Application of Convolution and Average Pressure Approximation for Solving Nonlinear Flow Problems — Constant Wellbore Pressure Case

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Abstract

"Real gas" flow problems (i.e., problems where the gas properties are specifically taken as implicit functions of pressure, temperature, and composition) are particularly challenging because the diffusivity equation for the "real gas" flow case is strongly non-linear. Whereas different methods exist which allow us to approximate the solution of the real gas diffusivity equation, all of these approximate methods have limitations (including numerical models).

The purpose of this work is to provide a direct solution mechanism for the case of time-dependent real gas flow which uses an approach that combines the so-called average pressure approximation (a convolution for the right-hand-side non-linearity) and the Laplace transformation. For reference, Mireles and Blasingame used a similar scheme to solve the real gas flow problem conditioned by the constant rate inner boundary condition.

In this work we provide a direct solution scheme to solve the constant pressure inner boundary condition problem. Our new semi-analytical solution was developed and implemented in the form of a direct (non-iterative) numerical procedure and successfully verified against numerical simulation.

Introduction

Mireles and Blasingame [Mireles and Blasingame (2003)] developed a closed form Laplace domain solution for the flow of a real gas from a well producing at a constant rate in a bounded circular reservoir. More importantly, they proposed a new approach that uses pseudopressure to linearize the spatial portion of the diffusivity equation (i.e., the left-hand-side (LHS)) as was done traditionally, but for the right-hand-side (RHS), (i.e., the "time" portion) Mireles and Blasingame used a special convolution formulation to account for the pressure-dependent, non-linear term. Consequently, the Mireles and Blasingame semi-analytical solution eliminates the use of pseudotime for this case.

Although being rigorous, the Mireles and Blasingame solution relies on evaluation of the non-linear term based on the average reservoir pressure predicted from material balance. They did not assess the nature and applicability of the average pressure approximation (APA), but exhaustively validated the APA approach using numerical simulation for the case of a constant rate inner boundary condition. The effort of Mireles and Blasingame should be considered to be an empirical demonstration of validity of the APA for all (realistic) values of pressure.

Again, the need for such a (direct) solution arises in the analysis of both gas well test data and gas well production data — where both analyses have traditionally used approximate methods such as the pressure or pressure-squared methods, [Rawlins and Schellhardt (1935), Aronofsky and Jenkins (1954)] or rigorous, but tedious pseudovariables [Al-Hussainy et al (1966) and Agarwal (1979)].

Mireles and Blasingame showed that for the constant rate inner-boundary condition (and the uniform pressure initial condition) that the diffusivity equation for time-dependent fluid flow in porous media is given as:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial m_D}{\partial r_D} \right] = \frac{\mu c}{\mu c_i} \frac{\partial m_D}{\partial t_D}
\]  \hspace{1cm} (1)

Where Eq. 1 is written in the appropriate dimensionless form. Mireles and Blasingame proposed the following Laplace domain result for relating the "liquid" and "gas" solutions:

\[
\bar{m}_{D,\text{gas}} = \bar{g}(u) \bar{m}_D(ug(u))
\]  \hspace{1cm} (2)

\[
\bar{m}_D(ug(u))
\]  is the solution of the "liquid" problem (which can be written in analytical form for the stated boundary conditions). The non-linear portion of the problem (g(u)) is defined as:

\[
L[g(t_D)] = \bar{g}(u) = \frac{1}{u} \left[ 1 - \frac{1}{\beta_T(t_D)} \right]
\]  \hspace{1cm} (3)

Where the APA method is used to evaluate the non-linear term \(\beta_T\), which is defined as:

\[
\beta_T = \frac{\mu c}{\mu c_i}
\]  \hspace{1cm} (4)
We note that in the case of the constant rate inner boundary condition that the $\tilde{q}(u)$ term depends only on the value of $\beta$-function calculated at average pressure. Since the rate is known, the average pressure and therefore $\beta_p$ can be evaluated directly.

The final step in the procedure given by Mireles and Blasingame was to propose functional and numerical models to represent the $\tilde{q}(u)$ term. A numerical model which included applying the Roumboutsos and Stewart algorithm [Roumboutsos and Stewart (1988)] to transform the data into Laplace domain. Finally, the Stehfest algorithm [Stehfest (1970)] was used to obtain the (numerical) inverse transformation of the Laplace domain solution.

Rationale for This Work:

A logical continuation of the work by Mireles and Blasingame is the development of a closed form Laplace domain solution for the flow of a real gas from a well producing at a constant pressure in a bounded reservoir — using pseudopressure, convolution, and the APA method. Since the approach should predict both pressures and rates, coupling the diffusivity equation with a (rate) deliverability equation must be used (a "predictor-corrector" type of approach).

Mireles and Blasingame gave suggestions to develop a solution for the constant pressure inner-boundary condition (i.e., the constant pressure production case). In particular, they questioned applicability of convolution (i.e., the superposition principle) as a means to relate the constant pressure inner boundary condition with the constant rate condition (this is the approach used for the liquid flow case). Although the approach based on convolution and the APA linearizes the diffusivity equation in a sense, convolution does not work for the constant production pressure case. Mireles and Blasingame suggested recasting the diffusivity equation specifically for the case of constant pressure condition and conjectured a need to start from fundamental flow theory and material balance.

In this work we provide a rigorous development of the solution methodology using the same fundamental principles in convolution theory and Laplace transformation as Mireles and Blasingame. The validity of our results relies upon how we approximate and reference the time-dependent inner boundary condition (e.g., a variable bottomhole pressure schedule).

Numerical simulation is used to validate our proposed methodology and as well as assess the accuracy of our solution for the case of a well producing at a constant bottomhole pressure in a bounded circular reservoir. We will also discuss the application of the proposed methodology to more complex cases — in particular, to gas flow systems with a time-dependent inner boundary condition (e.g., a variable flowing bottomhole pressure schedule).

Some possible applications of the new approach are as follows:

- Validation of numerical simulator results — assessment of time and space discretization.
- Generation of production rates and well test analysis pressures — especially for the case of a general rate/pressure schedule. This would provide a semi-analytical "reservoir simulator."
- Computation of pressure distributions at any time given a prescribed rate history.
- Application to gas injection/gas displacement problems.

We also discuss the advantages of the method over alternative methods such as:

- **Numerical Simulation**: In case of a constant production rate the solution is robust and direct (it can be evaluated at any time), and has no temporal or spatial discretization (as shown by Mireles and Blasingame). For the case of constant wellbore pressure production, the solution is robust and requires only minor temporal discretization.

In addition, the constant wellbore pressure solution is competitively fast compared to numerical simulation — and we believe future implementation of this solution may approach the "instantaneous" speed of the constant rate solution. By comparison, numerical simulation solutions must be calibrated in terms of spatial and temporal discretizations.

