Abstract

Double-porosity/naturally-fractured reservoir (NFRs) models have traditionally been used to represent the flow and pressure behavior for highly fractured carbonate reservoirs. Given that unconventional reservoirs such as shale oil/gas reservoirs may or may not be considered to be multi-porosity media, the use of the traditional/classical "double-porosity" models may not be adequate (or appropriate). The recent development of anomalous diffusion models has opened the possibility of adapting double-porosity models to estimate reservoir (and related) parameters for unconventional reservoirs. The primary objective of this work is to develop and demonstrate analytical reservoir models that provide (possible) physical explanations for the anomalous diffusion phenomenon.

The models considering anomalous diffusion in reservoirs with Euclidean shape are developed using a convolved (i.e., time-dependent) version of Darcy's law. The use of these models can yield a power-law (straight-line) behavior for the pressure and/or rate performance — similar to the fractal reservoir models. The main advantage of using anomalous diffusion models compared to models considering fractal geometry is the reduction from two parameters (i.e., the Fractal Dimension and the Conductivity Index) to only one parameter (i.e., the anomalous diffusion exponent). However, the anomalous diffusion exponent does not provide information about the geometry or spatial distribution of the reservoir properties.

To provide an alternative explanation for the anomalous diffusion phenomenon in petroleum reservoirs, we have developed double-porosity models considering matrix blocks with fractal geometry and fracture networks with either radial or fractal fracture networks. The flows inside the matrix blocks and the fractal fracture network assume that Darcy's Law is valid in its space-dependent (fractal) form, whereas the classical version of Darcy's Law is assumed for the radial fracture network case. The transient interporosity transfer is modeled using the classical convolution schemes given in the literature.

For the system composed by a radial fracture network and fractal matrix blocks, we have investigated three cases by changing the producing conditions for the well and the flowing conditions for the matrix blocks: These three cases are:

1. A well producing at a constant-rate and closed matrix blocks,
2. A well producing at a constant-rate and "infinite-acting" matrix blocks, and
3. A well producing at a variable-rate (time-dependent inner boundary condition) and "infinite-acting" matrix blocks.

We have defined the matrix blocks to be "infinite-acting" in order to represent the nano/micro permeability of shale reservoirs. For the system defined by a fractal fracture network and infinite-acting fractal matrix blocks, we have investigated the influence of the fractal parameters (both matrix and fracture network) in the pressure- and rate-transient performance behaviors. We have defined the periods of flow that can be observed in these sorts of systems and we have developed analytical solutions for pressure-transient analysis. We demonstrate that the use of the convolved version of Darcy's Law results in a model very similar to the diffusivity equation for double-porosity systems (which incorporates transient interporosity flow).

In performing this work, we establish the following observations/conclusions using our new solutions:

1. We have found that the assumption of a well producing at variable-rate (time-dependent inner boundary condition) has a more significant impact on the pressure (and derivative) functions and hinders the effects of the properties of the reservoir.
2. We have demonstrated that the anomalous diffusion phenomena in unconventional reservoirs can be related to their multi-porosity nature.
3. The pressure and pressure derivative responses may be used in the diagnosis of flow periods and the evaluation of reservoir parameters in unconventional reservoirs.

Introduction

The use of double-porosity models to depict the pressure-transient behavior of NFRs has been studied since the 1960s. These models idealize reservoirs with multiple porous media (e.g., micro/nano fractures, micro/nano vugs, matrix blocks, etc.) as a fracture network with matrix blocks — where these models can utilize either the transient or pseudosteady-state interporosity transfer condition, as specified by the analyst. In this work we only use the transient interporosity transfer condition.

In 1976, de Swaan presented a double-porosity analytical model with transient interporosity transfer which considered a radial fracture network and either slabs or spherical matrix blocks. The transient interporosity transfer condition was modeled with a convolution integral of the flux from the matrix blocks weighted by the derivative of the pressure in the radial network of fractures with respect to time. The flux of the matrix blocks was determined by solving the diffusivity equation for either linear or spherical systems considering constant-pressure at the matrix-fracture interface.

The periods of flow exhibited by the dual porosity model with transient interporosity transfer conditions are (see Fig. 1):

- **Period 1**: (radial fracture network flow) This period exhibits a flat slope (radial flow) in the pressure derivative function that represents that the flow is dominated by the properties of the radial fracture network.
- **Period 2**: (interaction between the radial fracture network and matrix blocks) This period can be subdivided into three subperiods: (1) a transition sub-period dominated by the early interaction of the matrix blocks, (2) a pseudoradial flow, and (3) a transition sub-period, dominated by the closed boundaries of the matrix blocks.
- **Period 3**: (single porosity reservoir flow) In this period of flow, the double-porosity reservoir has achieved equilibrium and behaves as a single porosity reservoir. However, this period is dominated by the geometry of the fracture network (radial) and exhibits a flat slope in the pressure derivative function.
In 1982, three independent research groups (i.e., Cinco-Ley et al., Serra et al., and Streltsova) modified de Swaan’s model and obtained asymptotic solutions in the real domain — in this work, we use the model presented by Cinco-Ley et al. (1982). Traditionally, fractal diffusivity models have been used to model highly heterogenous reservoirs (e.g., NFRs and shale reservoirs). In double-porosity systems (fractures and matrix blocks), such models are used to describe the flow of fluids within the network of natural fractures (e.g., Chang et al., 1990, Olairevaju, 1996 and Valdes-Perez et al., 2013). Given that the porosity of the matrix blocks in shale oil/gas reservoirs is a combination of multiple organic and inorganic porosities, we believe that it is appropriate (in terms of geological evidence) to model the matrix blocks as fractal objects.

Proposed Models

Assumptions. The models in this work consider two porous media: (1) the matrix blocks and (2) the fracture network. All the models consider the matrix blocks to be fractal, whereas the fracture network is considered to be either radial or fractal. The model assumptions are summarized in Table 1. The dimensionless variables for the models presented in this paper are summarized in Table 2.

