Without feelings of respect,  
what is there to distinguish men from beasts?  
— Confucius (6th century B.C.)

**Topic:** Second Ordinary Differential Equations

**Objectives:** (things you should know and/or be able to do)

**Standard Form of Second Ordinary Differential Equations:**

- General Form of a Constant Coefficient 2nd Order ODE:
  \[ a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x) \]

**Solutions of Second Ordinary Differential Equations:**

- Be able develop the *homogeneous* (or *complementary*) solution of a 2nd order ordinary differential equation (ODE) using \( y = e^{mx} \) as a trial solution.
- Be able develop the *particular* solution of a 2nd order ordinary differential equation (ODE) using the method of undetermined coefficients.

**Application of the Runge-Kutta Method:**

- Be able to apply the Runge-Kutta methods to numerically solve 1st order ordinary differential equations given a general 1st order relation of the form:
  
  1. Given \( a_0 \frac{dy}{dt} + a_1 y = r(t) \), we must rearrange to yield the following form:

  \[ \frac{dy}{dt} = \frac{1}{a_0} [r(t) - a_1 y] \]

  2. We also require the "initial" conditions: \( t_i \) and \( y_i = y(t_i) \), where \( t_i \) is usually set equal to zero (but does not have to be set to zero).

- Be able to apply the Runge-Kutta methods to numerically solve 2nd order ordinary differential equations given a general 2nd order relation of the form:
  
  1. Given \( a_0 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = r(t) \), we must rearrange to yield the following form:

  \[ \frac{d^2y}{dt^2} = \frac{1}{a_0} [r(t) - a_1 \frac{dy}{dt} - a_2 y] \text{ or } \frac{d^2y}{dt^2} = \frac{1}{a_0} [r(t) - a_1 \frac{dy}{dt} - a_2 y], \text{ where } v = \frac{dy}{dt} \]

  2. For 2nd order equations, we again require "initial" conditions, but we now include a first derivative term. In this case we require: \( t_i, y_i = y(t_i) \), and \( v_i = \frac{dy}{dt}(t_i) \) where again, \( t_i \) is usually set equal to zero (but does not have to be set to zero).
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Math Lecture 4a — Solutions of Second Ordinary Differential Equations

Unrest of spirit is a mark of life.
— Karl Menninger (1958)

**Topic:** Solutions of Second Ordinary Differential Equations

**Objectives:** (things you should know and/or be able to do)

- Be able develop the **homogeneous** (or **complementary**) solution of a 2nd order ordinary differential equation (ODE) using \( y = e^{mx} \) as a trial solution.
- Be able develop the **particular** solution of a 2nd order ordinary differential equation (ODE) using the method of undetermined coefficients.

**Lecture Outline:**

- **General Form of a Constant Coefficient 2nd Order ODE:**
  \[
  a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = r(x)
  \]

- **Homogeneous** (or **Complementary**) Solution of a Constant Coefficient 2nd Order ODE: (solution for \( r(x) = 0 \))
  - Use \( y = e^{mx} \) as a trial solution, solve the resulting polynomial for the \( a \) coefficients.
  - Possible solutions
    1. All roots real and distinct.
    2. Complex roots (solution in terms of \( \sin x \) and \( \cos x \) via the Euler formula).
    3. Real and repeated roots.
- **Particular** Solution of a Constant Coefficient 2nd Order ODE: (solution for \( r(x) \neq 0 \))
  - Method of Undetermined Coefficients—trial solution approach.
  - Method of Variation of Parameters—arbitrary coefficient approach, coefficients determined by simplification of derivatives, expanded using arbitrary coefficients.
- **Example Solution:** Problems 3.18 and 3.19, Schaum's Outline
  - Homogeneous (or complementary) solution by \( y = e^{mx} \) as a trial solution.
  - Particular solution by the Method of Undetermined Coefficients.

**Reading Assignment:** Focus on the solution of ordinary differential equations (ODE's)

- Review attached notes.
- Solution of Second Order Ordinary Differential Equations
- Chapters 2 (DE's), 3, (Linear ODE's), and 4 (Laplace Transforms) of the Schaum's Outline text, *Advanced Mathematics for Engineers and Scientists*, by Spiegel.

