Deconvolution of Variable-Rate Reservoir-Performance Data Using B-Splines

D. Ilk, P.P. Valkó, SPE, and T.A. Blasingame, SPE, Texas A&M U.

Summary
We use B-splines for representing the derivative of the unknown unit-rate drawdown pressure and numerical inversion of the Laplace transform to formulate a new deconvolution algorithm. When significant errors and inconsistencies are present in the data functions, direct and indirect regularization methods are incorporated. We provide examples of under- and over-regularization, and we discuss procedures for ensuring proper regularization.

We validate our method using synthetic examples generated without and with errors (up to 10%). Upon validation, we then demonstrate our deconvolution method using a variety of field cases, including traditional well tests, permanent downhole gauge data, and production data. Our work suggests that the new deconvolution method has broad applicability in variable rate/pressure problems and can be implemented in typical well-test and production-data-analysis applications.

Introduction
The constant-rate drawdown pressure behavior of a well/reservoir system is the primary signature used to classify/establish the characteristic reservoir model. Transient-well-test procedures typically are designed to create a pair of controlled flow periods (a pressure-drawdown/buildup sequence) and to convert the last part of the response (the pressure buildup) to an equivalent constant-rate drawdown by means of special time transforms. However, the presence of wellbore storage, previous flow history, and rate variations may mask or distort characteristic features in the pressure and rate responses.

With the ever-increasing ability to observe downhole rates, it has long been recognized that variable-rate deconvolution should be a viable option to traditional well-testing methods because deconvolution can provide an equivalent constant-rate response for the entire time span of observation. This potential advantage of variable-rate deconvolution has become particularly obvious with the appearance of permanent downhole instrumentation.

First and foremost, variable-rate deconvolution is mathematically ill-conditioned; while numerous methods have been developed and applied to deconvolve “ideal” data, very few deconvolution methods perform well in practice. The ill-conditioned nature of the deconvolution problem means that small changes in the input data cause large variations in the deconvolved constant-rate pressures. Mathematically, we are attempting to solve a first-kind Volterra equation (see Lamm (2000)) that is ill-posed. However, in our case the kernel of the Volterra-type equation is the flow-rate function (i.e., the generating function); this function is not known analytically but, rather, is approximated from the observed flow rates. In practical terms, this issue adds to the complexity of the problem (Stewart et al. 1983).

In the literature related to variable-rate deconvolution, we find the development of two basic concepts. One concept is to incorporate an a priori knowledge regarding the properties of the deconvolved constant-rate response. The observations of Coats et al. (1964) on the strict monotonicity of the solution led Kuchuk et al. (1990) to impose a “nonpositive second derivative” constraint on pressure response. In some respects, this tradition is maintained in the work given by von Schroeter et al. (2004), Levitan (2003), and Gringarten et al. (2003) when they incorporate non-negativity in the “encoding of the solution.” We note that in the examples given, this concept (non-negativity/monotonicity of the solution) requires less-straightforward numerical methods (e.g., nonlinear least-squares minimization).

The second concept is to use a certain level of regularization (von Schroeter et al. 2004; Levitan 2003; Gringarten et al. 2003), where “regularization” is defined as the act or process of making a system regular or standard (smoothing or eliminating nonstandard or irregular response features). Regularization can be performed indirectly, by representing the desired solution with a restricted number of “elements,” or directly, by penalizing the non-smoothness of the solution. In either case, the additional degree of freedom (the regularization parameter) has to be established, where this is facilitated by the discrepancy principle (effectively tuning the regularization parameter to a maximum value while not causing intolerable deviation between the model and the observations). In some fashion, each deconvolution algorithm developed to date combines these two concepts (non-negativity/monotonicity of the solution or regularization).

In addition to these main features, individual deconvolution algorithms may have other distinctive features. From a numerical standpoint, recent deconvolution methods tend to use advanced techniques to solve the underlying system of linear equations [e.g., singular value decomposition, the equivalent concept of pseudo-inverse, etc.—for example, see Cheney and Kincaid (2003)].

One class of approaches makes extensive use of transformations (i.e., the Laplace transform, the Fourier transform, etc.). Several cases consider numerical Laplace transformation of observed tabulated data combined with numerical inversion (Roumboutsos and Stewart 1988; Onur and Reynolds 1998). In addition, Cheng et al. (2003) consider the application of the Fourier transformation to solve the deconvolution problem.

Error-minimization methods, typically driven by some type of least-squares algorithm, include von Schroeter’s total least squares method [von Schroeter et al. (2004); Gringarten et al. (2003); Levitan (2003)], as well as a group of methods that rely on the use of spline functions (in various steps of the deconvolution algorithm) [e.g., Guillo and Horne (1986) and Mendes et al. (1989)].

Formally, it would be easy to place our proposed method into existing categories. Our proposed approach uses B-splines; numerical inversion of the Laplace transform is also used for certain components of the solution, and regularization is provided indirectly (i.e., by the number of knots used in the selected B-spline) and directly (by penalizing the nonsmoothness of the logarithmic derivative of the reconstructed constant-rate response). However, in detail our approach is radically different from any of the existing methods—in contrast to previous methods, our approach does not fit B-splines to observed data. Rather, B-splines are used to represent the unknown response solution (i.e., as a linear combination of B-splines). With respect to the Laplace transform, our approach does not use the Laplace transform to transform the observed pressure data series, nor does our approach use numerical inversion to provide the objective response (i.e., the constant-rate response), as do other spectral methods. In our approach, the application of
numerical inversion of the Laplace transform is restricted to the construction of the sensitivity matrix for the least-squares problem.

Our new technique represents the derivative of the unknown constant-rate drawdown pressure response as a weighted sum of B-splines, using logarithmically distributed knots. For the discrete flow-rate function, we use piecewise constant, piecewise linear, or any other appropriate representation for which the Laplace transform of the convolution kernel: input signal, the Laplace transform of the convolution theorem is applicable. Therefore, the convolution integral equation of the first kind (i.e., the convolution integral), where the role of the convolution kernel is represented by the sandface rate.

