Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 4 — Pseudosteady-State Flow in a Circular Reservoir

The great tragedy of life is not that men perish, but that they cease to love.
— W. Somerset Maugham (1938)

**Topic:** Pseudosteady-State Flow in a Circular Reservoir

**Objectives:** (things you should know and/or be able to do)

- Be familiar with and be able to derive the single-phase, pseudosteady-state flow relations for compressible liquids in a radial flow system. In particular, you should be able to derive the following:
  - $p_r - p_{wf}$ flow relation:
    $p_r = p_{wf} + \frac{1}{c_r} \frac{qB\mu}{kh} \left[ \frac{r_e^2}{r_w^2} \ln \frac{r_e}{r_w} - \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} + s \right]
    $ (Well Centered in a Circular Reservoir)

  - $p - p_{wf}$ flow relation:
    $p = p_{wf} + \frac{1}{c_r} \frac{qB\mu}{kh} \left[ \ln \frac{r_e}{r_w} - \frac{3}{4} + s \right]
    $ (General Formulation)

where:

$\gamma = 0.577216...$ (Euler's Constant)

$C_A = $ Dietz "shape factor" (e.g., $C_A = 31.62$ for a well in a circular reservoir)

- $p(r,t)$ solution for pseudosteady-state flow conditions:
  $p_r = p_i - \frac{qB\mu}{2\pi kh} \left[ \ln \frac{r_e}{r} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} - \frac{3}{4} \right] - \frac{qB}{V_p c_t} t
  $ (Darcy Units)

$\frac{p_r}{p_i} = 1 - 141.2 \frac{qB\mu}{kh} \left[ \ln \frac{r_e}{r} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} - \frac{3}{4} \right] - 5.615 \frac{qB}{V_p c_t} t
  $ (Field Units)

where for field units we use $t$ in days, and $V_p$ in ft$^3$.

- $p(r_w,t)$ solution for pseudosteady-state flow conditions:
  $p_{wf} = p_i - 141.2 \frac{qB\mu}{kh} \left[ \ln \frac{r_e}{r_w} - \frac{3}{4} + s \right] - 5.615 \frac{qB}{V_p c_t} t
  $ (Field Units)

**Table of Units Conversions:** (for the equations given above)

<table>
<thead>
<tr>
<th>Variable</th>
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</tbody>
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- Be able to derive the $r(\bar{p})$ result for a radial system and be able to relate this result to reservoir performance.
- Be able to sketch pressure distributions during steady-state and pseudosteady-state flow conditions for a radial system.
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Lecture Outline:

- Development of pseudosteady-state flow relations (attached notes)
  - Material balance considerations.
  - $p_r-p_{wf}$ flow relation.
  - $\bar{p}-p_{wf}$ flow relation.
  - $p(r,t)$ solution for pseudosteady-state flow conditions.
- Review illustrative plots of pseudosteady-state flow performance.

Reading Assignment:

- Review the attached notes.
  - Derivation of the Pseudosteady-State Flow Relations for a Radial System.
  - Derivation of the Pseudosteady-State Flow Relations for a Radial System.
Exercises: For your own practice/skills building—do NOT turn in!

- You are to derive the following relations: (given in the attached notes)
  - $p_r - p_{wf}$ flow relation.
  - $\bar{p} - p_{wf}$ flow relation.
  - $p(r,t)$ solution for pseudosteady-state flow conditions.

- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- **Problem:**
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?

- **Assumptions and Limitations:**
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?

- **Practical Applications:**
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how *could* the solutions/results be used in practice?

- **Discussion:**
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.

- **Recommendations/Extensions:**
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Derivation of the Pseudosteady-State Flow Relations for a Radial System

- Physical Considerations
- Material Balance Considerations
- Pseudosteady-State Solutions of the Radial Flow Diffusivity Equation
  - $p_r - p_{wf}$ Formulation
  - $\bar{p} - p_{wf}$ Formulation
  - $\bar{p} - r$ Concept (i.e., $r(\bar{p})$)
  - $p(r,t)$ Solution for Pseudosteady-State Flow Conditions

(from Petroleum Engineering 412 Course Notes — 1997)
Physical Considerations

The physical concept of pseudosteady-state is defined as the condition where the pressure at all points in the reservoir changes at the same rate. Mathematically, this condition is given by:

\[ \frac{d}{dt} [p(r, t)] \bigg|_r = \text{constant} \]  

(1)

Physically, this condition is illustrated by

\[ q = \text{constant} \]

where,

- \( P_{wi} \): wellbore pressure at time \( t \)
- \( P_i \): average reservoir pressure at time \( t \)
- \( P_e \): external boundary pressure at time \( t \)

Objectives: (for pseudosteady-state flow conditions)

1. Derive a pressure change relation (i.e., \( \frac{dp}{dt} \)) using the material balance relation.
2. Derive a relation between the average reservoir pressure, \( P \), and the wellbore flowing pressure, \( P_{wi} \).
3. Derive a pressure-radius-time (i.e., \( p(r, t) \)) solution of the radial flow diffusivity equation.
Material Balance Considerations

Recalling the material balance relation for a slightly compressible liquid, we have

$$\bar{P} = P_i' - \frac{B}{NB_i'ct} Np$$  \hspace{1cm} (2)

or, noting that $NB_i'=V_P$ we obtain

$$\bar{P} = P_i' - \frac{B}{V_Pct} Np$$  \hspace{1cm} (3)

For a cylindrical reservoir, we have

$$Np = \phi h \pi (r_e^2 - r_w^2)$$  \hspace{1cm} (4)

Substituting Eq. 4 into Eq. 3 gives us

$$\bar{P} = P_i' - \frac{B}{\phi h \pi (r_e^2 - r_w^2)ct} Np$$  \hspace{1cm} (5)

Recalling the definition of the cumulative production, $Np$, we have

$$Np = \int_0^t q(t) \, dt$$  \hspace{1cm} (6)

Therefore,

$$\frac{dNp}{dt} = q$$  \hspace{1cm} (7)

Taking the derivative of Eq. 5 with respect to time

$$\frac{d\bar{P}}{dt} = -\frac{B}{\phi h \pi (r_e^2 - r_w^2)ct} q$$  \hspace{1cm} (8)

Note: all derivations are in "cary" Units unless otherwise noted.
Pseudosteady-State Flow Solutions for the Radial Flow Diffusivity Equation

The governing partial differential equation for flow in porous media is called the "diffusivity" equation. The diffusivity equation for a "slightly compressible liquid" is given (without derivation) as

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \frac{\phi \mu c_f}{k} \frac{dp}{dt} \quad (9)$$

The significant assumptions made in Eq. 9 are:
- slightly compressible liquid (constant compressibility)
- constant fluid viscosity
- single-phase liquid flow
- gravity and capillary pressure are neglected
- constant permeability
- horizontal radial flow (no vertical flow)

If we assume that the flowrate, \( q \), is constant, then \( dp/dt \) is also constant—hence \( dp/dt \) is constant as well. Assuming \( q \) is constant, then

$$\frac{dp}{dt} = \frac{dp}{dt} = -\frac{B}{\phi \pi \left(r_o^2 - r_h^2\right) c_f} \quad q = \text{constant} \quad (10)$$

Substituting Eq. 10 into Eq. 9 (we note that partial derivatives are now expressed as ordinary derivatives), this gives

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \left[ \phi \mu c_f \right] \left[ -\frac{B}{\phi \pi \left(r_o^2 - r_h^2\right) c_f} \right] q$$

or, reducing

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -\frac{q B \mu}{\pi k \left(r_o^2 - r_h^2\right)} \quad (11)$$
Defining
\[ c = \frac{gB\mu}{\pi h (r_e^2-r_w^2)} \] (12)

Substituting Eq. 12 into Eq. 11 we have
\[ \frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -c \] (13)

Separating
\[ d \left[ r \frac{dp}{dr} \right] = -cr \, dr \]

Integrating (indefinite integration)
\[ \int d \left[ r \frac{dp}{dr} \right] = -c \int r \, dr \]

Completing
\[ r \frac{dp}{dr} = -c \frac{r^2}{2} + c_1 \] (14)

Multiplying through Eq. 14 by \( 1/r \) gives us
\[ \frac{dp}{dr} = -c \frac{r}{z} + c_1 \frac{1}{r} \] (15)

For pseudosteady-state we assume a closed reservoir, that is
\[ \left[ \frac{dp}{dr} \right]_{r_e} = 0 \]

or
\[ \left[ \frac{dp}{dr} \right]_{r_e} = 0 = -c \frac{r_e}{z} + c_1 \frac{1}{r_e} \]

Solving for \( c_1 \) gives
\[ c_1 = \frac{c}{z} r_e^2 \] (16)
Substituting Eq. 16 into Eq. 15 gives

\[ \frac{dp}{dr} = \frac{c}{2} \left[ \frac{r^2}{r} \right] \]  

(17)

Multiplying through Eq. 17 by \( dr \) gives us

\[ dp = \frac{c}{2} \left[ \frac{r^2}{r} \right] \, dr \]

Integrating across the reservoir, we have

\[ \int_{r_{wf}}^{r} dp = \frac{c}{2} \int_{r_{w}}^{r} \left[ \frac{r^2}{r} \right] \, dr \]  

(18)

Completing the integration

\[ Pr - P_{wf} = \frac{c}{2} \left[ \frac{r^2 \ln(r)}{r_{w}} - \frac{r^2}{2} \right] \]

or

\[ Pr - P_{wf} = \frac{c}{2} \left[ \frac{r^2 \ln\left(\frac{r}{r_{w}}\right)}{r_{w}} - \frac{1}{2} \left( r^2 - r_{w}^2 \right) \right] \]  

(19)

Recalling Eq. 12

\[ c = \frac{q_{wf}}{\pi kh \left( r^2 - r_{w}^2 \right)} \]  

(20)

Substituting Eq. 20 into Eq. 19, we obtain

\[ Pr - P_{wf} = \frac{q_{wf}}{2\pi kh \left( r^2 - r_{w}^2 \right)} \left[ \frac{r^2 \ln\left(\frac{r}{r_{w}}\right)}{r_{w}} - \frac{1}{2} \left( r^2 - r_{w}^2 \right) \right] \]

(21)

Expanding through with the \( 1/(r^2 - r_{w}^2) \) term gives

\[ Pr - P_{wf} = \frac{q_{wf}}{2\pi kh \left( r^2 - r_{w}^2 \right)} \left[ \frac{r^2}{r_{w}} \ln\left(\frac{r}{r_{w}}\right) - \frac{1}{2} \left( \frac{r^2 - r_{w}^2}{r_{w}^2} \right) \right] \]

(21)

Eq. 21 is our final result (in "Darcy" units).
Development of a $\bar{p}$-$q_{wf}$ Relation—Pseudostate-State Flow

In this section we develop the relationship between the average reservoir pressure, $\bar{p}$, and the wellbore flowing pressure, $p_{wf}$. The definition of the average reservoir pressure is given as

$$\bar{p} = \frac{\int_{r_w}^{r} p_r \, dv}{\int_{r_w}^{r} dv} \quad (22)$$

and for a cylindrical reservoir, we have

$$v = \phi h \pi (r^2 - r_w^2) \quad (23)$$

$$dv = \phi h \pi (2r) \, dr \quad (24)$$

Substituting Eq. 24 into Eq. 22 gives

$$\bar{p} = \frac{\phi h 2\pi}{\phi h \pi (r^2 - r_w^2)} \int_{r_w}^{r} p_r \, r \, dr$$

which reduces to

$$\bar{p} = \frac{z}{(r^2 - r_w^2)} \int_{r_w}^{r} p_r \, r \, dr \quad (25)$$

Solving Eq. 21 for $p_r$ gives us

$$p_r = p_{wf} + \frac{q_{wf}}{2\pi kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{z} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} \right]$$
Substituting Eq. 26 into Eq. 25 gives

$$\bar{\rho}_r = \frac{z}{(r^2-x^2)} \int_{r_w}^{r} \left[ \rho_{\text{ef}} + \frac{q_8 u}{\pi k h} \frac{r_e^2}{(r_e^2-x^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{z} \frac{(r^2-x^2)}{(r^2-x^2)} \right] r \, dr \quad (27)$$

Separating

$$\bar{\rho}_r = \frac{z}{(r^2-x^2)} \int_{r_w}^{r} \rho_{\text{ef}} r \, dr$$

$$+ \frac{z}{(r^2-x^2)} \frac{q_8 u}{\pi k h} \frac{r_e^2}{(r_e^2-x^2)} \int_{r_w}^{r} r \ln \left[ \frac{r}{r_w} \right] \, dr$$

$$- \frac{z}{(r^2-x^2)} \frac{q_8 u}{\pi k h} \frac{1}{z} \frac{1}{z} \frac{r_e^2}{(r_e^2-x^2)} \int_{r_w}^{r} r^3 \, dr$$

$$+ \frac{z}{(r^2-x^2)} \frac{q_8 u}{\pi k h} \frac{r_w^2}{z} \frac{1}{z} \frac{1}{z} \frac{r_e^2}{(r_e^2-x^2)} \int_{r_w}^{r} r \, dr \quad (28)$$

Isolating terms and evaluating each integral, we have

$$\int_{r_w}^{r} r \, dr = \frac{1}{z} \frac{(r^2-x^2)}{(r^2-x^2)} \quad (29)$$

$$\int_{r_w}^{r} r^3 \, dr = \frac{1}{4} (r^4-x^4) \quad (30)$$

$$\int_{r_w}^{r} r \ln \left[ \frac{r}{r_w} \right] \, dr = ?$$

Obviously, the integral of the logarithm term will require a little work to resolve; we could simply look up the appropriate result in a suitable text—but deriving the required result will be enlightening.
Starting with the fundamental form of the logarithm integral, we have

\[ \int x \ln(x/c) \, dx \]