- **Other Approaches**: The Average Pressure Approximation (APA) is applicable for the complete range of pressure for a given problem. This is compared to the $p$- and $p^2$-methods which have very limited validity (i.e., specific pressure ranges of validity). In addition, the APA approach is direct and non-iterative, as compared to iterative pseudopressure-pseudotime approach currently used in certain commercial implementations for the variable-pressure solution.

### Development of an Analytical Pressure Solution for the Case of a Constant Pressure Inner Boundary Condition

#### General Approach

In this work we consider a non-linear parabolic partial differential equation of the form:

$$\nabla^2 y = \beta(y) \frac{\partial y}{\partial t} \tag{5}$$

We show that such an equation effectively describes the flow of real (compressible) gases through porous media. Whereas similar linear equations ($\beta(y)=$constant) are readily solved using the Laplace transformation, in our case, the application of the Laplace transformation is subject to reconstruction of the right-hand side of Eq.5 into a more suitable form.

Since $y$ is a function of both spatial and temporal variables (i.e., $y=y(r,t)$) and $\beta(y)$ is a composite function, then $\beta(y)$ is necessarily a function of both spatial and temporal variables. Considering spatial variables as independent, we can write:

$$\beta(t) \frac{\partial y}{\partial t} = \int_0^t \frac{\partial y}{\partial \tau} g(t-\tau) d\tau \tag{6}$$

We note that the $g(t)$ function must adhere to similar mathematical conditions as $\beta(t)$ (e.g., continuity, boundedness, derivative formulations, etc.). As convolution is commutative, Eq. 6 becomes:

$$\beta(t) \frac{\partial y}{\partial t} = g(t) \star \frac{\partial y}{\partial t} \tag{7}$$

(* is the convolution operation)
Another well-known property of the convolution is that the Laplace transform of the convolution of two functions is the product of the Laplace transforms of the functions — applying this identity to Eq. 7 gives:

$$\mathcal{L}\left[ g(t) \ast \frac{\partial v}{\partial t}(t) \right] = \mathcal{L}[g(t)] \mathcal{L}\left[ \frac{\partial v}{\partial t} \right]$$ ...................................................... (8)

where $\mathcal{L}[g]$ represents the non-linearity of the problem. Using the identities proposed using Eqs., 6-8, and combining these with Eq. 5, we obtain:

$$\nabla^2 y = g(t) \ast \frac{\partial v}{\partial t}(t)$$ ...................................................... (9)

Where $y$, its temporal derivative, and the $g(t)$-function are all functions of both spatial and temporal variables. Taking the Laplace transform of Eq. 9, we obtain:

$$\nabla^2 \mathcal{L}[y] = \mathcal{L}[g(t)][g(u) - y(t = 0)]$$ ...................................................... (10)

Where, in Eq. 10, $u$ is the Laplace transform parameter. Coupled with the appropriate boundary conditions, Eq. 10 can be solved, and the solution (obviously) depends on $\mathcal{L}[g]$. If we can then calculate the $g(t)$-function for every position in space and time (or $\mathcal{L}[g(t)]$ for every position in space and $u$) within the region of interest, then we will obtain the Laplace domain solution of Eq. 10.

The primary value of this work is that we have developed and validated a direct (non-iterative) method of calculating the $g(t)$-function. Moreover, we show that using an "average (pressure) value" approximation (i.e., the APA approach) we can eliminate the dependence of $g(t)$ on spatial variables. Specifically, $g(t)$ can be assessed solely as a function of the average (reservoir) pressure — which is implicitly a function of time.

Real-Gas Systems

For clarity we will formulate the problem in terms of effective "gas" porosity and neglect residual water compressibility. Also, rock compressibility (i.e., porosity dependence on pressure) will be neglected. It will be evident that our method could be generalized to account for the compressibility effects as stated above. Our assumptions are reasonable because residual water and rock compressibility related effects are typically of second order compared to gas expansion (for gas reservoirs) — with the noted exception being abnormally pressured gas reservoirs. Finally, permeability is assumed to be constant and independent of pressure. The interested reader is referred to Appendix A of Zhakupov [Zhakupov (2005)] for complete development of this formulation.

Using the following dimensionless groups:

$$t_D = t_{DC} \frac{k}{\phi \mu \varepsilon_0 r_w^2}$$ ...................................................... (11)

Where $t_{DC}$ equals 0.0002637 (for $t$ in hours) or 0.00633 (for $t$ in days) — $t$ in days is the basis used in this work.

$$r_D = r/r_w$$ ...................................................... (12)

$$m_D = \frac{m_i - m}{m_i - m_{wf}}$$ ...................................................... (13)

Where the pseudopressure definition ($m_D$) is characteristic to constant pressure production. In addition, for this work we use the "normalized" pseudopressure function defined as:

$$m(p) = \frac{\mu z_i}{\rho_l \rho_{base} \mu z} \int_0^p \frac{p}{p_D} dp$$ ...................................................... (14)

Combining the continuity equation, Darcy’s law, and the definition of isothermal compressibility (for a gas), we obtain the diffusivity equation for real gases in dimensionless form:

$$\frac{1}{r_D^2 \frac{\partial}{\partial r_D}} \left[ r_D \frac{\partial m_D}{\partial r_D} \right] = \frac{\mu z}{\mu_0 z_0} \frac{\partial m_D}{\partial r_D}$$ ...................................................... (15)

We note that although the definition of the dimensionless form of the pseudopressure for a constant pressure inner boundary condition is different (i.e., Eq. 13), the dimensionless form of the diffusivity equation (Eq 15) is the same for the case of a constant rate boundary condition. The initial and boundary conditions applied to Eq. 15 in this work are as follows:

Initial condition: (uniform pressure distribution)

$$m_D(t_D = 0) = 0$$ ...................................................... (16)

Inner boundary condition: (constant bottomhole pressure)

$$\left[ m_D \right]_{r_D = 1} = 1$$ ...................................................... (17)

Outer boundary condition: (no-flow boundary)

$$\left[ r_D \frac{\partial m_D}{\partial r_D} \right]_{r_D = r_{eD}} = 0$$ ...................................................... (18)

Relating Eq. 15 to Eq. 5, we define:

$$\beta = \frac{\mu z}{\mu_0 z_0}$$ ...................................................... (19)

The $\beta$-function depends on pressure — and thus, in the context of production from a reservoir, then the $\beta$-function depends on both position and time. Comparing Eq. 15 to Eq. 6, and using Eq. 16 we have:

$$\beta(t_D) \frac{\partial m_D}{\partial t_D}(t_D) = \int_0^{t_D} \frac{\partial m_D}{\partial \tau}(\tau) g(t_D - \tau) d\tau$$ ...................................................... (20)

Using a process similar to the one used to construct Eq. 10, we have for this case:

$$\frac{d^2 m_D(u)}{dr_D^2} + \frac{1}{r_D} \frac{d m_D}{d r_D} = \left[ \bar{m}_D(u) - m_D(t_D = 0) \right] \bar{g}(u)$$ ...................................................... (21)