Lamm (2000) shows that for \( q(t-\tau)_{\tau=t=0} = 0 \)—that is, when the rate does not jump instantaneously at the first moment—the left side of the equation is “smoothing at least of order two,” and hence, the problem is severely ill-conditioned. In addition, the sandface flow rate is also observed in discrete points and is always corrupted in practice by some form of error; therefore, the results will further vary depending on the way in which we make the transition from discrete observations into the convolution kernel: \( q(t-\tau) \).

As noted by Lamm, the convolution operation has a natural smoothing effect; therefore, significantly different \( K(t) \) functions can (and may) reconstruct essentially the same \( \Delta p(t) \) response for a given rate history. Ideally, \( K(t) = 0 \) and decreases monotonically [Coats et al. (1964) showed that derivatives of a higher order must remain monotonic]. Unfortunately, even using downhole measurements, measured (observed) flow rates contain effects that may cause deviation from monotonic behavior (e.g., wellbore storage or other variable-rate conditions). If the response contains a skin effect and no wellbore storage, then the impulse response, \( K(t) \), will contain a Dirac delta component (van Everdingen and Hurst 1949). Therefore, the conceptual representation of the \( K(t) \) function is not trivial, a point often established by numerous failed attempts to represent this function. In particular, many algorithms substitute the convolution integral in Eq. 1 by the trapezoidal rule (or some similar approach), which is difficult to justify near the origin.

**Development of a New B-Spline-Based Deconvolution Method**

**Splines Over Logarithmically Distributed Knots.** Spline functions (Cheney and Kincaid 2003) are piecewise polynomial functions that are defined on subintervals connected by points called knots \( t_i \) and have the additional property of continuity (of the function and its derivatives). In this work, we represent the unknown \( K(t) \) function as a second-order spline with logarithmically spaced knots. To reveal characteristic reservoir behavior, the number of knots should be on the order of at least 2 to 6 knots per log cycle [for typical kernels of interest (i.e., input flow-rate functions observed in reservoir engineering). Therefore, we never use more knots, but we may be forced to reduce the number of knots in cases in which the data quality does not justify the pursuit of sophisticated (reservoir pressure) signatures. A spline function can be represented effectively with a linear combination of basic spline functions called B-splines (Cheney and Kincaid 2003). Once the knots (or continuity points) are set, generation of B-splines is easy because of their intrinsic recurrence relation. Distributing the knots logarithmically, we have \( t_i = b^i \), \( b > 1 \) \( i = 0, 1, 2, \ldots \) with a suitable selected \( b > 1 \) basis, the B-spline of degree 0 is defined by

\[
B_0^0(t) = \begin{cases} 
1 & t_i < t < t_{i+1} \\
0 & \text{otherwise} 
\end{cases} 
\]

Higher-degree B-splines are generated recursively using

\[
B_l^k(t) = \left( \frac{t - t_i}{t_{i+l} - t_i} \right) B_{l-1}^{k-1}(t) + \left( \frac{t_{i+l+1} - t}{t_{i+l+1} - t_{i+l}} \right) B_{l-1}^{k-1}(t_i), 
\]

where \( k = 1, 2, \ldots \). The \( k \)-th B-spline has a nonzero value (support) between \( t_i \) and \( t_{i+k+1} \), as is illustrated in Appendix B. Any \( k \)-th degree spline function (over the given system of knots) can be represented as a linear combination of \( k \)-th order B-splines.

\[
S(t) = \sum_{i=0}^{n} c_i B_i(t),
\]

where the number of B-splines involved is \( n-k+1 \). For second-order splines having support in an interval \( (t_{i}, t_{i+2}) \), we obtain \( l = n_i - 2 \), \( u = n_i - 1 \), and the number of B-splines involved is \( n = n_i - i_i + 2 \). By linearity, the Laplace transform of Eq. 4 is

\[
\mathcal{L}(S(s)) = \sum_{i=0}^{n} c_i \mathcal{L}(B_i(s)).
\]

We will represent \( K(t) \) as a weighted sum of B-splines of degree 2, defined over logarithmically evenly spaced knots:

\[
K(t) = \sum_{i=0}^{n} c_i B_i^2(t).
\]

Substituting Eq. 6 into Eq. 1, we have:

\[
\Delta p(t) = \int_0^t \sum_{i=0}^{n} c_i B_i^2(\tau) q(t - \tau) \, d\tau.
\]
If we have \( m \) drawdown observations collected in a vector \( \Delta \mathbf{p} \) that were observed at times \( (t_1, t_2, \ldots, t_n) \), then the problem can be written as an overdetermined system of linear equations:

\[
\mathbf{Xc} = \Delta \mathbf{p},
\]

where \( \mathbf{X} \) is the \( m \times n \) sensitivity matrix (or “design” matrix). Before solving Eq. 8 in the least-squares sense, each row of the overdetermined system is multiplied by a weight factor. Typically, we use the reciprocal of the observed value; in other words, we assume the responses are corrupted by a constant level of relative errors. Also, additional rows are added; we will discuss this operation in the section on regularization.

Theoretically, the elements of the sensitivity matrix are defined as

\[
X_{ij} = \int_0^1 B_i'(\tau)q(t_j - \tau) \, d\tau.
\]

In contrast to other methods, we do not assume that the rate function is given in a stepwise manner. Rather, we allow for a general rate function with a Laplace transform of \( q(s) \). We calculate the elements of the \( \mathbf{X} \) matrix using numerical inverse Laplace transformation:

\[
X_{ij} = \int_0^1 B_i'(\tau)q(t_j - \tau) \, d\tau.
\]

Here, \( B_i'(s) \), the Laplace transform of the \( i \)th B-spline, is known in closed form (which is given in Appendix A). We exclude B-splines with indices \( i \) for which \( b' > t_j \) because those splines “start” later than the time of the given observation.