... integration by parts \( \quad \int udv = uv - \int vdu \)

\[ u = \ln(x/c) \quad \quad dv = x \, dx \]

\[ du = \frac{1}{x} \, dx \quad \quad v = \frac{1}{2} x^2 \]

Then

\[ \int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{2} \int x \, dx \]

Reducing

\[ \int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{4} x^2 \]

Therefore

\[ \int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = \left[ \frac{1}{2} r^2 \ln \left( \frac{r}{r_w} \right) - \frac{1}{4} r^2 \right]_{r_w}^{r} \]

\[ = \frac{1}{2} r^2 \ln \left( \frac{r}{r_w} \right) - \frac{1}{4} r^2 - \left[ \frac{1}{2} r_w^2 \ln \left( \frac{r_w}{r_w} \right) - \frac{1}{4} r_w^2 \right] \]

or finally, we have

\[ \int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = \frac{1}{2} r^2 \ln \left( \frac{r}{r_w} \right) - \frac{1}{4} (r^2 - r_w^2) \]
Substituting Eqs. 29-31 into Eq. 28 gives

\[
\bar{\rho} = \frac{z}{(r^2 - r_w^2)} \rho_{wf} \frac{1}{Z} (r^2 - r_w^2) \\
+ \frac{z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{r_e^2}{r^2 - r_w^2} \left[ \frac{1}{Z} \frac{r^2 \ln \left[ \frac{r}{r_w} \right]}{r_w} - \frac{1}{4} \frac{(r^2 - r_w^2)}{r_w} \right] \\
- \frac{z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{1}{Z(r^2 - r_w^2)} \frac{1}{4} \frac{(r^4 - r_w^4)}{r_w^2} \\
+ \frac{z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{Z(r^2 - r_w^2)} \frac{1}{Z} \frac{(r^2 - r_w^2)}{r_w^2}
\]

Reducing

\[
\bar{\rho} = \rho_{wf} + \frac{g_8 u}{2 \pi k h} \frac{z}{(r^2 - r_w^2)} \frac{r_e^2}{(r^2 - r_w^2)} \left[ \frac{1}{Z} \frac{r^2 \ln \left[ \frac{r}{r_w} \right]}{r_w} - \frac{1}{4} \frac{(r^2 - r_w^2)}{r_w} \right] \\
- \frac{g_8 u}{2 \pi k h} \frac{z}{(r^2 - r_w^2)} \frac{1}{Z(r^2 - r_w^2)} \frac{1}{4} \frac{(r^2 - r_w^2)(r^2 + r_w^2)}{r_w^2} \\
+ \frac{g_8 u}{2 \pi k h} \frac{z}{(r^2 - r_w^2)} \frac{1}{Z(r^2 - r_w^2)} \frac{r_w^2 (r^2 - r_w^2)}{Z}
\]

Collecting

\[
\bar{\rho} = \rho_{wf} + \frac{g_8 u}{2 \pi k h} \frac{r_e^2}{(r^2 - r_w^2)} \left[ \frac{r^2}{(r^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{Z} \right] \\
- \frac{g_8 u}{2 \pi k h} \frac{(r^2 + r_w^2)}{4(r^2 - r_w^2)} + \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{Z(r^2 - r_w^2)}
\]

or "finally"

\[
\bar{\rho} = \rho_{wf} \\
+ \frac{g_8 u}{2 \pi k h} \left[ \frac{r_e^2}{(r^2 - r_w^2)} \left[ \frac{r^2}{(r^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{Z} \right] - \frac{(r^2 + r_w^2)}{4(r^2 - r_w^2)} + \frac{r_w^2}{Z(r^2 - r_w^2)} \right] \\
(32)
\]
Eq. 32 (which is given in "Darcy" units) is our fundamental linking relation between the wellbore and average reservoir pressures during pseudosteady-state flow. However, \( \bar{p} \) (the average reservoir pressure at a given radius, \( r \)) is of little use except as a rigorous "linking" relation for pressures in the reservoir.

In contrast, if we consider \( \bar{p} \) (i.e., \( \bar{p} \) at \( r=r_e \)) we obtain the average reservoir pressure based on the entire reservoir volume. Such a result can be directly coupled with the material balance equation to develop a time-pressure relation for pseudosteady-state flow.

Evaluating Eq. 32 at \( r=r_e \) we have
\[
\bar{p} = \bar{p}_e = \bar{p}_{wf} + \frac{4 \pi \mu}{2 \pi k h} \left[ \frac{r_e^2}{r_e^2-r_w^2} \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} - \frac{1}{4} \left( \frac{r_e^2 + r_w^2}{r_e^2-r_w^2} \right) + \frac{r_w^2}{2(r_e^2-r_w^2)} \right] \quad (33)
\]

Assuming that \( r_e \gg r_w \), then
\[
\frac{r_e^2}{(r_e^2-r_w^2)} \approx 1 \quad ; \quad \frac{(r_e^2+r_w^2)}{(r_e^2-r_w^2)} \approx 1 \quad ; \quad \frac{r_w^2}{(r_e^2-r_w^2)} \approx 0
\]

Substituting these expressions into Eq. 33, we obtain
\[
\bar{p} = \bar{p}_{wf} + \frac{4 \pi \mu}{2 \pi k h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} - \frac{1}{4} \right]
\]

or
\[
\bar{p} = \bar{p}_{wf} + \frac{4 \pi \mu}{2 \pi k h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] \quad (34)
\]
Summarizing our results so far (using generalized units systems)

Pressure at any radius:

\[
P_r = P_{wf} + \frac{Q_B u}{c_r kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{(r_e^2 - r_w^2)}{2(r_e^2 - r_w^2)} \right]
\]  \hspace{1cm} (35)

Average Reservoir Pressure at any Radius:

\[
\bar{P}_r = P_{wf} + \frac{Q_B u}{c_r kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \left[ \frac{r^2}{(r_e^2 - r_w^2)^2} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \right] - \frac{(r_w^2 + r_e^2)}{4(r_e^2 - r_w^2)} + \frac{r_w^2}{2(r_e^2 - r_w^2)} \right]
\]  \hspace{1cm} (36)

Average Reservoir Pressure at \( r_e \) (volumetric average pressure):

\[
\bar{P} = P_{wf} + \frac{Q_B u}{c_r kh} \left[ \ln \left[ \frac{r_e}{r_w} \right] - \frac{3}{4} \right]
\]  \hspace{1cm} (37)

For a general reservoir geometry, Eq. 38 becomes

\[
\bar{P} = P_{wf} + \frac{Q_B u}{c_r kh} \left[ \frac{1}{2} \ln \left[ \frac{4 \pi A}{e^2 r_e^2 r_w^2 C_A} \right] \right]
\]  \hspace{1cm} (38)

where

\[ \gamma = 0.577216... \text{ Euler's Constant} \]
\[ C_A = \text{Dietz 'shape factor' (e.g. } C_A = 31.62 \text{ for circular reservoir}) \]

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\( \bar{r} - \bar{r} \) Concept

An interesting (and possibly useful) result is the concept of \( \bar{r} \), which would be the location of the average reservoir pressure, \( \bar{p} \). This development is only rigorously valid for a vertical well centered in a bounded circular reservoir. A graphical illustration of this concept is shown below.

Mathematically, \( \bar{r} \) is defined by equating the \( \bar{p} \) relation (Eq. 35) with the average reservoir pressure identity, \( \bar{p} \), (Eq. 37). Equating Eqs. 35 and 37 gives

\[
\frac{r_e^2}{(r_e^2-r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{(r_e^2-r_w^2)}{z(r_e^2-r_w^2)} = \ln \left[ \frac{r_e^2}{r_w^2} \right] - \frac{3}{4}
\]

We now assume \( \frac{r_e^2}{(r_e^2-r_w^2)} \approx 1 \) and \( \frac{r_w^2}{(r_e^2-r_w^2)} \approx 0 \) gives

\[
0 = \ln \left[ \frac{r}{r_w} \right] - \ln \left[ \frac{r_e}{r_w} \right] - \frac{r_e^2}{z(r_e^2-r_w^2)} + \frac{3}{4}
\]
Assuming that \( r_e \gg r_w \) (i.e., \((r_e^2-r_w^2)\approx r_e^2\)) and rearranging, we have

\[
\ln\left(\frac{r_D^2}{r_e^2}\right) - \frac{r_D^2}{2r_e^2} + \frac{3}{4} = 0
\]  \hspace{1cm} (39)

Defining a dimensionless radius, \( r_D \), we obtain

\[
r_D = \frac{r}{r_e}
\]  \hspace{1cm} (40)

Substituting Eq. 40 into Eq. 39 gives us

\[
\ln(r_D) - \frac{r_D^2}{2} + \frac{3}{4} = 0
\]  \hspace{1cm} (41)

Solving Eq. 41 for \( r_D \), we obtain

\[
r_D = 0.54928... \quad \text{(or } r = 0.54928... \times r_e)\]

### Development of a \( p(r,t) \) Relation for Pseudosteady-State Flow

Our last objective is to develop a \( p(r,t) \) relation for pseudosteady-state flow in a bounded circular reservoir. Recalling the material balance relation (Eq. 5) we have

\[
\vec{p} = p_i - \frac{8}{\phi h \pi (r_e^2-r_w^2) c_t} Np
\]  \hspace{1cm} (5)

For a constant flow rate, \( q \), we have

\[
Np = \int_0^t q(t) dt = qt
\]  \hspace{1cm} (43)

Substituting Eq. 43 into Eq. 5,

\[
\vec{p} = p_i - \frac{48}{\phi h \pi (r_e^2-r_w^2) c_t} t \quad \text{(Darcy units)}
\]  \hspace{1cm} (44)
Recalling the average reservoir pressure identity for a well centered in a bounded circular reservoir, we have

$$p = p_w + \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] \quad \text{(Darcy)} \quad (45)$$

Substituting Eq. 45 into Eq. 44 gives

$$p_w + \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] = p_i - \frac{q_8}{\phi \pi (r_e^2 - r_w^2) c t}$$

Rearranging,

$$p_i - p_w = \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] + \frac{q_8}{\phi \pi (r_e^2 - r_w^2) c t} \quad (46)$$

or

$$p_i - p_w = \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] + \frac{q_8}{\nu p c t} \quad (47)$$

Recalling the wellbore-reservoir pressure relation (Eq. 26), we have (upon slight rearranging)

$$p_i - p_w = \frac{q_0 u}{2\pi kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r}{r_w} \right) - \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{(r_e^2 - r_w^2)} \right) \right] \quad (48)$$

Subtracting Eq. 48 from Eq. 47 and solving for $p_i$ gives us

$$p_i = p_i - \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} - \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r}{r_w} \right) + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{(r_e^2 - r_w^2)} \right) \right]$$

$$- \frac{q_8}{\nu p c t} \quad (49)$$

Assuming that $r_e \gg r_w$ (i.e., $(r_e^2 - r_w^2) \approx r_e^2$) gives us

$$p_i = p_i - \frac{q_0 u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{3}{4} \right] - \frac{q_8}{\nu p c t} \quad (50)$$
Summarizing, we have the following relations in Darcy units:

\[ \eta = \eta_l - \frac{98u}{2\pi Kh} \left[ \ln \frac{r_e}{r_w} \right] - \frac{3}{4} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) \ln \frac{r_e}{r_w} + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) \right] - \frac{98}{V_p c t} \]  \hspace{1cm} (49)

and

\[ \eta = \eta_l - \frac{98u}{2\pi kh} \left[ \ln \frac{r_e}{r_w} + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{3}{4} \right] - \frac{98}{V_p c t} \]  \hspace{1cm} (50)

In Field Units we have

\[ \eta = \eta_l - 141.2 \frac{98u}{kh} \left[ \ln \frac{r_e}{r_w} \right] - \frac{3}{4} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) \ln \frac{r_e}{r_w} + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) \right] \]

\[ - \frac{5.615 \frac{98}{V_p c t}}{t} \] \hspace{1cm} (51) \hspace{1cm} \text{in days,} \ V_p \text{ in ft}^2

and

\[ \eta = \eta_l - 141.2 \frac{98u}{kh} \left[ \ln \frac{r_e}{r_w} + \frac{1}{2} \left( \frac{r_e^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{3}{4} \right] - \frac{5.615 \frac{98}{V_p c t}}{t} \] \hspace{1cm} (52) \hspace{1cm} \text{in days,} \ V_p \text{ in ft}^2

For \( t \) in hours we use 5.615/24 = 0.23395. \hspace{1cm} \text{Vp in ft}^2

Finally for conditions at the well, we have:

\[ \text{Darcy Units:} \quad \eta_w = \eta_l - \frac{98u}{2\pi Kh} \left[ \ln \frac{r_e}{r_w} \right] - \frac{3}{4} \right] - \frac{98}{V_p c t} \] \hspace{1cm} (53)

\[ \text{Field Units:} \quad \eta_w = \eta_l - 141.2 \frac{98u}{kh} \left[ \ln \frac{r_e}{r_w} \right] - \frac{3}{4} \right] - \frac{5.615 \frac{98}{V_p c t}}{t} \] \hspace{1cm} (54) \hspace{1cm} \text{in days,} \ V_p \text{ in ft}^2

\[ \text{in hours,} \ V_p \text{ in ft}^2 \]

\[ \eta_w = \eta_l - 141.2 \frac{98u}{kh} \left[ \ln \frac{r_e}{r_w} \right] - \frac{3}{4} \right] - 0.23395 \frac{98}{V_p c t} \] \hspace{1cm} (55)

Recall that the pore volume, \( V_p \), is given by:

\[ V_p = \phi h A (r_e^2 - r_w^2) = \phi h A \] \hspace{1cm} (4)
Illustrations of Pseudosteady-State Performance in Radial Flow Systems

(from Petroleum Engineering 412 Course Notes — 1997)

Petroleum Engineering 620
Fluid Flow in Reservoirs
Fig. 5.2 Radial flow under semi steady state conditions.