Taking the Laplace transform of the inner boundary condition (Eq. 17), we obtain:

$$\bar{m}_D = \frac{1}{u}$$ ...................................................... (22)

Taking the Laplace transform of the outer boundary condition (Eq. 18), yields:

$$\left[ r_D \frac{d \bar{m}_D}{d r_D} \right]_{r_D = r_{eD}} = 0$$ ...................................................... (23)

One of the most important points of this work is that it can be shown by induction that the following identity is valid: (see [Warren and Root (1963) and Thompson et al (1991)])

$$\bar{m}_D(u) = \bar{g}(u) \int_0^\infty P_D(t_D) e^{-u \bar{g}(u) t_D} dt_D$$ ...................................................... (24)

Or, taking the Laplace transform of Eq. 24, we have:

$$\bar{m}_{D,gas}(u) = \bar{g}(u) \bar{m}_D(\bar{g}(u))$$ ...................................................... (25)
Where $\tilde{m}_D$ is the dimensionless solution in the Laplace domain for the linear (i.e., "liquid" or "slightly compressible") formulation. In this work we construct a solution in analytical form for our boundary conditions. However, in order to implement this formulation we must establish a relationship between $\tilde{\beta}(r_D,t_D)$ and $\tilde{g}(r_D,u)$.

As noted by Mireles and Blasingame [Mireles and Blasingame (2005)] the proposed form of our solution (Eq. 25) is equivalent in form to the analytical solution developed for the case of a naturally fractured (dual porosity) reservoir [Warren and Root (1963)]. In reviewing Eq. 25, it is obvious that we obtain the $\tilde{g}(r_D,u)$ function independently — this is straightforward (and rigorous) for the constant rate case, but becomes problematic for the case of production at a constant bottomhole flowing pressure (i.e., we must develop a methodology which computes rate and average reservoir pressure simultaneously). This is discussed in detail in other sections of this paper.

Key Relation

As with the work of Mireles and Blasingame, we use the gas material balance to establish a relationship between $\tilde{\beta}(r_D,t_D)$ and $\tilde{g}(r_D,u)$. This leads to the APA approach — i.e., referencing the non-linear terms to the average reservoir pressure:

$$\tilde{\beta}_P = \frac{\tilde{w}}{\mu c_i}$$

Eliminating the dependence of the $\beta$-function on position, then the right-hand side of Eq. 15 depends on only one time. In Appendix A we provide the complete mathematical developments for the following relationship:

$$q(t_D) = \int_0^{t_D} \frac{g(\tau)}{\tilde{\beta}_P(\tau)} g(t_D - \tau) d\tau$$

Taking the Laplace transformation of Eq. 27, the $\tilde{g}(u)$ function is obtained as:

$$L[\tilde{g}(t_D)] = \tilde{g}(u) = \frac{1}{L[\tilde{\beta}_P(t_D)]} L[q(t_D')]$$

Note that (by construction), our expression for $\tilde{g}(u)$ (Eq. 28) contains the defining expression for the constant rate boundary condition as a particular case (i.e., $q(t_0)=1$). Further, as Eq. 28 contains the rate function explicitly, the strategies proposed by Mireles and Blasingame to calculate $\tilde{g}(u)$ are not applicable for the case of a constant flowing bottomhole pressure.

In the case addressed by Mireles and Blasingame the average reservoir pressure (and therefore $\tilde{\beta}_P$) can be directly obtained because the rate is known. In our case (i.e., the constant pressure inner boundary condition), we must calculate the $\tilde{g}(u)$ term using time discretization. In the next section we present a strategy for obtaining the $\tilde{g}(u)$ term, which in turn will enable us to apply Eq. 25.

Validation of the Proposed Constant Pressure Solution and the Average Pressure Approximation: A Direct Procedure for Performance Prediction

We propose a two-fold approach. First, as Eq. 28 is implicit in rate, we will generate the rate history for a particular reservoir system using a finite-difference simulator, and calculate the average pressure versus time relationship using gas material balance equation, and lastly we will calculate the $\beta_P$-function versus time to obtain the $\tilde{g}(u)$ function. This process allows us to compute the real-domain solution of Eq. 15 (utilizing the identity given by Eq. 25). As a validation, we then generate profiles of dimensionless pseudopressure versus dimensionless radius for several different times. Our goal is to compare these profiles to those generated by the numerical simulator for exactly the same production schedules.

Second, we suppose that the computed pseudopressure profiles coincide — our conclusion is that if we can generate the correct rate history, then the computed pressure profiles are also correct. In other words, achieving the correct pressure profiles with our computational approach would constitute a necessary condition of correctness for the proposed rate calculation. As such, we concentrate on the computation of the flowrate profiles as a mechanism to validate our method — again, we will check both the computed rate profiles and the computed pressure profiles against results from numerical simulation.

Both parts of the validation computations are implemented in Matlab [Mathworks (2002)] — which, due to its precision and computational structure, is a convenient environment for this particular work. We performed numerous checks for accuracy, and we optimized the speed of execution with Matlab using its internal settings.

The most effective mechanism to validate our method (and, in particular, the APA concept) lies in comparison of results to those of a numerical simulator. We chose an in-house simulator known as Gassim [Wattenbarger (2002)]. The Gassim module has a long development and validation history, and we systematically tested for convergence and accuracy for our problem by refining spatial and temporal discretizations. We refer to Gassim as the numerical simulator for the remainder of this paper.

Verification Based on Numerical Simulator Outputs — Validation of Pressure Profiles:

Using the numerical simulator for a known reservoir configuration, we can generate the gas rate and average pressure history corresponding to a particular constant pressure inner boundary condition. Using the rate and pressure histories generated by numerical simulation, we compute the solution using the diffusivity equation for the case of a gas well being produced at a constant bottomhole flowing pressure (this process employs Eqs. 25 and 28). The semi-analytical solution provides us with the pressure profiles in the reservoir for different times, which we then compare to the pressure profiles generated by the finite-difference simulator.

We note that instead of using the average reservoir pressure history generated by the simulator, we can generate this history directly from the rate history and the material balance equation. We expect these profiles to be essentially equal — and all of our test cases confirmed this hypothesis.

The reservoir-production system described in Table 1 is used to test the new semi-analytical solution, where we have generated pseudopressure profiles based on these properties.
Figs. 1 and 2 show very good matches between the dimensionless pseudopressure functions generated by our method and using the numerical simulator. In Fig. 2 we note that the profiles generated by the numerical simulator tend towards the profiles generated by our method when a finer temporal discretization (i.e., time grid) is used in the numerical simulator.

