac{1}{c_f} \frac{1}{R} \frac{\partial R}{\partial d^2}, \quad \frac{\partial p}{\partial R} = \frac{1}{c_f} \frac{1}{R} ,$$

(7)

where $d^2 \equiv x, y, z$ in cartesian coordinates, or $d^2 \equiv r, \theta, z$ in cylindrical coordinates.

Expansion of the left-hand side of Eq. 1 and substitution for $p$ yields

$$\frac{\partial}{\partial d^2} \left[ k \frac{\rho}{\mu} \left( \frac{\partial p}{\partial d^2} - \rho g \frac{\partial z}{\partial d^2} \right) \right]$$

$$= \frac{\partial}{\partial d^2} \left[ k \frac{\rho_0}{\mu c_f} \frac{\partial R}{\partial d^2} - k \frac{\rho_0^2 R^2 z_d}{\mu} \right],$$

(8)

where $z_d$ is the elevation gradient along the $d^2$ direction.

Expansion of the time derivative in the right-hand side of Eq. 1 and substitution for $\rho$ and $\partial p/\partial R$ yields

$$\frac{\partial}{\partial t} (\rho \phi) = \rho_0 \left( \phi + R \frac{\partial \phi}{\partial R} \frac{\partial R}{\partial t} \right) = \rho_0 C_T \frac{\partial R}{\partial t} ,$$

(9)

where $C_T = \phi + (1/c_f) (\partial \phi/\partial p)$.

The porosity $\phi$ is a weak function of pressure. For consolidated formations the variation of pore volume with pressure is accounted for by

$$\phi = \phi_0 \exp[c_{fm}(p - p_0)],$$

(10)

From Eqs. 6 and 10,

$$\phi = \phi_0 \left( 1 + c_{fm} \frac{c_f}{c_f} \right) R^{(c_{fm}/c_f)} ,$$

(11)

leading to

$$C_T = \phi_0 \left( 1 + c_{fm} \frac{c_f}{c_f} \right) R^{(c_{fm}/c_f)} .$$

(12)

For large pressure differentials of compressible liquids in compressible reservoirs, $R \neq 1$ and Eq. 12 has to be used in its most general form. For small pressure differentials, $R \approx 1$ and $C_T$ is reduced to the constant

$$C_T \approx \phi_0 \left( 1 + c_{fm} \frac{c_f}{c_f} \right) .$$

(13)

For nearly incompressible formations $C_T \approx \phi_0$.

Substitution in Eq. 1 and division by the constant $\rho_0$ yields the general equation of compressible liquid flow in cartesian coordinates as

$$\frac{\partial}{\partial x} \left[ k_x \frac{\partial R}{\partial x} - \frac{k_x g z \rho_0 R^2}{\mu} \right]$$

$$+ \frac{\partial}{\partial y} \left[ k_y \frac{\partial R}{\partial y} - \frac{k_y g z \rho_0 R^2}{\mu} \right]$$

$$+ \frac{\partial}{\partial z} \left[ k_z \frac{\partial R}{\partial z} - \frac{k_z g z \rho_0 R^2}{\mu} \right] = C_T \frac{\partial R}{\partial t} + q ,$$

(14)

where $q = \dot{q}/\rho_0$. In cylindrical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r k_r \frac{\partial R}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ k_\theta \frac{\partial R}{\partial \theta} \right]$$

$$+ \frac{\partial}{\partial z} \left[ k_z \frac{\partial R}{\partial z} - \frac{k_z g z \rho_0 R^2}{\mu} \right] = C_T \frac{\partial R}{\partial t} + q .$$

(15)

Inspection of the $R$ formulation of Eqs. 14 and 15 reveals the following:

1. The nonlinearity of the product $\rho \partial p/\partial d^2$, ($d^2 \equiv x, y, z$ in cartesian coordinates, or $d^2 \equiv r, \theta, z$ in cylindrical coordinates) in the original PDE (Eq. 1) has been removed.

2. The nonlinearity of $\rho^2$ in the gravitational component is maintained in $R^2$.

3. For areal simulations and all cases where gravitational effects are not included, the flow Eqs. 14 and 15 are linear if $C_T$ can be approximated by Eq. 13 within a desired degree of accuracy.

4. When gravity effects are included or $C_T$ cannot be adequately described by Eq. 13, Eqs. 14 and 15 are weakly nonlinear.

Eqs. 14 and 15 are theoretically correct, but present potentially serious computational problems which severely limit the usefulness of the $R$ formulation even if the nonlinearity in $R^2$ is not considered. If solutions in terms of $R$ are sought, near-singular matrices result almost invariably. Such behavior is more pronounced if the pressure changes are small. This is caused by (a) the inherent machine accuracy limitations, and (b) the proximity of $R$ to the value of 1 and its relative insensitivity even to very significant pressure variations, due to the small compressibility of the reservoir liquids. Significant changes in pressure (thousands of psi) may result in minuscule decreases in $R$ which may vary within a narrow range (i.e. between 0.95 and 1). Solutions in $R$ may thus suffer from substantial errors.

