We note that the spacing and the optimal design for the drainage aspect ratio is a useful parameter for establishing the optimum well permeability reservoir systems. (and does) dominate the well performance for very low radial flow. In a practical sense, the elliptical flow regime can bilinear and/or formation linear flow and the onset of pseudo-transitional flow that occurs between the end of "shale gas" sands. Conceptually, the elliptical flow period is <0.001 md) formations (often referred to as "tight gas" or producing from low permeability (<0.01 md) and ultra-low conductivity and as a function of the elliptical boundary curves are generated for different values of the fracture pressure solution obtained using an analytical method. The system consisting of a hydraulic fracture at the center of an elliptical reservoir. The curves are generated from the traditional well test analysis of fractured wells.1 Elliptical flow also occurs also in circular anisotropic reservoirs.

Elliptical flow has long been considered as a transitional flow pattern that occurs between linear and pseudoradial flow in the reservoir behavior for incompressible and compressible fluid cases. Prats proposed the representation of a fracture with an equivalent effective well radius (this was a convenience of the times (early 1960's)).

Kuchuk developed the analytical solution for the transient elliptical flow problem resulting from an infinite conductivity fracture producing from an elliptical, or an anisotropic-radial reservoir. The behavior of composite, elliptically-shaped reservoirs has also been studied by Obut, and Ertekin — assuming a variety of boundary conditions. Perhaps the most "analytical" treatment of the problem was provided by Riley who developed an analytical solution for the case of a vertical well with a finite conductivity vertical fracture in an infinite-acting reservoir system.

In addition to attempts to obtain analytical solutions for the elliptical flow problem, there are numerous studies which consider numerical modeling of the elliptical flow problem. Hale applied type curve solutions (dimensionless plots) derived from numerical simulation of the elliptical problem as a mechanism to interpret well tests in tight gas reservoirs. Liao developed a general numerical model for the elliptical flow case which is capable of accounting for the simultaneous effects of wellbore storage and fracture face skin effects on the behavior of pressure transient tests.

Based on practical experience as well as numerical/analytical models, it is apparent that elliptical flow behavior dominates the performance of fractured wells in low/ultra-low permeability gas reservoir systems. The primary objective of this paper is to develop and validate a series of "type curve" solutions for a system consisting of a hydraulically fractured well in a bounded elliptical shape reservoir. The "type curves" are pre-
sented (and implemented) as a diagnostic tool — which is used to assess the flow behavior based on production data. Type curves are presented for different values of the fracture conductivity \((F_E = 1, 10, 100, \) and \(1000\) (very low to very high fracture conductivity cases)), as well as a function of the elliptical boundary characteristic parameter \((\xi)\).

**Development of the Model**

Elliptical flow is considered to be the governing flow regime for low permeability gas reservoirs — and, as such, we assume that the reservoir has some sort of an elliptical outer (or perhaps, inner) boundary. Cases such as production from an elliptical wellbore, an elliptical fracture, or a circular wellbore in an anisotropic reservoir system can be considered to be examples of an elliptical inner boundary. An elliptic reservoir surrounded by an (elliptic) aquifer is a relevant example of an elliptical outer boundary — as is also the case of a very low permeability gas reservoir that achieves only a limited drainage area pattern (usually modeled by an ellipse).

Our model analytical consists of a hydraulically fractured well in the center of a reservoir with a closed elliptical boundary. The schematic illustration for an elliptical reservoir with a hydraulic fracture is shown in Fig. 1.

![Figure 1 — Schematic of the elliptical reservoir model.](image)

The following assumptions are made in order to develop an analytical solution for this case:

- The reservoir is assumed to be a single-layer system that is isotropic, horizontal, and of uniform thickness with constant reservoir characteristics.
- Production is obtained from a hydraulic fracture which intersects the wellbore. The fracture is assumed to have elliptical shape — but we can also assume that the fracture is very narrow compared to the length (and area) of the fracture, and in such cases, we assume a zero-width fracture and the fracture transforms into a line whose length is equal to the focal length of the elliptic system.
- The elliptical outer boundary is assumed to have a focal length equal to the fracture length.
- Fracture properties are assumed constant — and we have considered the cases of both infinite and finite conductivity fractures.
- Wellbore storage and skin effects are neglected for the sake of simplicity (as we are focusing on production data analysis, this issue not significant).
- The flowing fluid is assumed to be a single-phase, slightly compressible fluid (although rigorous transformations can be used for gas flow cases).
- Fluid flow in the formation and in fracture is laminar, and Darcy's law is presumed valid.