**Exercises:** For your own practice/skills building—do NOT turn in!

- **Solved Problems** in Chapter 3 of the Schaum's Outline text, *Advanced Mathematics for Engineers and Scientists*, by Spiegel.
  - Method of Undetermined Coefficients: 3.18, 3.19, 3.20, 3.21
  - Method of Variation of Parameters: 3.22
- **Solved Problem** 2.50 in Chapter 2 of the Schaum's Outline text (*Advanced Mathematics for Engineers and Scientists*) using the Method of Undetermined Coefficients.
- **Solved Problems** in the Schaum's Outline text, *Differential Equations*, 2nd edition, by Bronson
  - 2nd Order Linear, Homogeneous Eqs.: 8.1, 8.2, 8.3, 8.4, 8.7, 8.8, 8.10, 8.15, 8.16
  - Method of Undetermined Coefficients: 10.1, 10.2, 10.4, 10.6
Solution of Second Order Ordinary Differential Equations

(from Petroleum Engineering 620 Course Notes -- 1999)
Solution of Second Order Ordinary Differential Equations

The general form of a second order ordinary differential equation is given as:

\[ a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = R(x) \]

Homogeneous Solution

In the present work we will assume that the \( a_i(x) \) coefficients are constant. The case of \( a_i(x) \) coefficients is quite tedious and is typically left to numerical methods (although some forms can be solved by analytical techniques).

Assuming constant \( a_i(x) \) coefficients,

\[ a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x) \]

Letting \( y = e^{mx} \) be our "trial solution," we have

\[ y = e^{mx} \quad \frac{dy}{dx} = me^{mx} \quad \frac{d^2y}{dx^2} = m^2 e^{mx} \]

Substituting these results into the differential equation (above), we have

\[ a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = R(x) \]

Letting \( R(x) = 0 \), we obtain the "characteristic" equation, the solution of which is called the "homogeneous" solution. Substituting \( R(x) = 0 \) and dividing through by \( e^{mx} \), we obtain

\[ a_0 m^2 + a_1 m + a_2 = 0 \]

... the "characteristic" equation.
Homogeneous Solution: (cont'd)

Applying the quadratic formula

\[ m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0} \]

We have 3 possible cases for the "homogeneous" solution, these are:

Case 1: All roots are real and distinct

\[ y_h(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} \]

Case 2: Complex roots

In this case we have:

\[ m = a + bi \quad b = \frac{1}{2a_0} \sqrt{a_1^2 - 4a_0 a_2} \]

which yields the following form for the homogeneous solution:

\[ y_h(x) = A e^{(a+bi)x} + B e^{(a-bi)x} \]

\[ = A e^{ax} e^{bix} + B e^{ax} e^{-bix} \]

\[ = e^{ax} [A e^{bix} + B e^{-bix}] \]

From Euler's Formula, we have

\[ e^{imu} = \cos(mu) + i \sin(mu) \]

which yields:

\[ e^{bix} = \cos(bx) + i \sin(bx) \]

\[ e^{-bix} = \cos(bx) - i \sin(bx) \]
Homogeneous Solution: (cont'd)

Combining these relations gives us

\[ y_h(x) = e^{ax} \left[ A \cos(bx) + B \sin(bx) + E \cos(bx) - E \sin(bx) \right] \]

\[ = e^{ax} \left[ (A+E) \cos(bx) + (A-E) i \sin(bx) \right] \]

or

\[ y_h(x) = e^{ax} \left[ c_1 \cos(bx) + c_2 \sin(bx) \right] \]

where

\[ c_1 = A+E \]
\[ c_2 = (A-E) i \]

Case 2: Real and Repeated Roots \( m_1 = m_2 \)

\[ y_h(x) = (c_1 + c_2 x) e^{m_1 x} \]

Particular Solution

In the case of the "homogeneous" solution we let \( R(x) = 0 \). For the "particular" solution we use \( R(x) \neq 0 \). The combination of the homogeneous and particular solutions is called the "general" solution and is given by:

\[ y = y_h + y_p \]

Two common techniques for determining the particular solution are:

a. Method of Undetermined Coefficients
b. Method of Variation of Parameters

In these notes we will only demonstrate the Method of Undetermined Coefficients.
Method of Undetermined Coefficients

The Method of Undetermined Coefficients considers the components (or "elements") of the $R(x)$ term, where a trial "particular" solution is written based on the individual elements in $R(x)$. This trial solution is then substituted into the original differential equation and the coefficients of the trial solution are established via cancellation.