Table 1—Assumptions used to develop the proposed reservoir models

<table>
<thead>
<tr>
<th>Medium</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>• Flow to the wellbore occurs only through the fracture network (radial or fractal).</td>
</tr>
<tr>
<td></td>
<td>• Pressure-squared gradients are negligible.</td>
</tr>
<tr>
<td></td>
<td>• Uniform initial pressure, ( p_i ).</td>
</tr>
<tr>
<td></td>
<td>• Single slightly-compressible fluid flow with constant compressibility, ( c_{so} ), and constant viscosity, ( \mu ).</td>
</tr>
<tr>
<td>Fractal matrix blocks</td>
<td>• Single size of the matrix blocks of ( D_{fru} )-dimension.</td>
</tr>
<tr>
<td></td>
<td>• Flow obeys modified Darcy’s Law for fractal systems of ( \theta_{fru} )-conductivity index.</td>
</tr>
<tr>
<td>Fractal matrix blocks</td>
<td>• Porosity and permeability vary according to power-law functions.</td>
</tr>
<tr>
<td></td>
<td>• Reference matrix porosity, ( \phi_{fru,m} ), and matrix permeability, ( k_{fru,m} ), are considered.</td>
</tr>
<tr>
<td></td>
<td>• The matrix blocks have constant compressibility, ( c_{m} ).</td>
</tr>
<tr>
<td>Radial fracture network</td>
<td>• Unbounded radial network.</td>
</tr>
<tr>
<td></td>
<td>• Flow obeys Darcy’s Law.</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Constant properties: compressibility, ( c_{b} ), permeability, ( k_{b} ), and porosity, ( \phi_{b} ).</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Unbounded fractal fracture network of ( D_{fr} )-dimension.</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Flow obeys modified Darcy’s Law for fractal systems of ( \theta )-conductivity index.</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Reference fractured bulk porosity, ( \phi_{fr,b} ), and bulk matrix permeability, ( k_{fr,b} ), are considered.</td>
</tr>
<tr>
<td>Fractal fracture network</td>
<td>• Fractal fracture network has constant compressibility, ( c_{fr,b} ).</td>
</tr>
</tbody>
</table>

Figure 1—Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a double-porosity reservoir considering slab and spherical matrix blocks.
Table 2—Dimensionless variables for the proposed models.

<table>
<thead>
<tr>
<th>Dimensionless Variables</th>
<th>Radial Fracture Network Model</th>
<th>Double-porosity Fractal Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure in the fracture network</td>
<td>$p_{efnD_c}(r_{D_c}, t_{D_c}) = \frac{k_f b_h [p_i - p_{efnD_c}(r, t)]}{q B \mu}$</td>
<td>$p_{ffnD_c}(R_{D_c}, t_{D_c}) = \frac{a v_f b_f [p_i - p_{ffn}(R, t)]}{q B \mu W^{1-\beta} \phi_0 b_f}$</td>
</tr>
<tr>
<td>Pressure in the fractal matrix blocks</td>
<td>$p_{maD_c}(r_{D_c}, t_{D_c}) = \frac{k_f b_h [p_i - p_{ma}(r, t)]}{q B \mu}$</td>
<td>$p_{maD_c}(R_{D_c}, t_{D_c}) = \frac{a v_f b_f [p_i - p_{ma}(R, t)]}{q B \mu W^{1-\beta} \phi_0 b_f}$</td>
</tr>
<tr>
<td>Time</td>
<td>$t_{D_c} = \frac{k_f b_h [\phi_f l_t]}{[\phi_f l_t]_{f_w}}^{\frac{1}{2}}$</td>
<td>$t_{D} = \frac{k_f b_h [\phi_f l_t]}{[\phi_f l_t]_{f_w}}^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>Radius</td>
<td>$r_{D_c} = \frac{r}{r_w}$</td>
<td>$r_{D} = \frac{r}{r_w}$</td>
</tr>
<tr>
<td>Storativity ratio</td>
<td>$\omega = \frac{\phi_i p_c f_h}{[\phi_f l_t]}$</td>
<td>$\omega = \frac{\phi_i p_c f_h}{[\phi_f l_t]}$</td>
</tr>
<tr>
<td>Fracture area</td>
<td>$A_{FD} = A_{fma} h_{ma}$</td>
<td>$A_{FD} = A_{fma} h_{ma}$</td>
</tr>
<tr>
<td>Hydraulic diffusivity of the fractal matrix blocks</td>
<td>$\eta_{maD} = \frac{k_{0ma} [c \phi_i l_t]}{c_{ma} \phi_{0ma} k_{fb} r_{ma}^{2}}$</td>
<td>$\eta_{maD} = \frac{k_{0ma} [c \phi_i l_t]}{c_{ma} \phi_{0ma} k_{fb} r_{ma}^{2}}$</td>
</tr>
<tr>
<td>Matrix block size</td>
<td>$h_{maD} = \frac{h_{ma}}{r_w}$</td>
<td>$h_{maD} = \frac{h_{ma}}{r_w}$</td>
</tr>
</tbody>
</table>

**Fluid Transfer Function considering Fractal Matrix Blocks.** The fluid transfer function from fractal matrix blocks to the fracture network, $F(\eta_{maD}, h_{maD}, t_D)$, is defined by the geometry and properties of the matrix blocks. To derive this function, we have used an analogous procedure as the one presented by de Swaan (1976) for repetitive elements of matrix blocks (see Fig. 2). We have investigated two scenarios considering: (1) closed (finite) matrix blocks and (2) infinite-acting matrix blocks. For the closed matrix blocks case, the fracture network function is defined by:

$$F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt[4]{\eta_{maD}}^2} \left[ h_{maD}^{\frac{1}{2}} \right]^{\theta_{ma} / 2} L^{-1} \left[ \frac{1}{\sqrt{u}} \left[ \Gamma(v_{ma}) \Gamma[1 - v_{ma}] \Gamma(v_{ma} - 1) \Gamma[\xi] - 2K v_{ma} - 1 \Gamma[\xi] \right] \right],$$  \hspace{1cm} (1)

where:

$$\xi = \frac{2}{\theta_{ma} + 2} \left[ h_{maD}^{\frac{1}{2}} \right]^{\theta_{ma} + 2} \sqrt{u} \sqrt[4]{\eta_{maD}},$$  \hspace{1cm} (2)

and $v_{ma} = [1 - \beta_{ma}] / [\theta_{ma} + 2]$. $\beta_{ma}$ is the spatial dimension of the matrix and is defined as $\beta_{ma} = D_{fma} - \theta_{ma} - 1$. The expression given by Eq. 1 applies for $v_{ma} \neq 0$. If $v_{ma} = 0$, the following expression should be used:

$$F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt[4]{\eta_{maD}}^2} \left[ h_{maD}^{\frac{1}{2}} \right]^{\theta_{ma} / 2} L^{-1} \left[ \frac{1}{\sqrt{u}} \left[ \Gamma(v_{ma}) \Gamma[1 - v_{ma}] \Gamma(v_{ma} - 1) \Gamma[\xi] \right] \right].$$  \hspace{1cm} (3)

Due to the extremely low permeability of the matrix blocks in shale oil/gas reservoirs, we modeled the behavior of the matrix blocks as infinite-acting (fully transient) media. The fluid transfer function for infinite-acting matrix blocks is:
Asymptotic Behavior. As pointed out by Cinco-Ley et al. (1982), the flow from the matrix blocks to the fracture network is linear at early and intermediate times (Periods 1 and 2) regardless the geometry of the matrix block. During these periods of flow, the behavior of the fluid transfer function for closed and infinite-acting matrix blocks are the same. Therefore, the fluid transfer function for all cases at early and intermediate times can be simplified to:

$$ F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt{\pi \eta_{maD}}} \left[ \frac{h_{maD}}{2} \right]^{-\theta_{ma}/2} \frac{1}{L^1} \left[ \frac{K_{vma} - 1}{K_{vma}} \right] \left[ \frac{1}{K_{vma}} \right]^{1/2} $$

At late times, the fluid transfer function for the infinite-acting matrix blocks case can be simplified to:

$$ F(\eta_{maD}, h_{maD}, t_D) = \left[ \frac{\theta_{ma} + 2}{\eta_{maD}} \right]^{-\frac{1}{2}} \left[ \frac{h_{maD}}{2} \right]^{-\frac{\theta_{ma}}{2}} t_D^{-\frac{1}{2}} $$

when $\nu_{ma} > 0$.