The analytical solutions for the case of an elliptical fracture and outer boundary have been discussed in prior literature.\(^4\) We also introduce a parameter called the elliptical boundary characteristic parameter \((\xi)\) — which is a variable that correlates all the aspects of the drainage area (e.g., the drainage aspect ratio \((a/b)\) and penetration ratio based on the fracture half-length \((x_f/a)\). The \(\xi\)-parameter helps us to characterize the drainage area using a single dimensionless parameter, similar in concept and application to the dimensionless radius for the circular bounded reservoir \((r_{re} = r/\xi)\) as introduced by Pratikno and Blasingame.\(^1\) In Appendix A we provide useful orientation and detail regarding the correlation between the elliptical boundary characteristic parameter \((\xi)\) and the drainage area size and shape.

**Elliptical Flow Model — Type Curves**

Given a volumetric reservoir acting under pseudosteady-state flow conditions (any well/reservoir configuration), we can write the following material balance/flow relation for this case:

\[
P_{D, pss} = b_{D, pss} + 2\pi D_A \tag{1}
\]

Where \(b_{D, pss}\) is the "pseudosteady-state constant" for a particular well/reservoir configuration. We can calculate a series of pressure solutions as functions of dimensionless time for different values of the fracture conductivity and the elliptical boundary characteristic parameter \((\xi)\) in order to estimate the \(b_{D, pss}\)-parameter for different configurations. Cartesian-style plots of \(p_{D, pss}\) versus \(t_{DA}\) are used to establish the \(b_{D, pss}\)-parameter values, and a sampling of \(b_{D, pss}\)-values is presented in Table 1 for the cases of \(F_E=1, 10, 100\), and \(1000\). The process of estimating the \(b_{D, pss}\)-parameter is discussed in detail in ref. 13.

![Table 1 — Values of \(b_{D, pss}\) from elliptical flow solution.](image)

<table>
<thead>
<tr>
<th>(\xi)</th>
<th>(F_E = 1)</th>
<th>(F_E = 10)</th>
<th>(F_E = 100)</th>
<th>(F_E = 1000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.8481</td>
<td>0.2150</td>
<td>0.1306</td>
<td>0.1220</td>
</tr>
<tr>
<td>0.50</td>
<td>0.9902</td>
<td>0.3337</td>
<td>0.2396</td>
<td>0.2298</td>
</tr>
<tr>
<td>0.75</td>
<td>1.1671</td>
<td>0.4609</td>
<td>0.3540</td>
<td>0.3426</td>
</tr>
<tr>
<td>1.00</td>
<td>1.3627</td>
<td>0.6109</td>
<td>0.4936</td>
<td>0.4812</td>
</tr>
<tr>
<td>1.25</td>
<td>1.5733</td>
<td>0.7880</td>
<td>0.6634</td>
<td>0.6501</td>
</tr>
<tr>
<td>1.50</td>
<td>1.7963</td>
<td>0.9884</td>
<td>0.8591</td>
<td>0.8453</td>
</tr>
<tr>
<td>1.75</td>
<td>2.0293</td>
<td>1.2067</td>
<td>1.0743</td>
<td>1.0602</td>
</tr>
<tr>
<td>2.00</td>
<td>2.2682</td>
<td>1.4363</td>
<td>1.3021</td>
<td>1.2877</td>
</tr>
<tr>
<td>3.00</td>
<td>3.2529</td>
<td>2.4084</td>
<td>2.7176</td>
<td>2.2570</td>
</tr>
<tr>
<td>4.00</td>
<td>4.2503</td>
<td>3.4040</td>
<td>3.2669</td>
<td>3.2522</td>
</tr>
<tr>
<td>5.00</td>
<td>5.2486</td>
<td>4.4021</td>
<td>4.2649</td>
<td>4.2502</td>
</tr>
</tbody>
</table>

Using the data in Table 1 and a non-linear regression method, we can estimate the \(b_{D, pss}\)-parameter as a function of the fracture conductivity and the elliptical boundary characteristic parameter as follows:

\[
b_{D, pss} = 1.00146\xi_0 + 0.0794849e^{-\xi_0} - 0.16703u + \frac{A}{B} - 0.754772 \tag{3}
\]
Where the auxiliary functions are:

\[ u = \ln(F_u) \] .......................................................... (4)

\[ A = a_1 + a_2 u + a_3 u^2 + a_4 u^3 + a_5 u^4 \] .................................. (5)

\[ B = b_1 + b_2 u + b_3 u^2 + b_4 u^3 + b_5 u^4 \] .................................. (6)

Where the coefficients are given as:

\[ a_1 = -4.7468 \quad b_1 = -2.4941 \]
\[ a_2 = 36.2492 \quad b_2 = 21.6755 \]
\[ a_3 = 55.0998 \quad b_3 = 41.0303 \]
\[ a_4 = -3.9831 \quad b_4 = -10.4793 \]
\[ a_5 = 6.07102 \quad b_5 = 5.6108 \]

The \( b_{Dpss} \) results for our work and Eq. 3 (the correlation equation) are presented graphically in Fig. 2.