The following table is used to prescribe the trial solution—based on the elements in the $R(x)$ term.

Table: Trial Solutions for the Methods of Undetermined Coefficients

<table>
<thead>
<tr>
<th>Element in $R(x)$-term</th>
<th>Trial Solution &quot;Family&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^m$</td>
<td>$x^m, x^{m-1}, \ldots, x^1$</td>
</tr>
<tr>
<td>$\sin(qx)$</td>
<td>$\sin(qx), \cos(qx)$</td>
</tr>
<tr>
<td>$\cos(qx)$</td>
<td>$\sin(qx), \cos(qx)$</td>
</tr>
<tr>
<td>$e^{px}$</td>
<td>$e^{px}$</td>
</tr>
</tbody>
</table>

Rules:

1. For any element, use the entire family below that element.

   For Example: $x^2$; use $Ax^2 + Bx + C$

2. For any element that appears in both $R(x)$ and the homogeneous solution, multiply the entire family (in the trial solution) by $x$, or the lowest integer power of $x$ where no member of the new family is in the homogeneous solution.
Particular Solution: (cont'd)

2. (cont'd) An exception to this rule occurs for the presence of $e^x$ or $\sin(x)$ in the homogeneous solution. This exception does not require modification of a family containing the product $e^x \sin(x)$, unless that product also appears in the homogeneous solution.

Example:

\[ R(x) = 2x + 1 \]
\[ y_h(x) = c_1 + c_2 e^x + c_3 e^{-x} \]

Element $2x \rightarrow$ Family $\{x, 1\}$

Element $1 \rightarrow$ Family $\{1, \}$

As the second family (i.e., $\{1, \}$) is contained in the first family (i.e., $\{x, 1\}$), we can discard the second family.

However, the product $1 \cdot x$ occurs in $y_h(x)$, therefore the first family becomes $\{x^2, x, 1\}$ and the particular solution is given as:

\[ y_p(x) = Ax^2 + Bx \]


The second order ordinary differential equation is given by:

\[ \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 8x^2 + 12e^{-x} + 10 \sin(3x) \]
Particular Solution: (cont'd)

The "characteristic" equation is obtained from using \( y = e^{mx} \) in the differential equation where \( R(x) = 0 \). This result is given as

\[ m^2 + 2m + 4 = 0 \]

It appears from inspection that this equation has complex roots. We note that the quadratic coefficients are:

\[ a_0 = 1, \quad a_1 = 2, \quad a_2 = 4 \]

Recalling the quadratic formula, we have

\[ m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0} \]

Substituting

\[ m = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} \]

\[ = -1 \pm \frac{\sqrt{-12}}{2} \]

\[ = -1 \pm \frac{1 \sqrt{-12}}{2} \]

\[ = -1 \pm \frac{1 \sqrt{4 \sqrt{3}} \sqrt{-1}}{2} \]

Or

\[ m = -1 \pm \sqrt{3} i \]

where this is of the form:

\[ m = a \pm bi \quad a = -1 \quad b = \sqrt{3} \]

This gives

\[ y_p(x) = e^{-x} \left[ c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \right] \]
Particular Solution: (cont'd)

Recalling

\[ R(x) = 8x^2 + 12e^{-x} + 10\sin(3x) \]

\[ y_h(x) = e^{-x} \left[ c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \right] \]

The \( R(x) \)-term has the following elements:

Element \( 8x^2 \) → Family \( \{x^2, x, 1\} \)

Element \( 12e^{-x} \) → Family \( \{e^{-x}\} \)

Element \( 10\sin(3x) \) → Family \( \{\sin(3x), \cos(3x)\} \)

Note that there are no terms which appear in the families for the particular solutions that require modification. The \( e^{-x} \) and \( \sin(3x) \) terms follow the exception to Rule 2 (i.e., the presence of \( e^x \) or \( \sin(x) \) in the homogeneous solutions). Using these results, our trial particular solution is written as:

\[ y_p(x) = Ax^2 + Bx + C + De^{-x} + E\sin(3x) + F\cos(3x) \]