Fig. 5.4 Radial flow under steady state conditions.

Figure 6.1 The Calculated History of the Steady-State Pressure Distribution (after Hurst, 1934).

Figure 6.3 Pressure Distribution in a Closed Circular Reservoir Produced at a Constant Pressure (after Hurst, 1934).

Figure 4 - Reservoir Pressure Distribution during Log Linear Rate Transient Flow Drawdown.
Figure 7 - Reservoir Pressure Distribution during Constant Wellbore Pressure Transient Flow Drawdown

- $r_1 = 32.2$ ft
- $r_2 = 88.4$ ft
- $r_3 = 195$ ft
- $r_4 = 413$ ft

1. $t_1 = 1.77$ hr
2. $t_2 = 13.3$ hr
3. $t_3 = 64.6$ hr
4. $t_4 = 291.7$ hr
Figure 52 - Reservoir Pressure Distribution During Constant Rate Post-Transient Flow Drawdown, Homogeneous Reservoirs
Figure 57 - Reservoir Pressure Distribution During Constant Wellbore Pressure Post-Transient Flow Drawdown, Homogeneous Reservoirs
Figure 3.3 -- Variation of radius corresponding to the average reservoir pressure with time.
(Note: These are numerical simulation results, not analytical results—hence the postulate that $r(p)/r_e = 0.54928 ...$ should be both correct and unique.)

References:


Petroleum Engineering 620
Fluid Flow in Reservoirs
Determination of Average Reservoir Pressure From Build-Up Surveys

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ABSTRACT

A method for determining average reservoir pressure is presented, which is simpler to apply than that devised by Matthews, Brons and Hazebroek. For bounded reservoirs, identical results are obtained if stabilized-flow conditions prevail. The present method yields inferior results in the transient state. The method can, with a slight modification, also be used for water-drive reservoirs.

INTRODUCTION

In the method proposed by Matthews, Brons and Hazebroek for determining average reservoir pressure in a multi-well reservoir, the cumulative production (the production time of each well) enters twice: once when the build-up is plotted against ln (t + Δt)/Δt to arrive at p*, and a second time when the correction, p*-p is determined with one of the several formulas for differently shaped drainage areas.

Once a steady state has been attained, however, the previous production history should be immaterial. The same pressure distribution could have been arrived at after different cumulative productions of the individual wells. In principle, therefore, it should be possible to determine average pressures without referring to cumulative productions.

In the following paragraphs an expression is presented for the difference between the pressure in a producing well and the average pressure of its drainage area. Then the build-up time needed to overcome this difference is indicated.

CASE OF A CIRCULAR DRAINAGE AREA AND A CENTRAL WELL

PRESSURE DISTRIBUTION BEFORE SHUT-IN

The general differential equation of radial flow (see Muskat, Eq. 10.2) may be written

$$\frac{k}{\mu} 2\pi h \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = 2\pi h \phi r^2 \frac{\partial p}{\partial t}. \quad (1)$$

In the steady state, the rate of production of the well is equal to the rate of expansion of the fluid contained in the drainage area; thus

$$q = -\pi r^2 \phi \frac{\partial p}{\partial t} \quad , \quad \ldots \quad (2)$$

and therefore, combining Eqs. 1 and 2 we have

$$\frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = -\frac{q\mu}{2\pi kh} \quad , \quad \ldots \quad (3)$$

which can be integrated to

$$\frac{\partial p}{\partial r} = -\frac{q\mu}{2\pi kh} \left( r + \frac{C_i}{r^2} \right) \quad , \quad \ldots \quad (4)$$

The boundary condition can be introduced as

$$\frac{\partial p}{\partial r} = 0 \quad at \quad r = r_s, \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

so that Eq. 4 can be rewritten

$$\frac{\partial p}{\partial r} = \frac{q\mu}{2\pi kh} \left( \frac{1}{r} - \frac{r}{r_s^2} \right) \quad , \quad \ldots \quad \ldots \quad (6)$$

A second integration gives

$$p = \frac{q\mu}{2\pi kh} \left( \ln r - \frac{r^2}{r_s^2} \right) + C_i \quad , \quad \ldots \quad \ldots \quad (7)$$

AVERAGE PRESSURE OF THE DRAINAGE AREA

The average pressure can be expressed as

$$p = \frac{1}{\pi r_s^2} \int_0^{r_s} p \ 2\pi r dr = \frac{2}{r_s^2} \int_0^{r_s} p r dr \quad , \quad \ldots \quad \ldots \quad (8)$$

Substitution of Eq. 7 and integration yields

$$p \approx \frac{q\mu}{\pi kh} \left\{ \frac{1}{2} \ln \frac{r_s}{2} - \frac{3}{8} \right\} + C_i \quad , \quad \ldots \quad \ldots \quad (9)$$

BUILD-UP

If Eq. 7 is applied to the wellbore at r_w, and if C_i is eliminated by the combination with Eq. 9, we have for the pressure at the well under steady-state conditions

$$p \approx p - \frac{q\mu}{2\pi kh} \left( \ln \frac{r_s}{r_w} - \frac{3}{4} \right) \quad , \quad \ldots \quad \ldots \quad (10)$$

After shut-in, and as long as the physical boundaries of the entire reservoir have no influence, the pressure increases according to the well-known expression

$$\Delta p = \frac{q\mu}{4\pi kh} E \left( -\frac{\phi \mu r_w^2}{4k \Delta t} \right) \quad , \quad \ldots \quad \ldots \quad (11)$$

Original manuscript received in Society of Petroleum Engineers office March 22, 1966. Revised manuscript of SPE 1166 received June 26, 1966.

References given at end of paper.

Discussion of this and all following technical papers is invited. Discussion in writing (three copies) may be sent to the office of the Journal of Petroleum Technology. Any discussion offered after Dec. 31, 1966, should be in the form of a new paper. No discussion should exceed 10 per cent of the manuscript being discussed.

AUGUST, 1966
which, for \( \phi \mu \epsilon r_c \sqrt{4k \Delta t} < 0.01 \), can be approximated by

\[
\Delta p_\ast = -\frac{q_\mu}{4\pi k h} \left( 0.5772 + \ln \frac{\phi \mu \epsilon r_c}{4k \Delta t} \right) . \tag{12}
\]

Addition of Eqs. 10 and 12 yields an expression for the straight-line part of the build-up curve if pressure is plotted against \( \ln \Delta t \), thus

\[
p_\ast (\Delta t) = \frac{q_\mu}{4\pi k h} \left[ 1n \frac{r^2}{r_w^2} - \frac{3}{2} + 0.5772 \right] + \ln \frac{\phi \mu \epsilon r_c}{4k \Delta t} . \tag{13}
\]

The well radius cancels out, as a skin-factor would have done if it had been introduced; therefore

\[
p_\ast (\Delta t) = \frac{q_\mu}{4\pi k h} \left[ 1n \frac{\phi \mu \epsilon r_c^2}{4k \Delta t} - 0.9228 \right] . \tag{14}
\]

On this straight line, or on its extrapolation, the value of \( \bar{p} \) is found at the point where the form in brackets vanishes. For this purpose, the line should be read at

\[
\Delta t_y = \frac{\phi \mu \epsilon r_c^2}{4k \left( 10.0/1.4 \right) \phi \mu A} = 31.6 k . \tag{15}
\]

\( k/\mu \) is known from the slope of the build-up curve, and \( \phi, c \) and \( A \) can be determined by known methods.

**DRAINAGE AREAS HAVING DIVERSE SHAPES AND WELL POSITIONS**

Eq. 15 is applicable only to a circular drainage area and a central well. Although similar expressions could be obtained in a like manner for other drainage areas, it is much more convenient to use some of the figures presented in Ref. 1. The curves of Figs. 2 through 8 in Ref. 1 become straight lines for sufficiently large values of \( t \). These straight-line parts can be represented by equations such as

\[
\frac{p_\ast - \bar{p}}{q_\mu/4\pi k h} = \ln \frac{C_4 kt}{\phi \mu A} , \tag{16}
\]

where \( C_4 \) is a constant dependent on the shape of the area and on the well position.

\( p_\ast \) is a point at \( \Delta t = \infty \) on a straight-line extrapolation of the build-up plot vs \( \ln (t + \Delta t)/\Delta t \). \( \bar{p} \) can be considered as a point at \( \Delta t_y = 0 \) on a straight-line extrapolation of the build-up plot vs \( \ln t/\Delta t \), since the early part of the build-up can be approximated by

\[
p_\ast (\Delta t) = p_\ast - \frac{q_\mu}{4\pi k h} \ln \frac{t + \Delta t}{\Delta t} \approx p_\ast - \frac{q_\mu}{4\pi k h} \ln \frac{t}{\Delta t} . \tag{17}
\]

Extrapolating the straight-line part of this type of plot is equivalent to using the latter expression beyond the validity of the above approximation. By definition, the point \( (p, \Delta t_y) \) is on the straight-line extrapolation, and this pair of values should satisfy Eq. 17, which leads to

\[
\bar{p} = p_\ast - \frac{q_\mu}{4\pi k h} \ln \frac{t}{\Delta t} . \tag{18}
\]

Combination of Eqs. 16 and 18 yields

\[
\Delta t_y = \frac{\phi \mu A}{C_4 k} , \tag{19}
\]

which is a generalized form of Eq. 15.

The shape factor \( C_4 \) can be obtained from Ref. 1 by the consideration that Eq. 16 is reduced to

\[
\frac{p_\ast - \bar{p}}{q_\mu/4\pi k h} = \ln C_4 \text{ for } \frac{kt}{\phi \mu A} = 1 . \tag{20}
\]

Therefore, \( \ln C_4 \) can be read from the straight-line parts of the curves in Figs. 2 through 8 of the reference, or from their extrapolations, at the abscissa value of \( kt/\phi \mu A = 1 \). The results are listed in Table 1.

For smaller values of \( t \), where the curves of Figs. 2 through 8 of Ref. 1 are not straight lines, Eq. 16 does not indicate an actual curve. In those cases, the \( \bar{p} \) following from Eq. 19, which depends on Eq. 16 representing an actual curve, cannot be corrected. The range of validity of the present method can therefore be established by observing where the graphs in Figs. 2 through 8 of Ref. 1 start to deviate from straight lines. The limits of validity thus found are presented in the last column of Table 1.

**APPLICATION OF METHOD**

1. Divide the reservoir on a map into drainage areas proportional to out take rates per unit sand thickness of wells, as prescribed in Ref. 1.

2. Plot pressure build-ups against \( \ln \Delta t \), and determine \( k/\mu \) from slopes.

3. Determine \( t, \phi, c \) and \( A \) by generally known methods.

4. Check applicability of method by comparing \( kt/\phi \mu A \) with required values in the table.

5. Select \( C_4 \) from the table and read straight-line parts of pressure build-ups or their extrapolations at \( \Delta t_y = \phi \mu A/C_4 k \) for \( \bar{p} \).

**DISCUSSION**

In a discussion with Matthews, Brons and Hazebroek and with other build-up experts, it was pointed out that the Matthews-Brons-Hazeboek method worked accurately also in the transient state, although the assumption of steady state is used in the division into drainage areas. The present method relies more heavily on this assumption, and, in the transient state, becomes increasingly inaccurate for smaller production times. Identical results are obtained in the steady state, and in this region the present method may be preferred for its simplicity.

**ADAPTATION TO WATER-DRIVE RESERVOIRS**

**COMPLETE WATER DRIVE**

Under complete water drive the pressure at any point tends to become constant. Drainage areas, defined in the usual sense, have very irregular shapes, each one having to be in contact with the advancing water front. In this case it is preferable to divide the reservoir as regularly as possible into what, in accordance with D. R. Horner, can be called associated reservoir areas allocated to the wells.

In a regularly drilled field the associated areas can be approximated by circles. The pressure distribution can be irregular along the boundary of such an area, but it should be constant with time. The pressure at any point in the area will be the sum of a pressure \( p_0 \) due to the pressure distribution at the boundary if the well were not producing, and a negative pressure \( p \), due to the withdrawal by the well, zero pressure being assumed along the boundary.