These problems of nonlinearity and nonsingularity are alleviated by reformulating Eqs. 14 and 15 in terms of $\Delta R$, i.e. the deviation of $R$ from the value of 1 corresponding to the reference pressure $p_0$. Substitution for

$$R = 1 + \Delta R, \quad R^2 = 1 + 2 \cdot \Delta R + (\Delta R)^2$$

(16)

in Eqs. 14 and 15 yields

$$\frac{\partial}{\partial x} \left[ \lambda_{x_1} \frac{\partial (\Delta R)}{\partial x} - \lambda_{x_2} \Delta R^2 \right] + \frac{\partial}{\partial y} \left[ \lambda_{y_1} \frac{\partial (\Delta R)}{\partial y} - \lambda_{y_2} \Delta R^2 \right]$$

$$+ \frac{\partial}{\partial z} \left[ \lambda_{z_1} \frac{\partial (\Delta R)}{\partial z} - \lambda_{z_2} \Delta R^2 \right] = C_T \frac{\partial (\Delta R)}{\partial t} + q ,$$

(17)

in cartesian coordinates, and

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \lambda_{r_1} \frac{\partial (\Delta R)}{\partial r} + \lambda_{r_1} \frac{\partial (\Delta R)}{\partial \theta} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \lambda_{\theta_1} \frac{\partial (\Delta R)}{\partial \theta} \right]$$

$$+ \frac{\partial}{\partial z} \left[ \lambda_{z_1} \frac{\partial (\Delta R)}{\partial z} - \lambda_{z_2} \Delta R^2 \right] = C_T \frac{\partial (\Delta R)}{\partial t} + q .$$

(18)
in cylindrical coordinates, where
\[ \lambda_{d_1} = \frac{k_{d_1}}{\mu c_f} \quad \text{and} \quad \lambda_{d_2} = \frac{k_{d_2}}{\mu g z_{d_2} \rho_0}, \] (19)
and \( d_\theta \equiv x, y, z \) or \( r, \theta, z \). Eqs. 17 and 18 are the fundamental equations of flow of compressible liquids through porous media in their most general form.

**THE LTFD NUMERICAL METHOD**

**Step 1: The Laplace Transform of the PDE**

The Laplace transform of Eq. 17 yields
\[
\sum_{\eta \equiv x, y, z} \frac{\partial}{\partial \eta} \left[ \lambda_{n_1} \frac{\partial \Psi}{\partial \eta} - \lambda_{n_2} \cdot \Psi - \lambda_{n_2} \cdot \frac{1}{s} \right] = C_T \cdot \left[ s \Psi - \Delta R(0) \right] + \ddot{q},
\] (20)

where \( s \) is the Laplace space parameter, \( \Delta R(0) \) is the distribution of \( \Delta R \) at \( t = 0 \),
\[
C_\beta = 2 + \Delta R,
\] (21)
\[
\ddot{q} = \mathcal{L}\{q\},
\] (22)
and
\[
\Psi = \mathcal{L}\{\Delta R\},
\] (23)
with \( \mathcal{L}\{\} \) denoting the Laplace transform of the quantity in braces. The \( q \) in Eq. 20 does not have to be time-invariant since Eq. 22 is general and allows the transformation of any time-variable \( q(t) \) which has a Laplace transform.

Applying the same procedures to a three-dimensional cylindrical coordinate system yields the following equation in the Laplace space:
\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \lambda_{r_1} \frac{\partial \Psi}{\partial x} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \lambda_{\theta_1} \frac{\partial \Psi}{\partial \theta} \right] + \frac{\partial}{\partial z} \left[ \lambda_{z_1} \frac{\partial \Psi}{\partial z} - \lambda_{z_2} C_\beta \Psi - \lambda_{z_2} \frac{1}{s} \right] = C_T \cdot \left[ s \Psi - \Delta R(0) \right] + \ddot{q}.
\] (24)

The term \( C_\beta \) is obtained from the linearization of
\[
2 \Delta R + (\Delta R)^2 = (2 + \Delta R) \Delta R \approx C_\beta \Delta R
\]
over the interval from \( t = 0 \) to \( t = t \), and
\[
\Delta R = \frac{1}{t} \int_0^t \Delta R \, dt = \frac{1}{t} \mathcal{L}^{-1}\{\Psi/s\}.
\] (25)

If small pressure differentials are involved, the approximation \( C_\beta \approx 2 \) can be used with negligible effects on accuracy.

**Step 2: The FD Scheme in the Laplace Space**

We employed a “block centered” grid\(^6\) method in the space discretization of the transformed PDE. In this paper, the subscripts \( i, j, k \) refer to the position in the three-dimensional grid, and define the \( i \), \( j \)th and \( k \)th grid-block center in the \( x, y \) and \( z \) direction respectively, with \( i = 1, \ldots, M_x, j = 1, \ldots, M_y \) and \( k = 1, \ldots, M_z \). The parameters \( M_x, M_y \) and \( M_z \) are the number of subdivisions of the \( x, y \) and \( z \) coordinate. This discretization divides the solution space into \( N_T = M_x \times M_y \times M_z \) locally uniform and homogenous grid-blocks.

Using a FD scheme with an appropriate space discretization, truncation errors are minimized and the continuous three-dimensional PDE’s (Eqs. 20 and 24) in the Laplace space are approximated by (and reduced to) the following algebraic form (see Appendix):
\[
A_X \Psi_{i-1} + B_X \Psi + C_X \Psi_{i+1} + G_X
+ A_Y \Psi_{j-1} + B_Y \Psi + C_Y \Psi_{j+1} + G_Y
+ A_Z \Psi_{k-1} + B_Z \Psi + C_Z \Psi_{k+1} + G_Z
= C_T \cdot \left[ s \Psi - \Delta R(0) \right] + \ddot{q}.
\] (26)

For simplicity, the subscripts \( i, j, k \) are omitted and only those needed to illustrate communication between gridblocks are mentioned.

Collecting and rearranging terms, the final form of the FD equation in the Laplace space is
\[
A_X \Psi_{i-1} + A_Y \Psi_{j-1} + A_Z \Psi_{k-1} + B
+ C_X \Psi_{i+1} + C_Y \Psi_{j+1} + C_Z \Psi_{k+1} = D,
\] (27)
where \( B = B_X + B_Y + B_Z - C_T \cdot s \), (28)
\[ D = -C_T \cdot \Delta R(0) + \ddot{q} + G_T, \] (29)
and
\[ G_T = -(G_X + G_Y + G_Z). \] (30)

**Step 3: The Solution in the Laplace Space**

The FD approximation of the PDE in the Laplace space results in \( N_T \) simultaneous equations. Written in matrix notation, the FD system of simultaneous equations in Eq. 27 becomes
\[
\mathbf{M} \tilde{\Psi} = \tilde{D},
\] (31)
where \( \mathbf{M} \) is the flow coefficient matrix, \( \tilde{\Psi} \) is the vector of the unknown transformed \( \Delta R \)‘s, and \( \tilde{D} \) the ‘known’ right-hand side vector.