Taking the derivatives \( y_p'(x) \) and \( y_p''(x) \), we have

\[ y_p'(x) = 2Ax + B - De^{-x} + 3E\cos(3x) - 3F\sin(3x) \]

\[ y_p''(x) = 2A + De^{-x} - 9E\sin(3x) - 9F\cos(3x) \]

Recalling the original differential equation, we have

\[ \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 8x^2 + 12e^{-x} + 10\sin(3x) \]
Particular Solution: (cont'd)

Substitution of the trial particular solution into the original differential equation gives:

\[
\frac{2A}{2A} + \frac{z}{A} + \frac{8}{B} + \frac{4}{C} + \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x} = \frac{B}{x^2}
\]

Expanding:

\[
\frac{2A}{2A} + \frac{z}{A} + \frac{8}{B} + \frac{4}{C} + \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x} = \frac{B}{x^2}
\]

Collecting:

\[
4A + (4A + 4B)x + (2A + 2B + 2C) + (1 - 2A)\frac{D}{x} + (-9E - 6F + 4E)\sin(3x) + (-9F + 6E + 4F)\cos(3x)
\]

Equating:

\[
\begin{align*}
4A &= B \
4A + 4B &= 0 \
2A + 2B + 2C &= 0 \
-5E - 6F &= 10 \
6E - 5F &= 0 \
-30E - 36F &= 60 \
30E - 25F &= 0 \
-25E - 30F &= 50 \
36E - 30F &= 0
\end{align*}
\]

\[
\begin{align*}
A &= 2 \
B &= -2 \
C &= 0 \
D &= 12 \
E &= -50/61 \
F &= -60/61
\end{align*}
\]
Particular Solution (cont'd)

Summarizing:

\[ A = 2 \quad D = 4 \quad E = \frac{-50}{61} \quad \text{Error in Schaum's} \]

\[ B = -2 \quad F = \frac{-60}{61} \quad \text{text:} \quad E = -\frac{10}{61} \]

\[ C = 0 \quad F = -\frac{12}{61} \]

The resulting particular solution is given as

\[ y_p(x) = 3x^2 - 2x + 4e^{-x} - \frac{50}{61} \sin(3x) - \frac{60}{61} \cos(3x) \]

The complete general solution is given by

\[ y(x) = y_h(x) + y_p(x) \]

\[ = e^{-x} \left[ c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \right] \]

\[ + 3x^2 - 2x + 4e^{-x} - \frac{50}{61} \sin(3x) - \frac{60}{61} \cos(3x) \]


The second order ordinary differential equation is given by:

\[ \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = x \quad y(0) = 1, \quad y'(0) = 0 \]

The characteristic equation \((\lambda(x) = 0, \ y = e^{mx})\) is

\[ m^2 - 3m + 2 = 0 \]

which factors into

\[ (m-1)(m-2) = 0 \]

or \(m = 1, \ m = 2\)
Particular Solution: (cont'd)

The homogeneous solution is given as:

\[ y_h(x) = c_1 e^x + c_2 e^{-x} \]

The \( R(x) \) term for this case is given by

\[ R(x) = x \]

The \( R(x) \) term has the following elements - families:

Element \( x \) → Family \( \{x, 1\} \)

There are no conflicting terms in the family and the homogeneous solution, so we simply write the particular solution as:

\[ y_p(x) = Ax + B \]

Taking the derivatives \( y'_p(x) \) and \( y''_p(x) \), we have

\[ y'_p(x) = A \]
\[ y''_p(x) = 0 \]

Recalling the original differential equation

\[ \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = x \]

Substitution of the trial particular solution into the differential equation gives

\[ 0 - 3A + 2(Ax + B) = x \]

or

\[ (-3A + 2B) + 2Ax = 0 + x \]

which gives

\[ 2A = 1 \quad (-3A + 2B) = 0 \]
Particular Solution: (cont'd)

Where these equalities yield

\[ A = \frac{1}{2}, \quad B = \frac{3}{2} A \]

or

\[ A = \frac{1}{2}, \quad B = \frac{3}{4} \]

