Topic: First Order Ordinary Differential Equations

Objectives: (things you should know and/or be able to do)
- Be able to classify the order of a differential equation (order of the highest derivative)
- Be able to verify a given solution of a differential equation via substitution of a given solution into the original differential equation. As simplistic as this seems, it may someday keep you out of serious trouble.
- Be able to sketch the general solution of an ordinary differential equation (ODE) and determine the particular solution for a given set of boundary conditions.
- Be familiar with and be able to solve first order ordinary differential equations using the method of separation of variables (or separable equations).
- Be able to derive the method of integrating factors for a first order ordinary differential equation.
- Be able to determine the solution of a first order ordinary differential equation using the method of integrating factors.
- Be able to apply the Euler and Runge-Kutta methods to numerically solve first order ordinary differential equations.

Lecture Outline:
- Classification of the order of a differential equation:
  - An ordinary differential equation has only one independent variable while a partial differential equation has two or more independent variables.
  - The order of a differential equation is the order of the highest derivative in the differential equation.
  - The initial condition is a known condition of the function or its derivatives at some starting time (not necessarily time zero).
- Verification of solutions via substitution:
  - When given a differential equation (ordinary or partial) and a proposed solution of that equation, you should:
    - Evaluate the derivatives of the solution as required by the differential equation and substitute these terms in the differential equation.
    - If the solution is exact, all terms will cancel, if not, you are left to determine whether the proposed solution is an approximation, or just plain wrong.
- Solution by separation of variables (or separable equations):
  - For many simple cases the differential equation can be rearranged, isolating like terms and integrated to yield the solution.
- Solution by separation of variables (or separable equations):
  - For many simple cases the differential equation can be rearranged to isolate like terms, then integrated to yield the solution.
- Method of Integrating Factors:
  - Definition of an exact differential equation.
  - Derivation of the integrating factor from the exact equations (Schaum's Outline text, Advanced Mathematics for Engineers and Scientists—Solved Problem 2.17, p.50)
  - Applications
Lecture Outline: (Continued)
- Numerical solution of ordinary differential equations:
  - Discussion of the Euler method (Schaum's Outline text, Advanced Mathematics for Engineers and Scientists—Solved Problem 2.44)
  - Discussion of the Picard's method (Schaum's Outline text, Advanced Mathematics for Engineers and Scientists—Solved Problem 2.47)
  - Discussion of the Runge-Kutta method (Schaum's Outline text, Advanced Mathematics for Engineers and Scientists—Solved Problems 2.48 and 2.50).

Reading Assignment:
- Review attached notes.
- Introduction to Ordinary Differential Equations
- Chapters 2 and 3 of the Schaum's Outline text, Advanced Mathematics for Engineers and Scientists, by Spiegel.

Exercises: For your own practice/skills building—do NOT turn in!
- Schaum's Outline text, Advanced Mathematics for Engineers and Scientists, "Solved Problems" in Chapter 2:
  - Basic ODE's: (introduction) 2.1, 2.2, 2.4, 2.8
  - Basic ODE's: (various) 2.10, 2.11, 2.18, 2.19, 2.27, 2.28
  - Numerical Methods: 2.44, 2.47, 2.48, 2.49, 2.50
Introduction to 
Ordinary Differential Equations 
(from Petroleum Engineering 620 Course Notes — 1999)

Petroleum Engineering 620 
Fluid Flow in Reservoirs
Introduction to Ordinary Differential Equations

Linear Classification

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x) \frac{dy}{dx} + a_0(x) y = R(x) \]

Rules for linearity

1. y and all derivatives of y are raised to no power other than 1.
2. Each coefficient \( a_i(x) \) is only a function of x.
3. No y and/or y-derivative products are allowed.

Nomenclature

1. Any violation of the "rules for linearity" labels the differential equation as non-linear.
2. If \( R(x) = 0 \), then the equation is called "reduced," "complimentary," or "homogeneous."
   If \( R(x) \neq 0 \), then the equation is called "complete" or "non-homogeneous."