**Rate Representation.** To obtain the Laplace transform of the rate function, we have various options. The one we prefer is to dissect the rate into \( n_q \) segments with starting times \( t_k \). \( k = 1, \ldots, n_q \). In each segment, we describe the incremental contribution by an exponential term and, hence, the total rate is obtained in the form

\[
q(t) = \sum_{i=1}^{n_q} \left( a^i + b^i \exp\left(-\frac{t-t^i}{c^i}\right) \right) \theta(t-t^i),
\]

where \( \theta(\cdot) \) is the Heaviside (unit-step) function. The parameters for each segment are found by linear least-squares fit combined by direct search on the nonlinear parameter \( c^i \). A reasonable range for this time constant is between \( t_i \) and \( t_{i-1} \). The fewest segments we use, the more we smooth the observations. A shut-in period is always represented as an individual segment. The selection of flow segments is completely independent from the location of the pressure observations.

With the special form of the rate given by Eq. 11, the elements of the sensitivity matrix are calculated as

\[
X_{ij} = \sum_{i=1}^{n_q} \int_0^1 B_i'(s)q(t_j - s) \, ds.
\]

For strongly varying flow rates, we can use a piecewise constant representation of the flow rates with many segments, as given in Eq. 14.

\[
q(t) = \sum_{i=1}^{n_q} (q_i - q_{i-1}) \theta(t-t^i).
\]

In such a case, however, there is no need for inverse Laplace transformation because Eq. 9 can be calculated by use of closed formulas in which the analytical integral of the second-order B-spline is available (see Appendix A).

**Numerical Laplace Inversion.** The success or failure of our proposed approach depends primarily on the numerical Laplace transform inversion. For the previous three decades, fixed precision computing has defined the status of numerical inversion [see Davies and Martin (1979) and Narayanan and Beskos (1982)]. Recently, several new algorithms were introduced using multi-precision methods (Abate and Valkó 2004).

In this work, we use the publicly available GWR algorithm. With this algorithm, it is possible to invert a large class of Laplace transforms with essentially any desired accuracy. For more details on the particular algorithm and Laplace transform inversion in general, see Appendix B.

**Decoupling the Unit Response and Its Derivative.** As we already mentioned, in general, the earliest part of the unknown \( K(t) \) function is critical, especially if the observed variable-rate response contains a skin effect [but minimal (or no) wellbore-storage effects]. In such cases, the \( K(t) \) function contains a Dirac delta component. To reconcile this condition, we add additional “anchor” B-splines to the left of the first observed time point. After considerable experimentation, we established that four additional B-splines are sufficient for virtually any scenario.

These anchor B-splines do not affect the reconstructed \( p_{u,d}(t) \) function (that is, the logarithmic derivative at the observation points) directly, but they do affect the reconstructed \( p(t) \) (constant-rate) response at all observation points [i.e., the integral of the \( p_{u,d}(t) \) function]. In fact, these additional B-splines are a convenient vehicle to represent the Dirac delta component of the function \( K(t) \).

**Regularization.** We found that using the concept of a pseudoinverse (a cutoff for small singular values) does not provide a sufficient regularization as the noise level in the data increases. Therefore, we require an additional regularization that makes sense in a physical context and ensures the relevance of the spline representation.

Appended to the overdetermined system (Eq. 8) to be solved by least squares are the following two conditions for each spline interval:

\[
\alpha' \left( t \sum_{i=1}^{n_q} c_i B_i(t) \right)_{p_{u,d}} - \left( t \sum_{i=1}^{n_q} c_i B_i(t) \right)_{p_{u,d}} = 0
\]

and

\[
\alpha' \left( t \sum_{i=1}^{n_q} c_i B_i(t) \right)_{p_{u,d}} - \left( t \sum_{i=1}^{n_q} c_i B_i(t) \right)_{p_{u,d}} = 0 \ldots \ldots (15)
\]

In other words, we require that the value of the logarithmic derivative of the constant-rate response differ only slightly between the knot and the middle location between knots. Because the entire system is overdetermined, these equations will not be satisfied exactly, and their influence on the solution will depend on how large \( \alpha \) is selected. When \( \alpha = 0 \), there is no regularization, and with generated data with practically no error, \( \alpha = 0 \) results in the required smooth solution for the constant-rate response and its logarithmic derivative. In the presence of random noise and/or other inconsistencies, a positive \( \alpha \) is selected on the basis of an informal interpretation of the discrepancy principle—that is, we increase the value of the regularization parameter until the calculated (model) pressure difference begins to deviate from the observed pressure difference in a specific manner. The mean and standard deviation of the arithmetic difference of the computed and input pressure functions are also computed, but algorithmic rules (e.g., L-curve method) for selecting \( \alpha \) are not recommended, for reasons discussed by von Schroeter et al. (2004) and Gringarten et al. (2003).

**Solution Using Pseudoinverse.** Once the augmented sensitivity matrix \( \mathbf{X}_a \) has been constructed, the estimate of the unknown parameter vector \( \mathbf{c} \) is obtained from

\[
\hat{\mathbf{c}} = \mathbf{X}_a^T \Delta \mathbf{p}.
\]

where the superscript plus sign denotes pseudoinverse calculated through singular value decomposition. (For simplicity, here we show the case in which all the weights are unity.) To create the pseudoinverse, small singular values are removed from the calculations; in our computations, the automatic cutoff selection of Mathematica (2005) was used. Using the \( \hat{c} \) coefficient estimates
obtained from this process, we reconstruct the constant-rate response as follows:

$$p_u(t) = \sum_{i=1}^{u} \hat{B}_u^2(t).$$

The logarithmic derivative of the constant-rate response is given as

$$p_w(t) = t \sum_{i=1}^{u} \hat{B}_w^2(t),$$

where the integral of the second-order B-spline is easily obtained in symbolic form (see Appendix A). Hence, the unit response and its logarithmic derivative can be calculated anywhere between the start and the end of the test sequence.