The resulting particular solution is given as

\[ y_p(x) = \frac{1}{2} x + \frac{3}{4} \]

The complete general solution is

\[ y(x) = y_h(x) + y_p(x) \]

\[ = c_1 e^x + c_2 e^{2x} + \frac{1}{2} x + \frac{3}{4} \]

The derivative of the general solution is

\[ y'(x) = c_1 e^x + 2c_2 e^{2x} + \frac{1}{2} \]

The initial conditions are:

\[ y(0) = 1, \quad y'(0) = 0 \]

Setting up \( y(x) \) and \( y'(x) \), we have

\[ c_1 e^{(0)} + c_2 e^{2(0)} + \frac{1}{2} (0) + \frac{3}{4} = 1 \]

\[ c_1 e^{(0)} + 2c_2 e^{2(0)} + \frac{1}{2} = 0 \]

or

\[ c_1 + c_2 = 1/4 \]
\[ c_1 + 2c_2 = -1/2 \]
Particular Solution: (cont'd)

Solving for $c_2$

$$
(1-2)c_2 = \frac{1}{4} - (-\frac{1}{2})
$$
$$
-c_2 = \frac{3}{4}
$$
or

$$
c_2 = -\frac{3}{4}
$$

Solving for $c_1$

$$
(2-1)c_1 = \frac{1}{2} - (\frac{1}{2})
$$
$$
c_1 = 1
$$

Therefore, the complete general solution is given as

$$
y(x) = e^x - \frac{3}{4} e^{2x} + \frac{1}{2} x + \frac{3}{4}
$$
Topic: Application of the Runge-Kutta Method

Objectives: (things you should know and/or be able to do)

- Be able to apply the Runge-Kutta methods to numerically solve 1st order ordinary differential equations given a general 1st order relation of the form:
  1. Given \( \frac{dy}{dt} + a_1y = r(t) \), we must rearrange to yield the following form:

\[
\frac{dy}{dt} = \frac{1}{a_0} [r(t) - a_1y]
\]

2. We also require the "initial" conditions: \( t_i \) and \( y_i = y(t_i) \), where \( t_i \) is usually set equal to zero (but does not have to be set to zero).

- Be able to apply the Runge-Kutta methods to numerically solve 2nd order ordinary differential equations given a general 2nd order relation of the form:

1. Given \( a_0 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2y = r(t) \), we must rearrange to yield the following form:

\[
\frac{d^2y}{dt^2} = \frac{1}{a_0} \left[ r(t) - a_1 \frac{dy}{dt} - a_2y \right] \text{ or } \frac{d^2y}{dt^2} = \frac{1}{a_0} \left[ r(t) - a_1v - a_2y \right], \text{ where } v = \frac{dy}{dt}
\]

2. For 2nd order equations, we again require "initial" conditions, but we now include a first derivative term. In this case we require: \( t_i, y_i = y(t_i) \), and \( v_i = v(t_i) \) where again, \( t_i \) is usually set equal to zero (but does not have to be set to zero).

Lecture Outline:

- Review attached notes.
  - Application of a Fourth-Order Runge-Kutta Formula (First and Second-Order ODE's)
    - Schaum's Solved Problem 2.48 (1st order ODE)
    - Schaum's Solved Problem 2.50 (2nd order ODE)

Reading Assignment:

Exercises: For your own practice/skills building—do NOT turn in!

- Modify the attached Runge-Kutta programs to solve the following problems:
  - Bessel's Differential Equation:
    \[
    t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - z^2)y = 0 \text{ or } \frac{d^2 y}{dt^2} = -\frac{y}{t} - y, \text{ where } \nu = \frac{dy}{dt} \text{ and } z = 0
    \]
    - Subject to the following initial conditions: \( t=0, y(0)=1, y'(0)=\nu(0)=0 \)
    - Exact solution: \( y(t)=J_0(t) \)

- Solution Plot for Bessel's Differential Equation:

![Bessel's Differential Equation \([J_0(t), z=0]\)
(Fourth Order Runge-Kutta Solution)](image)