The solution of Eq. 31 necessitates computation of \( \mathbf{M} \) and \( \tilde{D} \), and requires arithmetic values of the \( s \) parameter of the Laplace space. For a desired observation time \( t \), these are provided by the first part of the Stehfest algorithm\(^7,8\) as
\[
s_\nu = \frac{\ln 2}{t} \nu, \quad \nu = 1, \ldots, N_S.
\] (32)
where \( N_S \) is the number of summation terms in the algorithm and \( N_S \) is an even number. Optimum values for \( N_S \) were discussed extensively by Moridis and Reddell\(^1/2\). Solution of Eq. 31 returns the set of \( N_S \) vectors
\[
\tilde{\Psi}_\nu = \mathbf{M}_\nu^{-1} \tilde{D}_\nu, \quad \nu = 1, \ldots, N_S.
\] (33)
To obtain a solution at a time \( t \), all vectors \( \tilde{\Psi}_\nu, \nu = 1, \ldots, N_S \) are needed, i.e. the simultaneous equations have to be solved \( N_S \) times.

**Step 4: The Numerical Inversion of the Laplace Solution**

The unknown vector \( \Delta \tilde{R} \) at time \( t \) is obtained by using the Stehfest algorithm\(^7,8\) to numerically invert the Laplace solutions \( \tilde{\Psi}_\nu \). The procedure is described by the following equations:
\[
\Delta \tilde{R}(t) = \frac{\ln^2 2}{t} \sum_{\nu = 1}^{N_S} W_\nu \cdot \tilde{\Psi}_\nu,
\] (34)
and

\[ W_\nu = F_\nu \sum_{k=0}^{L_M} \frac{\nu^{2k} (2\nu)! \nu^{2k} (2\nu)!}{(\kappa - \frac{1}{2})! (1 + \nu \kappa)! (1 + \nu \kappa)! (1 + \nu \kappa)!} \],

(35)

where \( F_\nu = (-1)^{N_S} \nu^2 + \nu \), \( L_0 = \frac{1}{2}(\nu + 1) \), and \( L_M = \min\{\nu, \frac{N_S}{2}\} \).

\( \vec{R} \) is then computed as

\[ \vec{R}(t) = I + \vec{t} \Delta R(t) \],

(36)

where \( \vec{I} \) is the unit vector. The pressure vector \( \vec{p} \) can be obtained from Eq. 6.

Although the accuracy of the method is theoretically expected to improve with increasing \( N_S \), Strehfuss\(^7\) indicated that with increasing \( N_S \) the number of correct significant figures increases linearly at first and then, due to roundoff errors, decreases linearly. Testing his algorithm against 50 equations with known inverse Laplace transforms, he determined the optimum \( N_S = 10 \) for single precision variables (8 significant figures) and \( N_S = 18 \) for double precision variables (16 significant figures). Investigating the performance of LTDFD in the simulation of flow of incompressible fluids, Mordis and Reddell\(^1,2\) determined that optimum results were obtained for a \( N_S \) between 16 and 20, but that the improvement in accuracy between \( N_S = 8 \) and \( N_S = 20 \) was marginal.

The use of step time functions for time-variable well rates and boundary conditions (i.e. step functions or pulses) poses no conceptual and mathematical problem in the Laplace space. The presence of such step functions may reduce the accuracy of the solution because of limitations of the inversion algorithm. In this case the value of \( N_S \) needed for a sufficiently accurate solution rises to \( 18 \leq N_S \leq 24 \). Although at such a high \( N_S \) some oscillations still persist in the immediate vicinity of the time \( t_s \) at which the step function or pulse occurs, we determined that these affect the solution by less than 1%, completely disappear for \( t \geq 1.15 t_s \), and have a negligible effect on the mass balance error.

In a standard FD method, there are three sources of error: the time-related truncation error, the space-related truncation error, and the roundoff error. In the LTDFD method (which has no time-related truncation error) the sources of error are the space-related truncation error, the error from the numerical inversion of the Laplace solution, and the roundoff error. It should be mentioned that heterogeneity and anisotropy in LTDFD have no effect on the performance of the method, and are treated in exactly the same way as in standard FD models. The numerical inversion creates very little error, and the limited number of operations involved significantly reduces the roundoff error. Thus, the LTDF solution is inherently more accurate than the standard FD method for the same grid system. It offers a stable, non-increasing roundoff error irrespective of the time of observation \( t_{obs} \) because calculations to have to be performed at this time only using a \( \Delta t = t_{obs} \). Moreover, the LTDF method does not increase the memory requirements over a standard FD method because the \( N_S \) sets of unknowns can be stored and summed in a single array. In the FD method this array is that of the unknowns at the end of the previous timestep.

**Inner Iterations.** These are required only when the approximations of \( \sigma_T \) in Eq. 13 and of \( C_T \) in Eq. 21 are not acceptably accurate. Under these circumstances, the LTDFD method is easily extended to cover these weak nonlinearities through an iteration procedure analogous to the treatment of such nonlinearities in FD. The process involves a limited number of inner iterations (2 to 4), during which the values of \( \sigma_T \) and \( C_T \) are updated and an improved solution \( \Delta R \) is obtained until a desired convergence criterion is met.