According to Frank and v. Mises, \( (\text{Eq. 12}),\)

\[
P_{\infty} = \bar{p} , \tag{21}
\]

As the total flow originates from the boundary, the
TABLE 1

<table>
<thead>
<tr>
<th>In bounded reservoirs</th>
<th>Stabilized conditions ( \frac{\phi}{\mu \text{CA}} )</th>
<th>In water-drive reservoirs</th>
<th>Stabilized conditions ( \frac{\phi}{\mu \text{CA}} )</th>
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</thead>
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<tr>
<td>( \ln C_1 )</td>
<td>( C_1 )</td>
<td>( \phi )</td>
<td>( \frac{\phi}{\mu \text{CA}} )</td>
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<td>0.5</td>
<td>3.22</td>
</tr>
</tbody>
</table>

In reservoirs of unknown production character
second pressure field is described by

\[ p_i = \frac{q\mu}{2\pi k h} \ln \frac{r}{r_i} \]  \hspace{1cm} (22) \]

Integration and division by \( c r_i \) lead to

\[ \bar{p}_i = -\frac{q\mu}{4\pi k h} \]  \hspace{1cm} (23) \]

Application of Eq. 22 to the well radius gives

\[ p_w = \frac{q\mu}{2\pi k h} \ln \frac{r_w}{r_i} \]  \hspace{1cm} (24) \]

The well pressure before closing in can be expressed as

\[ p_w = \bar{p} + \frac{q\mu}{4\pi k h} \left( 1 + 2 \ln \frac{r_w}{r_i} \right) \]  \hspace{1cm} (25) \]

and after closing in as

\[ p_w(\Delta t) = \bar{p} + \frac{q\mu}{4\pi k h} \left( 1 + 2 \ln \frac{r_w}{r_i} \right) \]

\[ - 0.5772 - \ln \frac{\phi \mu c r_i}{4k \Delta t} \]  \hspace{1cm} (26) \]

from which it follows that

\[ \Delta t = \frac{\phi \mu c r_i^2}{4k \varepsilon \omega - c} = \frac{\phi \mu c r_i^2}{6.1 k} - \frac{\phi \mu c A}{19.1 k} \]  \hspace{1cm} (27) \]

APPLICATION OF PROPOSED METHOD TO RESERVOIRS OF UNKNOWN PRODUCTION CHARACTER

The usual purpose of average pressure determinations is to calculate by material balance the strength of the water drive. It may therefore appear that the problem has entered a vicious circle. As shown in the Appendix, bounded reservoirs also can be analyzed after division into associated reservoir areas, rather than into drainage areas. The same constant \( C_s = 31.6 \) for a circle is applicable in the latter case. When there is doubt about the amount of water drive, it is recommended that the reservoir be divided regularly into associated reservoir areas and that the intermediate value \( C_s = 25 \) be used for the circularized area. The inevitable range of uncertainty from 19.1 to 31.6 should lead to errors no worse than those from the uncertainty in the compressibility. These errors in \( \Delta t_b \) have but little influence on \( \bar{p} \), because \( \bar{p} \) is read from a semi-logarithmic plot.

NOMENCLATURE

The formulas are suitable for any consistent system of units. The units indicated below will serve as an example.

\[ A \quad \text{drainage area or associated reservoir area, sq cm} \]

\[ C_s \quad \text{constant dependent on shape of area, position of well and on production character} \]

\[ C_{1}, C_{2} \quad \text{integration constants, atm} \]

\[ E_i \quad \text{exponential integral defined by} \]

\[ E_i(-x) = \int_{\infty}^{x} e^{-u} du \]

\[ c \quad \text{effective compressibility of reservoir fluid, atm}^{-1} \]

\[ h \quad \text{sand thickness, cm} \]

\[ k \quad \text{permeability, darcies} \]

\[ p \quad \text{reservoir pressure dependent on place and time, atm} \]

\[ p_w \quad \text{well pressure, atm} \]

\[ \bar{p} \quad \text{average pressure in drainage area or associated reservoir area, atm} \]

\[ p^* \quad \text{closed-in pressure linearly extrapolated on plot against \( 1n(1 + \Delta t)/\Delta t \) for infinite closed-in time, atm} \]

\[ p_1 \quad \text{well field due only to pressure distribution on boundary, atm} \]

\[ p_2 \quad \text{pressure field due only to withdrawal by well, atm} \]

\[ p_v \quad \text{pressure field due only to oil expansion, atm} \]

\[ \Delta p_2 \quad \text{pressure increase at the well since closing in, atm} \]

\[ \frac{dp}{dt} \quad \text{general rate of pressure drop in stabilized reservoir, atm sec}^{-1} \]

\[ q \quad \text{production rate of well before closing in, cc sec}^{-1} \]

\[ q_w \quad \text{rate of oil expansion in associated reservoir area, cc sec}^{-1} \]

\[ r \quad \text{distance to well center, cm} \]

\[ r_o \quad \text{outer radius of circular drainage area or associated reservoir area, cm} \]

\[ r_w \quad \text{well radius, cm} \]

\[ i \quad \text{corrected production time, defined as cumulative well production divided by rate before closing-in, sec} \]

\[ \Delta t \quad \text{closed-in time, sec} \]

\[ \Delta t_b \quad \text{defined by} \quad \Delta t_b \neq \frac{p_1}{p} \text{ on linear extrapolation of plot against \( 1n(1 + \Delta t)/\Delta t \), sec} \]

\[ \mu \quad \text{viscosity of reservoir fluid, cp} \]

\[ \phi \quad \text{porosity, fraction} \]

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REFERENCES


APPENDIX

A CIRCULAR ASSOCIATED RESERVOIR AREA IN A BOUNDED RESERVOIR

We introduce \( q_w \) as the rate at which oil is expanding within the associated area because of the general pressure decline \( \frac{dp}{dt} \). This rate will differ from the production rate \( q \) of the well to the extent that the associated area differs from the drainage area. In the case of a non-produc-
ducing (observation) well, the outward flow through the boundary of the associated area would be \( q_w \).

The pressure field is now split into three simpler ones: \( p_i \) due to the pressure distribution along the boundary at
the time of closing in, while this distribution is considered constant with time; \( \rho \) (negative) caused by the withdrawal by the well, constant zero pressure being assumed along the boundary; and \( \rho \), as it would occur for a uniform rate of pressure drop, if the well were not producing and if the pressure everywhere along the boundary just reached zero.

The first two fields are identical to \( \rho \) and \( \rho \), discussed for water-drive reservoirs. Therefore,

\[
\rho_{w} = \rho \quad \cdots \cdots \cdots \cdots \cdots \quad (21)
\]

\[
\bar{\rho} = -\frac{q_\mu}{4\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (23)
\]

\[
\rho_{w} = \frac{q_\mu}{2\pi k h} \ln \frac{r_w}{r_i} \quad \cdots \cdots \cdots \cdots \cdots \quad (24)
\]

The third field is governed by

\[
\frac{k}{\mu} 2\pi \hbar \frac{\partial}{\partial r} \left( r \frac{\partial \rho}{\partial r} \right) = 2\pi \hbar \phi \frac{\partial \rho}{\partial r} \quad \cdots \cdots \cdots \cdots \cdots \quad (28)
\]

which can be rewritten as

\[
\frac{\partial}{\partial r} \left( r \frac{\partial \rho}{\partial r} \right) = -\frac{q_\mu}{\pi k h} \frac{r}{r_i} \quad \cdots \cdots \cdots \cdots \cdots \quad (29)
\]

Integration gives

\[
\frac{\partial \rho}{\partial r} = -\frac{q_\mu}{2\pi k h} + \frac{C_i}{r} \quad \cdots \cdots \cdots \cdots \cdots \quad (30)
\]

The condition at the inner boundary

\[
\frac{\partial \rho}{\partial r} = 0 \quad \text{for} \quad r = 0 \quad \cdots \cdots \cdots \cdots \cdots \quad (31)
\]

reduces Eq. 30 to

\[
\frac{\partial \rho}{\partial r} = -\frac{q_\mu}{2\pi k h r} \quad \cdots \cdots \cdots \cdots \cdots \quad (32)
\]

Further integration leads to

\[
\rho = -\frac{q_\mu}{4\pi k h} r + C_i \quad \cdots \cdots \cdots \cdots \cdots \quad (33)
\]

which, after introduction of the outer boundary condition

\[
\rho = 0 \quad \text{for} \quad r = r_o \quad \cdots \cdots \cdots \cdots \cdots \quad (34)
\]

becomes

\[
\rho = -\frac{q_\mu}{4\pi k h} \frac{r_o^2 - r^2}{r_i^2} \quad \cdots \cdots \cdots \cdots \cdots \quad (35)
\]

The average value is then found to be

\[
\bar{\rho} = \frac{1}{r_i} \int_{r}^{r_i} \rho \, 2\pi r dr = \frac{q_\mu}{8\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (36)
\]

and the value at the well

\[
\rho_w = \frac{q_\mu}{4\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (37)
\]

Addition of the three fields yields the average pressure

\[
\bar{p} = p_\rho - \frac{q_\mu}{4\pi k h} + \frac{q_\mu}{8\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (38)
\]

and the well pressure before shut-in

\[
p_w = \bar{p} + \frac{q_\mu}{2\pi k h} \ln \frac{r_w}{r_i} + \frac{q_\mu}{4\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (39)
\]

or

\[
p_w = \bar{p} + \frac{q_\mu}{4\pi k h} \left( 1 + 2 \ln \frac{r_w}{r_i} \right) + \frac{q_\mu}{8\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (40)
\]

After closing in, the well pressure can now be expressed by

\[
p_w(\Delta t) = \bar{p} + \frac{q_\mu}{4\pi k h} \left\{ 1 + 2 \ln \frac{r_w}{r_i} - 0.5 \right\}
\]

\[
- \ln \left( \frac{4\mu c r_i^2}{4k \Delta t} \right) + \frac{q_\mu}{8\pi k h}
\]

\[
\begin{align*}
\bar{p} + \frac{q_\mu}{4\pi k h} \left( \frac{3}{2} + 2 \ln \frac{r_w}{r_i} \right) - 0.5772 \\
- \ln \left( \frac{4\mu c r_i^2}{4k \Delta t} \right) + \frac{q_\mu}{8\pi k h}
\end{align*}
\]

\[
\begin{align*}
\Delta p_{\text{average}} = & \frac{4\mu c A}{31.6 k} \quad \cdots \cdots \cdots \cdots \cdots \quad (41)
\end{align*}
\]

The bracket form again vanishes for

\[
\Delta p_{\text{average}} = \frac{4\mu c A}{31.6 k}
\]

and the true average may be expressed as

\[
p_{\text{average}} = \bar{p} + \Delta p_{\text{average}} = \frac{(q_\mu - q) \mu}{8\pi k h} \quad \cdots \cdots \cdots \cdots \cdots \quad (42)
\]

The indicated correction is but a minor one. Moreover, in the determination of the average pressure of the entire reservoir, it cancels out because

\[
\Sigma q = 2q \quad \cdots \cdots \cdots \cdots \cdots \quad (43)
\]

***
Properties of Homogeneous Reservoirs, Naturally Fractured Reservoirs, and Hydraulically Fractured Reservoirs From Decline Curve Analysis

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ABSTRACT

This paper proposes analysis techniques for post-transient flow at constant bottomhole pressure. Rate-time decline curves approximate this flow regime. Reservoir characteristics for homogeneous reservoirs, vertically-fractured reservoirs, and naturally-fractured reservoirs can be obtained using these techniques. These analysis techniques are based on the exponential, post-transient, constant-pressure radial flow solutions for each case. We show that theory predicts a linear relation between log (rate) and time for these curves. Thus, a straight line on a semi-log rate vs. time plot may be the line predicted by the analytical solution for that reservoir. If so, important formation characteristics can be estimated analytically. Reservoir pore volume is determined directly while other reservoir characteristics are calculated indirectly. These new techniques are a very powerful extension of transient well testing.

DESCRIPTION OF PROPOSED DECLINE CURVE ANALYSIS METHODS

The need for accurate estimates of formation properties from decline curves led us to develop analysis techniques for post-transient production at constant bottomhole pressure (BHP). We have developed methods for homogeneous reservoirs, naturally-fractured reservoirs, and vertically-fractured reservoirs; these methods are derived in the Appendices and illustrated with examples in this paper. All cases exhibit an exponential rate decline for post-transient flow conditions; however, the reservoir characteristics which can be determined vary from case to case. Knowledge of these reservoir characteristics, which include drainage area, reservoir permeability, fracture half-length, natural fracture porosity volume and storage, and the natural fracture dimensionless matrix/fracture permeability ratio give insight into well spacing efficiency, the need for reservoir development, and well stimulation efficiency.

Each of the methods employs the rate-time plot used in decline curve analysis, and each was rigorously developed from the constant rate pseudo-steady-state flow equation using superposition. These methods are also exact in a material balance sense. This means the same results would be obtained from these methods as would be obtained from more tedious average reservoir pressure material balance calculations. Also, our methods use periodically measured or estimated flowrates instead of formal "test" data, thus eliminating the need to shut-in the well.

To use these methods, one must have measurements of flowrates. For homogeneous reservoirs, the slope and intercept of the decline curve plot are used to estimate reservoir pore volume. However, estimates of skin factor and permeability are required to calculate the reservoir shape factor from either the slope or intercept of the decline curve plot. For naturally-fractured reservoirs, there are too many unknowns to allow us to solve for pore volume, so it must be assumed. Reservoir shape is also assumed to be circular. Also, the skin factor must be known to estimate fracture and matrix properties. Therefore, both the homogeneous and naturally-fractured cases require that a short buildup test be performed prior to obtaining the production data so that the skin factor and permeability can be estimated. For vertically-fractured reservoirs the pore volume can be calculated directly from the slope of the decline curve plotted. The fracture half-length and reservoir fracture shape factor can be estimated from either the slope or intercept of the decline curve plot and an empirical correlation.

Each method requires production at constant bottomhole pressure and post-transient flow conditions. Each of the three methods is rigorous for constant BHP production. The major limitations of these methods are that the exponential solutions derived are applicable only to single-phase (oil or gas) flow and that measurements or estimates of flowrates are required in the post-transient production period.
DEVELOPMENT OF ANALYSIS TECHNIQUES

The exponential rate decline has long been known to be an analytical solution for post-transient constant pressure production. However, only recently has this solution been considered for well testing purposes. There are several possible approaches to the derivation of this solution, but the use of the Laplace transformation is the simplest. In the Appendix, we develop the constant pressure solutions using Laplace transformation and the constant rate solutions for pseudosteady-state. This approach was originally suggested by van Everdingen and Hurst and later used by Ehlig-Economides and Ramey. Each derivation is rigorous for a single well producing a single liquid of small and constant compressibility. Frain and Wattenbarger showed that pseudopressure and pseudotime formulations will allow liquid equations to be used for gas.