Double-porosity Model considering a radial fracture Network and Fractal Matrix Blocks
To model the flow within the radial fracture network, we have considered the diffusivity equation presented by Cinco-Ley et al. (1982). In its dimensionless form, this result is given by:

$$ \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial P_{frid,cr}}{\partial r_D} \right] = \frac{\partial P_{frid,cr}}{\partial t_D} + (1 - \omega) \frac{A_{frinD,cr} P_{frid,cr} t_D^{1/2}}{\eta_{maD}} \left[ \frac{1}{\eta_{maD}} \right]^{1/2} F(\eta_{maD}, h_{maD}, t_D) - \frac{1}{\eta_{maD}} P_{frid,cr} (r_D, \tau) d\tau $$

The well-known solution of Eq. 7 in the Laplace domain considering a constant flowrate at the wellbore and an infinite fracture network is given by:

$$ P_{frid,cr}(r_D, \tau) = \frac{K_0 \sqrt{\tau}}{u \sqrt{\tau} K_1} $$

$$ F(\eta_{maD}, h_{maD}, t_D) = \frac{K_0 \sqrt{\tau}}{u \sqrt{\tau} K_1} $$

$$ P_{frid,cr}(r_D, \tau) = \frac{K_0 \sqrt{\tau}}{u \sqrt{\tau} K_1} $$
where:

\[
f(u) = \omega + [1- \omega] \frac{A_f D \eta_{maD}}{h_{maD}} \frac{1}{\eta_{maD}} \cdot u \tag{9}
\]

To obtain the performance of Eq. 8 in the real domain, a numerical inversion is required. In this work, we have applied the Gaver-Wynn-Rho algorithm implemented in Mathematica. Additionally, we have developed asymptotic solutions, evaluated at the wellbore, for the main periods of flow depicted in this model. These solutions are summarized in Table 3.

Table 3—Asymptotic constant-rate solutions in the real domain for the double-porosity model considering a radial fracture network.

<table>
<thead>
<tr>
<th>Asymptotic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early times</td>
</tr>
<tr>
<td>[ p_{wd,cr}(t_{Dr}) = \frac{1}{2} \ln \left[ \frac{\exp[\gamma/2]}{\omega} \cdot t_{Dr} \right] ]</td>
</tr>
<tr>
<td>Intermediate times</td>
</tr>
<tr>
<td>[ p_{wd,cr}(t_{Dr}) = \frac{1}{4} \ln[t_{Dr}] + \frac{1}{2} \ln \left[ \frac{\exp[\gamma/2]}{[1- \omega]} \cdot \frac{h_{maD}}{A_f D} \cdot \frac{\eta_{maD}}{2} \cdot \theta_{ma} \right]^{1/2} ]</td>
</tr>
<tr>
<td>Late times (for closed matrix blocks)</td>
</tr>
<tr>
<td>[ p_{wd,cr}(t_{Dr}) = \frac{1}{2} \ln[\exp[\gamma/2] t_{Dr}] ]</td>
</tr>
<tr>
<td>Late times (for infinite-acting matrix blocks)</td>
</tr>
<tr>
<td>[ p_{wd,cr}(t_{Dr}) = \frac{\nu_{ma}}{2} \ln[\exp[\gamma/2] t_{Dr}] \left[ \frac{A_f D}{h_{maD}} \cdot \left[ \frac{\theta_{ma} + 2}{\Gamma[1 - \nu_{ma}]} \cdot \frac{1 - 2\nu_{ma}}{\Gamma[\nu_{ma}]} \cdot \frac{2}{h_{maD}} \right] \right] ]</td>
</tr>
</tbody>
</table>

**Analogy to the Anomalous Diffusion Model.** The 2D diffusivity model presented by Raghavan (2012) included the anomalous diffusion effects by using a version of Darcy’s Law expressed in terms of a fractional derivative. The definition of the fractional derivative involves a convolution term where the derivative of the pressure with respect to time is weighted by a power-law function of time. A similar diffusivity model can be obtained from the dual porosity model considering a radial fracture network and fractal matrix blocks. Such a model is obtained by neglecting the storativity ratio (\(\omega = 0\)) and considering the asymptotic behavior for late times of the fluid transfer function for infinite-acting fractal matrix blocks with positive \(\nu_{ma}\)-values (Eq. 6). The diffusivity equation for this case is:

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial p_{f D} \cdot cr}{\partial r_D} \right] = \alpha_1 \frac{t_D}{\Gamma[1 - \nu_{ma}]} \int_0^{t_D - \tau} [t_D - \tau]^{-\nu_{ma}} \cdot d\tau \tag{10}
\]

where:

\[
\alpha_1 = \frac{A_f D h_{maD}}{2} \left[ \frac{\theta_{ma} + 2}{\Gamma[1 - \nu_{ma}]} \cdot \frac{1 - 2\nu_{ma}}{\Gamma[\nu_{ma}]} \cdot \frac{\eta_{maD}}{2} \right]^{\gamma \beta_{ma} - 1} \tag{11}
\]

To provide a model that is comparable to the anomalous diffusion model presented by Raghavan (2012), we have considered a time-dependent inner boundary condition modeled by a power-law function mathematically defined by:

\[
\left[ r_D \frac{\partial p_{f D}}{\partial r_D} \right]_{r_D=1} = t_D^{-\nu_{ma} - 1} \tag{12}
\]
where \( v_0 \) is an arbitrary reference exponent (note that we have assumed a unit value of this parameter (i.e., \( v_0 = 1 \)).

The line source approximation in the Laplace domain of Eq. 10 considering the variable flowrate case given by Eq. 12, and an infinite radial fracture network is given by:

\[
\bar{p}_{rfnDcr}(r_D,u) = \frac{\Gamma[2-v_{ma}]K_0}{u^{2-v_{ma}}} \left[ \frac{\beta p_{fnDcr}}{\beta r_D} \right]^{1/2}
\]

where:

\[
g(u) = a_0 u^{v_{ma}-1}
\]

Double-Porosity Model Considering Fractal Fracture Network and Fractal Matrix Blocks (Double-Porosity Fractal Model)

The motivation of the double-porosity fractal model is to provide a reservoir model to depict the transient performance behavior of highly heterogeneous multi-porosity systems such as shale oil and shale gas reservoirs. The objective of this model is to take into account the randomness and heterogeneity of the natural fractures and the matrix blocks.