Validation

In this section, we provide validation of the new method by comparing the results from the deconvolution procedure with the exact responses obtained from a known reservoir model. As a base case (specifically, the case of a reservoir model with strong characteristic behavior), we selected a dual-porosity reservoir with circular boundary ($\lambda = 1 \times 10^{-5}, \omega = 5 \times 10^{-5}$, and $r_{wf} = 1,600$; see Table 1 for details). To generate the synthetic pressure histories required for the validation phase, we convolve the known unit-rate pressure solution with the flow history, which includes two variable-rate-production and two pressure-buildup (PBU) periods.

In our validation, we use the following variable-rate profile:

$$q(t) = 405.23 - 736.79 e^{-t/5} + 368.39 e^{20 - 3.684 t} \cdots$$

Because we ultimately intend to process permanent gauge data, this selected rate profile was used to mimic typical pressure-drawdown/buildup test sequences. In the validation process, we increase the complexity of the reservoir model step by step to assess what influence the underlying reservoir model may have on the performance of our deconvolution procedure. We first consider a model without wellbore storage and skin effects (Fig. 1); we then incorporate a skin factor of 5 (Fig. 2), and, finally, we use a dimensionless wellbore-storage coefficient of 100 and a skin factor of 5 (Fig. 3).

In the first example, we perform deconvolution with the input data shown in Fig. 1. Fig. 4 presents our deconvolution results for this case, and we note that when the deconvolution results are compared with the exact input response, one hardly sees any difference. All the characteristic features of a dual-porosity reservoir model are present in the results, including the boundary-dominated flow regime.

<table>
<thead>
<tr>
<th>Reservoir Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wellbore radius, $r_w$</td>
</tr>
<tr>
<td>Net pay thickness, $h$</td>
</tr>
<tr>
<td>Formation permeability, $k$</td>
</tr>
<tr>
<td>Formation compressibility, $c_t$</td>
</tr>
<tr>
<td>Porosity, $\phi$</td>
</tr>
<tr>
<td>Storativity ratio, $\omega$</td>
</tr>
<tr>
<td>Interporosity flow parameter, $\lambda$</td>
</tr>
<tr>
<td>Outer boundary radius, $r_o$</td>
</tr>
<tr>
<td>Initial reservoir pressure, $p_i$</td>
</tr>
<tr>
<td>Wellbore-storage coefficient, $C_D$</td>
</tr>
<tr>
<td>Skin factor, $s_x$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fluid Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid viscosity, $\mu$</td>
</tr>
<tr>
<td>Formation volume factor, $B$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Production Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal production rate, $q_{nominal}$</td>
</tr>
</tbody>
</table>
In Fig. 5, we present the deconvolution results for the case in which a skin factor of 5 is included in the input pressure response; we note an additional increase in the unit-rate drawdown pressure response (from the positive skin factor), and we see that the well-testing derivative function remains essentially identical to the trends observed in Fig. 4 (as would be expected).

In Fig. 6, we present the deconvolution results when wellbore-storage and skin effects are incorporated into the base model. We note that the deconvolution shown in Fig. 6 is performed using “surface” rates, which do not reflect wellbore-storage conditions (although wellbore storage effects are, obviously, included in the observed pressure data).

Fig. 6 is of particular practical interest because we clearly must recognize that although deconvolution to remove wellbore-storage effects is a worthwhile goal, in current practice, it is very unlikely that we would be able to obtain the downhole flow-rate data with sufficient accuracy such that we could remove wellbore-storage effects. However, given a general variable-rate/variable-pressure data sequence, we can use deconvolution to reconstruct the constant-rate drawdown response, which would include wellbore-storage effects (as shown quite effectively in Fig. 6).

In the validation cases considered to this point, only ideal data have been used as input (i.e., noiseless/consistent data), and regularization was not necessary. As noted, the deconvolved results are essentially the same as the (input) ideal model responses. Regularization is not required for synthetic cases, but for most (if not all) field cases (which usually have inconsistencies and significant measurement errors), we do require some sort of regularization procedure.

We provide a synthetic case in which we have added random and systematic data errors (Fig. 7); we note that we have used the same production history and reservoir model with wellbore-storage and skin effects [base performance (no error) is shown in Figs. 3 and 6]. In this particular example, we have added 10% random error to both the pressure and rate data, and we also seeded the data with a small systematic error in pressure (see Fig. 7 for a summary of the input data).

We can still perform deconvolution on these data, but in this case, we must use regularization to address the combined effects of random and systematic errors in the data.

The value of the regularization parameter has a profound effect on the solution for this case. Figs. 8 through 10 illustrate the discrepancy principle by which the regularization parameter, α, is chosen. When regularization is not used (Fig. 8), there is a random error between the observed response (Δp) and its reconstructed

Fig. 4—B-spline deconvolution results for Validation Case 1 (all data used); no wellbore-storage or skin effects.

Fig. 5—B-spline deconvolution results for Validation Case 2 (all data used); no wellbore-storage effects, skin factor (s_x)=+5.

Fig. 6—B-spline deconvolution results for Validation Case 3 (all data used); dimensionless wellbore-storage coefficient (C_D)=100, and skin factor (s_x)=+5.

Fig. 7—Validation Case 4: input data with systematic error and noise; dimensionless wellbore-storage coefficient (C_D)=100, and skin factor (s_x)=+5.
value ($X_c$). If a higher value of regularization parameter is used, oversmoothing is obvious from the one-sided deviations (Fig. 10).

The optimum value of the regularization parameter (in this case, $\alpha = 0.0007$) is obtained as the largest value still not causing systematic discrepancy between observed and reconstructed responses (Fig. 9).