**Verification and Test Problems**

The LTDFD numerical method was tested using three test problems which represented increasing levels of complexity. Analytical solutions exist for the first two problems. The LTDFD solution was verified through comparison with the analytical solutions. No analytical solutions exist for the other two test problems. In all three cases the results obtained with LTDFD were tested against results obtained using a standard, commercially available implicit FD simulator\(^9\) with the same space discretization. A direct banded-matrix solver was used to solve the system of simultaneous equations arising in both the LTDFD and FD methods. Double precision variables with 16 and 20 significant figures were used in all simulations. A variable timestep \( \Delta t \) was used for all test cases in the FD simulator, given by the recursive formula

\[ \Delta t = \min\{M_L \times \Delta t_{t-1}, \Delta t_{max}, \Delta t_{\Delta t_{max}}\} \]

(37)

where \( M_L \) is a multiplier, \( \Delta t_{max} \) is the maximum permissible \( \Delta t \), and \( \Delta t_{\Delta t_{max}} \) is the \( \Delta t \) corresponding to a maximum permissible pressure change \( \Delta p \). Table 1 shows the \( M_L, \Delta t_{t}, \Delta t_{max} \), and \( \Delta t_{\Delta t_{max}} \) used in the three test problems, as well as the number of timesteps and matrix solutions (i.e. the number of times the system of simultaneous equations in the FD simulator had to be solved) in the simulations. Because of internal iterations, the number of matrix solutions is significantly larger than the number of timesteps.

**Verification and Test Problem 1**

Test Problem 1 was a problem of flow to a well in a circular bounded reservoir, for which a number of exact and approximate analytical solutions exist. The fluid properties are: \( p = 5000 \) psi, \( B = 1.2 \), \( \mu = 1 \) cp, and \( C_f = 1.5 \times 10^{-1} \) psi\(^{-1}\). Reservoir properties and dimensions, and discretization information for the FD and LTDFD simulations appear in Table 2. We used the equation provided by van Everdingen and Hurst\(^10\) for the computation of the analytical solutions.

We investigated three subproblems. In Problem 1a we compared the LTDFD and the analytical solutions over the whole reservoir at 4 observation times. We used \( N_S = 8 \) in all the LTDFD simulations of Problem 1a. Fig. 1 shows that the two solutions coincided regardless of the magnitude of observation time. This gives the measure of the accuracy of LTDFD, and shows the complete insensitivity of the method to the size of the time increment.

In Problem 1b we compared the LTDFD solution with \( N_S = 8 \) to a) the analytical solution, and b) 4 FD solutions (see Table 1) for \( t = 60 \) days. Fig. 2 shows in a dimensionless formulation the percent difference between the analytical and the numerical solutions. The LTDFD solution exhibited a very small deviation from the analytical solution, and was consistently superior to the FD difference solution. The superiority of LTDFD persisted even when a very fine time discretization (65 timesteps, and a total of 136 matrix solutions) was used in the FD simulation, indicating that for 1/17th of the computational load (8 vs 136) LTDFD returned a more accurate solution than the FD. With an increasing number of timesteps (corresponding to smaller \( \Delta \), the FD solutions tended asymptotically towards the analytical and the LTDFD solutions.

In Problem 1c we studied the effect of \( N_S \) on the performance of LTDFD. For a \( N_S \) ranging between 6 and 20, we observed a negligible difference between the analytical and the LTDF solutions (Fig. 3) at \( t = 1000 \) days. The minimum absolute difference was observed for \( N_S = 6 \), and increased with an increasing \( N_S \) until \( N_S = 12 \). The
solutions for an $N_S$ between 12 and 20 were identical and exhibited arithmetic differences in the 7th decimal place and beyond.

The implications of the results in Fig. 3 are that a) the accuracy of LTDF for this one-dimensional problem is practically insensitive to the value of $N_S$, and b) the number of summation terms $N_S$ for an accurate solution may be 6, far smaller than the $N_S = 18$ which Stehfest suggested for double precision variables. This drastically reduces the execution time and makes the LTDF method even more efficient than theoretically predicted.

**Verification and Test Problem 2**

Test Problem 2 involved flow towards a well at the center of a bounded square reservoir with an infinite conductivity vertical fracture at the center of the square. The fluid properties remain as in Test Problem 1. Fig. 4 shows the reservoir geometry, and Table 3 presents reservoir properties and dimensions, and the grid discretization used in the FD and LTDF simulations. We used the analytical solution of Gringarten et al\textsuperscript{11} (which predicts the unsteady-state pressure at the well) as a reference. We investigated two subproblems.

In Problem 2a we compared the LTDF and analytical solutions at the well for 9 observation times. We let $N_S = 8$ in the LTDF simulations. We observed a pattern similar to the one in Test Problem 1. The two solutions coincided regardless of the magnitude of observation time (Fig. 4). This testifies to the power and accuracy of LTDF, and confirms the complete insensitivity of the method to the size of the time increment.

Analytical solutions of the pressure distribution in the formation do not exist for this problem. In Problem 2b we compared the LTDF solution ($N_S = 8$) to 4 FD solutions along the $y$ axis at $x = 0.025$ ft at $t = 365$ days. Fig. 5 shows a dimensionless formulation the difference between the LTDF and the FD solutions. We observed an identical pattern: with an increasing number of $\Delta t$'s the FD solutions a) tended to the LTDF solution, and consequently b) the difference between the two solutions decreased. Using a very fine time discretization (79 timesteps, and a total of 231 matrix solutions), the FD yielded a solution within 0.1 psi of the LTDF solution, but with a computational load 29 times larger.

In Problem 2c we studied the effect of $N_S$ on the performance of LTDF along the same axis and at the same time. In Fig. 6 we show the absolute difference between the LTDF solutions for different $N_S$ values from a reference, taken as the solution obtained with $N_S = 8$. A slightly different pattern emerged. With the exception of $N_S = 6$, all other solutions exhibited extremely small pressure differences from each other, i.e. 0.1 to 0.2 psi when $p$ was in excess of 3000 psi. The solution for $N_S = 6$ was consistently about 5 psi higher than the rest of the solutions, which, while significantly larger than all the other deviations, was still very small when compared to the maximum pressure change of 1835 psi. In light of the significant difference in the computational load, a $N_S = 6$ could easily be used when extreme accuracy is not essential. These results confirm our previous observations that the accuracy of LTDF is virtually insensitive to the value of $N_S$ for $8 \leq N_S \leq 20$. Slightly lower accuracy (insignificant for most practical applications) may be obtained for $N_S = 6$.