Examples

<table>
<thead>
<tr>
<th>Case</th>
<th>Differential Equation</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x \frac{dy}{dx} + y \frac{dy}{dx} = 0 )</td>
<td>1st order, linear</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0 )</td>
<td>2nd order, linear</td>
</tr>
<tr>
<td>3*</td>
<td>( \frac{dy}{dx} = x \left( \frac{dy}{dx} \right)^{1/2} )</td>
<td>1st order, non-linear (Bernoulli-type Eq.)</td>
</tr>
<tr>
<td>4</td>
<td>( \left( y \frac{d^2 y}{dx^2} \right) - 2 \frac{dy}{dx} = x + 1 )</td>
<td>2nd order, non-linear</td>
</tr>
</tbody>
</table>

* The general Bernoulli differential equation:
  \[ \frac{dy}{dx} + a(x) y = b(x) y^x \quad x \neq 0, 1 \]
  can be "linearized" using an "Integrating Factor."
Legendre's Differential Equation

\[
(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \left[ n(n+1) - \frac{m^2}{1-x^2} \right] y = 0
\]

Solution:

\[ y = C_1 P_n^m(x) + C_2 Q_n^m(x) \]

where,

\[ C_1, C_2 = \text{arbitrary constants} \]

\[ P_n^m(x) = \text{Legendre polynomials} \]

\[ Q_n^m(x) = \text{Legendre functions of the second kind} \]

\[ n = \text{degree of the Legendre polynomials/functions} \]

\[ m = \text{order of the Legendre polynomials/functions} \]

Special case: \( m = 0 \)

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad \text{(Rodrigue's formula)} \]

\[ Q_n(x) = \frac{1}{2} P_n(x) \ln \left[ \frac{1+x}{1-x} \right] - \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(x) P_{n-m}(x) \]

where

\[ \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(x) P_{n-m}(x) = 0, \text{ if } n \neq 0 \]

Bessel's Differential Equation

\[
x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2)y = 0
\]

Solution:

\[ y = C_1 J_v(x) + C_2 Y_v(x) \]

where,

\[ C_1, C_2 = \text{arbitrary constants} \]

\[ J_v(x) = \text{Bessel functions of the first kind} \]

\[ Y_v(x) = \text{Bessel functions of the second kind} \]

\[ v = \text{order of the Bessel function (standard notation: } \nu \text{ for real values, } n \text{ for integers}) \]
Functional relations and approximations for $J_0(x)$ and $Y_0(x)$ (as well as for $J_n(x)$ and $Y_n(x)$) can be found in Chapters 9 and 10 of Abramowitz and Stegun, "Handbook of Mathematical Functions," Dover Publications (1972).

**Bessel’s Modified Differential Equation**

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + v^2) y = 0$$

**Solution:**

$$y = c_1 J_0(x) + c_2 K_0(x)$$

where,

$c_1, c_2$ = arbitrary constants

$J_0(x)$ = Modified Bessel functions of the first kind

$K_0(x)$ = Modified Bessel functions of the second kind

As with the Bessel functions $J_0(x)$ and $Y_0(x)$, we can also obtain functional relations and approximations for the Modified Bessel functions $J_0(x)$ and $K_0(x)$ in Chapters 9 and 10 of the Abramowitz and Stegun reference.