Fig. 11 presents the results of deconvolution when errors and inconsistencies are present in the data. Clearly, some artifacts caused by inconsistent data are seen (particularly in the well-testing derivative), but deconvolution reconstructs the characteristic features of the reservoir model for the duration of the entire test sequence. (From another point of view, deconvolution without regularization can effectively reveal inconsistencies present in the data.)

From our validation experiments, we can conclude that very accurate (essentially exact) results can be obtained using any major event in the test sequence (i.e., a production test, a shut-in, or the complete history). We also note that when the input data have errors or inconsistencies, we need some form of regularization to provide meaningful results. Later, we show that our deconvolution approach can address relatively high levels of noise and other data inconsistencies.

In the deconvolution process, the initial pressure is required to compute the observed pressure drop response ($\Delta p$). Therefore, the initial reservoir pressure is an input parameter for the B-spline deconvolution algorithm, and an accurate estimate of the initial pressure is a critical component of deconvolution. Using an erroneous estimate of initial reservoir pressure will cause a systematic error in the deconvolved, equivalent constant-rate pressure response. In addition, there is disagreement between the base pressure data and the reconstructed pressures. For example, using an initial pressure larger than the actual initial pressure will result in an apparent pseudosteady-state flow regime at the end of the deconvolved test sequence or production data set. Thus, we note that an apparent pseudosteady-state flow regime is a possible artifact of using an incorrect estimate of the initial reservoir pressure.

It is possible to recover the initial reservoir pressure using deconvolution [see von Schroeter et al. (2004) and Levitan et al. (2004)]. We also are able to recover the initial reservoir pressure using an iterative process in which we start with an initial estimate to minimize the error between base pressure data and reconstructed pressures. This is not an automated process; user intervention is required to realize the discrepancy between reconstructed pressures and the base pressure data. Nevertheless, we conclude that this procedure is an effective way to estimate initial reservoir pressure.

**Validation 2: Test of the New Deconvolution Method Using Only the PBU Data Portion of the Test Sequence**

In this section, we will test the methodology using individual parts of the well-test sequence. For our purposes, we have generated two synthetic cases in which we select a simple homogeneous reservoir model with wellbore-storage effects for simplicity. The properties for the base reservoir model are provided in Table 2.
In the first case, a constant-rate production of 50 STB/D is held for 5 hours and followed by 500 hours shut-in. We specifically set the duration of the buildup to be 100 times larger than the duration of the drawdown to establish the difference between deconvolution and conventional PBU analysis when the durations of production sequences are not consistent.

We recognize that this is an extreme test of the algorithm (i.e., to use only the data from the PBU portion, plus the production history, to yield the equivalent constant-rate drawdown response by deconvolution). However, in practice, the primary utility of our algorithm is likely to be the resolution of PBU data. While obvious, it is worth restating that an accurate production history (i.e., accurate flow rates) is absolutely critical for deconvolution.

Fig. 12a presents the input data for deconvolution and PBU analysis. In Fig. 12b, we immediately note artifacts in the very-late-time pressure derivative of the PBU case (conventional PBU plot), where these artifacts are caused by the fact that the reservoir pressure has completely built up to the limiting value of the average reservoir pressure.

However, our deconvolution method is not affected by this behavior. Using only the PBU data for this case, we perform deconvolution, and we then construct the constant-rate pressure-drawdown response for the entire history. We present the results of the deconvolution as well as the conventional PBU data functions in Fig. 12b, and we compared these data functions to the correct model responses for each case.

As shown in Fig. 13a, for our second case, we add another production and shut-in period to the well-test sequence. The duration ratio between both drawdown and buildup sequences is preserved (shut-in duration is 100 times larger than the duration of the drawdown).

In this case, we use only the data from the final PBU portion for the deconvolution process. As noted previously, we can construct the constant-rate pressure-drawdown response for the duration of the buildup using deconvolution.

In our work, we set the weights of the input pressure data to zero, except for the data from the buildup portion [or any individual portion of a test in which data are consistent; PBU data (zero rate case) are the most “consistent” data because they are unaffected by flow rate during the test]. As such, we use the data from the second (final) PBU portion and set the weights of prior data to zero. The results for this case are presented in Fig. 13b.

It is worth noting that using only the PBU data from the second (final) PBU test, we can recover the equivalent constant-rate pressure-drawdown response for the entire well-test sequence using deconvolution.

Applications of the New Deconvolution Technique

Having verified our new deconvolution technique, we can now apply the same procedure to field data. For convenience, we classify our field examples into four groups:

- Wellbore-storage distorted pressure data.
- Well-test data, including multiple flow sequences.
- Permanent downhole gauge data (several flow sequences).
- Long-term production data (rates and pressures).

These examples illustrate the use of B-spline deconvolution for analyzing various events (production/shut-in sequences) and should be considered typical of cases that could be encountered in field operations.

Field Case 1: B-Spline Deconvolution of Pressure-Transient Data Distorted by Wellbore Storage. The most straightforward application of deconvolution is to deconvolve PBU data to eliminate the effects of wellbore storage. Fetkovich and Vienot (1984) published data on an oil well that was hydraulically fractured at initial completion. The early-time pressure response is distorted by wellbore-storage effects, but sandface rates are available. Using this input, we perform deconvolution, the results of which are shown in Fig. 14.
drawdown and its logarithmic derivative, and the model matches for the deconvolved response. We used two models to match the deconvolved pressure response: the uniform flux fracture (Fetkovich and Vienot used this model in their analysis) and the finite-conductivity vertical fracture model. Our deconvolution results compare extraordinarily well with these model responses and serve as confirmation of previous interpretations.

In this case, we have used deconvolution to eliminate wellbore-storage effects, and we suggest that this case serves to verify our contention that our method is sufficiently accurate for use in eliminating wellbore-storage effects (when sandface rates are available).

Field Case 2: B-Spline Deconvolution of Conventional Well Tests Including Multiple Flow Periods (Complex Gas Reservoir). In Fig. 15a, we present the rate and pressure data obtained during the four-point test of a gas well, which was followed by an extended shut-in.