**Example 1: East Texas (US) — Tight Gas (good production)**

This case is taken from ref. 13, and all of the relevant data for this case can be found in that reference. The diagnostic plots for this case are shown in Figs. 3-5. The operator used "better-than-average" data acquisition practices, and the results confirm the validity of the elliptical flow model.

**Application of the Elliptical Flow Model — Illustrative Examples**

In this section we provide diagnostic examples to demonstrate/illustrate value of the elliptical boundary model. The purpose of this exercise is not to obtain "answers" (although several analyses are performed), but rather, to illustrate the quality matches obtained using the elliptical boundary model.
Example 2: North Texas (US) — Very Tight Gas
This case has reasonably good correlation of time-rate-pressure data as shown in Fig. 6. All analyses for this case yield a permeability of < 0.001 md. The log-log diagnostic and type curve matches are shown in Figs. 7 and 8 (respectively) — elliptical reservoir match is reasonable (Fig. 8).

Figure 6 — Example 2: Time-Pressure-Rate (TPR) history plot. Only limited well history was provided, but the data do appear reasonably correlated.

Figure 7 — Example 2: Log-log (rate function) diagnostic plot. Data functions suggest a high to very high conductivity vertical fracture.

Figure 8 — Example 2: "Type curve" plot. Good to possibly excellent match of data functions with elliptical boundary model. Transition flow regime is apparent at late times.

Example 3: North Texas (US) — Tight Gas
This is a "sister" case to Example 2 — but appears to be less well-stimulated ($F_{fr}$). The data are reasonably well-correlated (Fig. 9) and the log-log plot (Fig. 10) clearly suggests a low conductivity vertical fracture. The data match (Fig. 11) is very good, indicating validity in the elliptical reservoir model.

Figure 9 — Example 3: Time-Pressure-Rate (TPR) history plot. Data correlation is fair — most significant features do correlate, but not all features are related.

Figure 10 — Example 3: Log-log (rate function) diagnostic plot. Data functions confirm low conductivity vertical fracture, reasonably good correlation of data.

Figure 11 — Example 3: "Type curve" plot. As noted above, good correlation with low conductivity vertical fracture case. Minor concern regarding data artifacts at late times (good data interpretation).
Example 4: Mexico — Very Tight Gas (long production)

First and foremost, this reservoir has a permeability of < 0.001 md (multiple analyses), and the well is the only well in the field. The long production history and high quality data yield "near textbook" quality diagnostic plots (Figs. 12-15). Fig. 15 confirms validity of the elliptical boundary model.

As a closure in this section we present the matching results for this work, as well as the "average" analysis results for these examples (Tables 2a and 2b). Perhaps the most important outcome of this work is not any specific result, model, etc. — but rather, the proof-of-concept for application of the elliptical boundary model. This observation of the usefulness of the elliptical boundary model as a diagnostic can (and should) lead to better production data analysis — including better "model-based" estimates of fluids in-place.

<table>
<thead>
<tr>
<th>Table 2a — Matching results from interpretation/analysis.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2b — &quot;Average&quot; analysis results.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

Summary and Conclusions

Summary: Simply put, the development of the elliptical boundary model for production data diagnostics and analysis is long overdue. It is often argued that the same concept can be applied using a rectangular or circular reservoir model as a surrogate for the elliptical boundary model. Given the examples reviewed in this work, it is difficult to justify a "surrogate" for the elliptical boundary model which appears (by far) to be the best model for evaluating fractured wells in low to ultra-low permeability gas reservoirs.

Conclusions:

1. The elliptical boundary model is an essential tool for the diagnosis and analysis of production data from hydraulically fractured wells in low to ultra-low permeability gas reservoirs.

2. The diagnostic matches of the production data presented in this work are excellent — despite the fact that the quality of these data, while reasonable, is certainly not comparable to the quality of data that can be acquired in the market (e.g., continuous downhole pressure and surface rate measurements). In practice, the elliptical boundary model should be appropriate for the analysis/diagnosis of production data for virtually all cases of hydraulically fractured wells in tight gas reservoirs.