**Hypergeometric Differential Equations**

**Kummer’s Equation:**

$$x \frac{d^2 y}{dx^2} + (b-x) \frac{dy}{dx} - ay = 0$$

**Solution:**

$$y = c_1 M(a,b,x) + c_2 U(a,b,x)$$

where,

$c_1, c_2$ = arbitrary constants

$M(a,b,x)$ = First Kummer function

$U(a,b,x)$ = Second Kummer function

$a, b$ = parameters in the Kummer functions
**Whittaker's Equation:**

\[ \frac{d^2y}{dx^2} + \left[ -\frac{1}{4} + \frac{k}{x} + \frac{(\mu^2 - \nu^2)}{x^2} \right] y = 0 \]

**Solution:**

\[ y = c_1 M_{k, \nu}(x) + c_2 W_{k, \nu}(x) \]

where,
- \( c_1, c_2 = \text{arbitrary constants} \)
- \( M_{k, \nu}(x) = \text{First Whittaker function} \)
- \( W_{k, \nu}(x) = \text{Second Whittaker function} \)
- \( k, \nu = \text{parameters in the Whittaker functions} \)

The details of these hypergeometric functions can be found in Chapter 13 of the Abramowitz and Stegun reference.

**Summary: Special Differential Equations**

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Legendre</td>
<td>( (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \left[ n(n+1) - \frac{m^2}{(1-x^2)} \right] y = 0 )</td>
</tr>
<tr>
<td>Bessel</td>
<td>( x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0 )</td>
</tr>
<tr>
<td>Modified Bessel</td>
<td>( x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0 )</td>
</tr>
<tr>
<td>Kummer</td>
<td>( x \frac{d^2y}{dx^2} + (b-x) \frac{dy}{dx} - ay = 0 )</td>
</tr>
<tr>
<td>Whittaker</td>
<td>( \frac{d^2y}{dx^2} + \left[ -\frac{1}{4} + \frac{k}{x} + \frac{(\mu^2 - \nu^2)}{x^2} \right] y = 0 )</td>
</tr>
</tbody>
</table>
### Summary: Special Differential Equations and Their Solutions

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential Equation</th>
<th>Solution Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Legendre</td>
<td>$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \left[ \frac{n(n+1)}{(1-x^2)} \right] y = 0$</td>
<td>(y = c_1 P_n^m(x) + c_2 P_n^{-m}(x))</td>
</tr>
<tr>
<td>Bessel</td>
<td>(x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2) y = 0)</td>
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</tr>
<tr>
<td>Whittaker</td>
<td>(\frac{d^2y}{dx^2} + \left[ -\frac{1}{4} + \frac{k}{x} + \frac{(k^2 - u^2)}{x^2} \right] y = 0)</td>
<td>(y = c_1 M_{k\mu}(x) + c_2 W_{k\mu}(x))</td>
</tr>
</tbody>
</table>

**Where:**

- \(P_n^m(x)\) = Legendre Polynomials
- \(Q_n^m(x)\) = Legendre Polynomials of the 2nd kind
- \(J_v(x)\) = Bessel Functions of the 1st kind
- \(Y_v(x)\) = Bessel Functions of the 2nd kind
- \(J_v(x)\) = Modified Bessel Functions of the 1st kind
- \(Y_v(x)\) = Modified Bessel Functions of the 2nd kind
- \(M(a,b,x)\) = First Kummer Function
- \(U(a,b,x)\) = Second Kummer Function
- \(M_{k\mu}(x)\) = First Whittaker Function
- \(W_{k\mu}(x)\) = Second Whittaker Function
Solution of First Order Ordinary Differential Equations

1. Method of Integrating Factors

The objective is to obtain a general solution of the generalized first order ordinary differential equation, given by:

\[ \frac{dy}{dx} + p(x)y = q(x) \quad (1) \]

Rearranging Eq. 1 gives us:

\[ [p(x)y - q(x)] \, dx + dy = 0 \quad (2) \]

We begin by assuming that the general differential equation can be written as

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \quad (3) \]

where \( M(x, y) \) and \( N(x, y) \) are “independent functions.”