**Test Problem 3**

Test Problem 3 was a two-dimensional (areal) simulation problem, and represented a reservoir with spatially variable properties and wells with variable flow rates. The fluid properties remain the same as in Test Problem 1. The reservoir geometry, formation thickness, and permeability distribution are depicted in the contour plots of Fig. 7. The reservoir $\phi = 0.1$, $c_{ftm} = 0$. The domain was discretized in $48 \times 36 = 1728$ uniform gridblocks in ($x, y$), with $\Delta x = \Delta y = 200$ ft. Table 4 lists the production rates for the 10 wells in the reservoir, of which 3 have constant rates, 3 have linearly declining (with time) rates, and 4 have exponentially declining rates. No analytical solution is available for this problem. A comparison of the LTDF and the FD solutions provided a measure of the power, validity and accuracy of the LTDF method.

Fig. 8 shows the pressure distribution in the reservoir for the LTDF solution ($N_S = 8$) at $t = 180$ days. We evaluated LTDF by comparing the pressure distribution along the $x$ axis at $y = 2900$ ft, corresponding to row $J = 15$ (Fig. 9). The accuracy of the LTDF solution was indicated by the fact that for a decreasing $\Delta t$ (which results in smaller truncation errors, more accurate solutions, and larger numbers of timesteps) the FD solution tended to approach the LTDF solution and the difference between the two solutions decreased. For 86 timesteps, a total of 329 matrix solutions, and 41 times the computational load of LTDF, the FD solution virtually coincides with the LTDF solution. FD cannot deliver the accuracy of LTDF unless a very large number of timesteps is taken, and then only at the expense of significantly increased CPU time. Fig. 9 demonstrates that LTDF can capture in detail the significant pressure changes due to the presence of wells and zones of different permeability. The well locations and permeability zones can be identified by the existence of peaks and sharp variations in drawdown. These variations decrease in magnitude with a decreasing $\Delta t$ size in the FD solution. For large $\Delta t$'s, the FD solutions exhibited significant deviations and insufficient accuracy. This was caused by the averaging effect used in the treatment of the time derivative. With smaller $\Delta t$'s, these deviations decreased, and the accuracy of FD the solution improved.

**Material Balance Error Considerations**

A very important measure of the performance of the LTDF method was provided by the magnitude of the material balance error as a percentage of the original fluid mass. Mass conservation is a necessary but not sufficient condition to guarantee accuracy in parabolic-type problems.

Fig. 10 shows an analysis of the behavior of the material balance errors for the three Test Problems for both the LTDF and the FD methods. The small magnitude (bordering machine precision) of the material balance error for the methods shows that they both conserve mass. The extremely small magnitude of the LTDF material balance error further testified to the power of the method. LTDF seemed to have an overall advantage over FD, indicated by the generally lower material balance errors. With the conditional exception of the computationally simple Test Problem 1, LTDF produced consistently a smaller error. This was more obvious in the more difficult problem 3. The $\Delta R$ formulation poses some computational difficulties since it involves matrices with coefficients that vary by up to 8 orders of magnitude. In small problems, LTDF will produce a very accurate solution, but the roundoff error involved may make FD infinitely or more accurate (i.e. by the difference between $10^{-11}$ and $10^{-12}$). For larger problems LTDF seems to have an advantage because of a) a smaller truncation error, equal to the truncation error of the space discretization only, and b) a smaller roundoff error due to fewer operations.

The material balance error of the LTDF method generally improved as $N_S$ increased. We determined that a $12 \leq N_S \leq 16$ yields the smallest material balance error. However, for practical purposes, the improvement between $N_S = 6$ and $N_S = 20$ was marginal. The implications of these observations are that a) for $N_S \leq 20$, the accuracy
of LTFD is practically insensitive to the value of $N_S$, and b) the number of summation terms $N_S$ for an accurate solution may be far fewer than the $N_S = 18$ suggested by Stehfest\textsuperscript{7} for double precision variables. Our experience in this area tends to suggest that the combination of the Finite Difference discretization with the Stehfest algorithm may act as a filter in the LTFD scheme, removing frequencies which could cause errors in the application of the Stehfest algorithm in the inversion of simple functions. A value of $6 \leq N_S \leq 10$ is adequate for LTFD simulations. This reduces significantly the execution time requirements.

The material balance error for LTFD method starts to deteriorate when $N_S > 16$. This is probably due to limitations imposed by the computer accuracy in the determination of the weighing factors $V_r$ in Eq. 35. The calculations involve operations with factorials, which may become inaccurate for large numbers. Although this deterioration may be eliminated in computers with more significant figures, this would be pointless and undesirable because Eq. 31 would have to be solved more times and would require longer execution times.

**CONCLUSIONS**

A new numerical method, the Laplace Transform Finite Difference (LTFD) method, was developed for the solution of the nonlinear, parabolic Partial Differential Equation (PDE) of transient, slightly compressible, single-phase liquid flow through porous media. The nonlinearity of the PDE is removed by using a variation of the density formulation, the $\Delta R$ formulation. Because LTFD uses a Laplace transform formulation to remove the time dependency and eliminate the need for time discretization, it allows an unlimited timestep size without loss of stability or accuracy.

To evaluate the LTFD method we investigated three Test Problems. LTFD requires solution of the resulting system of simultaneous equations $N_S$ times, one for each of the $N_S$ different approximations of the Laplace space variable $s$. In essence, this means that the solution at any time is obtained by solving the system of simultaneous equations in LTFD $N_S$ times and algebraically combining the solutions. We established that a $6 \leq N_S \leq 10$ is sufficient to provide an extremely accurate solution. Although the accuracy increases with increasing $N_S$ for $N_S \leq 16$, the improvement is marginal and insufficient to justify the additional execution time.