Recommendations/Comment: We note that the solution for the elliptical boundary model (Appendix A) is extremely tedious — at present, the solution is not-suited for dynamic analysis/history matching (i.e., computational times for this solution can take days on a personal computer). The only practical mechanisms for application of this solution are the preparation of data tables (for interpolation) and the generation of an appropriate suite of "type curves" (see Appendix C for example type curves).
## Nomenclature

### Field Variables
- $c_t$ = Total system compressibility, psi\(^{-1}\)
- $h$ = Pay thickness, ft
- $k$ = Permeability, md
- $k_f$ = Fracture permeability, md
- $k_R$ = Reservoir permeability, md
- $p$ = Pressure, psia
- $p_f$ = Fracture pressure, psia
- $p_i$ = Initial pressure, psia
- $p_p$ = Pseudopressure function, psia
- $p_R$ = Reservoir pressure, psia
- $p_{wf}$ = Flowing bottomhole pressure, psia
- $q$ = Flowrate, STB/D
- $q_g$ = Gas flowrate, MSCF/D
- $r_w$ = Wellbore radius, ft
- $t$ = Time, hr
- $x_f$ = Fracture half-length, ft

### Dimensionless Variables
- $b_{D_{pss}}$ = Dimensionless pseudosteady-state constant
- $F_E$ = Elliptical fracture conductivity
- $P_D$ = Laplace transform of dimensionless pressure
- $p_D$ = Dimensionless pressure
- $p_{Dd}$ = Dimensionless pressure derivative
- $q_D$ = Dimensionless flowrate
- $q_{Dd}$ = Dimensionless rate integral
- $q_{Did}$ = Dimensionless rate integral derivative
- $t_D$ = Dimensionless time (wellbore radius)
- $t_{Da}$ = Dimensionless time (drainage area)
- $t_{Dxf}$ = Dimensionless time (fracture half-length)

### Mathematical Functions and Variables
- $A_{2n}^{2n}$ = Mathieu function Fourier coefficients
- $a$ = Long axis of the elliptical system
- $b$ = Short axis of the elliptical system
- $B_n$ = Coefficient in solution series
- $c_{e2n}$ = $\pi$-periodic angular Mathieu function
- $C_{e2n}$ = Radial Mathieu function
- $D_n$ = Coefficient in solution series
- $E_{ek2n}$ = Radial Mathieu function
- $F_{ek2n}$ = Radial Matthieu function
- $H$ = Angular function
- $I_{2r}$ = Bessel function of order $rr$
- $K_{2r}$ = Bessel function of order $r$
- $m$ = Positive integer
- $n$ = Positive integer
- $r$ = Positive integer
- $R_{2n}$ = Coefficient in series solution
- $s$ = Laplace domain parameter
- $X$ = Radial function
- $x$ = Cartesian coordinate
- $y$ = Cartesian coordinate
- $Y_{2r}$ = Bessel function of order $r$

### Greek Symbols
- $\alpha$ = regression coefficient
- $\beta$ = Coefficient in fracture pressure series
- $\delta$ = Dirac delta function
- $\epsilon_n$ = $1+$ Kronecker delta function
- $\phi$ = porosity, fraction
- $\gamma$ = coefficient in reservoir pressure series
- $\eta$ = Angular elliptical coordinate
- $\kappa_D$ = Diffusivity ratio, dimensionless
- $\mu$ = Viscosity, cp
- $\Omega$ = Kernel of sum equation
- $\xi$ = Radial elliptical coordinate
- $\xi_{0}$ = Elliptical boundary characteristic variable

### Subscript
- $D$ = Dimensionless
- $f$ = Fracture
- $i$ = Integral function or initial value
- $id$ = Integral derivative function
- $pss$ = Pseudosteady-state
- $m$ = Positive integer
- $n$ = Positive integer
- $r$ = Positive integer
- $R$ = Reservoir

### Gas Pseudofunctions:
- $p_p = \frac{\mu_{i}z_i}{p_i} \int_{p_{base}}^{p_{p}} \frac{dp}{p}\int_{t}\mu_2d$
- $t_a = \frac{\mu_{gi}c_{gi}}{q(t)} \int_{0}^{t} q(t) dt$
- $t_{mba,gas} = \frac{\mu_{gi}c_{gi}}{q(t)} \int_{0}^{t} q(t) dt$
References


Appendix A: Development of the Model

For a system containing a single-phase, slightly compressible fluid, the diffusivity equation is expressed by Eq. A.1

$$\nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \quad \text{............................................................ (A.1)}$$

Where:

- \(p\) = pressure
- \(t\) = time
- \(k\) = permeability
- \(\phi\) = porosity
- \(\mu\) = fluid viscosity
- \(c_t\) = total compressibility

Assuming uniformity in the \(z\)-axis, and neglecting any gravity effects, we can write the diffusivity equation in Cartesian coordinates as:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \quad \text{................................................. (A.2)}$$

Eq. A.2 can be transformed into elliptical coordinates using the following transform correlations:

$$x = x_f \cdot \cosh \xi \cdot \cos \eta \quad \text{............................................................ (A.3)}$$

$$y = x_f \cdot \sinh \xi \cdot \sin \eta \quad \text{............................................................ (A.4)}$$

Where \(x_f\) is defined as the fracture half-length.