Multiplying through Eq. 3 by an “integrating factor,” \( \mu \), gives

\[ \mu M(x, y) \, dx + \mu N(x, y) \, dy = 0 \quad (4) \]

If Eq. 4 is exact (as we assume it is), then

\[ \frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] \quad (5) \]

Expanding Eq. 5, we have

\[ \mu_y M + \mu M_y = \mu M_x + \mu N_x \quad (6) \]

where

\[ \mu_y = \frac{\partial \mu}{\partial y}, \quad \mu_x = \frac{\partial \mu}{\partial x}, \quad M_y = \frac{\partial M}{\partial y}, \quad N_x = \frac{\partial N}{\partial x} \]
We will also assume that the "integrating factor" \( \mu \) is only a function of \( x \), that is, \( \frac{d\mu}{dy} = 0 \). This reduces Eq. 6 to

\[ \mu M_y = N\overline{x} + N\mu \]  

Solving for \( \mu_x \) gives

\[ \mu_x = \frac{d\mu}{dx} = \mu \left( \frac{M_y - N\overline{x}}{N} \right) \]

or

\[ \frac{1}{\mu} \frac{d\mu}{dx} = \left( \frac{M_y - N\overline{x}}{N} \right) dx \]

Integrating, we have

\[ \ln(\mu) = \int \left( \frac{M_y - N\overline{x}}{N} \right) dx \]

or

\[ \mu = \exp \left[ \int \left( \frac{M_y - N\overline{x}}{N} \right) dx \right] \]  \( \text{(7)} \)

Recalling Eqs. 2 and 3, we have

\[ \left[ p(x) \overline{y} - q(x) \right] dx + dy = 0 \]  \( \text{(2)} \)

\[ M(x,y) \overline{y} + N(x,y) dy = 0 \]  \( \text{(3)} \)

we note by inspection that

\[ M(x,y) = p(x) \overline{y} - q(x) \]  \( \text{(8)} \)

\[ N(x,y) = 1 \]  \( \text{(9)} \)

Therefore,

\[ \frac{M_y}{dy} \left[ M(x,y) \right] = \frac{1}{dy} \left[ p(x) \overline{y} - q(x) \right] = p(x) \]  \( \text{(10)} \)
Method of Integrating Factors (cont'd)

\[ N_x = \frac{d}{dx} [N(x,y)] = \frac{d}{dx} [1] = 0 \]  

(11)

Substituting Eqs. 9-11 into Eq. 7 gives

\[ m = \exp \left[ \int \left[ \frac{p(x)}{q(x)} \right] dx \right] \]

which reduces to:

\[ m = \exp \left[ \int p(x) \, dx \right] = e^{\int p(x) \, dx} \]  

(12)

Multiplying Eq. 1 by the "integrating factor," given by Eq. 12, we have

\[ e^{\int p(x) \, dx} \frac{dy}{dx} + e^{\int p(x) \, dx} p(x) \, y = e^{\int p(x) \, dx} q(x) \]  

(13)

By inspection, we note that

\[ \frac{d}{dx} \left[ e^{\int p(x) \, dx} \right] = \frac{d}{dx} \left[ \int p(x) \, dx \right] e^{\int p(x) \, dx} \]

or

\[ \frac{d}{dx} \left[ e^{\int p(x) \, dx} \right] = p(x) e^{\int p(x) \, dx} \]  

(14)

Substituting Eq. 14 into Eq. 13 gives

\[ e^{\int p(x) \, dx} \frac{dy}{dx} + \frac{d}{dx} \left[ e^{\int p(x) \, dx} \right] \, y = e^{\int p(x) \, dx} q(x) \]  

(15)

By inspection, we note that (using the product rule)

\[ \frac{d}{dx} \left[ e^{\int p(x) \, dx} \, y \right] = e^{\int p(x) \, dx} \frac{dy}{dx} + \frac{d}{dx} \left[ e^{\int p(x) \, dx} \right] \, y \]  

(16)
Method of Integrating Factors (cont'd) ⑨

Substituting Eq. 16 into Eq. 15 gives

\[
\frac{d}{dx} \left[ e^{\int p(x) \, dx} \cdot y \right] = e^{\int p(x) \, dx} \cdot q(x)
\]  \hspace{1cm} (17)