The presence of natural fractures in shale reservoirs is significantly lower than in carbonate reservoirs. Therefore, the porosity of the natural fractures in shale reservoirs is very small, and the storativity ratio should be negligible. However, it is important to consider the presence of the natural fractures because these provide a significant conduit for the flow of fluids towards hydraulic fractures and/or the wellbore. To model the flow within the fractal fracture network, we consider the diffusivity equation presented by Valdes-Perez (2013). Such an equation in its dimensionless form is defined by:

\[
\frac{1}{R_D^{Df-1}} \frac{\partial}{\partial R_D} \left[ R_D^D \frac{\partial p_{fnDcr}}{\partial R_D} \right] = \omega_0 A_fD_f \eta_{maD} \frac{\partial p_{fnDcr}}{\partial D_f} \int_0^{T_{Df}} \frac{\partial p_{fnDcr}}{\partial \tau} F(\eta_{maD},\eta_{maD},T_{Df}-\tau)d\tau
\]

The solution of Eq. 15 in the Laplace domain considering constant-rate at the wellbore and an infinite fracture network is given by:

\[
\bar{p}_{f_{Df-1}}(R_D,u) = \frac{K_v \left[ 2R_D \sqrt{u f(u)} \right]}{u \sqrt{u f(u)} K_{v-1} \left[ 2 \frac{\sqrt{u f(u)}}{\theta + 2} \right]}
\]

where \( V = [1-\beta]/(\theta + 2) \). The transfer function \( f(u) \) has the same shape as the one presented in Eq. 9. For this model, we have considered only the behavior of the fluid transfer function of the matrix blocks as infinite-acting (Eq. 4) due to their extremely low permeability in shale oil/gas reservoirs. For systems with a negligible storativity ratio and considering the fluid transfer function at intermediate times (Eq. 5), the model given by Eq. 15 can be simplified to yield:

\[
\frac{1}{R_D^{Df-1}} \frac{\partial}{\partial R_D} \left[ R_D^D \frac{\partial p_{fnDcr}}{\partial R_D} \right] = \frac{\alpha_0}{\sqrt{\pi}} \int_0^{T_{Df}} \frac{\partial p_{fnDcr}}{\partial \tau} [T_{Df}-\tau]^{-1/2} d\tau,
\]

where:

\[
\alpha_0 = A_fD_f \eta_{maD} \left[ \frac{h_{maD}}{2} \right]^{\theta_{maD}/2-1}
\]
A similar model for radial fracture networks and Euclidean matrix blocks was developed by Cinco-Ley et al. (1982). By analogy, for late times, substituting Eq. 6 in Eq. 14 and neglecting storativity ratio, the following result is obtained:

\[
\frac{1}{R_{D}^{D_{f}} - 1} \frac{\partial}{\partial R_{D}} \left[ R_{D}^{\beta} \frac{\partial^{2} P_{D}}{\partial R_{D}^{2}} \right] = \frac{\alpha_{l}^{\prime}}{\Gamma(1 - \nu_{ma})} \int_{0}^{t} \frac{\partial P_{D}}{\partial \tau} \left[ t_{D}^{\nu} - \tau \right]^{\nu_{ma}} d\tau \quad (19)
\]

Eq. 19 has the same shape as the models assuming the anomalous diffusion phenomena (Camacho-Velazquez et al., 2008 and Raghavan, 2012), but this model takes into account physical properties related to the geology of the the matrix blocks. The asymptotic constant-rate solutions in the real domain of Eq. 11 and Eq. 17 are summarized in Table 4.

<table>
<thead>
<tr>
<th>Table 4—Asymptotic constant-rate solutions in the real domain for the double-porosity fractal model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptotic solution</td>
</tr>
<tr>
<td>Early-Intermediate times</td>
</tr>
<tr>
<td>Late-Intermediate times</td>
</tr>
<tr>
<td>Late times</td>
</tr>
</tbody>
</table>

**Inclusion of Wellbore Storage.** The wellbore storage effects can be included using the classic (constant) wellbore storage relation in the Laplace domain:

\[
\bar{P}_{wD}(u, C_D) = \frac{\bar{P}_{wD}(u, s, C_D = 0)}{1 + C_D u^2 \bar{P}_{wD}(u, s, C_D = 0)} \quad (20)
\]

**Constant-Pressure Solution.** We have applied Duhamel's principle (convolution) to obtain the models for rate-transient analysis (i.e., the constant pressure solution obtained from the constant rate solution). Duhamel's principle relates the constant rate dimensionless pressure solution and the constant pressure dimensionless rate solution in the Laplace domain as follows:

\[
\bar{q}_{wD}(u) = \frac{1}{u^2 \bar{P}_{wD}(u)} \quad (21)
\]

**Results and Discussion**

**Double-porosity Model considering a radial fracture Network and Fractal Matrix Blocks**

**Sensitivity to the \( D_{fma} \)-Parameter: Closed Matrix Blocks.** Fig. 3 exhibits the effect of the fractal dimension of the matrix \( (D_{fma}) \) on the pressure transient performance behavior of a dual porosity reservoir considering a radial fracture network (Eq. 8) and closed matrix blocks (Eq. 1 for \( D_{fma} \neq 2 \) and Eq. 3 for \( D_{fma} = 2 \)). This plot shows no strong differences among the pressure-transient signatures when the fractal dimension of the matrix \( (D_{fma}) \) is varied. However, the influence of the \( D_{fma} \)-parameter is clear in the pressure derivative function. For all cases, the pressure derivative function exhibits two radial flow regimes.

The first radial flow regime (i.e., the fracture network flow) occurs at early times and represents the geometry and the fluid expansion within the radial fracture network. The second radial flow regime (i.e., the single porosity reservoir flow) occurs at late times and demonstrates that the radial fracture network and
the matrix blocks behave as a single porosity system where the geometry of the fracture network prevails. At intermediate times (during the interaction between the fracture network and the matrix blocks), we have observed that the interaction between the fracture network and the matrix blocks and the boundary effects of the matrix blocks occur faster when the matrix blocks have higher values of the fractal dimension (keeping the other parameters unchanged). This model can reproduce the behavior of the transfer functions developed by de Swaan (1976) when the conductivity index of the matrix is zero (i.e., $\theta_{ma}=0$) and $D_{fma}=1$ for slabs and $D_{fma}=3$ for spheres.

Fig. 4 shows the rate-transient case for the example presented in Fig. 3. Based on the rate derivative function in this plot, we observed that is not possible to distinguish the first period of flow (i.e., the radial fracture network flow). However, the influence of the fractal dimension of the matrix ($D_{fma}$) is also observed in this plot—i.e., the higher the $D_{fma}$-value, the sooner the boundary effects of the matrix will appear.
Sensitivity to the $\theta_{ma}$-Parameter: Closed Matrix Blocks. Fig. 5 presents the sensitivity analysis of the conductivity index of the matrix ($\theta_{ma}$) in the pressure transient performance behavior for the radial fracture network and closed matrix blocks case (Eq. 1 for $\theta_{ma} \neq 1$ and Eq. 3 for $\theta_{ma} = 1$). For the $\theta_{ma}$-parameter, we observe two phenomena: (1) there is a delay in the interaction between the fracture network when the matrix blocks have higher $\theta_{ma}$-values and (2) the boundary effects of the matrix blocks occur at earlier times for high values of the conductivity index of the matrix. High $\theta_{ma}$-values imply that the permeable sites inside the matrix blocks are poorly connected.