Table 1 — System Parameters for Pressure Profiles Validation Cases.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reservoir properties:</strong></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>50 md</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.15</td>
</tr>
<tr>
<td>$h$</td>
<td>50 ft</td>
</tr>
<tr>
<td>$r_w$</td>
<td>0.25 ft</td>
</tr>
<tr>
<td>$r_e$</td>
<td>1200 ft</td>
</tr>
<tr>
<td><strong>Gas properties:</strong></td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>200ºF</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.7 (air=1)</td>
</tr>
<tr>
<td>$p_i$</td>
<td>3000 psia</td>
</tr>
<tr>
<td><strong>Production parameters:</strong></td>
<td></td>
</tr>
<tr>
<td>$p_{wf}$</td>
<td>2500 psia</td>
</tr>
</tbody>
</table>

We note that in Fig. 1 we do not provide values for comparison for $r_D < 2.04$ since this value corresponds to the center of the inner-most gridblock in the spatial discretization that we used in the numerical simulator. Nevertheless, we note that for $r_D$ tending to 1 (wellbore surface), $m_D$ calculated by our method tends to the correct value which is 1 (by definition).

We do not present an exhaustive comparison in this section, but rather, we prefer to do so in the next section because the correctness of pressure profiles (shown in this section) is a necessary condition for the correctness of rates (shown in the next section).

On the other hand, our prior statement regarding a necessary condition is empirical if we base this statement only on the comparison of numerical results. The question is — how can we demonstrate this statement theoretically? This is one direction for improving this work. An idea for such a demonstration is as follows — since the rate history is correct, then at every time (step) the pressure and its gradient at the wellbore should be correct, we can then construct a suite of concentric cylinders and propagate the constructed pressure profile towards higher $r_D$-values.

**Forward Modeling (Performance Prediction)**

We now consider that the rate history is not available and we want to use our approach to predict both rates and reservoir pressure profiles at any time — this is what we will call performance prediction. To achieve this goal we obviously need to couple our pressure solution with a deliverability equation (or possibly deliverability equations), since there will be more than one flow regime encountered during the production. For the purposes of this discussion, a "deliverability equation" relationship between rate and pressure drop. We note that only Darcy’s law is universally valid — but we will test Darcy’s law as well as other (simple) formulations for robustness and correctness.

In order to properly calculate flowrates during the transient period (as well as during pseudosteady-state (PSS) period), we will use the deliverability equation given by Darcy’s law (radial geometry):

$$q = \frac{1}{25.148 \frac{B \mu}{r_p}} \frac{\partial p}{\partial r}$$

In Eq. 29, the thermodynamic properties and pressure gradient are calculated at the wellbore surface (i.e., $r_D = 1$). In particular, the gas FVF ($B_g$) and the gas viscosity ($\mu_g$) are evaluated at $p_{wf}$.

The proposed general procedure is direct and consists of the following steps:

1. Timesteps (i.e., the temporal grids) are chosen so that the rate is assumed constant during a particular time interval.
2. The average reservoir pressure is calculated at the end of the timestep using the gas material balance equation.
3. The computed average reservoir pressure is used to estimate the non-linear term ($\beta_p$) — and using the $\beta_p$ history and the rate history (computed prior), we can obtain $\beta(u)$ using Eq. 28.
4. We then generate a pressure profile in the vicinity of the wellbore surface and use this to calculate the pressure gradient term required in Darcy’s law (the near-well gradient is estimated using a finite-difference approximation).
Attention should be paid when calculating the very first rate (at \( t = 0 \)) as this is the moment where a discontinuity in pressure exists. Obviously, there will not be an \( \zeta \) such that for all \( r_D < 1 + \zeta \) the pressure profile will be essentially linear (if we use a first-order finite-difference formulation) — for \( r_D \) tending to 1 the rate will tend to infinity. This issue is resolved in a practical sense by using a larger argument difference for the first rate calculation, and/or by using a very small first timestep (for the case of approximating the gradient at the first timestep, we recommend a difference of \( -2 \sim 5 r_D \), whereas for subsequent timesteps, a difference smaller than \( -0.02 r_D \) gives excellent results).

We test our method on two sets of reservoir properties. Following Mireles and Blasingame we will use fluid properties given in Table 2 because these represent a typical set of conditions.

**Table 2 — Gas Properties.**

| \( T \) | 200°F |
| \( \gamma \) | 0.7 (air=1) |
| \( p_i \) | 5000 psia |

As to reservoir properties, we use the Mireles and Blasingame high porosity/low permeability set (Table 3) as well as a lower porosity/higher permeability set (Table 4). The difference between these two sets is that dimensionless time as defined by Eq. 11 is 100 times "slower" for the reservoir properties given in Table 3, compared to Table 4. In addition, the equation for radius of investigation given by:

\[
r_{inv} = 0.032 \frac{24kt}{\phi \mu c} \tag{30}
\]

shows that the transient period for Mireles and Blasingame set will be substantially longer — compare

\[
r_{inv} = 30.98 \sqrt{t} \tag{31}
\]

to

\[
r_{inv} = 309.8 \sqrt{t} \tag{32}
\]

In Eqs. 31 and 32 we estimated the thermodynamic properties at \( p_i = 5000 \) psia (same value is used for all "rate" validation cases).

**Table 3 — Reservoir Properties (Type 1) (from Mireles and Blasingame).**

| \( k \) | 1 md |
| \( \phi \) | 0.3 |
| \( h \) | 30 ft |
| \( r_w \) | 0.25 ft |

**Table 4 — Reservoir Properties (Type 2).**

| \( k \) | 50 md |
| \( \phi \) | 0.15 |
| \( h \) | 50 ft |
| \( r_w \) | 0.25 ft |

For both sets we will study three reservoir sizes: \( r_{DW} = 10^2, 10^3, 10^4 \). Each reservoir will be produced at high, low and intermediate rates (Table 5). This scheme (5 drawdowns \( p_{off}/p_i \), 3 reservoir sizes, and 2 types of reservoir) gives us 30 cases.

**Table 5 — Flowing Bottomhole Pressure Schedules (Used in Performance Prediction).**

<table>
<thead>
<tr>
<th>Case</th>
<th>( p_{off}/p_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>0.98</td>
</tr>
<tr>
<td>Intermediate 1</td>
<td>0.8</td>
</tr>
<tr>
<td>Intermediate 2</td>
<td>0.5</td>
</tr>
<tr>
<td>Intermediate 3</td>
<td>0.8</td>
</tr>
<tr>
<td>High</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Time and rate are made "dimensionless" using the "dimensionless decline time" and "dimensionless decline rate" definitions proposed by Fetkovich [Fetkovich (1980)] and modified by Carter [Carter (1985)]:

\[
t_{Dd} = 0.00633 \frac{k}{\phi \mu c} \frac{1}{r_w^2} \left[ \frac{r_c}{r_w} \right]^2 \frac{1}{\ln \left( \frac{r_c}{r_w} \right)}^2 \frac{1}{2} t \tag{33}
\]

\[
q_{Dd} = 25.148 \frac{\mu B}{kh} \left( p_i - m_{off} \right) \left[ \ln \left( \frac{r_c}{r_w} \right) \right] \frac{1}{2} q(t) \tag{34}
\]

Fetkovich [Fetkovich (1980)] and Carter [Carter (1985)] showed that, using the definitions given above, the rate histories for different types of reservoir properties (Type 1 and 2 in our context), but the same production constraint (i.e., \( p_{off} \) value), will "collapse" to a common trend for all times if the reservoirs are the same size. This is another validation criterion for our approach, as this "collapse" of solutions will be true only if our method is fundamentally correct. Without understanding how the APA works, we cannot expect these solutions to collapse a priori. Of course, our rate-time curves must also coincide with corresponding numerical simulator results.