Homogeneous Reservoirs

In Appendix A, we develop the post-transient constant pressure solution for the homogeneous reservoir case. The solution is of the diffusivity equation describing the radial flow of a slightly compressible liquid

$$q = \frac{\mu}{2 \pi \mu} \frac{dP}{dt} + \frac{1}{2} \frac{dP}{dI} \frac{6A}{e^{C_{2}A} t_{w}^{2}}$$

(1)

The pseudosteady-state solution for constant rate is given by Ramey and Cobb as

$$P_{bh} = 2\pi \frac{e^{C_{2}A} t_{w}}{4A} \ln \frac{6A}{e^{C_{2}A} t_{w}}$$

(2)

Using the Laplace transformation, we developed the following expression for pseudosteady-state constant pressure flow:

$$q_{bh} = \frac{2}{1 A} \frac{e^{C_{2}A} t_{w}}{e^{C_{2}A} t_{w}} \exp (-\frac{6A}{e^{C_{2}A} t_{w}})$$

(3)

Note that when we take the logarithm of the terms in Eq. 3, we obtain the equation of a straight line. Figure 3a illustrates this linear relationship. This suggests that if we plot log q vs. t we can obtain reservoir characteristics from the slope and intercept of the straight line. In field units the slope is

$$P_{h} = -0.001439 = \frac{k}{\phi \mu c_{e} A \ln \frac{6A}{e^{C_{2}A} t_{w}}^{2}}$$

(4)

and the intercept is

$$q_{bh} = \frac{kh}{70.6 \phi \mu c_{e} A \ln \frac{6A}{e^{C_{2}A} t_{w}}^{2}}$$

(5)

If we rearrange Eqs. 4 and 5 and solve for the reservoir drainage area, A, we obtain

$$A = -0.1016 \frac{q_{bh}}{D_{h}} \frac{1}{\phi h c_{e} (p_{h} - P_{w})}$$

(6)

Similarly, the reservoir shape factor, C_{r}, can be calculated using either Eq. 4 or 5. Eq. 4 gives

$$C_{r} = \frac{2.246 A}{e^{C_{2}A} t_{w}} \exp \left(\frac{-0.001439 k}{e^{C_{2}A} t_{w}} \right)$$

(7)

while Eq. 5 gives

$$C_{r} = \frac{2.246 A}{e^{C_{2}A} t_{w}} \left(\frac{kh}{P_{h} - P_{w}} \right) \exp \left(\frac{-70.6 q_{bh}}{e^{C_{2}A} t_{w}} \right)$$

(8)

The relations show that an exponential decline for the homogeneous reservoir can be decomposed into reservoir characteristics. The reservoir drainage area, A, is calculated directly from the slope and intercept of the plot while the reservoir shape factor, C_{r}, is calculated from the exponential of either the slope or intercept. This means that any error in slope or intercept becomes exponentiated also. Therefore, we would not expect the calculated reservoir shape factor, C_{r}, to be as accurate as the calculated drainage area, A.

Vertically-Fractured Reservoirs

In Appendix B, we develop the post-transient solution for constant BHP production of a well with an infinite-conductivity vertical fracture. Several authors studied the effect of finite and infinite-conductivity vertical fractures on reservoir pressure behavior, and they developed pseudo-radial flow solutions for these cases. The constant-rate pseudosteady-state solution is given by Gringarten as

$$P_{dr} = 2\pi \frac{e^{C_{2}A} t_{w}}{4A} \ln \frac{6A}{e^{C_{2}A} t_{w}}$$

(9)

Earlougher developed an alternative form of Eq. 9; however, the Gringarten equation is more general and less ambiguous. Using Eq. 9 and the methods presented in Appendix A, we developed the following expression for post-transient flow at constant BHP:

$$q_{dr} = \frac{2}{1 A} \frac{e^{C_{2}A} t_{w}}{e^{C_{2}A} t_{w}} \exp (-\frac{6A}{e^{C_{2}A} t_{w}})$$

(10)

Note that if we take the logarithm of Eq. 10, we obtain the equation of a straight line, as Figure 3b shows. This suggests that if we plot log q vs. t we can obtain reservoir characteristics from the slope and intercept of the straight line. In field units the slope is

$$P_{h} = -0.001439 = \frac{k}{\phi \mu c_{e} A \ln \frac{6A}{e^{C_{2}A} t_{w}}^{2}}$$

(11)
and the intercept is

\[ q_{\text{vf}} = \frac{kh(p_f - p_{nf})}{70.6 \mu \ln \left( \frac{24A}{\kappa \chi_f} \right)} \]  \hspace{1cm} (12)

If we rearrange Eqs. 11 and 12 and solve for the reservoir drainage area, \( A \), we obtain

\[ A = -0.1016 \frac{q_{\text{vf}}}{v_{\text{vf}}} + \frac{B}{\kappa \chi_f (p_f - p_{nf})} \]  \hspace{1cm} (13)

Similarly, the reservoir-fracture shape factor, \( C_f \), can be determined by using either Eq. 11 or 12. Using Eq. 11 gives

\[ C_f = \frac{2.246 A}{\exp(-0.001439 k \chi_f)} \chi_f^2 \]  \hspace{1cm} (14)

while using eq. 12 yields

\[ C_f = \frac{2.246 A}{\exp(-0.001439 k \chi_f)} \chi_f \]  \hspace{1cm} (15)

Eqs. 14 and 15 require that the fracture half-length, \( \chi_f \), be known, and, more than likely, it is not. Therefore, we need a general relationship between the fracture shape factor, \( C_f \), and the fracture half-length, \( \chi_f \). We developed a correlation for an infinite conductivity fracture in a square reservoir using the data of Gringarten et al. and the method of Earlougher. A more general solution was recently developed by Gringarten, but since these results were given graphically we chose the earlier tabulated results. This correlation is shown for each variable in Figures 1 and 2. In order to use the correlation we must rearrange Eqs. 14 and 15 slightly and solve for \( C_f \chi_f^2/A \). Rearranging eq. 14 yields

\[ \frac{C_f \chi_f^2}{A} = \frac{2.246}{\exp(-0.001439 k \chi_f)} \]  \hspace{1cm} (14a)

while rearranging eq. 15 yields

\[ \frac{C_f \chi_f^2}{A} = \frac{2.246}{\exp(-0.001439 k \chi_f)} \]  \hspace{1cm} (15a)

These relations show that an exponential decline curve for a vertically-fractured well can be decomposed into reservoir characteristics. The reservoir drainage area, \( A \), is calculated directly from the slope and intercept of the plot while the fracture shape factor, \( C_f \chi_f \), is calculated from the exponential of either the slope or intercept. This means that any error in slope or intercept is also exponentiated. Therefore, we would not expect the calculated fracture shape factor, \( C_f \chi_f \), to be as accurate as the calculated drainage area, \( A \).

Naturally-Fractured Reservoir

In Appendix C, we develop the post-transient solution for constant BHP production from a naturally fractured reservoir. There are two solutions: the first describes fracture depletion and the second describes matrix depletion. The constant-rate, pseudosteady-state solution for fracture depletion is given by Naylor and Cinco-Ley as

\[ P_{\text{Dff}} = \frac{2 p_{\text{Dff}}}{\omega} + \ln \frac{r_{\text{ed}} - \frac{3}{4}}{r_{\text{ed}}} \]  \hspace{1cm} (16)

Using Eq. 16 and the methods presented in Appendix A we developed the following expression for post-transient flow at constant BHP:

\[ q_{\text{Dff}} = \frac{1}{2} \exp\left( -\frac{2}{\ln \frac{r_{\text{ed}} - \frac{3}{4}}{r_{\text{ed}}} \omega} \right) \]  \hspace{1cm} (17)

DaPrat et al. gave the solution for matrix depletion during the constant-pressure, post-transient flow as

\[ q_{\text{Dfn}} = \frac{1}{2} \left( \frac{r_{\text{ed}} - 1}{(1-\omega) \omega} \right) \]  \hspace{1cm} (18)

Using Eq. 17 as the methods presented in Appendix A, we developed the following expression for constant rate pseudo-steady-state flow:

\[ P_{\text{Dfn}} = \frac{2 p_{\text{Dfn}}}{(r_{\text{ed}} - 1)(1-\omega) + \frac{2}{(r_{\text{ed}} - 1)}} \]  \hspace{1cm} (19)

Though Eq. 19 is not used in this paper, we present it for those interested in constant rate production. Note that, when we take the logarithm of each term in Eqs. 17 and 18, we obtain the equations of straight lines. Figure 3c illustrates these lines. This suggests that if we plot \( \log q \) vs. \( t \), we can obtain fracture and matrix characteristics from the slopes and intercepts of the two straight lines. In field units the slope of the fracture depletion stem is

\[ D_{\text{Dff}} = -\frac{0.000229}{k_f} \]  \hspace{1cm} (20)

and the intercept of the fracture depletion stem is

\[ q_{\text{Dff}} = 141.2 B \mu (\ln \frac{r_{\text{ed}} - \frac{3}{4}}{r_{\text{ed}}}) \]  \hspace{1cm} (21)

In field units the slope of the matrix depletion stem is

\[ D_{\text{Dfn}} = 0.0001165 \]  \hspace{1cm} (22)
and the intercept of the matrix depletion stream is

\[ q_{\text{inf}} = \frac{k_{t} h \left( p_{1} - p_{w} \right) \left( r_{e} \right)^{2} - 1}{282.4 B \mu} \]  \hspace{1cm} (23)

If we solve Eq. (21) for the fracture permeability, \( k_{f} \), we obtain

\[ k_{f} = 141.2 \frac{q_{\text{inf}} B}{h \left( p_{1} - p_{w} \right) \left( \ln r_{e} - 0.8 \right)} \]  \hspace{1cm} (24)

If we solve Eqs. 20 and 21 for the fracture storage, \( (\psi c)_{f} \), we obtain

\[ (\psi c)_{f} = -0.03234 \frac{q_{\text{inf}}}{h \left( p_{1} - p_{w} \right) \left( r_{e} \right)^{2}} \]  \hspace{1cm} (25)

where

\[ (\psi c)_{f} = \omega (\psi c)_{f} + (\psi c)_{m} \]  \hspace{1cm} (26)

If we solve Eqs. 21 and 23 for the dimensionless matrix/fracture permeability ratio, \( \lambda \), we obtain

\[ \lambda = 2 \frac{q_{\text{inf}}}{\frac{B}{h \left( p_{1} - p_{w} \right) \left( r_{e} \right)^{2}}} \]  \hspace{1cm} (27)

If we solve Eqs. 20-23 for the total reservoir storage, \( (\psi c)_{f} + (\psi c)_{m} \), we obtain

\[ \frac{(\psi c)_{f} + (\psi c)_{m}}{D_{\text{inf}} + D_{\text{mf}}} \]  \hspace{1cm} (28)

Finally, we can estimate the dimensionless fracture storage, from the following definition:

\[ \omega = \frac{(\psi c)_{f}}{(\psi c)_{f} + (\psi c)_{m}} \]  \hspace{1cm} (29)

If we combine Eqs. 21 and 28, we obtain

\[ \omega = \frac{k_{f} h \left( p_{1} - p_{w} \right)}{141.2 \frac{q_{\text{inf}} B}{h \left( p_{1} - p_{w} \right) \left( r_{e} \right)^{2}}} \]  \hspace{1cm} (30)

These relations show the decline curve for a naturally fractured reservoir can be decomposed into fracture and matrix characteristics. The drainage area shape is assumed to be circular and the drainage area size is estimated. All the derived properties are calculated directly so there should be no error due to exponentiation as there was for the homogeneous and vertically-fractured wells.

Table 3 summarizes the variables which can be calculated from the decline curve plot for each of the constant pressure cases.

### APPLICATIONS TO DECLINE CURVES

In this section we will present an example analysis of each of the three reservoir types. The appropriate analysis technique will be used and the calculated results will be compared to the formation properties used to construct the hypothetical decline curves analyzed.

### Homogeneous Reservoir Case

#### Example 1:

In this example we simulated constant pressure production with a fully implicit, single-phase radial reservoir simulator. This simulator was verified with the analytical solution of Hulbert\(^\text{a}\) for this case.