The elliptical system is characterized by series of confocal ellipses (focal radii equal to \(x_f\)), which are normal to a family of hyperbolas of the same focal radii. Fig. A.1 shows a schematic of elliptical coordinates.

**Figure A.1 — Elliptical coordinates.**

The diffusivity equation in elliptical coordinates is given as:

$$\frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2} = \frac{\phi \mu c_t}{2k} \left[\cosh(2\xi) - \cos(2\eta)\right] \frac{\partial p}{\partial t} \quad \text{............................................................ (A.5)}$$

An elliptical system with a boundary at \(\xi = 0\) is transformed into a \((\xi \times 2\pi)\) rectangle in elliptical coordinates. The fracture transforms to a line of the width \(x_f\) from 0 to \(2\pi\). Fig. A.2 shows how the transformation works. It should be noted that according to the symmetry of the problem, we have just shown the transform in one-quarter of the plane. That is the reason \(\eta\) is taken from 0 to \(\pi/2\).

**Figure A.2 — Elliptical system after transformation.**

### Initial Condition and the Boundary Conditions

After defining the governing equation for the model we need to introduce initial and boundary conditions to solve the given partial differential equation (PDE).

#### Initial Condition:

We assume a uniform initial pressure \(p_i\) for reservoir at the start of production

$$p(\xi, \eta, 0) = p_i \quad \text{............................................................ (A.6)}$$

#### Boundary Conditions:

For defining boundary conditions we need orientation from the geometry of the problem. In the angular direction \(\eta\) we use the following boundary conditions which are based on the symmetry of the model:

$$\left.\frac{\partial p}{\partial \eta}\right|_{\eta = 0} = 0 \quad \text{............................................................ (A.7)}$$

$$\left.\frac{\partial p}{\partial \eta}\right|_{\eta = \frac{\pi}{2}} = 0 \quad \text{............................................................ (A.8)}$$

It is obvious that the solution should be \(\pi\)-periodic solution because of the symmetry in the flow from two sides of the fracture. In the radial direction our first B.C. is given as:

$$p(0, \eta, t) = p_{wf} \quad \text{............................................................ (A.9)}$$

It should be noted that we have assumed the fracture to be an infinite-conductivity conduit of zero width lying at \(\xi = 0\).

Depending on whether we are dealing with infinite-acting or finite reservoir case, the second B.C. is defined as a constant pressure at infinity for the infinite-acting model, and using a "no-flow" boundary for the bounded case:

$$p(\xi, \eta, t) \xi \rightarrow \infty = p_i \quad \text{............................................................ (A.10)}$$

$$\left.\frac{\partial p}{\partial \xi}\right|_{\xi = \xi_0} = 0 \quad \text{............................................................ (A.11)}$$

Applying the following dimensionless variables enables us to transform the diffusivity equation and the boundary and initial conditions into dimensionless form.

$$p_D = \frac{1}{141.2} \frac{k h}{q B \mu} (p_i - p) \quad \text{............................................................ (A.12)}$$

$$t_D = 0.0002637 \frac{k}{\phi \mu c_t x_f^2} t \quad \text{............................................................ (A.13)}$$
\[ q_D = 141.2 \frac{qB\mu}{k h} \left( \frac{1}{(P_i - P_w)} \right) \]  

(A.14)

Using these definitions, we can write the dimensionless form of the diffusivity equation, the I.C., and the B.C.s as follows:

**Diffusivity Equation:**

\[ \frac{\partial^2 p_D}{\partial \xi^2} + \frac{\partial^2 p_D}{\partial \eta^2} = \frac{s}{2} [\cosh(2\xi) - \cos(2\eta)] \frac{\partial p_D}{\partial t_D} \]  

(A.15)

**Initial Condition:**

\[ p_D(\xi, \eta, 0) = 0 \]  

(A.16)

**Boundary Conditions:**

\[ \frac{\partial p_D}{\partial \eta} \mid_{\eta = 0} = 0 \]  

(A.17)

\[ \frac{\partial p_D}{\partial \eta} \mid_{\eta = \pi/2} = 0 \]  

(A.18)

\[ p_D(0, \eta, t) = 1 \]  

(A.19)

\[ p_D(\xi, \eta, t) \xi \rightarrow \infty = 0 \]  

(infinite-acting model)  

(A.20)

\[ \frac{\partial p_D}{\partial \xi} \mid_{\xi = \xi_0} = 0 \]  