In this case, the PBU data seem to be of sufficient duration for a conventional PBU analysis; therefore, we simply could proceed and analyze the individual PBU portion of these data. However, our intention is not to replace PBU analysis by deconvolution but, rather, to implement our deconvolution algorithm as an option for well-test analysis.

We choose to analyze all the available data, including the production periods. First, the plot of the base pressure data and the pressures generated by the B-spline model indicate that pressure data and reconstructed pressures by deconvolution are consistent. Therefore, we note that deconvolution using the entire test sequence is successful in this case (Fig. 15b). We proceed and construct the unit-rate pressure and well-testing derivative functions in Fig. 16, together with model matches of the deconvolved pressure drawdown.
Because the flowing fluid is gas, we must perform the pseudopressure transform to adhere to theoretical considerations (we note that pseudotime should also be considered, but to be theoretically rigorous, pseudotime requires the average reservoir pressure history, which is never available in practice). The deconvolved pressure response suggests that a complex, channel-type reservoir geometry is possible.

Therefore, we proceed and match the deconvolved response with two models to verify this possibility. The models considered include a well in a homogeneous reservoir bounded by parallel faults and a well in a homogeneous reservoir with closed rectangular boundaries. The results of our model matches validate the complexity of the reservoir. We do observe the familiar “half-slope” trend of the pressure and pressure-derivative functions on the log-log plot, which indicates a channel-type reservoir geometry.

On the other hand, we also note that the signature of a closed reservoir is evident in the pseudopressure drop and pseudopressure-drop derivative functions at late times.

In this application, we have used deconvolution to extract information from a time interval larger than just the PBU test sequence. Analysis of the entire testing sequence using deconvolution may reveal reservoir characteristics not “seen” by a single event in the sequence.

Field Case 3: B-Spline Deconvolution of Conventional Well Tests Including Multiple Flow Periods (Low-Permeability Oil Reservoir). In our third example, we perform deconvolution using the data from a selected flow period during a well-test sequence, as shown for the selected oil well in Fig. 17a. This test sequence includes 170 hours of data, where we have a 50-hour constant-rate production (at 200 STB/D), followed by a shut-in for 16 hours; then, production resumes at a constant rate of 160 STB/D for 24 hours and is then shut in for 80 hours.

Inconsistencies are observed; in particular, we suspect that in the first production period, the given rate is not correct (we note fluctuations in the drawdown pressure profile). We have two options in this case: first, we can deconvolve the entire sequence, but we have to use a high value of regularization parameter to overcome the effects of inconsistencies that most likely will introduce bias. We also can use the data from the final buildup portion to perform deconvolution but take the preceding flow history into account. We choose the second option and perform deconvolution using only the PBU data.

In Fig. 17b, reconstructed pressures obtained from the B-spline model and base pressure data are shown together. It is obvious that the data are not consistent. We note that for construction of Fig. 16b, we set the weights of pressure data to zero except for the data from the last buildup; as expected, we have a very good match of the reconstructed pressures for both sets of PBU data that are assumed consistent. These results suggest that, most probably, reported rates are wrong, and performing deconvolution using the entire set of test data will not produce meaningful results. Nevertheless, the deconvolved pressure response (obtained using the final PBU data, which are considered consistent with the given rates) and a model match are shown in Fig. 18. The deconvolved well-testing pressure-derivative function suggests that the well has a finite-
Field Case 4: B-Spline Deconvolution of Permanent Downhole Gauge (PDG) Data. Our next example is taken from an oil well equipped with a permanent downhole measurement system, which is used to provide continuous (downhole) pressure and rate data. In this case, a tremendous amount of data is available for a very short time—10 hours (as shown in Fig. 19).

Nevertheless, PBU analysis or other traditional-type analysis is not a realistic option because the builds are very short and, in addition, other phenomena (most likely temperature change and/or phase resegregation) cause nonintuitive behavior (e.g., between 2 and 4 hours, the well is shut in, but the observed pressure is decreasing).

Fig. 20 presents the results of deconvolution for this case. The apparent oscillations in the deconvolved pressure-derivative function indicate that the data are somewhat inconsistent, but we can proceed and match the deconvolved pressure with simple models [a well in an infinite-acting homogeneous reservoir with wellbore storage and skin, and a well in a homogeneous reservoir with a single sealing fault (also including wellbore-storage and skin effects)].

Using analytic simulation, we reproduce the pressure response relative to the observed rate history for the entire test sequence, as shown in Fig. 21. The pressures generated by the model fail to honor the observed data at some points (especially in the intervals in which we have already established that inconsistencies exist in the data); however, these simulations do honor the observed pressures at later times, in agreement with our expectations.

Field Case 5: B-Spline Deconvolution Applied to Production Data (Tight Gas Reservoir). Our deconvolution method is not limited to analyzing well tests; we can also apply this methodology to analyze and interpret traditional (long-term) production data. Production data are regarded as “low-frequency” and “low-resolution” data. For these reasons, rigorous variable-rate/variable-pressure analysis of production data is rarely attempted. Our objective is to convert the entire production sequence (including poor-quality data) into an equivalent constant flow-rate pressure response. We corroborate our results with material-balance deconvolution (Johnston and Lee 1991), which includes plotting \( \Delta p_q/q_g \) (rate-normalized pseudopressure) vs. material-balance pseudotime and comparing our deconvolution results and the material-balance deconvolution results with a reservoir model established from the deconvoluted data.

This example considers a tight gas reservoir (Pratikno et al. 2003). The data are of good (to excellent) quality, and we note the effect of shut-ins as well as daily production fluctuations (Fig. 22a).

Fig. 22b presents the reconstructed pressures by B-spline deconvolution compared with base pressure data. As seen, pressures obtained using the B-spline model honor the measured data over a considerable portion of the production history. We can conclude that production data are consistent and that our B-spline deconvolution based on the entire long-term production history was successful.