We typically predict performance until \( q_{Dd} \) reaches values of order of \( 10^3-10^4 \), which corresponds to a gas flowrate on the order of 10 MSCF/D for reservoir Type 2 cases. An extreme example for a reservoir Type 2 case is a reservoir size of \( r_c = 10^3 r_w \) produced at \( p_{off} = 100 \) psia (for comparison, the initial rate of such a system is of order of 100,000 MSCF/D).

As would be expected, the actual (real) time needed for the flowrate to decrease to such small values depends on the reservoir properties. For \( r_{DW} = 10^4 \) and an extreme drawdown (\( p_{off} = 100 \) psia), the time needed for a reservoir of Type 1 is of order of 50,000 days, while for a reservoir of Type 2 depletion to the same state would occur in only 500 days.

This approach (based on Eq. 29 (i.e., Darcy’s Law)) is computationally intense — and more difficult to implement than approaches based on PSS-type deliverability equations. However, the application of Eq. 29 yields excellent results (Figs. 3-7), and production curves for the different reservoir types coincide almost identically with the corresponding numerical simulator results. The APA approach seems to be a uniquely correlative concept as it allows us to correctly estimate the near-wellbore pressure gradient and subsequently obtain very accurate estimates of the gas production rate.
In Figs. 3-5 we impose the Arps empirical hyperbolic decline curves (and their limiting cases — exponential and harmonic) [Arps (1945)] along with the results of numerical simulation and our semi-analytical approach — and we note generally good to excellent matches of the numerical and semi-analytical results (except at very late times where the numerical solution begins to fail).

It is interesting to note an essentially exponential decline for production at the lowest drawdown ($p_{wf} = 4900$ psia) — this pressure range is within the zone of validity of the so-called pressure method where the gas behaves like a "slightly compressible" fluid (i.e., a liquid). For larger pressure drawdowns we note that the computed solutions (both numerical and semi-analytical) are not hyperbolic. In practice, the hyperbolic rate decline model is very popular — particularly for the analysis of gas production data, but this observation suggests that the hyperbolic rate relation should not be used as a general model for gas flow behavior at boundary-dominated flow conditions.

In Fig. 7 we "zoom in" on large $t_{Dd}$ region (where reservoir size is irrelevant), and we note that, as would be expected, the numerical and semi-analytical production trends differ only by the pressure drawdown.

Fig. 7 shows the rate performance for all production times (i.e., all $t_{Dd}$). We note (as expected) that the numerical and semi-analytical production trends differ only by the reservoir size (i.e., $r_e/r_w$) during the early (transient) production period. We note that during transition from transient to boundary-dominated flow that all of the production trends yield an "apparent" exponential decline regime. The basis for this behavior is that, during transient flow, the $\mu c$-product does not vary significantly, and we obtain "liquid"-like performance — i.e., the non-linearity does not significantly affect the solution, and the "linear" (or liquid) flow solution is approximately valid (which in this case is the exponential decline).
From a practical standpoint, our analytical solution has "constraints," if not limitations — if the timestep is too large, then we overestimate the cumulative production at the end of the timestep — and consequently, average reservoir pressure well be underestimated.

Figure 6 — Fetkovich-Carter "composite" rate type curve — zoom view (Type 1 and 2 cases, superposition of $r_{eq} = 10^2, 10^3, 10^4$).

We recognize that a scheme should be developed to optimize the the timestep size as a function of time — and in fact, it turns out that this corresponds to the slope of the $\mu c$ product versus pressure. From a practical standpoint, as shown in Fig. 8, the use of finer timesteps during late times does yield a more accurate rate calculation.

Figure 7 — Fetkovich-Carter "composite" rate type curve — global view (Type 1 and 2 cases, $r_{eq} = 10^2, 10^3, 10^4$).

Finally, we note the "start-up" effect (Fig. 7) caused by the discontinuity at $t = 0$ of the pressure distribution. Regardless of the case, the rate solution becomes correct in one or two timesteps.

Alternative Deliverability Equations:

Let us assume that the pseudosteady-state (PSS) deliverability equation is valid for our purposes:

$$q = \frac{1.204 \times 10^{-7} (\frac{r_w}{T})^{1/2} (\frac{r_w}{T})^{1/4} \ln \left( \frac{r_w}{T} \right)}{r_{eq}^{3/2} (T + 459.67) \left( \frac{T}{T} \right) \ln \left( \frac{r_w}{T} \right) - \frac{3}{4}}$$

Timesteps are chosen such that the average reservoir pressure can be assumed to be constant during a particular timestep interval. Thus, for the very first timestep, we can assume average reservoir pressure to be equal to the initial pressure. Eq. 35 can be used to estimate the corresponding rate at that time. Using the rate and average reservoir pressure we can calculate $\bar{g}(u)$ function according to Eq. 28 and we can then generate the pressure profile for this timestep. We can compute the average pressure at the end of the timestep in two ways — either using the rate and material balance equation, or by calculating the average reservoir pressure from the synthetic (i.e., calculated) pressure profile.

A "deliverability" equation which uses the average reservoir
pressure profile (e.g., Eq. 35) results in a smaller computational burden because we eliminate the need for the Laplace domain computations at every timestep. However; such deliverability equations are only valid during pseudosteady-state (PSS) flow regime.

Fig. 9 compares the rate histories generated using Darcy's law (i.e., Eq. 29) and the PSS relation (i.e., Eq. 35). We clearly note that the pseudosteady-state (PSS) flow relation (Eq. 35) is not accurate during transient flow (as expected), but we also note that this relation is consistent for all times.

In Fig. 10 we compare the pressure profiles generated using our methodology based on Darcy and PSS rate histories above — and we note (categorically) that Darcy's law is the more rigorous approach. The reservoir system we used is of Type 2 ($r_{pD} = 10^4$ and $p_{wf} = 4500$ psia). We note in Fig. 11 that cumulative error in pressure profiles decreases with time. However, this cumulative error is unique due to the fact that the PSS deliverability equation does not correctly predict rate during transient period. If we elect to use Darcy's law during the transient flow period, and then switch to the PSS equation, the error should be eliminated. However, such a proposal would require a "switch" and may be prone to uniqueness issues.