The disadvantage of having to solve the system of simultaneous equations 6 to 10 times for a single timestep is outweighed by a) an unlimited timestep size without any loss of accuracy, b) a superior accuracy, and c) a stable, non-increasing roundoff error. Therefore, calculations in a LTFD scheme are necessary only at the desired observation times, thus allowing 'snapshots' in time. On the other hand, in a standard FD method calculations are needed at all the intermediate times of the discretized time domain. Moreover, the LTFD method does not increase the computer memory requirement over a standard implicit FD method.

With a smaller timestep size and more timesteps, the FD solution tended to approach the LTFD solution. The LTFD method provided a solution generally more accurate than the FD solution for the same space discretization. This was expected because a) the elimination of the time discretization limited the truncation error to that of the space discretization only, and b) the roundoff error was reduced due to the limited number of operations needed. An unlimited timestep size with a stable, non-increasing error is thus possible.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_d$</td>
<td>flow coefficient defined in Eq. A-7</td>
</tr>
<tr>
<td>$B$</td>
<td>formation volume factor</td>
</tr>
<tr>
<td>$B_d$</td>
<td>flow coefficient defined in Eq. A-9</td>
</tr>
<tr>
<td>$c_f$</td>
<td>fluid compressibility</td>
</tr>
<tr>
<td>$c_{fm}$</td>
<td>rock compressibility</td>
</tr>
<tr>
<td>$C_d$</td>
<td>flow coefficient defined in Eq. A-8</td>
</tr>
<tr>
<td>$C_f$</td>
<td>$c_f + c_{fm}$</td>
</tr>
<tr>
<td>$C_T$</td>
<td>defined in Eq. 12</td>
</tr>
<tr>
<td>$C_{B}$</td>
<td>defined in Eq. 21</td>
</tr>
<tr>
<td>$d^o$</td>
<td>$x$, $y$, $z$</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$G_{d0}$</td>
<td>defined in Eq. A-10</td>
</tr>
<tr>
<td>$h$</td>
<td>formation thickness</td>
</tr>
<tr>
<td>$I$, $J$, $K$</td>
<td>gridblock indices in the FD and LTFD models</td>
</tr>
<tr>
<td>$k$</td>
<td>absolute permeability</td>
</tr>
<tr>
<td>$L$</td>
<td>length of horizontal well</td>
</tr>
<tr>
<td>$M_X$</td>
<td>gridblocks in the $x$ direction (similarly for $y$, $z$)</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
</tr>
<tr>
<td>$PD$</td>
<td>dimensionless pressure $= 2\pi k h (p_{i0} - p)/(Q B \mu)$</td>
</tr>
<tr>
<td>$P_w$</td>
<td>well pressure</td>
</tr>
<tr>
<td>$q$</td>
<td>volumetric flow rate per unit volume</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>defined in Eq. 22</td>
</tr>
<tr>
<td>$Q$</td>
<td>volumetric flow rate</td>
</tr>
<tr>
<td>$r$</td>
<td>radius</td>
</tr>
<tr>
<td>$r_w$</td>
<td>well radius</td>
</tr>
<tr>
<td>$r_D$</td>
<td>dimensionless radius $= r/r_w$</td>
</tr>
<tr>
<td>$R$, $\Delta R$</td>
<td>defined in Eqs. 5 and 16</td>
</tr>
<tr>
<td>$s$</td>
<td>Laplace transform parameter</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$t_D$</td>
<td>dimensionless time $= kt/(\pi \mu c_1 r_w^2)$</td>
</tr>
<tr>
<td>$t_{DA}$</td>
<td>dimensionless time $= kt/(4 \mu c_1 x_c y_c)$</td>
</tr>
<tr>
<td>$w$</td>
<td>mass flow rate per unit volume</td>
</tr>
<tr>
<td>$x_R$</td>
<td>reservoir half length</td>
</tr>
<tr>
<td>$x_f$</td>
<td>fracture half length</td>
</tr>
<tr>
<td>$y_R$</td>
<td>reservoir half width</td>
</tr>
<tr>
<td>$z$</td>
<td>vertical coordinate</td>
</tr>
<tr>
<td>$z_{d0}$</td>
<td>elevation gradient in the $d^o$ direction</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$= 1/B$</td>
</tr>
<tr>
<td>$\hat{d}^o$</td>
<td>$= \frac{1}{2} \Delta d^o$</td>
</tr>
<tr>
<td>$\Delta d^o$</td>
<td>$= \Delta x$, $\Delta y$, $\Delta z$: Mesh spacing in $x$, $y$, and $z$</td>
</tr>
<tr>
<td>$\Delta d^o_{\kappa+1}$</td>
<td>$= \frac{1}{2} (\Delta d^o_{\kappa} + \Delta d^o_{\kappa+1})$</td>
</tr>
<tr>
<td>$\Delta p$</td>
<td>difference between the LTFD and the FD solution</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>timestep</td>
</tr>
<tr>
<td>$\Delta t_{max}$</td>
<td>maximum allowable timestep</td>
</tr>
<tr>
<td>$\Delta t_{max}$</td>
<td>defined after Eq. 38</td>
</tr>
<tr>
<td>$\Delta \Psi$</td>
<td>defined in Eq. 21</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$= i$, $j$, $k$</td>
</tr>
<tr>
<td>$\lambda_{d^1,2}$</td>
<td>defined in Eq. 19</td>
</tr>
<tr>
<td>$\mu$</td>
<td>dynamic viscosity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>fluid density</td>
</tr>
<tr>
<td>$\phi$</td>
<td>porosity</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>defined in Eq. 23</td>
</tr>
</tbody>
</table>

**Subscripts**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>initial state</td>
</tr>
<tr>
<td>$d^*$</td>
<td>$X$, $Y$, $Z$</td>
</tr>
<tr>
<td>$i$, $j$, $k$</td>
<td>gridblock indices in the $x$, $y$, and $z$ direction</td>
</tr>
<tr>
<td>$STC$</td>
<td>standard conditions</td>
</tr>
</tbody>
</table>
ACKNOWLEDGEMENTS
This work was supported by the Texas Agricultural Experiment Station under grant 6993/91, and by the Director, Office of Energy Research, Office of Basic Energy Sciences, U.S. Department of Energy under Contract No. DE-AC03-76SF00098. The authors thank Drs. Curt Oldenburg and Sally Benson for their useful review comments.