In Fig. 23, we present the deconvolved responses (our method and material-balance time) compared with an appropriate reservoir model (a well with a finite-conductivity vertical fracture in a bounded reservoir); we note good agreement for this case.

We note some anomalies in the well-testing pressure-derivative function at early times, where these anomalies were presumably caused by inconsistencies in the pressure data (or simply a scarcity of data at early times). From the derivative trend, we find that boundary-dominated flow is in transition to full development, and there does exist a slight difference in our interpretation of the data and model selected for this portion of the analysis.

Field Case 6: B-Spline Deconvolution Applied to Production Data (Low-Productivity Gas Reservoir). The second field example is also a gas field case (Palacio et al. 1993). The daily production data are quite erratic (Fig. 24a) because of liquid loading; in spite of these features, the B-spline deconvolution method performs reasonably well (see Fig. 24b).

In this case, the margin of error between data and reconstruction is acceptable, and we expect that errors will result in the form of artifacts, especially in the deconvolved well-testing derivative function. In Fig. 25, we again compare our deconvolution results with material-balance results (Fig. 26) and note some discrepancies.
with traditional production analysis based on material-balance time deconvolution.

Artifacts caused by inconsistencies are seen at early times in the well-testing pressure-derivative function, again (we believe) because of sparse/erratic data. The late-time well-testing pressure-derivative function trend confirms that the boundary-dominated flow regime has been achieved. The solution for a homogeneous reservoir with a closed circular boundary matches all of the deconvolved pressure-response functions reasonably well, where this observation also confirms our analysis.

We believe that B-spline deconvolution can be applied successfully to production data (regularly measured flow-rate and pressure data). Processing low-quality production data with deconvolution can provide additional insight into reservoir performance-based analysis.

It is worth noting that numerous other field cases of long-term flow-rate and (surface) pressure data have been analyzed and interpreted successfully using the proposed B-spline deconvolution technique. In fact, we believe that B-spline deconvolution may be used more often for the analysis of production data, compared to its potential use for well-test data. This hypothesis arises from the fact that while often of poor quality, production data are generally available.

**Conclusions**

Our proposed deconvolution technique is robust because of the construction of the coefficient (sensitivity) matrix of the underlying least-squares problem using B-splines. Because of the (semilogarithmic) B-spline representation of the unknown impulse response and the high reliability of the numerical Laplace transform inversion technique used, the elements of the coefficient matrix are computed accurately, and the resulting deconvolution process is stable. On the other hand, modeling of the rates requires a significant effort for accurate computation of the sensitivity matrix. Rates are required to be in functional form, and if the rate history indicates discontinuities, it should be decomposed into smooth segments. An increasing number of segments will increase the computational time. With the appropriate description of rates, the proposed technique is robust and reliable.
For the deconvolution of data in practice, it is necessary to incorporate a physically sound regularization scheme into our proposed deconvolution method. Regularization might be seen as an “add-on” to the proposed algorithm, but in a practical sense (i.e., for field applications), our proposed deconvolution scheme is likely to perform poorly without the use of regularization methods.

We have successfully demonstrated the use of B-spline deconvolution for the analysis of a wide range of field data, from traditional PBU data, to permanent downhole gauge data, to traditional production data.

Nomenclature

\[
\begin{align*}
\Delta p &= \text{drawdown (with respect to } p) \\
\lambda &= \text{interporosity flow parameter} \\
\mu &= \text{viscosity, cp} \\
\tau &= \text{dummy variable} \\
\phi &= \text{porosity, fraction} \\
\omega &= \text{storativity parameter} \\
\end{align*}
\]

Superscripts

\[
\begin{align*}
^e &= \text{estimated (from least squares)} \\
^o &= \text{observed} \\
^+ &= \text{Laplace transform} \\
^{(\dagger)} &= \text{pseudoinverse} \\
\tau &= \text{transpose of a matrix} \\
\end{align*}
\]

Special Functions

\[
K_{\alpha}(t) = \text{modified Bessel function of the second kind, zero order}
\]

References


Appendix A—B-Splines Over Logarithmically Distributed Knots and Their Laplace Transforms

The 0th quadratic B-spline (defined over logarithmically spaced knots with basis $b>1$) is given by Eq. A-1, where $\theta(x)$ denotes the Heaviside function. The Laplace transform of the above B-spline can be written in a compact form (Eq. A-2). In Fig. A-1, we show two B-splines (of order 2, basis $b=2$ and indices 0 and 1); we note that the 0th index B-spline is nonzero only in the interval $(1;8)$. An interesting property is that the Heaviside function can be represented as the sum of all B-splines with indices from negative to positive infinity (this is the “partition of unity” property). In addition, an analytical integral of a second-order B-spline is available and given by Eq. A-3.

\[
B_k^1(x) = \frac{(x - 1) \left[ \theta(x - b) - \theta(x - b^2) \right]}{b^2 - 1}, \quad x \geq 1
\]

\[
B_k^2(x) = \frac{(x - 1) \left[ \theta(x - b) - \theta(x - b^2) \right]}{b^2 - 1} + \frac{b^2 - 1}{b^3 - b}
\]

\[
B_k^3(x) = \frac{(x - b) \left[ \theta(x - b) - \theta(x - b^2) \right]}{b^2 - b} + \frac{b^3 - b}{b^3 - b}
\]

\[
B_k^4(x) = \frac{(b - x) \left[ \theta(x - b) - \theta(x - b^2) \right]}{b^2 - b} + \frac{(b^2 - 1) \left[ \theta(x - b) - \theta(x - b^2) \right]}{b^3 - b} - \frac{2(e^{(b^2-1)x} - 1)}{(b - 1)(b^2 + b + 1)x^2}
\]

\[
\mathcal{L}\{B^1\}(s) = \frac{2e^{-1-b^2-b^3} \left[ (b^3 - b^2 - b) e^{b^2 + b^3} + (1 + b + b^2) e^{b^2 + b^3} \right]}{(b - 1)(b^2 + b + 1)x^2}
\]

\[
\mathcal{L}\{B^2\}(s) = \frac{2e^{1-b^2-b^3} \left[ (e^{b^2 + b^3} + 1 + b + b^2) e^{b^2 + b^3} \right]}{(b - 1)(b^2 + b + 1)x^2}
\]

\[
\mathcal{L}\{B^3\}(s) = \ldots
\]

\[
\mathcal{L}\{B^4\}(s) = \ldots
\]