- Use the polynomial approximations for \( J_0(t) \) from *Handbook of Mathematical Functions*, M. Abramowitz and I. Stegun, Dover Publications (1972); p. 369-370, Eqs. 9.4.1 and 9.4.3. Be sure to check your subroutine against Table 9.1, p. 390-396.
SUBROUTINE RKJST (X0, XN, Y0, YN, DDER, VDER)

C PROGRAM: RKJST (PETRO30) = (1st Order Runge-Kutta) C
C DATE: 3 OCTOBER 1994 C
C AUTHOR: THOMAS A. BLAsINGAME (TECHS AM UNIVERSITY) C
C CHARACTER: 100 NEEDED SUBROUTINE C
C CALL: CALL RKJST C
C CALL DESIGN (Y0, X0, XN, IEDER, L) C
C MESSAGES FOR THE OUTPUT FILES, DECISION HEADER WRITE(6,200) 
200 FORMAT(' 40X, 7X, 'Y(i)CALC', ') C
& 9X, 12X, 'Y(i)CALC', ') C
C MESSAGES INITIAL CONDITION, EXACT SOLUTION, AND ERROR 
VERROR = VEXACT(Y0) C
ERROR = (DSN137N(YERROR)(TERRORD)) C
WRITE(9,204) TERRORD, VERROR, ERROR C
C TRAP INTEGRAL 
INVL = 0.001
100 FORMAT(YERROR - TERRORD) C
C INITIALIZE TIME AND FUNCTION VARIABLES 
XCAL = NATION YCAL = NATION XDER = NATION YDER = NATION C
C MESSAGE TISK SOLUTION LOOP ON DATA C
C DO 1 I = 1, INVL C
1 XCAL = XCAL + INVL C
C CALL ENGINE'S EXACT SOLUTION FUNCTION 
VCAL = PFORD(XCAL, YCAL, XDER, YDER)
C UPGRADE GRID AND GRID VARIABLE 
NCAL = NCAL + 1.
C COMPUTE EXACT SOLUTION AND ERROR 
VERROR = VEXACT(YCAL) C
ERROR = (DSN137N(VERROR)(TERRORD)) C
WRITE(9,204) TERRORD, VERROR, ERROR C
C MESSAGE TISK SOLUTION, EXACT SOLUTION, AND ERROR 
WRITE(6,200) XCAL, YCAL, XDER, YDER, ERROR C
C SUBROUTINE CONFL C
C SUBROUTINE: CONFL -- OPEN FILES SUBROUTINE C
C DATE: 3 OCTOBER 1994 C
C AUTHOR: THOMAS A. BLAsINGAME (TECHS AM UNIVERSITY) C
C CHARACTER: 50 NEEDED SUBROUTINE C
C Pegas READ AND WRITE FILES SUBROUTINE C
C C GENERAL CASE: C
C READ SUBROUTINE #1, FORMAT STATEMENT #5 # VARIABLES C
C WRITE SUBROUTINE #1, FORMAT STATEMENT #5 # VARIABLES C
C FOR FILE NAME C
C C FOR M1R SYSTEM C
C C READ AND WRITE INTEGER # 1 C
C C READ AND WRITE INTEGER # 2 C
C C READ AND WRITE INTEGER # 3 C
C C WRITE*,111
111 FORMAT( 'PROGRAM RKJST', '/ ', & 
& ' = FIRST ORDER RUNGE-KUTTA PROGRAM', '/ ', & 
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& ' = FIRST ORDER RUNGE KUTTA PROGRAM', '/ ', & 
& ' = FIRST ORDER RUNGE KUTTA PROGRAM', '/ ', & 
& ' = FIRST ORDER RUNGE KUTTA PROGRAM'; / }}}}
Example: Problems 2.44 and 2.48, Schaum's Outline text, Advanced Mathematics for Engineers and Scientists.

\[ \frac{dy}{dt} = 2t + y, \quad y(0)=1, \quad \Delta t=0.5 \]

### Input Data:

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<th>( \Delta t )</th>
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### Output Data:

#### PROGRAM RC_POT
- FIRST ORDER RUNGE KUTTA METHOD

---

#### INPUT DATA DECK

- Initial y-value = 1.00000
- Initial x-value = 0.00000
- Final x-value = 10.00000
- x-value increment = 0.50000