APPENDIX – DERIVATION OF THE FLOW COEFFICIENTS
Using a Taylor series, the \(d^s\)-derivative \((d^s \equiv x, y, z)\) on the left side of Eq. 20 is approximated by

\[
\frac{\partial}{\partial d^s} \left[ \sum_{\lambda=1}^{2} \lambda^{d^s} \frac{\partial \Psi}{\partial d^s} - \lambda^{d^s} C_\beta \Psi + \frac{\lambda^{d^s}}{s} \right] \approx \left( \frac{\lambda^{d^s}}{s} \right)_{\kappa - \frac{1}{2}} \left( \frac{\lambda^{d^s}}{s} \right)_{\kappa - \frac{1}{2}}
\]

\[
\frac{1}{\Delta d^s \kappa} \left[ \sum_{\lambda=1}^{2} \frac{\Delta \Psi^\pm}{\Delta d^s \kappa} - \left( \frac{\lambda^{d^s}}{s} \right)_{\kappa - \frac{1}{2}} \right], \quad (A-1)
\]

where

\[
\Delta \Psi^\pm = \Psi^\pm_{\kappa - \frac{1}{2}} - \Psi^\pm_{\kappa - \frac{1}{2}}
\]

\[
\left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa \pm \frac{1}{2}} = \frac{\delta d^s_{\kappa}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa \pm 1} + \frac{\delta d^s_{\kappa - \frac{1}{2}}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa}
\]

\[
\left( \frac{\lambda^{d^s}}{s} \right)_{\kappa \pm \frac{1}{2}} = \frac{\delta d^s_{\kappa}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa \pm 1} + \frac{\delta d^s_{\kappa - \frac{1}{2}}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa}
\]

\[
\left( \frac{\lambda^{d^s}}{s} \right)_{\kappa \pm \frac{1}{2}} = \frac{\delta d^s_{\kappa}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa \pm 1} + \frac{\delta d^s_{\kappa - \frac{1}{2}}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa}
\]

\[
\left( \frac{\lambda^{d^s}}{s} \right)_{\kappa \pm \frac{1}{2}} = \frac{\delta d^s_{\kappa}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa \pm 1} + \frac{\delta d^s_{\kappa - \frac{1}{2}}}{\Delta d^s_{\kappa - \frac{1}{2}}} \left( \lambda^{d^s} C_\beta \Psi \right)_{\kappa}
\]

\[
A_{\kappa} \cdot \Psi_{\kappa - \frac{1}{2}} + B_{\kappa} \cdot \Psi_{\kappa} + C_{\kappa} \cdot \Psi_{\kappa + \frac{1}{2}} + G_{\kappa} \cdot \Psi_{\kappa}, \quad (A-6)
\]

\[
A_{\kappa} = \frac{(\lambda^{d^s} C_\beta \Psi)_{\kappa \pm 1} - \delta d^s_{\kappa - \frac{1}{2}} \cdot (\lambda^{d^s} C_\beta \Psi)_{\kappa}}{\Delta d^s \kappa \cdot \Delta d^s \kappa - \frac{1}{2}}, \quad (A-7)
\]

\[
C_{\kappa} = \frac{(\lambda^{d^s} C_\beta \Psi)_{\kappa \pm 1} - \delta d^s_{\kappa + \frac{1}{2}} \cdot (\lambda^{d^s} C_\beta \Psi)_{\kappa}}{\Delta d^s \kappa \cdot \Delta d^s \kappa + \frac{1}{2}}, \quad (A-8)
\]

\[
B_{\kappa} = -(A_{\kappa} + C_{\kappa}), \quad (A-9)
\]

\[
G_{\kappa} = \frac{1}{\Delta d^s \kappa} \left[ \frac{(\lambda^{d^s}}{s})_{\kappa - \frac{1}{2}} - \frac{(\lambda^{d^s}}{s})_{\kappa + \frac{1}{2}} \right], \quad (A-10)
\]

\[
d^s \equiv x, y, z.
\]

REFERENCES

128 SPE Advanced Technology Series, Vol. 2, No. 2
SI METRIC CONVERSION FACTORS

bbl × 1.589873 E-01 = m³
cp × 1.0 E-03 = Pa·s
ft × 3.048 E-01 = m
md × 9.869233 E-04 = μm²
psi × 6.894757 E+00 = kPa

Authors

George Moridis is a Staff Scientist in the Earth Sciences Division at the Lawrence Berkeley Laboratory. His research interests include new modeling methods, numerical simulation and experimental investigations of mass and energy flow and transport through porous media. He holds a BS and a ME from Metsovion National Technical University of Athens, and a ME and a PhD from Texas A&M University. Duane McVay is the Vice President for Reservoir Simulation Services of S. A. Holditch and Associates in College Station, TX. He holds BS and MS degrees in Petroleum Engineering from Texas A&M University. Donald Reddell is a professor and head in the Department of Agricultural Engineering at Texas A&M University. His teaching and research interests include simulation of heat, single- and multi-phase flow, and contaminant transport through porous media. He holds a BS from Texas Tech University, and a MS and a PhD from Colorado State University. Thomas Blasingame is an Assistant Professor of Petroleum Engineering at Texas A&M University, from which he holds BS, MS and PhD degrees in Petroleum Engineering.