This evaluation confirms (again) empirically that the correctness of the computed pseudopressure profiles is a necessary condition of correctness for the rate history.

We note that other deliverability equations may also be considered, but any acceleration device (i.e., something that permits larger timesteps (especially, in case of an extreme drawdown) or eliminates the Laplace domain computations) must be thoroughly validated.

**Extensions of the Proposed Solution**

The work of Mireles and Blasingame [Mireles and Blasingame (2003)] for the constant rate inner boundary condition and the present work for a constant pressure inner boundary condition both demonstrate that the Average Pressure Approximation (APA) is both original and robust. The question is how robust — what are the limitations? In particular, can this concept be used to develop a variable rate/variable pressure solution?

Following the logic of Appendix A (i.e., the development of the $g(t)$ function), we can test our concept for cases where linear (or liquid) solutions are well-established — for example, different non-circular geometries, arbitrary position of the well, horizontal wells, naturally fractured reservoirs. Another consideration would be to add the non-Darcy term and the skin factor into this problem — we note that such additions appear to be straightforward.

We would like to assess the feasibility of using our solution as a gas reservoir "simulator" — specifically, can we model changing flowrates and/or changing wellbore flowing pressures? In this section we demonstrate a possible approach which should allow us to generalize the application of our methodology for the case of a well with a changing well bottomhole pressure.
Initial Attempts to Model a Variable Pressure Schedule:

Two initial attempts were made to solve the variable pressure problem in a simplistic fashion, these are:

1. At the moment of the change in wellbore pressure, we propose that the "initial pressure" for the new period (or event) to be equal to the average reservoir pressure at the end of the previous period (or event). In this case, the problem is reduced to a suite of solutions developed and validated earlier in this work (Darcy's law is used for all periods). In this scenario, the inner boundary condition is not time-dependent for any problem of this type. This approach yields poor results — particularly, if bottomhole pressure successively decreases this approach underestimates the flowrate profiles. This is a consequence of non-uniform initial condition — which is our base assumption for this approach.

2. In our second attempt we consider a pseudosteady-state (PSS) deliverability equation for successive events (but not the first event). The results are poor at least in the beginning of a particular event (i.e., during transient flow — as would be expected using the PSS equation). Another characteristic of this solution approach is that if the bottomhole pressure successively decreases, then this approach will always overestimate the rate (and cumulative production).

To demonstrate these concepts we have chosen to generate example cases of using the approaches discussed above. The reservoir system we used for these cases is Type 2 — example cases of using the approaches discussed above. The problem in a simplistic fashion, these are:

| Table 6 — Flowing bottomhole pressure schedules (used in initial attempts to model a variable pressure schedule). |
|---|---|
| Time (D) | Flowing Bottomhole Pressure (psia) |
| 5 | 4500 |
| 10 | 4400 |

Time-Dependent Inner Boundary Condition (Variable Pressure Schedule):

A more theoretical approach is to specifically consider a time dependent inner boundary condition — however, this approach requires a complete recasting of the solution method. This concept seems to be the most logical and we present the developments in the following section.

As a start, the linear (or liquid) Laplace domain solution is cast in the form of a time-dependent inner boundary condition. Fortunately, the nature of the $\bar{g}(u)$ term is invariant with respect to bottomhole pressure schedule — actually $\bar{g}(u)$ is taken specifically to be a function of the average reservoir pressure — which (obviously) is a function of both time and the bottomhole pressure schedule.

We begin with the dimensionless diffusivity equation in terms of pseudopressure and time:

$$
\frac{1}{r_D^3} \frac{\partial}{\partial r_D} \left( r_D^3 \frac{\partial \sigma_D}{\partial r_D} \right) = \frac{\mu c}{\mu_D} \frac{\partial \sigma_D}{\partial t_D} \quad \text{............... (36)}
$$

Where dimensionless pseudopressure is defined by:

$$
\sigma_D = \frac{m_j - m_D}{m_i - m_\omega} \quad \text{............... (37)}
$$

In our previous developments the dimensionless pseudopressure ($\sigma_D$) was constrained between 1 ($m = m_\omega$) and 0 ($m = m_i$). And the inner boundary condition was specifically defined as:

$$
\sigma_D |_{k_D} = 1 \quad \text{............................. (38)}
$$

Where the condition given by Eq. 38 is independent of time. Generalizing, we can write the inner boundary condition for the case of a varying bottomhole pressure schedule as follows:

$$
\sigma_D |_{k_D} = \sigma(t_D) \quad \text{............... (39)}
$$

Where $\sigma(t_D)$ is, in general, a superposition of Heaviside functions (i.e., step-changes in the flowing bottomhole pressure). Applying the Laplace transformation and using the initial condition, we have:

$$
\frac{d^2 \bar{m}_D(u)}{dr_D^2} + \frac{1}{r_D} \frac{d \bar{m}_D(u)}{dr_D} = \left[ u \bar{m}_D(u) - m_D(t_D = 0) \right] \bar{g}(u) \quad \text{............... (40)}
$$

In this process, the inner boundary becomes:

$$
\bar{m}_D = \bar{g}(u) \quad \text{............... (41)}
$$

And the outer boundary condition remains:

$$
\left[ r_D^3 \frac{d \bar{m}_D}{dr_D} \right] |_{r_D = r_D} = 0 \quad \text{............... (42)}
$$

As in our previous development, the solution form of Eq. 40 is given as:

$$
\bar{m}_D(u) = \bar{g}(u) \int_0^\infty P_D(t_D)e^{-u \bar{g}(u)d} dt_D \quad \text{............... (43)}
$$

Or, taking the Laplace transform of Eq. 43, we have

$$
\bar{m}_{D,\text{Gas}}(u) = \bar{g}(u)\bar{m}_D(u) \quad \text{............... (44)}
$$

The linear (or liquid) form of the general solution for this problem is defined by:

$$
\bar{m}_D(r_D, u) = A \bar{m}_0(r_D \sqrt{u}) + B \sqrt{k_0(r_D \sqrt{u})} \quad \text{............... (45)}
$$
Using our boundary conditions, we have:

\[ A = \frac{K_1(r_e \sqrt{u})}{K_0(\sqrt{u})I_1(r_e \sqrt{u}) + K_1(r_e \sqrt{u})I_0(\sqrt{u})} \]  \hspace{1cm} (46)

\[ B = \frac{K_1(r_e \sqrt{u})}{K_0(\sqrt{u})I_1(r_e \sqrt{u}) + K_1(r_e \sqrt{u})I_0(\sqrt{u})} \]  \hspace{1cm} (47)

Substituting Eqs. 46 and 47 into Eq. 45 yields:

\[ m = \frac{1}{u} K_1(r_e \sqrt{u})I_0(r_e \sqrt{u}) + K_0(r_e \sqrt{u})I_1(r_e \sqrt{u}) \]  \hspace{1cm} (48)

We now substitute Eq. 48 into Eq. 44 to obtain our final form of the solution in the Laplace domain:

\[ m_D(t, u) = \frac{1}{u} K_1(r_e \sqrt{u})I_0(r_e \sqrt{u}) + K_0(r_e \sqrt{u})I_1(r_e \sqrt{u}) \]  \hspace{1cm} (49)

Performance testing of Eq. 49 yields transient responses in the beginning of every period — but the rates were incorrect — except (of course) for the first period. We believe that this phenomenon is related to numerical problems — the APA typically works for much worse conditions (see Appendix A).