Separating and integrating gives

\[
\int d \left[ e^{\int p(x) \, dx} \cdot y \right] = \int \left[ e^{\int p(x) \, dx} \cdot q(x) \right] \, dx + c
\]

Completing the integration, we have

\[
y \cdot e^{\int p(x) \, dx} = \int \left[ e^{\int p(x) \, dx} \cdot q(x) \right] \, dx + c
\]

Solving for \( y \), gives the following final form

\[
y = e^{-\int p(x) \, dx} \cdot \int \left[ e^{\int p(x) \, dx} \cdot q(x) \right] \, dx + c \cdot e^{-\int p(x) \, dx}
\]  \hspace{1cm} (18)


\[
\frac{dy}{dx} - y = 2x \quad y(0) = 1
\]

Recalling our general form

\[
\frac{dy}{dx} + p(x) y = q(x)
\]

Therefore

\[ p(x) = -1 \quad q(x) = 2x \]

Calculating the "integrating factor," \( m \), we have

\[ m = e^{\int p(x) \, dx} = e^{-\int -1 \, dx} = e^x \]
The general solution is given as

\[ y = \frac{1}{m} \int m q(x) \, dx + \frac{c}{m} \]

where

\[ m = e^{\int p(x) \, dx} \]

Recalling

\[ p(x) = -1; \quad q(x) = 2x; \quad m = e^{-x} \]

Substituting these into the general solution gives

\[ y = \frac{1}{e^{-x}} \int e^{-x} (2x) \, dx + \frac{c}{e^{-x}} \]

or

\[ y = 2e^{x} \int x e^{-x} \, dx + ce^{x} \]

Unfortunately, \( \int x e^{-x} \, dx \) requires a bit of effort to reduce — in this case, integration-by-parts

\[ I = \int x e^{-x} \, dx \quad S u d u = m v - \int v d m \]

\[ M = x \quad d v = e^{-x} \, dx \]

\[ d M = d x \quad v = -e^{-x} \]

\[ I = -xe^{-x} - \int (-e^{-x}) \, dx \]

\[ = -xe^{-x} + \int e^{-x} \, dx \]

\[ = -xe^{-x} - e^{-x} \]

Substituting

\[ y = 2e^{x} \left[ -(1+x)e^{-x} \right] + ce^{x} \]

\[ I = -(1+x)e^{-x} \]
Method of Integrating Factors (Cont'd) 10 October 1999

\[ y = -2(1+x) + ce^x \]

Recalling the initial condition
\[ y(x=0) = 1 \]

At \( x=0 \), we have
\[ 1 = -2(1+0) + ce^0 \]
\[ c = 3 \]

Therefore the particular solution is
\[ y = -2(1+x) + 3e^x \]

2. Separation of Variables

The objective is to use "separation of variables" to consolidate like terms — then integrate.

Given:
\[ B(y) \frac{dy}{dx} = A(x) \]

We simply consolidate (or "separate") like terms, which gives
\[ B(y)dy = A(x)dx \]

We then integrate to obtain the \( x \) and \( y \) forms
\[ \int B(y)dy = \int A(x)dx + C \]
Separation of Variables (cont'd)


\[
\frac{dy}{dx} - y = 2x \quad y(0) = 1
\]

The required form is

\[
R(y)\,dy = A(x)\,dx
\]

This problem can not be solved using "separation of variables."

Example 2: "Simple case"

\[
\frac{dy}{dx} = 4x^2 + 2
\]

Rearranging into the required form:

\[
dy = (4x^2 + 2)\,dx
\]

Integrating

\[
\int dy = \int (4x^2 + 2)\,dx + C
\]

\[
y = \frac{4}{3}x^3 + 2x + C
\]
Example 3: "Irreducible case"

\[
\frac{dy}{dx} = \frac{\cos(x) + z}{\sin(y) + y}
\]

Rearranging into the required form:

\[
(\sin(y) + y)\,dy = (\cos(x) + z)\,dx
\]

\[
\int (\sin(y) + y)\,dy = \int (\cos(x) + z)\,dx + c
\]

\[-\cos(y) + \frac{1}{2}y^2 = \sin(x) + zx + c\]

This result can not be reduced into a simple \(y\) form (note \(\cos(y)\) term).