(no-flow boundaries model)  

(A.21)

**Solution to the Model**

The resulting PDE can be solved in the Laplace domain where the time derivative is reduced to a linear dependency with the Laplace parameter \((s)\). Taking the Laplace transform of the diffusivity equation and the B.C.s yields:

**Diffusivity Equation:**

\[ \frac{\partial^2 p_D}{\partial \xi^2} + \frac{\partial^2 p_D}{\partial \eta^2} = \frac{s}{2} [\cosh(2\xi) - \cos(2\eta)] p_D \]  

(A.22)

**Boundary Conditions:**

\[ \frac{\partial p_D}{\partial \eta} \mid_{\eta = 0} = 0 \]  

(A.23)

\[ \frac{\partial p_D}{\partial \eta} \mid_{\eta = \pi/2} = 0 \]  

(A.24)

\[ p_D(0, \eta) = \frac{1}{s} \]  

(A.25)

\[ p_D(\xi, \eta) \xi \rightarrow \infty = 0 \]  

(infinite-acting model)  

(A.26)

\[ \frac{\partial p_D}{\partial \xi} \mid_{\xi = \xi_0} = 0 \]  

(no-flow boundaries model)  

(A.27)

Eq. A.22 can be solved by using the method of separation of variables. In this method we assume that the solution can be written as the product of two independent functions — written in terms of each independent parameter.

\[ p_D(\xi, \eta) = X(\xi)H(\eta) \]  

(A.28)

Using the separation of variables assumption (Eq. A.28) with Eq. A.22, and rearranging results:

\[ \frac{1}{X} \frac{\partial^2 X}{\partial \xi^2} - \frac{s}{2} \cosh(2\xi) = - \frac{1}{H} \frac{\partial^2 H}{\partial \eta^2} - \frac{s}{2} \cos(2\eta) = a \]  

(A.29)

This leads to two separate Ordinary Differential Equations (ODEs) as follows:

\[ \frac{\partial^2 H}{\partial \eta^2} + (a + \frac{s}{2} \cos(2\eta))H = 0 \]  

(Angular Equation)  

(A.30)

\[ \frac{\partial^2 X}{\partial \xi^2} - (a + \frac{s}{2} \cosh(2\xi))X = 0 \]  

(Radial Equation)  

(A.31)

These kinds of equations generally have series solutions in terms of a special function called the Mathieu Function. The characteristics and behavior of the different kinds of Mathieu functions are discussed extensively in McLachlan. The symmetry of the problem leads us to select a \(\pi\)-periodic, even function for the angular equation. Referring to the table of solutions we will find the cosine series solution as the adequate form of the solution for our equation:

\[ H_n(\eta) = c\cos_{2n}(\eta, -\frac{s}{4}) \]  

(A.32)

Where

\[ c\cos_{2n}(\eta, -\frac{s}{4}) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^2 \cos(2r \eta) \]  

(A.33)

The \(A_{2n}^2\) terms are called Fourier Coefficients, and are functions of the Laplace parameter as well as the order of Mathieu Function \((n)\). The subscript \(n\) reminds us the fact that infinite series solution exists — \(n\) corresponds to the \(n\)th characteristic value of the equation.

Using McLachlan's table, we find three types of Mathieu functions which are acceptable as solutions to the radial equation:

\[ C_{2n}(\xi, -\frac{s}{4}), F_{2n}(\xi, -\frac{s}{4}), \text{and } F_{2n}(\xi, -\frac{s}{4}) \]

These functions are defined by the following series:

\[ C_{2n}(\xi, -\frac{s}{4}) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^2 I_{2r}(\sqrt{s} \cos(\xi)) \]  

(A.34)

\[ F_{2n}(\xi, -\frac{s}{4}) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^2 I_{2r}(i\sqrt{s} \cosh(\xi)) \]  

(A.35)

\[ F_{2n}(\xi, -\frac{s}{4}) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^2 K_{2r}(\sqrt{s} \cosh(\xi)) \]  

(A.36)

Where \(I_{2r}, Y_{2r}, \text{and } K_{2r}\) are corresponding Bessel Functions of order \(r\).
Solution for the infinite-acting case: (radial flow)

The outer boundary condition (Eq. A.26) implies that a finite value of pressure occurs as \( \xi \to \infty \). Among the three aforementioned solutions, only the solution

\[
Fek_{2n}(\xi, -\frac{S}{4})
\]

meets the necessary requirements of the outer boundary condition. So the radial solution is given as:

\[
X_n(\xi) = B_n Fek_{2n}(\xi, -\frac{S}{4})
\] ............................. (A.37)