TABLE 1 - TIME DISCRETIZATION FOR THE FD METHOD

<table>
<thead>
<tr>
<th>Test #</th>
<th>Δt₀ (days)</th>
<th>Δt max (days)</th>
<th>M_L</th>
<th>ΔP max (psi)</th>
<th>Num. of Δt</th>
<th>Num. of MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>60</td>
<td>1</td>
<td>3000</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>30</td>
<td>5</td>
<td>1000</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>10</td>
<td>5</td>
<td>500</td>
<td>9</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>1</td>
<td>5</td>
<td>250</td>
<td>65</td>
<td>136</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>365</td>
<td>5</td>
<td>1000</td>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>100</td>
<td>2</td>
<td>500</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>50</td>
<td>3</td>
<td>500</td>
<td>13</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>5</td>
<td>5</td>
<td>250</td>
<td>79</td>
<td>231</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>100</td>
<td>5</td>
<td>1000</td>
<td>10</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>50</td>
<td>5</td>
<td>500</td>
<td>18</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>25</td>
<td>5</td>
<td>500</td>
<td>21</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>2</td>
<td>5</td>
<td>250</td>
<td>86</td>
<td>329</td>
</tr>
</tbody>
</table>

MS: Matrix Solution

TABLE 2 - RESERVOIR PROPERTIES, GEOMETRY, AND DISCRETIZATION IN TEST PROBLEM 1

<table>
<thead>
<tr>
<th>φ</th>
<th>k</th>
<th>Q</th>
<th>r_w</th>
<th>r_e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>10 md</td>
<td>50 bbl/D</td>
<td>0.25 ft</td>
<td>2,500 ft</td>
</tr>
<tr>
<td>c_m</td>
<td>0.0 psi⁻¹</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h</td>
<td>30 ft</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Discretization: Mₓ = 62

Δr (in ft) 0.00033, 0.00066, 0.00131, 0.00262, 0.00328, 0.0105,
3.281, 3.281, 4.921, 4.921, 6.672, 1.3438, 2.6877,
3.281, 3.281, 4.921, 1.3438, 2.6877,
37.139, 3.077, 2.582, 0.843, 6.562, 3.281, 1.64, 0.984, 0.328,
0.164, 0.164

TABLE 3 - RESERVOIR PROPERTIES, GEOMETRY, AND DISCRETIZATION IN TEST PROBLEM 2

<table>
<thead>
<tr>
<th>φ</th>
<th>c_m</th>
<th>h</th>
<th>k_x = k_y</th>
<th>Q</th>
<th>x_l</th>
<th>x_R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.0 psi⁻¹</td>
<td>30 ft</td>
<td>10 md</td>
<td>Q</td>
<td>500 ft</td>
<td>1000 ft</td>
</tr>
<tr>
<td>fracture k_x</td>
<td>10⁻⁷ md</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Discretization: Mₓ = 40, Mᵧ = 24

Δx (in ft) 0.005, 0.002, 0.025, 0.05, 0.25, 0.5, 1, 5, 2, 5, 7, 5, 15,
25, 25, 6, 40, 2, 20, 10, 5, 1, 5, 10, 1, 20, 30,
40, 6, 50, 7.7, 5

Δy (in ft) 0.001, 0.004, 0.015, 0.03, 0.05, 1, 0.3, 0.5, 1, 3, 5, 10, 20,
30, 50, 80, 80, 100

TABLE 4 - WELL INFORMATION IN PROBLEM 3

<table>
<thead>
<tr>
<th>Well</th>
<th>Q₀ (bbl/D)</th>
<th>Q₁ (bbl/D)</th>
<th>Q₂ (bbl/D)</th>
<th>Q₃ (1/D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>-15</td>
<td>0.0041</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>-10</td>
<td>0</td>
<td>-50</td>
<td>0.0037851</td>
</tr>
<tr>
<td>22</td>
<td>-15</td>
<td>-60</td>
<td>0.0189254</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>-20</td>
<td>-45</td>
<td>0.0037851</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>31</td>
<td>-15</td>
<td>-30</td>
<td>0.0037851</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>-40</td>
<td>0.0137</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>38</td>
<td>-20</td>
<td>0.0055</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>42</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

General well rate equation: Q = Q₀ + Q₁ t + Q₂ exp(-Q₃ t)
Figure 1 - Comparison of analytical and LTFD solutions of $p_D$ for various $t_D$'s in Test Problem 1a.

Figure 2 - % Difference of the LTFD and the FD solutions of $p_D$ from the analytical solution in Test Problem 1b.

Figure 3 - Effect of $N_s$ on the difference (%) of the LTFD solution from the analytical solution of $p_D$ in Test Problem 1c.

Figure 4 - Comparison of the analytical and the LTFD solutions of $p_D$ for various $t_{DA}$'s in Test Problem 2a. The inset depicts the reservoir geometry.

Figure 5 - Test Problem 2b-Difference $\Delta p$ between the LTFD and the FD solutions along $y$ at $x=0.025$ ft ($t=1$), $t=365$ days. The pressure distribution along the same axis is shown.

Figure 6 - Effect of $N_s$ on the performance of LTFD, indicated by the absolute difference of the LTFD solutions for $N_s = 6,...,18$ from the LTFD solution for $N_s = 8$ (Test Problem 2c).
Figure 7 - Reservoir geometry and property profile of the reservoir in Test Problem 3: (a) formation thickness, (b) permeability.

Figure 8 - LTFD solution of pressure distribution in the reservoir at \( t = 180 \) days (Test Problem 3).

Figure 9 - Difference (% \( \Delta p \)) between the LTFD and the FD solutions along \( x \) at \( y=2900 \) ft (\( J=15 \)), \( t=180 \) days (Test Problem 3).

Figure 10 - Material balance errors: (a) LTFD, (b) FD.