In this case the bottomhole pressure schedule makes use of Heaviside functions, consequently the Laplace domain solution includes Laplace transforms of Heaviside functions — which consists of discontinuous steps. Numerical inversion of this formulation is challenging to say the least — for example, Fourier-based inversion methods generally provide a good representation of "discrete steps" but often require so many terms that computational precision of the Fourier series is compromised (numerically).

The Laplace transform of a Heaviside function is analytical (as is the inversion), but when the Heaviside function is multiplied by another function in the Laplace domain, the inverse Laplace transformation often becomes problematic in a computational sense (this is a common issue).

With these presumed numerical problems in mind, we attempted (as before) to validate the reservoir pressure profiles computed for a given rate history (which is generated by a numerical simulator) for a specified schedule of flowing bottomhole pressures. We note that these results are excellent compared to those obtained for the single constant bottomhole flow case (Fig. 13). We believe that any differences which arise in this comparison are due to the use of the Laplace transformation and inverse transformation of the Heaviside functions used to represent the variable pressure case.

But we note (again) that the results shown in Fig. 13 are excellent. For this comparison we use a reservoir of Type 2 (\(r_e = 10^4\) and the pressure schedule is given in Table 7).

In simple terms, the formulation presented in this section is correct — at least for the computation of pressure profiles. However; this formulation does not yield accurate rate estimates — this case remains a prime candidate for further study.

**Summary, Conclusions and Recommendations**

**Summary:**
We have developed a new semi-analytical solution approach for solving the gas diffusivity equation for the case of a gas well producing at a constant flowing well bottomhole pressure in a homogeneous, bounded circular reservoir. We were inspired by the Mireles and Blasingame approach for the case of a constant (sandface) flowrate condition. Our new approach follows by also considering the viscosity-compressibility product to be only time-dependent — where we have assumed (then validated) that the \(\mu e\) product evaluated at average reservoir pressure uniquely represents the time-dependent character of this problem.

Incorporating the average pressure approximation (APA) requires recasting of the right-hand-side of the dimensionless diffusivity equation into a convolution of two independent functions, as well as the subsequent use of the Laplace transformation. The proposed solution is evaluated using the liquid flow solution, coupled with the "real gas" function \(\overline{g}(u)\) in the Laplace domain (or \(g(t)\) in the real domain, where this is defined by Eq. 28) — where specifically, \(\overline{g}(u)\) represents the non-linear behavior (i.e., pressure dependency of the thermodynamic properties of the gas).

The new solution approach was implemented and verified by comparison with numerical results from a finite-difference simulator. We considered a variety of reservoir and fluid properties, as well as various production schedules. We note excellent comparisons of pressure profiles and rates for all cases — which empirically validates the applicability of our
approach to gas flow systems.

The approach works well for all of the scenarios we tested — where this observation leads us to conclude that the APA assumption is not restrictive — but in fact, it is the appropriate basis for such solutions. In order to support this empirical verification by theory, we propose an insight into rigorous determination of limits of validity of the APA for both the constant pressure and constant rate inner boundary condition problems. We consider the validity of the APA as an intrinsic mathematical property of this problem, and we believe that the goal of clarifying this issue is crucial.

In summary, the primary theoretical difference between this work for a constant flowing wellbore pressure and the original work of Mireles and Blasingame for the constant flowrate problem is the specific formulation of the Laplace domain functional for gas (i.e., the $\tilde{g}(\mu)$ term). In addition, since this work requires "projection" of flowrate at a particular timestep, we used a rate projection based on Darcy's law (which is both rigorous and accurate).

Conclusions:
The most important conclusion of this work is that the APA concept is valid for the constant pressure inner boundary condition for solving the nonlinear gas flow problem. We also believe this concept to be valid for the general case of variable-rate/variable pressure, but this work is ongoing.

At present, we have only showed a proof of concept and our approach may be adopted for simple flow problems, as well as for verifying numerical simulator results, or more precisely for assessing adequacy of spatial and temporal discretizations used in the numerical simulator.

Recommendations:
We believe that investigation of the following topics will assist in the development the proposed approach into a reliable and robust tool for use in well test and production data analysis, and for modelling the performance of gas reservoirs:

- **Average Pressure Approximation (APA):** We recommend that an analytical proof of the APA concept be developed (as opposed to our empirical efforts). This will provide a common theoretical basis for constant pressure and constant rate production regimes.

- **Proof-of-Concept:** We have established (empirically) that the correctness of the computed pressure profiles (i.e., $p(r)$) is a necessary condition to prove the correctness of rate history.

- **Extension to a Generic Production Schedule:** A robust and functional solution for the case of a general production schedule (i.e., arbitrary rates and pressures) must be developed.

- **Other Applications:** The proposed solution methodology should be extended to other applications such as more complex geometries, non-Darcy flow, and eventually, to the case of multiphase flow.

Nomenclature

Field Variables
- $B$ = Gas formation volume factor at $p_0$, rcf/SCF
- $c$ = Gas compressibility, psi$^{-1}$
- $c_i$ = Gas compressibility at $p_0$, psi$^{-1}$
- $\tau$ = Gas compressibility at average reservoir pressure, psi$^{-1}$
- $G$ = Gas-in-place, SCF
- $G_p$ = Cumulative gas production, SCF
- $h$ = Pay thickness, ft
- $k$ = Effective permeability to gas, md

Greek Symbols
- $\beta$ = Non-linear viscosity-compressibility function, dimensionless
- $\beta_T$ = Non-linear viscosity-compressibility function at average reservoir pressure, dimensionless
- $\gamma$ = Gas gravity (air = 1)
- $\mu$ = Gas viscosity, cp
- $\mu_i$ = Gas viscosity at initial reservoir pressure, cp
- $\mu_f$ = Gas viscosity at average reservoir pressure, cp
- $\sigma$ = Inner boundary condition schedule function
- $\phi$ = Effective "gas" porosity, fraction
- $\tau$ = Dummy time-type variable, days

Special Signs and Operators
- $\text{L}$ = Referring to average conditions
- $\nabla$ = Gradient
- $\Delta$ = Argument increment