Combining Eqs. A.32 and A.37 we obtain the preliminary solution for the dimensionless pressure in the Laplace domain:

\[
\frac{1}{s} = \sum_{n=0}^{\infty} B_n c e_{2n}(\eta, -\frac{S}{4}) Fek_{2n}(0, -\frac{S}{4})
\] ............................. (A.38)

To fully solve Eq. A.37 we need to find the value of \( B_n \). So the radial solution can be written in the form of a linear combination of the two functions as:

\[
X_n(\xi) = B_n Fek_{2n}(\xi, -\frac{S}{4}) + D_n C e_{2n}(\xi, -\frac{S}{4})
\] ............................. (A.46)

Applying the outer boundary condition leads to:

\[
B_n Fek'_{2n}(\xi, -\frac{S}{4}) + D_n C e'_{2n}(\xi, -\frac{S}{4}) = 0
\] ............................. (A.47)

To calculate the \( D_n \) coefficients, we utilize the inner boundary condition similar to that for the infinite-acting case:

\[
\frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n \frac{2 A_n^2}{s} Fek_{2n}(0, -\frac{S}{4})
\] ............................. (A.48)

Finite Conductivity Fracture

The case of a well with a finite conductivity fracture becomes necessary as the pressure drop within the fracture becomes significant compared to the total pressure drop of the fracture and reservoir. The case of finite conductivity elliptical fracture has been studied extensively by Riley and this portion of the work borrows heavily from Riley’s work. The governing equation in the fracture can be written as the following in the Laplace domain:

\[
\frac{\partial^2 \bar{P}_{f}}{\partial \eta^2} + 2 \frac{\partial \bar{P}_{R}}{\partial \xi \xi} = -\kappa_D \frac{E}{2} (1 - \cos(2\eta)) \bar{P}_{f}
\] ............................. (A.49)

\[
\kappa_D = \frac{(\phi_1) f k_R}{(\phi_1) R k_f}
\]

represents the Dirac Delta function. In practical cases \( \kappa_D \) is a small number in the order of \( 10^{-7} \) to \( 10^{-10} \) the effect of fracture diffusivity in the governing equation is neglected. Hence, the final form of the governing equation for the fracture is reduced to:

\[
\frac{\partial^2 \bar{P}_{f}}{\partial \eta^2} + 2 \frac{\partial \bar{P}_{R}}{\partial \xi \xi} = -\frac{\pi}{s^{FE}} \delta(\eta - \frac{\pi}{2})
\] ............................. (A.50)

With the boundary conditions:

\[
\left. \frac{\partial \bar{P}_{f}}{\partial \eta} \right|_{\eta=0} = \left. \frac{\partial \bar{P}_{f}}{\partial \eta} \right|_{\eta=\frac{\pi}{2}} = 0
\] ............................. (A.51)

Solution for the case of a bounded reservoir: (radial flow)

In this case, we have two types of Mathieu functions that satisfy the outer boundary condition of the problem (Eq. A.21). So the radial solution can be written in the form of a linear combination of the two functions:
The solution to Eq. A.50 has been discussed extensively by Riley for the case of an infinite-acting reservoir. The solution is recast slightly to obtain a suitable solution for the bounded case. In brief, Riley proposed a solution in the form of cosine series as:

\[
\bar{p}_f(\eta,s) = \sum_{r=0}^{\infty} \beta_{2r} \cos(2r\eta) \quad \text{(A.52)}
\]

Introducing the fracture solution and the reservoir solution for a bounded reservoir into Eq. A.50 we have:

\[
\sum_{r=0}^{\infty} (-1)^r (-4r^2) \beta_{2r} \cos(2r\eta) + \frac{2}{F_E} \sum_{n=0}^{\infty} (-1)^n n^2 F_{2n} a_{2n} R_{2n}^2 \left(\frac{\eta}{s} - \frac{\pi}{4}\right) = -\frac{\pi}{sF_E} \delta(\eta - \frac{\pi}{2})
\]

\[
\text{.......................... (A.53)}
\]

Where:

\[
R_{2n} = F_{k'2n} \left(\xi_0, \frac{\eta}{4}\right) c_{e2n} \left(0, \frac{\eta}{4}\right) - c_{e2n} \left(0, \frac{\eta}{4}\right) F_{k'2n} \left(0, \frac{\eta}{4}\right).
\]

\[
\text{...................................... (A.54)}
\]

Using the definition of the Mathieu function \(c_{e2n}(\eta, -\frac{s}{4})\) (Eq. A.33), Riley showed that we can write Mathieu function as a cosine series or vice versa if we write it in form of the cosine series we would have:

\[
2F_E \beta_{2r} - \sum_{p=0}^{\infty} \epsilon_p \Omega_{2p} \beta_{2r} = \frac{2}{2\epsilon_r} \quad \text{(A.55)}
\]