Integrals of second-order B-spline functions are calculated by the following formula below (Cheney and Kincaid 2003):

\[
B_{c(t)}^j(t) = \int_{t=a}^{t=b} B_j^i(x)dx = b \left( \frac{b^3 - 1}{3} \right) \sum_{j=0}^{n} B_j^i(t).
\]

Appendix B—Comparison of Laplace Inversion Algorithms

The GWR algorithm for numerical inversion of Laplace transforms is publicly available from http://library.wolfram.com/database/MathSource/4738/. Because the algorithm relies on multiprecision computing, it can be realized only in software systems supporting variable precision methods for arithmetic operations and for special functions. The significance of built-in automatic control of variable precision cannot be overemphasized [see Stehfest (1970) and Valkó and Vajda (2002)]. Here, we show a comparison of the GWR algorithm with the standard Stehfest (1970) algorithm realized in double precision. Such an approach has been used to generate the overwhelming majority of results used in petroleum engineering well-testing applications so far. For illustrative purposes, we use the Theis function (i.e., the constant-rate homogeneous reservoir).

It can be seen from Table B-1 that for this simple inversion problem, the standard double-precision Stehfest algorithm gives reasonable accuracy (five digits) if the Stehfest parameter $n$ is selected optimally. Unfortunately, the use of this procedure does not provide any systematic way to estimate the number of significant digits or the optimum value of $n$. This fact has been the origin of considerable confusion in the past 30 years. The results given above also show that the GWR algorithm performs somewhat better even when the Stehfest algorithm is realized in an ideal infinite-precision computational environment. This is caused by the nonlinear sequence acceleration used in GWR. The particular realization of the GWR algorithm used here automatically selects the variable precision needed for the internal calculations; there-
fore, the user can increase the parameter \( M \) and stop when the required significant digits are already stabilized. For instance, in the example given above, if we require 10 significant digits, then a calculation with \( M = 16 \) is satisfactory, and a check with \( M = 20 \) proves this fact, without ever knowing the true value of the function.

**SI Metric Conversion Factors**

<table>
<thead>
<tr>
<th>Conversion</th>
<th>SI Unit</th>
<th>US Customary Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>bbl ( \times ) 1.589 873</td>
<td>E−01 = m³</td>
<td></td>
</tr>
<tr>
<td>cp ( \times ) 1.0</td>
<td>E−03 = Pa·s</td>
<td></td>
</tr>
<tr>
<td>ft ( \times ) 3.048*</td>
<td>E−01 = m</td>
<td></td>
</tr>
<tr>
<td>psi ( \times ) 6.894 757</td>
<td>E+00 = kPa</td>
<td></td>
</tr>
</tbody>
</table>

*Conversion factor is exact.*

### Table B-1—Numerical Inversion of \( F(s) = K_s (\sqrt{s})/s \) at \( t = 0.25 \)

<table>
<thead>
<tr>
<th>GWR (Stehfest 1970)</th>
<th>Stehfest (Stehfest 1970; Valkó and Vajda 2002)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>Built-in Variable Precision</td>
</tr>
<tr>
<td>4</td>
<td>0.11018283839223893798</td>
</tr>
<tr>
<td>8</td>
<td>0.10969291077594668650</td>
</tr>
<tr>
<td>12</td>
<td>0.10969195273404421135</td>
</tr>
<tr>
<td>16</td>
<td>0.10969196718149629167</td>
</tr>
<tr>
<td>20</td>
<td>0.1096919671976714063</td>
</tr>
<tr>
<td>24</td>
<td>0.10969196719776081799</td>
</tr>
<tr>
<td>28</td>
<td>0.10969196719776013771</td>
</tr>
<tr>
<td>32</td>
<td>0.10969196719776013884</td>
</tr>
</tbody>
</table>

*True value up to 20 digits* 0.10969196719776013684.

*Infinity precision means the value is calculated with as many digits as necessary not to affect the shown digits.

*True value is calculated from \( f(t) = \frac{-E t}{(1/4 + t)} / 2 \).

Dilhan Ilk is a PhD student in the Dept. of Petroleum Engineering at Texas A&M U. e-mail: dilhan@tamu.edu. His research interest includes well-test/production-data analysis, numerical analysis, and signal processing. He holds a BS degree from Istanbul Technical U., Turkey, and an MS degree from Texas A&M U., both in petroleum engineering. Peter P. Valkó is an associate professor of petroleum engineering at Texas A&M U. e-mail: p-valko@pe.tamu.edu. Previously, he taught at academic institutions in Austria and Hungary and worked for the Hungarian Oil Co. (MOL). His research interests include design and evaluation of hydraulic fracture stimulation treatments, rheology of fracturing fluids, and performance of advanced and stimulated wells. Valkó has coauthored three books and published several dozen research papers. He holds BS and MS degrees from Veszprém U., Hungary, and a PhD degree from the Inst. of Catalysis, Novosibirsk, Russia. Tom Blasingame is a professor of petroleum engineering at Texas A&M U., where he works in the areas of reservoir engineering and analysis of reservoir performance. e-mail: t-blasingame@tamu.edu. He holds BS, MS, and PhD degrees from Texas A&M U., all in petroleum engineering. Blasingame is a Distinguished Member of SPE and received the 2005 SPE Distinguished Service Award. He has chaired SPE Forums, Applied Technology Workshops (ATWs), and committees.