Based on Fig. 6, we conclude that better connected matrix blocks (lower $\theta_{ma}$-values) yield higher flowrates. Similar to the fractal dimension of the matrix case, the boundary effects of the matrix appear at earlier times for high $\theta_{ma}$-values.
Sensitivity to the $D_{fma}$-Parameter: Infinite-Acting Blocks. In Fig. 7, we present the sensitivity analysis of the fractal dimension of the matrix ($D_{fma}$) in the pressure transient performance behavior for the radial fracture network considering infinite-acting matrix blocks (Eq. 4). This plot shows that fractal matrix blocks with lower $D_{fma}$-values yield a higher pressure drop for a double-porosity system. The pressure derivative function in this plot exhibits the same behavior as the closed fractal matrix case at early times (fracture network flow) and intermediate times (interaction between the fracture network and the matrix blocks). However at late times, the flow is dominated by the fluid transfer from the matrix to the fractures. For $D_{fma}$-values greater than 2, the pressure derivative function yields a negative slope equal to the $v_{ma}$-parameter. For cases where $1 < D_{fma} < 2$, the pressure derivative function stabilizes to a constant value equal to $v_{ma}/2$.

The rate-transient performance behavior observed in Fig. 8 shows that fractal matrix blocks with higher $D_{fma}$-values yield higher flowrates. In addition, the rate derivative function has a response at early and intermediate times similar to the closed matrix blocks case. At late times, the infinite-acting nature of the matrix blocks dominates.
Sensitivity to the $\theta_{ma}$-Parameter: Infinite-Acting Matrix Blocks. The plot in Fig. 9 presents the sensitivity analysis of the conductivity index parameter for the matrix ($\theta_{ma}$) in the pressure transient performance behavior for a double-porosity reservoir considering a radial fracture network and infinite-acting matrix blocks. The example shown in this plot considers a small storativity ratio ($\omega=10^{-10}$) and, therefore, the flat portion of the pressure derivative function for dimensionless times from $10^0$ to $10^7$ corresponds to the interaction between the fracture network and the matrix blocks. In the rate-transient case, this implies that low $\theta_{ma}$-values yield higher flow-rates (see Fig. 10).

Also in Fig. 9, we observe that poorly-connected permeable sites inside the matrix blocks (i.e., high $\theta_{ma}$-values) accelerate the appearance of the single system behavior. Similar to the fractal dimension case, the pressure derivative function yields a negative slope equal to the $\nu_{ma}$-parameter for $\theta_{ma} > 1$. If $\theta_{ma} > 1$, then the pressure derivative function stabilizes to a constant value equal to $\nu_{ma}/2$. Therefore, by increasing the $\theta_{ma}$-parameter to very high values, the pressure derivative function approaches a constant value of 0.5.

Wellbore Storage Effects. The wellbore storage effects for all the models presented in this section are incorporated using the standard scheme presented by Eq. 20. Figs. 11 and Fig. 12 exhibit the impact of
wellbore storage for the same cases as Figs. 7 and Fig. 9, respectively. These plots show that wellbore storage effects tend to delay the appearance of the radial fracture flow regime when the storativity ratio is relatively high (Fig. 11) and shorten the transition period between the radial fracture flow and the interaction between fractures and matrix blocks period when the storativity ratio is relatively low (Fig. 12).

**Analogy to the Anomalous Diffusion Model: Sensitivity to the $D_{fma}$-Parameter.** Fig. 13 shows the influence of the fractal dimension of the matrix ($D_{fma}$) on the pressure drop and pressure drop derivative functions for the case of a dual porosity model considering a radial fracture network and infinite-acting matrix blocks, assuming a time-dependent inner boundary condition. We have observed that none of the cases presented in Fig. 13 show the characteristic signature of radial flow (i.e., the "0-slope" period of the pressure drop derivative function), although the geometry of the fracture network was assumed to
be radial. Instead, the responses of the pressure drop and pressure drop derivative functions for this model yield power-law behaviors equal to the exponent of the inner boundary condition as shown in Fig. 14. We conclude that the higher the $D_{f_{\text{in}}}$-parameter (i.e., lower $v_{\text{ma}}$) the steeper the log-log straight-line.

Fig. 13 presents the constant-pressure version for the example presented in Fig. 13. The rate and rate derivative functions yield power-law behaviors, the slope of which on the log-log plot is equal to the negative
value of slope in the constant-rate case (i.e., $v_{ma}-1$). This is confirmed by the $\beta$-rate derivative (Fig. 16). We have observed that at dimensionless times greater than $10^{-1}$ the fractal matrix blocks with higher $D_{fma}$-values yield higher flowrates. However, an unrealistic scenario is observed at earlier times ($t_D < 10^{-1}$), where the highest flow-rates are yielded by fractal matrix blocks with lower $D_{fma}$-values. The reason why this scenario is unrealistic is because high $D_{fma}$-values essentially represent more permeable sites within the matrix blocks.

Figure 15—Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).

Figure 16—Log-log plot of the $\beta$-rate derivative function for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the fractal dimension of the matrix ($D_{fma}$).
**Analogy to the Anomalous Diffusion Model: Sensitivity to the \( \theta_{ma} \)-Parameter.** In Fig. 17, we present the sensitivity analysis of the conductivity index of the matrix (\( \theta_{ma} \)). At late times (\( t_D > 10^9 \)), this plot shows a behavior similar to the one observed for the \( D_{fma} \)-parameter — i.e., the signatures of the pressure drop and pressure drop derivative functions yield power-law behaviors equal to the exponent of the inner boundary condition (i.e., 1-\( v_{ma} \)). This is confirmed by the \( \beta \)-pressure derivative function presented in Fig. 18. At early and intermediate times (\( t_D < 10^9 \)), the \( \beta \)-pressure derivative function exhibits a variable (i.e., non-constant) behavior for all of the cases presented. We conclude that the better connected permeable sites inside the matrix (i.e., lower \( \theta_{ma} \)- and consequently \( v_{ma} \)-values), the steeper the log-log straight-line of the pressure and pressure derivative functions.

![Figure 17](image1.png)

**Figure 17**—Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix (\( \theta_{ma} \)).

![Figure 18](image2.png)

**Figure 18**—Log-log plot of the \( \beta \)-rate derivative function for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix (\( \theta_{ma} \)).

Based on our observations of the behaviors shown in Fig. 19, we conclude that better connected permeable sites inside the matrix blocks (low \( \theta_{ma} \)-values) always yield higher flowrates. Similar to the fractal dimension
of the matrix \( (D_{ma}) \), the \( \beta \)-rate derivatives presented in Fig. 20 show that rate and rate derivative functions exhibited in Fig. 19 yield power-law behaviors with slope equal to \( v_{ma} = -1 \), for \( \theta_{ma} \)-values equal to 5 (only after \( t_D > 10^7 \)) and 10. The non-constant behavior of the \( \beta \)-rate derivative for \( \theta_{ma} \) equal to 100 indicates that for this case rate-transient performance does not correspond to a power-law behavior.