#### System Properties:

**Geometric - Single Well Producing From the Center of a Bounded Circular Reservoir**

- **Drainage area, A**: 40 acres
- **Drainage radius, rₐ**: 744.7 ft
- **Net pay thickness, t**: 30.0 ft
- **Reservoir permeability, k**: 1.0 md
- **Reservoir porosity, \( \phi \)**: 0.3
- **Wellbore radius, rₙ**: 0.2 ft
- **Oil viscosity, \( \mu \)**: 0.4 cp
- **Oil Formation volume factor, B**: \( 400 \text{ pds} \)
- **Total Compressibility, C**: 15x10\(^{-5}\) psid
- **Initial reservoir pressure, Pᵢ**: 4800 psia
- **Flowing bottomhole pressure, Pₕ**: 4000 psia

Using least squares on the straight-line portion in Figure 4, the following slope and intercept were obtained:

\[ q_{\text{th}} = 39.95 \text{ STB/D} \]

\[ D_{h} = -3.045x10^{-5} \text{ STB/D/ft} \]

The reservoir drainage area, A, is estimated from Eq. 6,

\[ A = -0.1016 \frac{q_{\text{th}} B}{D_{h} h} \frac{1}{\phi C} \left( p_{1} - p_{w} \right) \]  \hspace{1cm} (31)

\[ = -0.1016 \frac{\left( 39.95 \right)}{(30) \left( 15x10^{-5} \right)} \]  \hspace{1cm} (32)

\[ = 1.744 \times 10^{6} \text{ ft}^{2} \]

\[ = 40.64 \text{ acres} \]
Solving for the shape factor, \( C_A \), from the intercept and Eq. 8,

\[
C_A = \frac{2.246 A}{\exp \left( \frac{2}{70.6 \phi_{sh} B \mu} \right)}
\]

\[
= \frac{2.246}{\left(1.744 \times 10^{-6}\right)} \exp \left( \frac{1}{(1.0)(20.0)(800)(0.2)\left(70.6 \phi_{sh} B \mu\right)} \right)
\]

\[
= 28.3911
\]

Alternatively, using the slope and Eq. 7,

\[
C_A = \frac{2.246}{\exp \left(-0.001439 \frac{k}{B \mu c_{t} A}\right)} \exp \left(\frac{1}{\left(-3.0451 \times 10^{-5}\right)\left(0.3\right)\left(0.4\right)\left(1.744 \times 10^{-6}\right)}\right)
\]

\[
= 28.3904
\]

Averaging the estimated shape factors gives

\[
C_A = \frac{28.3904}{28.3908}
\]

Comparison of Results:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Input</th>
<th>This Work</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>40.00 acres</td>
<td>40.04 acres</td>
<td>0.094</td>
</tr>
<tr>
<td>( C_A )</td>
<td>31.6206</td>
<td>28.3908</td>
<td>-10.21</td>
</tr>
<tr>
<td>( \phi_{sh} )</td>
<td>0.1338</td>
<td>0.1329</td>
<td>-0.67</td>
</tr>
<tr>
<td>( D_h )</td>
<td>-0.8409</td>
<td>-0.8347</td>
<td>0.73</td>
</tr>
</tbody>
</table>

To explain the relatively large error in the estimate of the shape factor, \( C_A \), we present the error for the intercept, \( C = -0.001439 \frac{k}{B \mu c_{t} A} \), and the slope, \( D_h \). Note that the error for \( C = -0.001439 \frac{k}{B \mu c_{t} A} \) is quite acceptable. However, when \( \phi_{sh} \) and \( D_h \) are used to calculate the shape factor, \( C_A \), some of the terms are exponentiated, which causes the large error in the \( C_A \) estimate.

Vertically-Fractured Reservoir Case

Example 2:

In this example, we used a fully implicit, single-phase numerical simulator to model the decline of a vertically fractured well. This is a difficult case to simulate accurately, so some error can be attributed to the simulator. All cases are shown in Figure 5.

System Properties:

Geometry - Single Vertically Fractured Well in the Center of a Bounded Square Reservoir

| Drainage Area, \( A \) | 80 acres |
| Distance to Wall Drainage Boundary, \( x \) | 933.4 ft |
| Net Pay Thickness, \( h \) | 65.0 ft |

Reservoir Permeability, \( k \) | 0.05 md |
Reservoir Porosity, \( \phi \) (fraction) | 0.16 |
Fracture Half-Length, \( x_f \) | 186.7 ft |
Oil Viscosity, \( \mu \) | 0.62 cp |
Oil Formation Volume Factor, \( B \) (# 3000 psia) | 1.434 RB/STB |
Total Compressibility, \( C_p \) | 15.0x10^{-6} psi^{-1} |
Initial Reservoir Pressure, \( P_i \) | 4000 psia |
Flowing Bottomhole Pressure, \( P_{wf} \) | 3000 psia |

From a least-squares fit on the straight line portion in Figure 6, we obtained the following slope and intercept for \( C_f = 500 \):

\[
q_{wf} = 13.36 \text{ STB/D}
\]

\[
D_{ivf} = 3.695 \times 10^{-6} \text{ STB/D/hr}
\]

A fracture with productivity this large is essentially infinitely conductive.

The reservoir drainage area, \( A \), is estimated from Eq. 13:

\[
A = \frac{q_{ivf}}{D_{ivf}} \frac{B}{\phi h c_{t} (P_i - P_{wf})}
\]

\[
= 0.1016 \frac{\frac{13.36}{(13.36)(1.434)}}{(-3.695 \times 10^{-6})(0.16)(65)(15 \times 10^{-6})(1000)}
\]

\[
= 3.378 \times 10^{6} \text{ ft}^2
\]

\[
= 77.55 \text{ acres}
\]

We will now solve for the \( C_f X_f^2 / A \) product. This will allow us to use our correlation and solve for \( C_f \) and \( X_f / A \). The \( C_f X_f^2 / A \) product can be estimated from the intercept using Eq. 15a.

\[
C_f X_f^2 = \frac{2.246}{A}\exp \left( \frac{2}{70.6 \phi_{sh} B \mu} \right)
\]

\[
= \frac{2.246}{\exp \left(-0.001439 \frac{k}{B \mu c_{t} A}\right)} \exp \left(\frac{1}{\left(-3.0451 \times 10^{-5}\right)\left(0.3\right)\left(0.4\right)\left(1.744 \times 10^{-6}\right)}\right)
\]

\[
= 0.0486
\]

Alternatively, using Eq. (14a) and the slope,

\[
C_f X_f^2 = \frac{2.246}{A}\exp \left(-0.001439 \frac{k}{B \mu c_{t} A}\right)
\]

\[
= \frac{2.246}{\exp \left(-0.001439 \frac{k}{B \mu c_{t} A}\right)} \exp \left(\frac{1}{\left(-3.0451 \times 10^{-5}\right)\left(0.3\right)\left(0.4\right)\left(1.744 \times 10^{-6}\right)}\right)
\]

\[
= 0.0486
\]

Using Figure 1 to determine the fracture reservoir shape factor, \( C_f \), we obtain, using cubic spline interpolation

\[
C_f = 7.466
\]
Using Figure 2 to determine \(X_e/A^{1/2}\), we obtain using cubic spline interpolation

\[X_e/A^{1/2} = 0.0790\]

We made similar calculations for the finite conductivity fractures and constructed the following table:

<table>
<thead>
<tr>
<th>(A_e), acres</th>
<th>(X_e/A^{1/2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>80.00</td>
<td>0.00</td>
</tr>
<tr>
<td>78.51</td>
<td>0.00</td>
</tr>
<tr>
<td>78.35</td>
<td>0.00</td>
</tr>
<tr>
<td>77.54</td>
<td>0.00</td>
</tr>
<tr>
<td>77.55</td>
<td>0.00</td>
</tr>
<tr>
<td>7.098</td>
<td>0.100</td>
</tr>
<tr>
<td>7.545</td>
<td>0.055</td>
</tr>
<tr>
<td>7.541</td>
<td>0.073</td>
</tr>
<tr>
<td>7.477</td>
<td>0.077</td>
</tr>
<tr>
<td>7.468</td>
<td>0.078</td>
</tr>
<tr>
<td>7.467</td>
<td>0.079</td>
</tr>
<tr>
<td>7.466</td>
<td>0.079</td>
</tr>
</tbody>
</table>

**Example 3:**

**Naturally Fractured Reservoir**

DePrat, et al. 

presented an example based on computer simulation. Insufficient data were given in the paper to allow us to estimate slope during matrix decline. However, since the total storage, \((\psi V)\), can be estimated from the DePrat type curve match discussed in the paper, the matrix decline slope during matrix decline was calculated from these data.

**System Properties:**

- **Geometry:** Single Well Centered in a Naturally Fractured, Bounded Circular Reservoir:
  - Reservoir Drainage Radius, \(r_e\): 1500 ft
  - Net Pay Thickness, \(h\): 480 ft
  - Wellbore Radius, \(r_w\): 0.25 ft
  - Skin Factor, \(S\): 0.094
  - Effective Wellbore Radius, \(r_{wD}'\): 15.0 ft

- **Effective Dimensionless Drainage Radius:** \(r_{wD}' = r_w h / \kappa\)

- **Oil Viscosity, \(\mu\):** 1.0 cp

- **Oil Formation Volume Factor, \(B\):** 1.0 RB/STB

- **Pressure Drop, \(\Delta p = p_i - p_f\):** 15.0 ft

- **Data Derived From Type Curve:**
  - Fracture Permeability, \(k_f\): 0.15 md
  - Total Storage, \((\psi V)_\text{total}\): \(2.805 \times 10^{-6} \text{ psi}^{-1}\)
  - Permeability Ratio, \(\lambda\): 5.0 x 10^{-6}
  - Dimensionless Fracture Storage, \(\omega\): 0.61

**Matrix Depletion:**

\[q_{\text{inflow}} = 77.68 \text{ STB/D}\]

\[D_{\text{inflow}} = -1.289 \times 10^{-4} \text{ STB/D/hr}\]

The fracture permeability, \(k_f\), is estimated from Eq. 24:

\[k_f = 141.2 \frac{q_{\text{inflow}}}{h \left(\frac{1}{r_{wD}'} - \frac{1}{r_w}ight)}\]

\[r_{wD}'\] must be used instead of \(r_w\) because of the non-zero skin factor.

\[k_f = 141.2 \frac{(850.7)(1.0)(1.0)}{(480)(6500)}\]

\[= 0.1484 \text{ md}\]

The fracture storage, \((\psi V)_f\), is estimated from Eq. 25:

\[\frac{(\psi V)_f}{(\psi V)_m} = 0.03234 \frac{q_{\text{inflow}}}{h \left(\frac{1}{r_{wD}'} - \frac{1}{r_w}ight)}\]

\[= 0.03234 \frac{(850.7)(1.0)}{(480)(6500)(1500)}\]

\[= 2.833 \times 10^{-11} \text{ psi}^{-1}\]

The dimensionless matrix/fracture permeability ratio, \(\lambda\), is estimated from Eq. 27:

\[\lambda = 2 \frac{q_{\text{inflow}}}{h \left(\frac{1}{r_{wD}'} - \frac{1}{r_w}ight)}\]

Substituting \(r_{wD}'\) for \(r_w\),

\[= 2 \frac{(71.64)}{(850.7)(1000)}\]

\[= 4.737 \times 10^{-6}\]

The total reservoir storage, \((\psi V)\), can be estimated from Eq. 28:

\[\frac{(\psi V)_m}{(\psi V)_f} = 0.03234 \frac{h}{\left(\frac{1}{r_{wD}'} - \frac{1}{r_w}ight)}\]

\[\frac{q_{\text{inflow}}}{q_{\text{inflow}}, \text{inflow}} = \frac{(\psi V)_m}{(\psi V)_f}\]

However, since we used the total storage, \((\psi V)_m\) * \((\psi V)_f\). To calculate the matrix decline slope, \(D_{\text{inflow}}\), this calculation serves only to verify our arithmetic.

\[\frac{(\psi V)_m}{(\psi V)_f} = 0.03234 \frac{(1.0)}{(480)(6500)(1500)}\]
Finally, we estimate the dimensionless fracture storage, $\omega$, from Eq. 29:

$$
\omega = \frac{(\omega_{c})_f}{(\omega_{c})_m} = -0.0103
$$

Comparison of Results:

<table>
<thead>
<tr>
<th>Variable</th>
<th>DaPrat, et. al.</th>
<th>This Work</th>
<th>Error, $%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_e$, md</td>
<td>0.15 $\times 10^{-11}$</td>
<td>0.1484 $\times 10^{-11}$</td>
<td>-1.07</td>
</tr>
<tr>
<td>$(\omega_{c})_f$, $\text{psi}^{-1}$</td>
<td>2.805 $\times 10^{-11}$</td>
<td>2.883 $\times 10^{-11}$</td>
<td>2.77</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>5.0 $\times 10^{-10}$</td>
<td>4.735 $\times 10^{-10}$</td>
<td>5.26</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.01</td>
<td>0.0103</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Naturally Fractured Reservoir - Field Case

Example 4:

Chen presented the following data for a naturally fractured well in the Austin Chalk formation. He performed his analysis without taking the skin factor, $S$, into account. For our analysis, we assume a skin factor, $S$, of -4.0, which is a reasonable estimate for a naturally fractured reservoir.