#### OUTPUT DATA DECK

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Example: Problem 2.50, Schaum's Outline text, *Advanced Mathematics for Engineers and Scientists*.

\[ \frac{d^2y}{dt^2} = t + 3 \frac{dy}{dt} - 2y \]

where,

\[ y(0)=1, \quad y'(0)=0, \quad \Delta t=0.5 \]

**Input Data:**

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**Output Data:**

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Application of a
Fourth-Order Runge-Kutta Formula
(First and Second-Order ODE's)

(from Petroleum Engineering 620 Course Notes -- 1995)
Application of a 4th Order Runge-Kutta Formula

T. A. Blasingame

10 27 October 1995

Solution of a First Order Differential Equation:

The numerical solution of an ordinary differential equation is not concerned with whether or not the equation is linear, but rather with the behavior of any non-linearities, and/or discontinuity conditions. In other words -- numerical methods are blind; they simply operate according to procedures, we can apply them to virtually any problem, as long as we recognize their limitations.

The 4th order Runge-Kutta formula for a 1st order differential equation is given as follows:

\[ f(t, y) = \frac{dy}{dt} \]

\[ y(t_i + \Delta t) = y(t_i) + \frac{\Delta t}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] \]

where

\[ k_1 = \Delta t \cdot f(t_i, y_i) \]
\[ k_2 = \Delta t \cdot f(t_i + \frac{1}{2}\Delta t, y_i + \frac{1}{2}k_1) \]
\[ k_3 = \Delta t \cdot f(t_i + \frac{1}{2}\Delta t, y_i + \frac{1}{2}k_2) \]
\[ k_4 = \Delta t \cdot f(t_i + \Delta t, y_i + k_3) \]

Example: \( \frac{dy}{dt} = f(t, y) = 2t + y \quad y(t=0) = 1 \quad \) Problem 2.48 Schaum's

exact solution: \( y(t) = 3e^{2t} - 2t - 2 \quad "\text{Adv. Math for Env., & Scientists}" \)

\( t_0 = 0, y_0 = 1, \Delta t = 0.5 \quad y(0.5) = 3 \exp(1) - 2(0.5) - 2 = 1.946164 \quad \text{(exact)} \)

\[ k_1 = \Delta t \cdot f(t_0, y_0) = (0.5)[2(0) + 1] = 0.5 \]
\[ k_2 = \Delta t \cdot f(t_0 + \frac{1}{2}\Delta t, y_0 + \frac{1}{2}k_1) = (0.5)[2(0.25) + 1(0.25) + 1(0.5)] = 0.875 \]
\[ k_3 = \Delta t \cdot f(t_0 + \frac{1}{2}\Delta t, y_0 + \frac{1}{2}k_2) = (0.5)[2(0.25) + 1(0.25) + 1(0.875)] = 0.96875 \]
\[ k_4 = \Delta t \cdot f(t_0 + \Delta t, y_0 + k_3) = (0.5)[2(0.5) + 1(0.5) + 1(0.96875)] = 1.484375 \]

\[ y(0) + (0.5)) = y(0) + \frac{\Delta t}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] \]

or

\[ y(0.5) = 1 + \frac{1}{6} \left[ (0.5) + 2(0.875) + 2(0.96875) + 1.484375 \right] = 1.945313 \]
Solution of a Second Order Differential Equation:

The 4th order Runge-Kutta formula for a 2nd order differential equation is given as

\[
\frac{dy}{dt} = f(t, y, v) \quad \frac{d^2y}{dt^2} = q(t, y, v)
\]

\[
y(t_0 + dt) = y(t_0) + dt \cdot v(t_0) + \frac{dt}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\]

where,

\[
k_1 = dt \cdot f(t_0, y_0, v_0)
\]
\[
k_2 = dt \cdot f(t_0 + \frac{dt}{2}, y_0 + \frac{1}{2}k_1, v_0 + \frac{1}{2}k_1)
\]
\[
k_3 = dt \cdot f(t_0 + \frac{1}{2}dt, y_0 + \frac{1}{2}k_2, v_0 + \frac{1}{2}k_2)
\]
\[
k_4 = dt \cdot f(t_0 + dt, y_0 + k_3, v_0 + k_3)
\]