![Figure 19](image1.png)

**Figure 19**—Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix \( (\theta_{ma}) \).

![Figure 20](image2.png)

**Figure 20**—Log-log plot of the \( \beta \)-rate derivative function for a double-porosity reservoir with a time-dependent inner boundary condition, considering a radial fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the matrix \( (\theta_{ma}) \).

Given that the scheme to include the wellbore storage effects was developed assuming a constant inner boundary condition, it is not appropriate to apply the wellbore storage condition to models which utilize a variable inner boundary condition.
Double-porosity Model considering Fractal Fracture Network and Fractal Matrix Blocks (Double-Porosity Fractal Model)

In general, the most significant parameters in a diffusivity model for a fractal object are the fractal dimension and the conductivity index. The fractal dimension can have values ranging from one to three, whereas the conductivity index can have values higher than zero. Both parameters have a similar influence in the slope of the pressure derivative function — therefore, we will restrict our sensitivity analyses only to the conductivity indexes (both fractures and matrix) and consider that the fractal dimension is equal to three for both media.

**Sensitivity to the θ-Parameter.** Fig. 21 presents the sensitivity analysis of the conductivity index of the fracture network (θ) on the pressure transient performance behavior of the double-porosity fractal model, considering both cases, with and without wellbore storage effects. In Fig. 22 we present the β-pressure derivative for the cases presented in Fig. 21; in the no-wellbore storage case, we observe that there are two periods of flow separated by a smooth transition period. The first period corresponds to the interaction between the fracture network and the matrix blocks, and the second period corresponds to single system behavior. In both Figs. 21 and 22, the first period is not observed when the wellbore storage effects are taken into account. This leads to the hypothesis that for practical applications in pressure transient analysis, the only period of flow that could be observed is the single system behavior period, and the fluid transfer function of the matrix could be reduced to the expression given by Eq. 6.
Figure 22—Log-log plot of the $\beta$-pressure derivative function for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).

In Fig. 23, we observe that a better connected fractal fracture network yields higher flowrates (i.e., the lower the $\theta$-value, the higher the flowrate). Given that standard wellbore storage effects should not affect the rate transient behavior (i.e., due to the constant pressure inner boundary condition), it is possible to observe the two periods of flow of the transient behavior for the double-porosity fractal reservoir. As in the pressure-transient case, the two periods of flow can be observed in the $\beta$-rate derivative functions presented in Fig. 24.

Figure 23—Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).
Fig. 24—Log-log plot of the $\beta$-rate derivative function for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta$).

Fig. 25 shows the influence of the conductivity index of the matrix ($\theta_{ma}$) in the pressure transient performance behavior of the double-porosity fractal model, considering cases with and without wellbore storage effects. Fig. 26 shows the $\beta$-pressure derivative of the cases presented in Fig. 25. Similar to the case of the conductivity index of the fracture network ($\theta$), we observe that in the no-wellbore storage case, two periods of flow are separated by a smooth transition period. In this case, the first period of flow is obscured by wellbore storage effects. The second period of flow in Fig. 25 yields a straight-line with slope equal to the product $v_{ma}v$. This is confirmed by their $\beta$-pressure derivative plots (Fig. 26).

Fig. 25—Log-log plot of the dimensionless pressure and dimensionless pressure derivative functions for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal matrix blocks ($\theta_{ma}$).
Figure 26—Log-log plot of the $\beta$-pressure derivative function for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal matrix blocks ($\theta_{ma}$).

**Sensitivity to the $\theta_{ma}$-Parameter.** Fig. 27 shows that the better connected permeable sites inside the fractal matrix blocks yield higher flowrates (i.e., the lower the $\theta_{ma}$-value, the higher flowrate). Fig. 28 shows two periods of flow for each one of the cases presented in Fig. 29. We observe that the first period of flow corresponds to the interaction between the fracture network and the matrix blocks. The behavior of the second period is dominated by the $\theta_{ma}$-value and we have observed that the higher the conductivity index of the matrix, the sooner the second flow period appears.

Figure 27—Log-log plot of the dimensionless rate and dimensionless rate derivative functions for a double-porosity reservoir considering a fractal fracture network and infinite-acting matrix blocks for selected values of the conductivity index of the fractal fracture network ($\theta_{ma}$).
Conclusions

The following conclusions have been derived from this work:

1. The anomalous diffusivity phenomenon in unconventional reservoirs can be related to the fractal geometry and the heterogeneities of the fractal matrix blocks.
2. The fractal dimension of the matrix blocks does not have a significant impact in the signature of the pressure and pressure-transient derivative functions when the matrix blocks are considered closed (small blocks and/or high hydraulic diffusivity).
3. When the matrix blocks behave as "infinite-acting," the pressure- and rate-transient performance behaviors at late times are sensitive to the combined effect of the properties of the fracture network and the matrix blocks.
4. The variable-rate assumption (time-dependent inner boundary condition) yields transient signatures (pressure and rate) which are dominated by the inner boundary condition. This effect hinders the response of the reservoir (i.e., distorts the reservoir pressure and rate signatures, thereby prohibiting diagnostic analyses).

Acknowledgments

The authors gratefully acknowledge the assistance of Prof. Peter Valkó at Texas A&M U. for providing the Gaver-Wynn-Rho algorithm to numerically invert the Laplace domain solutions presented in this paper.

Nomenclature

\[ A_{fma} = \text{Fracture area per unit of matrix volume, } L^{-1} \text{[m}^{-1}] \text{ or [ft}^{-1}] \]
\[ B = \text{Oil formation volume factor, [bbl/STB] or [Rm}^3/Sm}^3] \]
\[ c_o = \text{Fluid compressibility (oil), (M/Lt}^2) \text{[Pa}^{-1}] \text{ or [psi}^{-1}] \]
\[ c_{ma} = \text{Matrix blocks compressibility, (M/Lt}^2) \text{[Pa}^{-1}] \text{ or [psi}^{-1}] \]
\[ c_{fb} = \text{Radial fracture network compressibility, (M/Lt}^2) \text{[Pa}^{-1}] \text{ or [psi}^{-1}] \]
\[ c_{tma} = \text{Natural fracture network total compressibility (either radial of fractal), (M/Lt}^2) \text{[Pa}^{-1}] \]
\[ c_{tma} = \text{Matrix blocks total compressibility, (M/Lt}^2) \text{[Pa}^{-1}] \text{ or [psi}^{-1}] \]
\[ h = \text{Formation thickness, L [m] or [ft]} \]
\[ h_{ma} = \text{Matrix block size, L [m] or [ft]} \]
\[ k_{0fb} = \text{Reference permeability for a fractal fracture network, } L^2 \text{[mD]} \text{ or [m}^2] \]
$k_{\text{fma}} = \text{Reference permeability for the fractal matrix blocks, } L^2 [\text{mD}] \text{ or } [\text{m}^2]$