Where,

\[
\Omega_{2p} = \sum_{m=0}^{\infty} \sigma_{2m} A_{2m} R_{2m}
\]

and

\[
\epsilon_n = 2 \text{ if } n = 0, \epsilon_n = 1 \text{ otherwise} \quad \text{(A.57)}
\]

We can solve the system of equations described in Eq. A.55 assuming a finite value for the series \(n\). It should be noted that \(\beta_{2r}\) oscillates at first and then steadily, but declines very slowly as \(r\) grows. We calculate \(\beta_{2r}\) for \(r\) values between 0 to \(n\) from the system of equation and use regression technique to obtain the values of \(\beta_{2r}\) for \(r\)'s greater than \(n\). Our observation shows that using a regression-based approach for the following model accurately describes the behavior of \(\beta_{2r}\) for \(r\) values greater than \(n\):

\[
\beta_{2r} = \frac{1}{2r^3} \quad \text{(A.58)}
\]

### Appendix B: The Elliptical Boundary Characteristic Variable \(\xi_0\)

If we assume the elliptical outer boundary has the same focal length as the hydraulic fracture length, we can write the following equations correlating all aspects of the drainage area to a single parameter \(\xi_0\) (See Fig. B.1 below).

**Figure B.1 — Schematic of the elliptical reservoir model**

\[
a = x_f \cosh(\xi_0) \quad \text{(B.1)}
\]

\[
b = x_f \sinh(\xi_0) \quad \text{(B.2)}
\]

\[
\text{Area} = \pi ab = \frac{\pi}{2} \sinh(2\xi_0) x_f^2 \quad \text{(B.3)}
\]

Drainage Aspect Ratio \(\frac{a}{b} = \coth(\xi_0) \quad \text{(B.4)}
\]

Penetration Ratio \(\frac{x_f}{a} = \cosh(\xi_0)^{-1} \quad \text{(B.5)}
\]

Fixing the boundary characteristic parameter, we can calculate every other aspects of the size and ratios of the drainage area as a function of fracture half length \(x_f\). Table B.1 shows the system parameters for some selected values of the elliptical boundary characteristic variable.

<table>
<thead>
<tr>
<th>(\xi_0)</th>
<th>Aspect Ratio</th>
<th>Penetration Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>4.0830</td>
<td>0.9695</td>
</tr>
<tr>
<td>0.50</td>
<td>2.1640</td>
<td>0.8868</td>
</tr>
<tr>
<td>0.75</td>
<td>1.5744</td>
<td>0.7724</td>
</tr>
<tr>
<td>1.00</td>
<td>1.3130</td>
<td>0.6481</td>
</tr>
<tr>
<td>1.25</td>
<td>1.1789</td>
<td>0.5295</td>
</tr>
<tr>
<td>1.50</td>
<td>1.1048</td>
<td>0.4251</td>
</tr>
<tr>
<td>1.75</td>
<td>1.0623</td>
<td>0.3374</td>
</tr>
<tr>
<td>2.00</td>
<td>1.0373</td>
<td>0.2658</td>
</tr>
<tr>
<td>3.00</td>
<td>1.0050</td>
<td>0.0993</td>
</tr>
<tr>
<td>4.00</td>
<td>1.0007</td>
<td>0.0366</td>
</tr>
<tr>
<td>5.00</td>
<td>1.0001</td>
<td>0.0135</td>
</tr>
</tbody>
</table>

### Appendix C: Type Curves \(q_D \text{ and } t_{DA} \text{ format}) for the Elliptical Boundary Model

The equivalent constant rate format type curves for this work are shown in \(q_D \text{ versus } t_{DA} \text{ format} — \text{ see Figs. C.1 to C.4, } F_E=1, 10, 100, 1000 \text{ (respectively).}
Figure C.1 — Type curve for a fractured well centered in a closed (homogeneous) elliptical reservoir — $F_e = 1$, various $\xi_0$-values; $q_0$ functions versus $t_{DA}$ format.

Figure C.2 — Type curve for a fractured well centered in a closed (homogeneous) elliptical reservoir — $F_e = 10$, various $\xi_0$-values; $q_0$ functions versus $t_{DA}$ format.
Figure C.3 — Type curve for a fractured well centered in a closed (homogeneous) elliptical reservoir — $F_0=100$, various $\xi_o$-values; $q_D$ functions versus $t_{DA}$ format.

Figure C.4 — Type curve for a fractured well centered in a closed (homogeneous) elliptical reservoir — $F_0=1000$, various $\xi_o$-values; $q_D$ functions versus $t_{DA}$ format.