System Properties:

Geometry - Single Well in the Center of a Naturally Fractured Bounded Circular Reservoir
- Reservoir Drainage Area, $A$ = 40 acres
- Reservoir Drainage Radius, $r_e$ = 744.7 ft
- Net Pay Thickness, $h$ = 40 ft
- Wellbore Radius, $r_w$ = 0.25 ft
- Skin Factor, $S$ = -4.0
- Effective Wellbore Radius, $r_{w,e} = 13.65$ ft
- Effective Dimensionless Drainage Radius, $r_{w,e} = 0.5456$
- Reservoir Temperature, $T_r$ = 225°F
- Initial Solution Gas-Oil Ratio, $P_{wi}$ = 1050 SCF/STB
- Oil Gravity, $\gamma_o$ = 40° API
- Oil Viscosity, $\mu_o$ = 0.24 cp
- Oil Formation Volume Factor, $B_o$ = 1.58 STB/STB
- Pressure Drop, $\Delta P = P_i - P_w$ = 3800 psi

Using least squares on the two straight lines in Figure 8, the following slopes were obtained:

Fracture Depletion

$$q_{inf} = 578.8 \text{ STB/D}$$

$$D_{nf} = -1.369 \times 10^{-4} \text{ STB/D/hr}$$

Matrix Depletion

$$q_{inf} = 126.7 \text{ STB/D}$$

$$D_{nf} = -3.701 \times 10^{-5} \text{ STB/D/hr}$$

The fracture permeability, $k_f$, is estimated from Eq. 24:

$$k_f = 141.2 \frac{q_{inf} B \mu}{h (P_i - P_{wf})} (\ln r_{eD} - \frac{3}{4})$$

Where $r_{eD}$ must be used in place of $r_{eD}$ because of the non-zero skin factor.

$$k_f = 141.2 \frac{(578.8) (1.58) (0.24) (\ln(54.56) - \frac{3}{4})}{(40)(3800)}$$

$$= 0.6625 \text{ md}$$

The fracture storage, $(\omega_{c})_f$, is estimated from Eq. 25:

$$(\omega_{c})_f = -0.03234 \frac{q_{inf} B}{h (P_i - P_{wf})} r_{eD}^2$$

$$= -0.03234 \frac{(578.8) (1.58)}{(-1.369 \times 10^{-4}) (40)(3800)} (744.7)^2$$

$$= 2.564 \times 10^{-6} \text{ psi}^{-1}$$

The dimensionless matrix/fracture permeability ratio, $\lambda$, is estimated from Eq. 27:

$$\lambda = 2 \frac{q_{inf}}{q_{nf} r_{eD}^2 (\ln r_{eD} - \frac{3}{4})}$$

Substituting $r_{eD}$' for $r_{eD}$:

$$= 2 \frac{(126.7)}{(578.8)(54.56)^2 (\ln(54.56) - \frac{3}{4})}$$

$$= 4.526 \times 10^{-5}$$

The total reservoir storage, $[(\omega_{c})_f + (\omega_{c})_m]$, is calculated from Eq. 28:

$$[(\omega_{c})_f + (\omega_{c})_m] = -0.03234 \frac{B}{h (P_i - P_{wf})} r_{eD}^2$$

$$= -0.03234 \frac{(1.58)}{(40)(3800)(744.7)^2}$$

$$= 4.639 \times 10^{-6} \text{ psi}^{-1}$$

Finally, we estimate the dimensionless fracture storage, $\omega$, from Eq. 29.
\[
\omega = \frac{\left(\frac{\phi_v c_f}{\rho_f g}\right)}{\left(\phi_v c_f + \phi_v c_m\right)} \\
= \frac{\left(2.564 \times 10^{-6}\right)}{\left(4.639 \times 10^{-6}\right)} \\
= 0.5527
\]

Summary of Results:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_f) (\phi_v) psd(^{-1})</td>
<td>0.6625 md</td>
</tr>
<tr>
<td>(\phi_v c_f) (\phi_v c_m) psd(^{-1})</td>
<td>2.564 \times 10^{-6}</td>
</tr>
<tr>
<td>(\phi_v c_m) psd(^{-1})</td>
<td>4.639 \times 10^{-6}</td>
</tr>
<tr>
<td>(\omega)</td>
<td>0.5527</td>
</tr>
</tbody>
</table>

In all cases, the results generated compare well with the input values. The results for the actual field case appear to be reasonable.

**RECOMMENDED TESTING AND ANALYSIS PROCEDURE**

We suggest the following testing and analysis procedure:

1. Measure or estimate both bottomhole pressures and flow rates as functions of time.
2. Plot log \(q\) vs. \(t\).
3. Calculate the slope, \(B\), and intercept, \(c\), of the best straight line on the graph. A least squares fit will give the best results.
4. Estimate reservoir characteristics from appropriate equations presented in Table 2.

This method is applicable only to constant pressure post-transient flow conditions; therefore, if the reservoir is not at the post-transient state, the estimated reservoir characteristics will be incorrect.

These methods are applicable only to systems of small and constant compressibility and single-phase flow. If water influx, solution gas evolution, or multi-phase flow is evident, these methods should not be used.

**SUMMARY AND CONCLUSIONS**

1. We have developed methods of analyzing post-transient constant pressure flow for homogeneous reservoirs, vertically-fractured reservoirs, and naturally-fractured reservoirs. These methods are rigorous and have been proved by comparison to simulated data.
2. Reservoir drainage area, \(A\), can be estimated for both homogeneous and vertically-fractured reservoirs using our new methods.
3. The reservoir shape factor, \(C_A\), and the reservoir fracture shape factor, \(C_f\), can be estimated using our methods for homogeneous and vertically-fractured reservoirs, respectively. However, error in data is magnified greatly due to the exponentiation required.

(4) We present methods to estimate reservoir characteristics for naturally-fractured reservoirs with known drainage area and circular reservoir geometry. The analysis combines the fracture and matrix depletion equations to solve for fracture permeability, \(k_f\), fracture storage \(\left(\phi_v c_f\right)\), dimensionless matrix/fracture permeability ratio, \(\lambda\), total reservoir storage, \(\left(\phi_v c_f + \phi_v c_m\right)\), and the dimensionless fracture storage, \(\omega\).

**NOMENCLATURE**

- \(A\) = Reservoir drainage area, acres \((\text{m}^2)\)
- \(B\) = Oil formation volume factor, \(\text{RB/STB (res \text{m}^3/\text{std \text{m}^3})}\)
- \(c\) = Compressibility, \(\text{psd}^{-1}\) \((\text{kPa}^{-1})\)
- \(C_f\) = Reservoir fracture shape factor, dimensionless
- \(C_A\) = Reservoir shape factor, dimensionless
- \(c_e\) = Porosity compressibility, \(\text{psd}^{-1}\) \((\text{kPa}^{-1})\)
- \(c_o\) = Oil compressibility, \(\text{psd}^{-1}\) \((\text{kPa}^{-1})\)
- \(C_T\) = Dimensionless fracture conductivity
- \(c_t\) = \(c_S + c_o + c_S + c_f\), \(\text{psd}^{-1}\) \((\text{kPa}^{-1})\)
- \(c_w\) = Water compressibility, \(\text{psd}^{-1}\) \((\text{kPa}^{-1})\)
- \(D\) = Slope of log \(q\) vs. \(t\) graph, \(\text{STB/dhr (std \text{m}^3/\text{dhr})}\)
- \(h\) = Net pay thickness, ft \((\text{m})\)
- \(k\) = Effective formation permeability, md
- \(A_p\) = \(P_D - P_{in}\) = Pressure drop, \(\text{psd}\) \((\text{psd})\)
- \(P_D\) = \(kh\left(P_{in} - P_{out}\right)\) \((\text{psd})\)
- \(P_{in}\) = Laplace transform of \(P_D\)
- \(P_{out}\) = Initial formation pressure, \(\text{psd}\) \((\text{psd})\)
\( P_{wf} \) = Flowing bottomhole pressure, psia (kPa)

\( q \) = Liquid flow rate, STB/D (std m^3/D)

\( q_D \) = \( \frac{q_B M}{kh} \), dimensionless rate

\( \tilde{q}_D \) = LaPlace transform of \( q_D \)

\( q_t \) = Intercept of log, q vs. t graph, STB/D (std m^3/D)

\( r_e \) = Drainage radius of the well, ft (m)

\( r_{ed} \) = \( \frac{r_e}{r_e'}, \) dimensionless drainage radius

\( r_{ed}' \) = \( \frac{r_e}{r_e'}, \) effective dimensionless drainage radius

\( r_w \) = Wellbore radius, ft (m)

\( r_w' \) = \( r_w^{-1/2}, \) effective wellbore radius, ft (m)

\( s \) = LaPlace transform variable

\( S \) = Skin factor, dimensionless

\( S_o \) = Oil saturation, fraction

\( S_g \) = Gas saturation, fraction

\( S_w \) = Water saturation, fraction

\( t \) = Flowing time, hr

\( t_D \) = \( \frac{0.0002637}{k_o u C_w^{2/3}} \), dimensionless time

\( t_{DA} \) = \( \frac{0.0002637}{k_o u A_0} \), dimensionless time based on drainage area

\( t_{Dnf} \) = \( \frac{0.0002637}{[(\theta Vc_c)_f + (\theta Vc_m)_f]} \frac{1}{u C_w} \), dimensionless time

\( T_R \) = Reservoir temperature, °R

\( V \) = Ratio of total volume of medium to bulk volume

\( w \) = Fracture width, ft (m)

\( X_e \) = Distance to reservoir boundary, vertically-fractured well, ft (m)

\( X_f \) = Fracture half-length, ft (m)

\( a \) = Interporosity flow shape factor, ft^2 (m^2)

\( \gamma \) = 0.577216, Euler's constant

\( \gamma_o \) = Oil gravity, "API

\( \mu \) = Oil viscosity, cp (Pa·s)

\( \phi \) = Porosity, fraction

\( (\theta Vc)_f \) = Fracture compressibility, psi^-1 (kPa^-1)

\( (\theta Vc)_m \) = Matrix compressibility, psi^-1 (kPa^-1)

\( [(\theta Vc)_f + (\theta Vc)_m] \) = Total reservoir compressibility, psi^-1 (kPa^-1)

\( \lambda \) = Dimensionless matrix/fracture permeability ratio

\( \omega \) = Dimensionless fracture storage

Subscripts

\( h \) = Homogeneous reservoir

\( w_f \) = Vertically fractured reservoir

\( nff \) = Naturally fractured reservoir, fracture depletion

\( nfm \) = Naturally fractured reservoir, matrix depletion

REFERENCES


APPENDIX A - DERIVATION OF THE POST-TRANSIENT SOLUTION FOR CONSTANT PRESSURE PRODUCTION IN A BOUNDED HOMOGENEOUS RESERVOIR

We start with the constant rate solution for pseudosteady-state from Ramey and Cobb.  

\[ P_{r} = \frac{2\pi r_{DA}}{2} \left( \frac{1}{2} \ln \frac{4A}{\pi r_{w}} \right) \]  

(A-1)

Taking the Laplace transform of Eq. A-1 gives

\[ P_{Dh} = \frac{2\pi}{s} + \frac{1}{s} \frac{1}{2} \ln \frac{4A}{\pi r_{w}} \]  

(A-2)

Where \( s \) is the Laplace transform variable. Next, ElHig-Economides and Ramey used the following relation for the constant rate and constant pressure cases in Laplace space:

\[ \frac{1}{s} \ln \left( \frac{4A}{\pi r_{w}} \right) \]  

(A-3)

Eq. A-3 was also proposed by van Everdingen and Hurst. Combining Eqs. A-2 and A-3 and solving for \( q_{Dh} \) yields

\[ q_{Dh} = \frac{1}{s} \left[ \frac{2}{s} + \frac{1}{2s} \ln \frac{4A}{\pi r_{w}} \right] \]  

(A-4)

Rearranging,

\[ q(a) = \frac{2}{s} \left[ \frac{2}{s} + \frac{1}{2s} \ln \frac{4A}{\pi r_{w}} \right] \]  

(A-5)

Now we will determine the inverse Laplace transform of Eq. A-5. Since Eq. A-5 is of the following form,

\[ q(a) = C \frac{1}{s} \]  

(A-6)

It has an inverse Laplace transform of the form

\[ q(a) = C e^{-\alpha x} \]  

(A-7)

Thus,

\[ q_{Dh} = \frac{2}{s} \left[ \frac{2}{s} + \frac{1}{2s} \ln \frac{4A}{\pi r_{w}} \right] \]  

(A-8)

If we take the natural logarithm of each term in Eq. A-8 we obtain the following slope and intercept:

\[ q_{Dh} = \frac{4\pi}{1n} \frac{4A}{\pi r_{w}} \]  

(A-9)
\[ q_{DWF} = \frac{2}{\ln \frac{4A}{e^{1/2}C_f X_f^2}} \exp \left( -\frac{-4\pi q_{DA}}{6A} \right) \]  \quad \text{(B-2)}

Taking the natural logarithm of each term in Eq. B-2, we obtain the following slope and intercept:

\[ D_{DWF} = \frac{-4\pi}{\ln \frac{4A}{e^{1/2}C_f X_f^2}} \]  \quad \text{(B-3)}

\[ q_{DWF} = \frac{2}{\ln \frac{4A}{e^{1/2}C_f X_f^2}} \]  \quad \text{(B-4)}

We now express Eqs. B-3 and B-4 in field units:

\[ D_{DWF} = \frac{8733}{k} \phi \mu c_e A \]  \quad \text{(B-5)}

where

\[ D_{DWF} = \frac{\log \left( \frac{q_{DA}}{q_i} \right)}{t_2 - t_1} \]  \quad \text{(B-6)}

\[ q_{DWF} = \frac{141.2}{kh} \frac{B \mu}{(P_L - P_{wf})} \]  \quad \text{(B-7)}

Solving Eqs. B-3 and B-4 simultaneously, we obtain,

\[ D_{DWF} = \frac{-2\pi q_{DWF}}{\ln \frac{4A}{e^{1/2}C_f X_f^2}} \]  \quad \text{(B-8)}

From Eqs. B-5, B-7, and B-8, we obtain

\[ A = -0.10160 \frac{q_{DWF}}{B \ln \frac{4A}{e^{1/2}C_f X_f^2}} \]  \quad \text{(B-9)}

From Eqs. B-5 and B-6, we obtain,

\[ C_f = \frac{2.246A}{\exp \left( -0.001439 \frac{k}{D_{DWF} \mu c_e A} \right)} \]  \quad \text{(B-10)}

From Eqs. B-7 and B-4, we obtain

\[ C_f = \frac{2.246A}{\exp \left( -0.001439 \frac{k}{D_{DWF} \mu c_e A} \right)} \]  \quad \text{(B-11)}

Eqs. B-10 and B-11 are suitable for estimating the reservoir-fracture shape factor, \( C_f \), when the fracture half-length, \( X_f \), is known. To make this possible, we developed an empirical correlation of \( C_f X_f^2 / A \) vs. \( C_f \) and \( X_f \). We solved Eqs. B-10 and B-11 for \( C_f X_f^2 / A \) and added calculated \( C_f X_f^2 / A \) from Eq. B-11 using solutions published by Gittinger, et al.