Gas Pseudofunctions:

- $m(p) = \text{Pseudopressure function, psia}$
- $m_i = \text{Initial pseudopressure, psia}$
- $m_{wp}(p) = \text{Flowing well pseudopressure, psia}$
- $p_i = \text{Initial reservoir pressure, psia}$
- $p = \text{Pressure, psia}$
- $p_{base} = \text{Base pressure in pseudopressure formulation, psia}$
- $p_{of} = \text{Flowing well pressure, psia}$
- $p = \text{Average reservoir pressure, psia}$
- $p_{of} = \text{Flowing bottomhole pressure, psia}$
- $q = \text{Gas flowrate, SCF/D}$
- $r_{avg} = \text{Radius of investigation, ft}$
- $r = \text{Radial distance, ft}$
- $r_e$ = Outer radius, ft
- $r_w$ = Wellbore radius, ft
- $T = \text{Temperature, } ^\circ\text{F}$
- $t = \text{Time, D}$
- $t_p = \text{Pseudotime (adjusted time), days}$
- $t_0 = \text{Initial time in pseudotime formulation, days}$
- $z = \text{Gas compressibility factor, dimensionless}$
- $z_i = \text{Gas compressibility factor at } p_0, \text{dimensionless}$

Dimensionless Variables
- $m_D = \text{Dimensionless pseudopressure function, dimensionless}$
- $p_D = \text{Dimensionless pressure, dimensionless}$
- $q_D = \text{Dimensionless flowrate, dimensionless}$
- $q_{Dd} = \text{Dimensionless flowrate (decline form), dimensionless}$
- $r_D = \text{Dimensionless radius, dimensionless}$
- $r_{reD} = \text{Dimensionless outer reservoir boundary radius}$
- $t_D = \text{Dimensionless time (based on } r_w), \text{dimensionless}$
- $t_{Dd} = \text{Dimensionless decline time, dimensionless}$

Mathematical Functions
- $g(t) = \text{Function representing the non-linearity of the problem}$
- $\Phi_i = \text{Modified Bessel function of first kind of order } i$
- $K_i = \text{Modified Bessel function of second kind of order } i$

References


Communications of the ACM Technical Conference and Exhibition, Denver, CO, 5-8 October.


Appendix A: Non-Linearity and Average Pressure Approximation

Non-Linear Term:

The non-linearity of this problem is represented by the \( g(u) \) term — the purpose of this section is the derivation of a mathematical representation of the \( g(u) \) term.

Using the chain rule for the differentiation of a multivariate function yields:

\[
\frac{\partial m_D}{\partial D} = \frac{\partial m_D}{\partial m} \frac{\partial m}{\partial p} \frac{\partial (p/z)}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial D} \tag{A-1}
\]

Where the individual terms on the right-hand side of Eq. A-1 are given as:

\[
\frac{\partial m_D}{\partial m} = \frac{1}{m_i - m_{ow}} \tag{A-2}
\]

\[
\frac{\partial m}{\partial p} = \frac{\mu_z i}{p_i \mu_z} \tag{A-3}
\]

\[
\frac{\partial (p/z)}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial D} = \frac{1}{z_i} \frac{\partial p}{\partial D} \tag{A-4}
\]

The gas material balance equation is given by:

\[
\frac{\partial p}{\partial D} = \frac{1}{c} \frac{\partial p}{\partial P} \tag{A-5}
\]

Differentiating the gas material balance relation (Eq. A-5) with respect to time, we obtain the following:

\[
\frac{\partial \varepsilon}{\partial t} = \frac{1}{z_i} \frac{\partial p}{\partial D} \int_0^t q(\tau) d\tau \tag{A-6}
\]

Where the definition of is: \( G_p(t) = \int_0^t q(\tau) d\tau \).

We will use the \( \varepsilon (p/z)/dt \) later as part of the structure of the non-linear term. The dimensionless time derivative (with respect to time) is given by:

\[
\frac{\partial \varepsilon}{\partial t} = \frac{1}{t DC} \frac{k}{\mu c \mu c} \tag{A-7}
\]

Combining Eqs. A-2 through A-7 we have:

\[
\frac{\partial m_D}{\partial t} = \frac{1}{t DC} \frac{k}{\mu c \mu c} \tag{A-8}
\]

Recall that we can write:

\[
\beta(t) \frac{\partial m_D}{\partial t} = \int_0^{t DC} g(t_D - \tau)d\tau \tag{A-9}
\]

And recall, that the definition of the \( \beta \)-function is given as:

\[
\beta = \frac{\mu c}{\mu c} \tag{A-10}
\]

Substitution of Eq. A-8 into Eq. A-9 yields:

\[
\frac{\mu c}{\mu c} = \int_0^{t DC} \frac{1}{t DC} \frac{k}{\mu c \mu c} q(t) \tag{A-11}
\]

And, after reduction of like terms we have:

\[
q = \int_0^{t DC} \frac{\mu c}{\mu c} q(t) \tag{A-12}
\]

We now introduce the average pressure approximation (or APA) approach for referencing the non-linear term (\( \beta \)) to the average reservoir pressure:

\[
\beta p = \frac{\mu c}{\mu c} \tag{A-13}
\]

As the right-hand side of the diffusivity equation depends only on time, it follows that:

\[
q(t_D) = \int_0^{t DC} \frac{\mu c}{\mu c} (t_D - \tau) \tag{A-14}
\]

Finally, the property of convolution provides the following ex-
pression (i.e., identity) for the \( \bar{g}(u) \) term:

\[
\mathcal{L}[g(t_D)] = \mathcal{L}(u) = \frac{1}{\mathcal{L}[q(t_D)]} \mathcal{L}[q(t_D)] \tag{A-15}
\]

**Nature and Limits of Validity of the Average Pressure Approximation (APA):**

Although one could think of the APA as "natural" and "sound" — such "faith" is not sufficient, a rigorous validation is required. For example, during the pseudosteady-state (PSS) flow regime, the average reservoir pressure is representative of the entire reservoir, this can serve as a basis (conceptually) for defining the "average pressure approximation" (or APA) approach, but more questions abound.

We present a schematic plot to show the real trend of the \( \beta \)-function (changing with respect to position at a specific time) in Fig. A-1. This plot is based on our informal studies on the behavior of the \( \beta \)-function which is a function of time and position as we noted before. Obviously, this plot suggests that the deviations are clear in the vicinity of the wellbore \( r_D = 1 \). However, we have shown in this work that with APA we can achieve acceptable results.

The question arises as to why does the APA approach work during transient period? In fact, the validity of our APA-based approach is due to a mathematical property of the diffusivity equation — we believe that this "property" is insensitive to spatial variations in the \( \beta \)-function.

![Figure A-1 — Typical profile of \( \beta \)-function during production (for both constant pressure and constant rate inner boundary conditions).](image)