and,

\[
l_1 = dt \cdot q(t_0, y_0, v_0)
\]
\[
l_2 = dt \cdot q(t_0 + \frac{dt}{2}, y_0 + \frac{1}{2}k_1, v_0 + \frac{1}{2}k_1)
\]
\[
l_3 = dt \cdot q(t_0 + \frac{1}{2}dt, y_0 + \frac{1}{2}k_2, v_0 + \frac{1}{2}k_2)
\]
\[
l_4 = dt \cdot q(t_0 + dt, y_0 + k_3, v_0 + k_3)
\]

where we also have

\[
y_p(t_0 + dt) = y(t_0) + \frac{dt}{6}(l_1 + 2l_2 + 2l_3 + l_4)
\]
\[
v(t_0 + dt) = v(t_0) + \frac{dt}{6}(l_1 + 2l_2 + 2l_3 + l_4)
\]

Example:

\[
\frac{d^2y}{dt^2} = -2 \frac{dy}{dt} + 2y = t \quad y(0)=1, \ y'(0)=0
\]

\[
y(t, y, v) = \frac{dy}{dt} = t - 2y + 3v \quad t_0 = 0, \ y_0 = 1, \ y'_0 = 0, \ dt = 0.5
\]

\[
k_1 = \Delta t \cdot v_0 = (0.5)(0) = 0
\]
\[
k_2 = \Delta t \cdot (v_0 + \frac{1}{2}k_1) = (0.5)(0 + \frac{1}{2}(-1)) = -0.25
\]
\[
k_3 = \Delta t \cdot (v_0 + \frac{1}{2}k_2) = (0.5)(0 + \frac{1}{2}(-2)) = -0.40625
\]
\[
k_4 = \Delta t \cdot (v_0 + k_3) = (0.5)(0 + (-1.984375)) = -0.984375
\]

\[
l_1 = \Delta t \cdot q(t_0, y_0, v_0) = (0.5)[(0) - 2(0) + 3(0)] = -1
\]
\[
l_2 = \Delta t \cdot q(t_0 + \frac{1}{2}dt, y_0 + \frac{1}{2}k_1, v_0 + \frac{1}{2}k_1) = (0.5)[(0) + \frac{1}{2}(0.5) - 2(0) + \frac{1}{2}(-1)] = -1.625
\]
\[
l_3 = \Delta t \cdot q(t_0 + \frac{1}{2}dt, y_0 + \frac{1}{2}k_2, v_0 + \frac{1}{2}k_2) = (0.5)[(0) + \frac{1}{2}(0.5) - 2(-1) + \frac{1}{2}(-0.25)] = -1.984375
\]
\[
l_4 = \Delta t \cdot q(t_0 + dt, y_0 + k_3, v_0 + k_3) = (0.5)[(0) + \frac{1}{2}(0.5) - 2(-1) + \frac{1}{2}(-1)] = -1.984375
\]
\[ L_4 = \Delta t \left( q_i \Delta t + q_i + k_2 + v_i + k_4 \right) = \frac{\Delta t}{4} \left[ \frac{1}{2} (o_i) \left( \frac{1}{2} \Delta t \right) + (o_i) \left( -\frac{1}{2} \Delta t \right) \right] + 3 (o_i) (-1.968750) \]
\[ = -3.296875 \]

\[ q_i (0.5) = q_i (0) + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]
\[ = 1 + \frac{1}{6} (1 + 2 (-0.25) + 2 (-0.406250) + 0.1) \]
\[ = 0.617188 \]

\[ v_i (0.5) = \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]
\[ = 0 + \frac{1}{6} \left( 1 + 2 \left( -1.125 \right) + 2 \left( -1.968750 \right) + 3.296875 \right) \]
\[ = -1.914063 \]

The exact solution is given by

\[ y(t) = \exp(t) - \frac{3}{4} \exp(2t) + \frac{1}{2} t + \frac{3}{4} \]

\[ v(t) = \exp(t) - \frac{3}{2} \exp(2t) + \frac{1}{2} t = \frac{d}{dt} \left[ y(t) \right] \]

\[ q_i (0.5) = 0.610010 \]

\[ v_i (0.5) = -1.935701 \]