Picard's Method

The objective is to use the \(dy/dx\) function as a basis for continued integration as a means for obtaining \(y(x)\). While the practical application of Picard's method is probably limited, the conceptual value of generating successively more accurate approximations is quite useful.

Picard's method is given as:

\[
f(x, y) = \frac{dy}{dx}
\]

\[
y_1(x) = y_0 + \int_{x_0}^{x} f(u, y_1)\,du
\]

\[
y_2(x) = y_0 + \int_{x_0}^{x} f(u, y_2)\,du
\]

\[
y_3(x) = y_0 + \int_{x_0}^{x} f(u, y_3)\,du
\]

\[
\vdots
\]

\[
y_n(x) = y_0 + \int_{x_0}^{x} f(u, y_{n-1})\,du
\]
Picard's Method (cont'd)  


\[
\frac{dy}{dx} - y = 2x \quad y(0) = y_0 = 1 \quad x_0 = 0
\]

Putting into proper form, we have

\[
f(x, y) = \frac{dy}{dx} = 2x + y
\]

First Approximation:

\[
y_1(x) = y_0 + \int_{x_0}^{x} f(u, y_0) \, du = y_0 + \int_{x_0}^{x} (2u + y_0) \, du
\]

\[
= y_0 + \left[ \frac{2u^2}{2} + uy_0 \right]_0^x
\]

which gives

\[
y_1(x) = y_0 + (x^2 - x_0^2) + (x - x_0)y_0
\]

\[
= y_0 + (x^2 - (0)^2) + (x - (0))(1)
\]

\[
= 1 + x^2 + x
\]

or

\[
y_1(x) = 1 + x + x^2
\]

Second Approximation:

\[
y_2(x) = y_0 + \int_{x_0}^{x} f(u, y_1) \, du = y_0 + \int_{x_0}^{x} (2u + y_1) \, du
\]

\[
= y_0 + \int_{x_0}^{x} (2u + 1 + u + u^2) \, du
\]

\[
= y_0 + \int_{x_0}^{x} (1 + 3u + u^2) \, du
\]

\[
= y_0 + \left[ u + \frac{3u^2}{2} + \frac{1}{3} u^3 \right]_0^x
\]

\[
= y_0 + \left[ (x - x_0) + \frac{3}{2} (x^2 - x_0^2) + \frac{1}{3} (x^3 - x_0^3) \right]
\]
Picard's Method: (cont'd)

\[ y_2(x) = (1) + \left[ (x - (0)) + \frac{3}{2} \left( x^2 - (0)^2 \right) + \frac{1}{3} \left( x^3 - (0)^3 \right) \right] \]

or

\[ y_2(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{3} x^3 \]

Third Approximation:

\[ y_3(x) = y_0 + \int_{x_0}^{x} f(m, y_2) \, dm = y_0 + \int_{x_0}^{x} (2m + y_2) \, dm \]

\[ = y_0 + \int_{x_0}^{x} \left( 2m + 1 + m + \frac{3}{2} m^2 + \frac{1}{3} m^3 \right) \, dm \]

\[ = y_0 + \int_{x_0}^{x} \left( 1 + 3m + \frac{3}{2} m^2 + \frac{1}{3} m^3 \right) \, dm \]

\[ = y_0 + \left( m + \frac{3}{2} m^2 + \frac{1}{3} m^3 + \frac{1}{4} m^4 \right) \bigg|_{x_0}^{x} \]

\[ = y_0 + \left[ (x - x_0) + \frac{3}{2} (x^2 - x_0^2) + \frac{1}{2} (x^3 - x_0^3) + \frac{1}{12} (x^4 - x_0^4) \right] \]

\[ = (1) + \left[ (x - (0)) + \frac{3}{2} (x^2 - (0)^2) + \frac{1}{2} (x^3 - (0)^3) + \frac{1}{12} (x^4 - (0)^4) \right] \]

\[ y_3(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{12} x^4 \]