$k_{fb} = \text{Radial fracture network permeability, } L^2 [\text{mD}] \text{ or } [\text{m}^2]$

$p = \text{Pressure, } M/L^2 [\text{Pa}] \text{ or } [\text{psi}]$

$p_i = \text{Initial reservoir pressure, } M/L^2 [\text{Pa}] \text{ or } [\text{psi}]$

$p_{wf} = \text{Wellbore flowing pressure, } M/L^2 [\text{Pa}] \text{ or } [\text{psi}]$

$r = \text{Radial distance, } L [\text{m}] \text{ or } [\text{ft}]$

$R = \text{Radial distance (fractal systems), } L [\text{m}] \text{ or } [\text{ft}]$

$r_w = \text{Wellbore radius, } L [\text{m}] \text{ or } [\text{ft}]$

$s = \text{Skin factor, dimensionless}$

$q = \text{Well Flowrate, } L^3/t [\text{m}^3/\text{sec}] \text{ or } [\text{ft}^3/\text{s}]$

$t = \text{Time, } t [\text{sec}]$

$u = \text{Laplace transform variable}$

**Dimensionless Variables:**

$A_{fD} = \text{Dimensionless natural fracture area}$

$D_f = \text{Fractal dimension of a fractal fracture network}$

$D_{fma} = \text{Fractal dimension of the matrix blocks}$

$C_D = \text{Dimensionless wellbore storage constant}$

$h_{maD} = \text{Dimensionless matrix block size}$

$p_{fmaD,cr} = \text{Dimensionless pressure in the fractal-fracture network (dimensionless variables for the constant-rate solution)}$

$p_{rfnD,cr} = \text{Dimensionless pressure in the radial-fracture network (dimensionless variables for the constant-rate solution)}$

$p_{rfnD,cr} = \text{Dimensionless pressure in the fractal matrix blocks (dimensionless variables for the constant-rate solution)}$

$p_{wD} = \text{Dimensionless wellbore flowing pressure}$

$r_D = \text{Dimensionless radius for a radial fracture network}$

$R_D = \text{Dimensionless radius for a fractal system}$

$q_{wD} = \text{Dimensionless flowrate}$

$t_D = \text{Dimensionless time (dimensionless variables for a radial fracture network)}$

$t_{DF} = \text{Dimensionless time (dimensionless variables for a fractal fracture network)}$

$v = \text{Grouping parameter of the fractal variables of a fractal fracture network}$

$v_{ma} = \text{Grouping parameter of the fractal variables of the fractal matrix blocks}$

**Greek Symbols:**

$\alpha_0 = \text{Grouping parameter of the properties the fractal matrix block, dimensionless}$

$\alpha_1 = \text{Grouping parameter of the properties the fractal matrix block, dimensionless}$

$\beta = \text{Spatial dimension of the fractal fracture network, dimensionless}$

$\beta_{ma} = \text{Spatial dimension of the fractal fracture network, dimensionless}$

$\gamma = \text{Euler's constant (0.5772456…)}$

$\theta = \text{Conductivity index of a fractal fracture network, dimensionless}$

$\theta_{ma} = \text{Conductivity index of the fractal matrix blocks, dimensionless}$

$\eta_{maD} = \text{Dimensionless hydraulic diffusivity of the fractal matrix, dimensionless}$

$\zeta = \text{Newtonian Viscosity, } M/Lt [\text{cp}] \text{ or } [\text{lb}_m/\text{ft}*\text{s}]$

$\zeta = \text{Grouping parameter, dimensionless}$

$\zeta_{RD} = \text{Function of } R_D, \text{ dimensionless}$
ϕ_0fb = Reference porosity for fractal fracture network, fraction
ϕ_0ma = Reference porosity for the fractal matrix blocks, fraction
ϕ_fb = Radial fracture network porosity, fraction
ω = Storativity ratio, dimensionless

Mathematical Functions:

I_v(x) = Modified Bessel Functions of the first kind, v-order
K_v(x) = Modified Bessel Functions of the second kind, v-order
Λ(x) = Gamma function

References

Appendix A

Development of the Fluid Transfer Function Considering Closed Matrix Blocks

To model the matrix blocks as fractal objects, consider the diffusivity model presented by Chang et al., (1990) in its dimensionless form:

\[
\frac{1}{R_D^D_{D \, f} \beta_{ma}^{-1}} \frac{\partial}{\partial R_D} \left[ R_D^{\beta_{ma}} \frac{\partial p_{maD,cP}}{\partial R_D} \right] = \frac{1}{\eta_{maD}} \frac{\partial p_{maD,cP}}{\partial t_D}.
\]  \hspace{1cm} (A.1)

The initial and the boundary conditions for this problem are:

\[
p_{maD,cP}(R_D, t_D = 0) = 0 \quad \text{(initial condition)}, \]
\[
\lim_{R_D \to 0} p_{maD,cP}(R_D, t_D) = 1 \quad \text{(inner boundary condition)}, \]
\[
\left[ \frac{\partial p_{maD,cP}}{\partial R_D} \right]_{R_D = \frac{h_{maD}}{2}} = 0 \quad \text{(outer boundary condition)}.
\]  \hspace{1cm} (A.2, A.3, A.4)

The particular solution of Eq. A.1 in the Laplace domain is:

\[
\tilde{p}_{maD,cP}(R_D, u) = \left[ \frac{2R_D}{h_{maD}} \right]^{\frac{1-\beta_{ma}}{2}} \left[ \frac{2K_{vma}}{2K_{vma} \xi + \Gamma vma \eta vma \xi} \right]^{\frac{1}{u}}
\]  \hspace{1cm} (A.5)

Where \( \xi \) is defined by Eq. 2 and:

\[
\xi = \frac{R_D^{\frac{\theta_{ma} + 2}{2}}}{\theta_{ma} + 2} \sqrt{\frac{u}{\eta_{maD}}} \]  \hspace{1cm} (A.6)

The fluid transfer function is defined by:

\[
F(\eta_{maD}, h_{maD}, t_D) = \left[ \frac{\partial p_{maD,cP}}{\partial R_D} \right]_{R_D = \frac{h_{maD}}{2}}
\]  \hspace{1cm} (A.7)

Taking the derivative with respect of \( R_D \) of Eq. A.5 and evaluating in \( h_{maD}/2 \), the fluid transfer function in the Laplace domain for this case is:

\[
\tilde{F}(\eta_{maD}, h_{maD}, u) = \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} \left[ \frac{1}{\sqrt{u \eta_{maD}}} \right] \left[ \frac{2K_{vma}}{2K_{vma} \xi + \Gamma vma \eta vma \xi} \right]^{\frac{1}{u}}
\]  \hspace{1cm} (A.8)

Development of the Fluid Transfer Function Considering Infinite-Acting Matrix Blocks

For this model consider the diffusivity model given by Eq A.1 and the conditions established by Eq. A.2 and Eq. A.3. The infinite-acting case is modeled by assuming infinite matrix blocks, i.e., the outer boundary condition for this model is defined by:

\[
\lim_{R_D \to \infty} p_{maD,cP}(R_D, t_D) = 0
\]  \hspace{1cm} (A.9)

The particular solution of Eq. A.1 considering this set of initial and boundary conditions is:
The fluid transfer function (defined by Eq. A.7) for this case is:

\[
\bar{p}_{maD, cp}(R_D, u) = \left[ \frac{2 R_D}{h_{maD}} \right]^{1 - \beta_{ma}/2} \frac{K_{\nu_{ma}} \xi R_D}{u K_{\nu_{ma}} [\xi]}
\]  
(A.10)

Asymptotic Behavior

At early and intermediate times (large values of the Laplace parameter \( u \)), the Modified Bessel function of second order \((K_v(x))\) can be approximated as:

\[
K_v(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}
\]  
(A.12)

Taking the approximation given by Eq. A.12, the fluid transfer function given in Eq. A.11 reduces to:

\[
\bar{F}(\eta_{maD}, h_{maD}, u) = \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} \frac{1}{\sqrt{\eta_{maD}}} K_{\nu_{ma}}^{-1}[\xi]
\]  
(A.13)

Applying inverse Laplace transform to Eq. A.13, an asymptotic expression for early and intermediate times of the fluid transfer function in the real domain is obtained:

\[
F(\eta_{maD}, h_{maD}, t_D) = \frac{1}{\sqrt{\pi \eta_{maD}}} \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} t_D^{-1/2}
\]  
(A.14)

At late times (small values of the Laplace parameter \( u \)), the Modified Bessel function of second order \((K_v(x))\) for positive \(v\)-values can be approximated as:

\[
K_v(x) \approx \frac{\Gamma[v]}{2} \left[ \frac{z}{2} \right]^{-v}
\]  
(A.15)

Taking the approximation given by Eq. A.15, the fluid transfer function given in Eq. A.11 becomes:

\[
\bar{F}(\eta_{maD}, h_{maD}, u) = \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma}/2} \frac{1}{\sqrt{\eta_{maD}}} \frac{\Gamma[1 - \nu_{ma}] \Gamma[\nu_{ma}]}{\Gamma[\nu_{ma}]} \left[ \frac{h_{maD}}{2} \right]^{\theta_{ma} + 2}/2 \left[ \frac{u}{\eta_{maD}} \right]^{1 + 2\nu_{ma}}
\]  
(A.16)

Rearranging:

\[
\bar{F}(\eta_{maD}, h_{maD}, u) = \frac{[\theta_{ma} + 2]^{1 - 2\nu_{ma}} \Gamma[1 - \nu_{ma}] \Gamma[\nu_{ma}]}{\Gamma[\nu_{ma}] \eta_{maD}^{\nu_{ma}}} \left[ \frac{h_{maD}}{2} \right]^{\beta_{ma}} u^{-[1 - \nu_{ma}]}
\]  
(A.17)

Applying inverse Laplace transform to Eq. A.17:

\[
F(\eta_{maD}, h_{maD}, t_D) = \frac{[\theta_{ma} + 2]^{1 - 2\nu_{ma}} \Gamma[\nu_{ma}]}{\Gamma[\nu_{ma}] \eta_{maD}^{\nu_{ma}}} \left[ \frac{h_{maD}}{2} \right]^{\beta_{ma}} t_D^{-\nu_{ma}}
\]  
(A.18)
Appendix B

Asymptotic Solution for Early-Intermediate Times

Consider the constant-rate solution of the double-porosity fractal model in the Laplace domain (Eq. 15), evaluated at wellbore:

\[ P_{wD,cr}(u) = \frac{K_v \left[ \frac{2\sqrt{uf(u)}}{\theta + 2} \right]}{u\sqrt{uf(u)}K_{v-1} \left[ \frac{2\sqrt{uf(u)}}{\theta + 2} \right]} \] (B.1)

Neglecting the storativity ratio \((\omega=0)\) and considering the fluid transfer function for early times (Eq. A.13), Eq. B.1 is rewritten as:

\[ P_{wD,cr}(u) = u^{-5/4} \frac{K_v \left[ \frac{2\sqrt{\alpha_0}}{\theta + 2} u^{1/4} \right]}{K_{v-1} \left[ \frac{2\sqrt{\alpha_0}}{\theta + 2} u^{1/4} \right]} \] (B.2)

where:

\[ \alpha_0 = \frac{A/\eta}{2} [\frac{h_{maD}}{2}]^{\theta_{ma}/2-1} \] (B.3)

For large arguments the Modified Bessel Functions can be approximated by the expression defined in Eq. A.12. Hence, Eq. B.2 reduces to:

\[ P_{wD,cr}(u) = \frac{u^{-5/4}}{\sqrt{\alpha_0}} \] (B.4)

Applying inverse Laplace transform:

\[ p_{wD,cr}(t_Df) = \frac{1/4}{\Gamma[5/4]\sqrt{\alpha_0}} \] (B.5)

Asymptotic Solution for Late-Intermediate Times

For this case, the \(f(u)\)-function used in the previous section should be considered. In addition, according to the property of the Modified Bessel function of second order \((K_v(x)):\)

\[ K_v(x) = K_{-v}(x) \] (B.6)

And the approximation defined by Eq. A.15, the following expression is derived:

\[ P_{wD,cr}(u) = \frac{(\theta + 2)^{2v-1}\Gamma[v]}{\alpha_0^v\Gamma[1-v]} u^{-[v/2+1]} \] (B.7)

Applying inverse Laplace transform:

\[ p_{wD,cr}(t_Df) = \frac{(\theta + 2)^{2v-1}\Gamma[v]}{\alpha_0^v\Gamma[1-v]\Gamma[v/2+1]} t_Df^{v/2} \] (B.8)
Asymptotic Solution for Late Times

The $f(u)$-function for this case is obtained by neglecting the storativity ratio ($i.e., \omega=0$) and by substituting Eq. A.17 in Eq. 9. The following equation results:

$$f(u) = \alpha_1 u^{-[1-v_ma]} \quad \text{(B.9)}$$

where $\alpha_1$ is given by Eq. 19. Substituting Eq. B.9 in Eq. B.1 and taking the asymptotic behavior given in Eq. A.15, the asymptotic solution in the Laplace domain is obtained:

$$\bar{p}_{wD,cr}(u) = \frac{[\theta + 2]^{2v-1} \Gamma[v] u^{-[v Ma+1]}}{\alpha_1^v \Gamma[1-v]} \quad \text{(B.10)}$$

Applying inverse Laplace transform:

$$p_{wD,cr}(t|D_t) = \frac{[\theta + 2]^{2v-1} \Gamma[v]}{\alpha_1^v \Gamma[1-v] \Gamma[v Ma + 1]} t^{v Ma} \quad \text{(B.11)}$$