Fourth Approximation:

\[ y_4(x) = y_0 + \int_{x_0}^{x} f(m, y_3) \, dm = y_0 + \int_{x_0}^{x} (2m + y_3) \, dm \]

\[ = y_0 + \int_{x_0}^{x} \left( 2m + 1 + m + \frac{3}{2} m^2 + \frac{1}{3} m^3 + \frac{1}{4} m^4 \right) \, dm \]

\[ = y_0 + \int_{x_0}^{x} \left( 1 + 3m + \frac{3}{2} m^2 + \frac{1}{3} m^3 + \frac{1}{4} m^4 \right) \, dm \]

\[ = y_0 + \left( m + \frac{3}{2} m^2 + \frac{1}{3} m^3 + \frac{1}{4} m^4 \right) \bigg|_{x_0}^{x} \]

\[ = y_0 + \left[ (x - x_0) + \frac{3}{2} (x^2 - x_0^2) + \frac{1}{2} (x^3 - x_0^3) + \frac{1}{4} (x^4 - x_0^4) \right] \]
Picard's Method: (Cont'd)

\[ y_4(x) = (1) + \left[ \frac{(x-0)}{2} + \frac{3(x^2-0)^2}{2} + \frac{1}{2} (x^3-0)^3 \right. \]
\[ \left. + \frac{1}{8} (x^4-0)^4 + \frac{1}{60} (x^5-0)^5 \right] \]

\[ y_4(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{8} x^4 + \frac{1}{60} x^5 \]

Continuing, we have

\[ y_5(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{8} x^4 + \frac{1}{40} x^5 + \frac{1}{360} x^6 \]

\[ y_6(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{8} x^4 + \frac{1}{40} x^5 + \frac{1}{240} x^6 + \frac{1}{2520} x^7 \]

\[ y_7(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{8} x^4 + \frac{1}{40} x^5 + \frac{1}{240} x^6 + \frac{1}{1680} x^7 + \frac{1}{20,160} x^8 \]

From our previous work (using the integrating factor), we found the exact solution for this problem. The exact solution is:

\[ y_{\text{exact}}(x) = -2(1+x) + 3e^x \]

and using Picard's method, our last approximation is

\[ y_{\text{approx}}(x) = 1 + x + \frac{3}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{8} x^4 + \frac{1}{40} x^5 \]
\[ + \frac{1}{240} x^6 + \frac{1}{1680} x^7 + \frac{1}{20,160} x^8 \]
Equating the exact and approximate results, we have:

\[-2(1+x) + 3e^x = -2 - 2x + 3e^x\]

\[1 + x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 + \frac{1}{40}x^5\]

\[+ \frac{1}{240}x^6 + \frac{1}{1680}x^7 + \frac{1}{20160}x^8\]

Or,

\[3e^x \approx 1 + x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 + \frac{1}{40}x^5\]

\[+ \frac{1}{240}x^6 + \frac{1}{1680}x^7 + \frac{1}{20160}x^8\]

Dividing through by 3, we obtain

\[e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6\]

\[+ \frac{1}{7!}x^7 + \frac{1}{8!}x^8\]

The Taylor series approximation for $e^x$ is given as

\[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5\]

\[+ \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9 + \ldots\]

These series only differ for the $x^8$ (and higher) terms.
Picard's Method: (cont'd)

As noted when initially discussing Picard's method, this approach can yield accurate results, but the generality (and value) may suffer as Picard's method results in polynomial expansions.

It is worthwhile to consider Picard's method as an approach to solving first order ordinary differential equations, but other techniques (such as Runge-Kutta formulas) will probably be given preference due to ease of programming.