Due to many possible geometries, we present this correlation only for a square reservoir.
However, using the general rectangular reservoir relations presented by Gringarten, other similar correlations can be constructed.

Rearranging Eq. B-10,

\[ C_f \frac{X_f^2}{A} = \frac{2.246}{\exp \left( -0.001439 \frac{X_f}{A} \right)} \]  \hspace{1cm} (B-12)

Rearranging Eq. B-11,

\[ C_f \frac{X_f^2}{A} = \frac{2.246 \exp \left( \frac{2}{4} \ln \frac{r_{eD}^2 - \phi}{r_{eD}^2} \right)}{\exp \left( 70.6 \frac{q_{ref} B}{w} \right)} \]  \hspace{1cm} (B-13)

Solving Eq. B-1 for \( C_f \frac{X_f^2}{A} \),

\[ C_f \frac{X_f^2}{A} = \frac{2.246}{\exp \left( 2 \left( \frac{1}{2} \ln \frac{r_{eD}^2 - \phi}{r_{eD}^2} \right) \right)} \]  \hspace{1cm} (B-14)

For a square reservoir,

\[ A = 4X_e^2 \]  \hspace{1cm} (B-15)

Therefore,

\[ X_f = \frac{X_f}{A^{1/2}} \]  \hspace{1cm} (B-16)

Using the Gringarten, et al. data for an infinite conductivity vertical fracture, we generated the following table. The Gringarten data could also be used to generate the correlation but this data is presented only graphically.

<table>
<thead>
<tr>
<th>( X_e/A )</th>
<th>( X_f/A^{1/2} )</th>
<th>( C_f )</th>
<th>( C_f \frac{X_f^2}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1</td>
<td>1/2</td>
<td>3.1524</td>
<td>0.7881</td>
</tr>
<tr>
<td>1/1.5</td>
<td>1/3</td>
<td>5.4396</td>
<td>0.6044</td>
</tr>
<tr>
<td>1/2</td>
<td>1/4</td>
<td>6.3500</td>
<td>0.3969</td>
</tr>
<tr>
<td>1/3</td>
<td>1/6</td>
<td>7.0336</td>
<td>0.1934</td>
</tr>
<tr>
<td>1/5</td>
<td>1/10</td>
<td>7.3980</td>
<td>0.07398</td>
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<tr>
<td>1/7</td>
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<td>1/10</td>
<td>1/20</td>
<td>7.5528</td>
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</tr>
<tr>
<td>1/15</td>
<td>1/30</td>
<td>7.5603</td>
<td>0.00882</td>
</tr>
</tbody>
</table>

This correlation is shown graphically in Figures 1 and 2. For non-zero skin factor, replace \( X_f \) by \( X_f - X_s^2 \).

**APPENDIX C - DERIVATION OF THE POST-TRANSIENT SOLUTION FOR CONSTANT PRESSURE PRODUCTION IN A NATURALLY-FRACTURED BOUNDED RESERVOIR**

In this appendix, we will develop a post-transient solution for constant pressure production of a well in a naturally-fractured reservoir. This section uses the same Laplace transformation technique as Appendix A.
\[ \omega = \frac{2 q_D \text{Dinf}}{(r_e^2 - 1) D_{\text{Dfm}}} \quad \cdots \cdots \cdots \quad (C-12) \]

If we assume \( r_b^2 = r_e^2 - 1 \) and solve Eqs. C-11 and C-12 simultaneously, we obtain

\[ r_e^2 = \frac{2 q_{\text{Dinf}}}{D_{\text{Dinf}}} \quad \cdots \cdots \cdots \quad (C-13) \]

Eq. C-13 is the material balance equation for the naturally fractured system. Though it ignores transient and transition production, Eq. C-13 gives us a relationship to estimate or verify pore volume.

We will now use Eqs. C-7 through C-13 to estimate specific reservoir properties. This requires the following definitions of dimensionless variables for a naturally fractured reservoir:

\[ q_D = 141.2 \frac{q \mu}{k_f (p_i - p_u)^2} \quad \cdots \cdots \cdots \quad (C-14) \]

\[ q_{\text{Dinf}} = 0.0002637 \frac{k_f t}{[(\phi V_c)_f + (\phi V_c)_m] \mu r_w^2} \quad \cdots \cdots \cdots \quad (C-15) \]

\[ r_e^2 = \frac{8733 D_{\text{Dnf}}}{k_f} \quad \cdots \cdots \cdots \quad (C-16) \]

\[ D_{\text{Dnf}} = \frac{\log (q_D/q_i)}{t_2 - t_1} \quad \cdots \cdots \cdots \quad (C-17) \]

We can solve for the fracture permeability, \( k_f \), by combining Eqs. C-7 and C-14:

\[ k_f = 141.2 \frac{q_{\text{Dinf}} \mu}{h (p_i - p_u)^2} \left( \ln r_e^2 - \frac{2}{3} \right) \quad \cdots \cdots \cdots \quad (C-18) \]

We can solve for the fracture storage \( (\phi V_c)_f \), by combining Eqs. C-5, C-8, C-14, and C-18:

\[ (\phi V_c)_f = -0.03234 \frac{h (p_i - p_u)}{r_e^2} D_{\text{Dinf}} \quad \cdots \cdots \cdots \quad (C-19) \]

We can solve for the dimensionless matrix/fracture permeability ratio, \( \lambda \), by combining Eqs. C-9, C-14, and C-18:

\[ \lambda = 2 \frac{q_{\text{Dinf}}}{q_{\text{Dinf}} r_e^2} \left( \ln r_e^2 - \frac{3}{2} \right) \quad \cdots \cdots \cdots \quad (C-20) \]

In addition to the fracture storage, \( (\phi V_c)_f \), we can also solve for the total reservoir storage, \( [(\phi V_c)_f + (\phi V_c)_m] \), using Eqs. C-13, C-14, and C-16:
Table 1
Summary of Constant Rate Pseudosteady-State Solutions
And Constant Pressure Post-Transient Solutions

<table>
<thead>
<tr>
<th>Case</th>
<th>Constant Rate Case</th>
<th>Constant Pressure Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneous</td>
<td>( P_{rh} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} + \frac{4a}{h} - \frac{4a}{h} \frac{a}{h} )</td>
<td>( \bar{q}_{w} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} )</td>
</tr>
<tr>
<td>Vertically Fractured</td>
<td>( P_{rh} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} + \frac{4a}{h} - \frac{4a}{h} \frac{a}{h} )</td>
<td>( \bar{q}_{w} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} )</td>
</tr>
<tr>
<td>Naturally Fractured</td>
<td>( P_{rh} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} + \frac{4a}{h} - \frac{4a}{h} \frac{a}{h} )</td>
<td>( \bar{q}_{w} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} )</td>
</tr>
</tbody>
</table>

For \( w = 2 \pi \) and \( \bar{q}_{w} = \frac{2}{\pi} \frac{4a}{h} \ln \frac{4a}{h} \)

Naturally Fractured

Table 2
Summary of Analysis Methods for Production at Constant BHP

<table>
<thead>
<tr>
<th>Homogeneous Reservoir</th>
<th>Naturally Fractured Reservoir (Bounded Circular Reservoir)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = 0.1016 \frac{h^2}{b} )</td>
<td>( k_s = 144.2 \frac{b}{h} \left( \frac{c_2 - c_{2f}}{c_2} \right) )</td>
</tr>
<tr>
<td>( C_1 = \frac{2.34}{A} \frac{b^2}{h} \ln \left( \frac{b^2}{h} \right) )</td>
<td>( (\frac{b}{h})<em>a = -0.0224 \frac{b}{h} \left( \frac{c_2 - c</em>{2f}}{c_2} \right) )</td>
</tr>
<tr>
<td>( C_2 = \frac{2.34}{A} \frac{b^2}{h} \ln \left( \frac{b^2}{h} \right) )</td>
<td>( c_{2f} = \frac{b}{h} \left( \frac{c_2 - c_{2f}}{c_2} \right) )</td>
</tr>
</tbody>
</table>

Use \( c_{2f} = c_{2f} \) for a non-zero skin factor.

Vertically Fractured Reservoir

<table>
<thead>
<tr>
<th>Vertically Fractured Reservoir (Infinite Conductivity Fracture)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = 0.1016 \frac{b^2}{h} )</td>
</tr>
<tr>
<td>( C_1 = \frac{2.34}{A} \frac{b^2}{h} \ln \left( \frac{b^2}{h} \right) )</td>
</tr>
<tr>
<td>( C_2 = \frac{2.34}{A} \frac{b^2}{h} \ln \left( \frac{b^2}{h} \right) )</td>
</tr>
</tbody>
</table>

Use \( c_{2f} = c_{2f} \) for a non-zero skin factor.

Use \( c_{2f} = c_{2f} \) for a non-zero skin factor.

*Use correlation in Figures 1 and 2 to estimate \( C_2 \) and \( b/\lambda^{1/2} \).*
Fig. 1—Reservoir-fracture shape factor correlation for a vertically fractured well (square reservoir, finite conductivity fractures).

Fig. 2—Fracture half-length/kink root of area correlation for a vertically fractured well (square reservoir, infinite conductivity fractures).

Fig. 3—Typical log q vs. t graphs for pseudo-steady flow.
Variable-Rate Reservoir Limits Testing
by T.A. Blasingame and W.J. Lee, Texas A&M U.
SPE Members

ABSTRACT

This paper presents a new method of estimating drainage area size and shape from production data (bottom-hole pressures and flowrates). The method is rigorously derived approximation for variable-rate flow in a closed reservoir. This method requires a graph of \(\Delta p/q_m\) vs. the superposition plotting function (which is easily calculated by hand). The slope and intercept of the graph are used to provide the desired estimates of drainage area size and shape.

The method that we propose is an approximation, however it has been proved to be very accurate for the constant rate, constant pressure, exponential rate, logarithmic rate, hyperbolic rate, sinusoidal rate, and discrete rate cases. The method also gives acceptable results for square wave rate and random rate cases.

The new method is derived for the time after the initial pressure transient has reached the outer boundary. The changes in flowrate cause additional transients, but we assume that this effect is negligible when compared to the influence of the outer boundary. Therefore, if the change in flowrate does not dominate the influence of the outer boundary, the new method should give acceptable results. Also, at present, this method is only derived for single-phase flow of a liquid of small and constant compressibility.

INTRODUCTION

The purpose of this paper is to present a simple, but accurate method of predicting reservoir drainage area size and shape from variable-rate production data. Previous works have dealt with constant or cyclically constant rate and constant bottom-hole pressure production. A summary of these methods is shown graphically in Figure 1. Ewrourger defined the cyclically constant or square wave rate case in Figure 2. Rather than focus on a particular rate scheme, we develop a general variable-rate approximation that should give accurate results for typical production situations.

Without a variable-rate solution we would have use the more tedious material balance methods that require average reservoir pressures to estimate reservoir pore volume. This would require the well to be shut-in, which results in lost revenue. However, with the new method, the reservoir pore volume and shape can be estimated directly from the production data, without shutting-in the well.

The problem of variable-rate flow in bounded systems is limited in the literature to the work by Ewrourger and the "stabilized flow" methods (which use average reservoir pressures). Though both approaches give acceptable results for their specific application, Ewrourger's case is not realistic and the "stabilized flow" methods again require the well to be shut-in for average reservoir pressure determinations. This suggests the need for a general solution for variable-rate flow in a bounded reservoir.

In the "Description of the New Method" section we will present the general variable-rate solution and the reservoir characteristics which can be derived from it. Also, we will verify the general variable-rate equation (Eq. 12) using analytical and finite-difference simulation. Then a step-by-step procedure for applying our method and a complete example will be shown in the "Method of Application" section. Finally, we will present the derivation of the exact solution for variable-rate flow in a bounded circular reservoir and the approximate solution for variable-rate flow in any shape reservoir in the Appendix of this report.

DESCRIPTION OF THE NEW METHOD

In the Appendix, we derive an equation describing the behavior of the bottom-hole pressure, \(p_{bh}\), as a function of time for a single well producing in a bounded reservoir with a...
variable rate. This solution assumes the following:

Radial flow into the well over the net pay thickness;
Homogeneous and isotropic porous medium;
Uniform net pay thickness;
Porosity and permeability constant (independent of pressure);
Fluid of small and constant compressibility;
Constant fluid viscosity;
Small pressure gradients;
Negligible gravity forces; and
Any rate schedule.

As shown in the Appendix, the complete solution for a circular drainage area is

$$\Delta p = P_t - P_w = 141.2 \frac{B_u}{kh} \int_0^1 \left[ \ln \frac{r_e}{r_w} - \frac{3}{4} + \frac{r_w^2}{2r_e^2} + \frac{S}{2} \right] + 2 \pi (0.0002637) \frac{k}{\phi \mu c_r A} \cdot Q_n$$

$$\sum_{j=1}^{m} \left( q_j - q_{j-1} \right) \sum_{n=1}^{m} \frac{\left( \frac{r_e}{r_n} \right)^m}{\phi \mu c_r A}$$

$$\exp \left( -\frac{K (0.0002637) - \frac{k}{\phi \mu c_r A} (t - t_{j-1})}{\sum_{n=1}^{m} \left( \frac{r_e}{r_n} \right)^m} \right)$$

$$\mbox{EXP} \left( \frac{-K (0.0002637) - \frac{k}{\phi \mu c_r A} (t - t_{j-1})}{\sum_{n=1}^{m} \left( \frac{r_e}{r_n} \right)^m} \right) \mbox{.........(1)}$$

If we assume that the infinite series in eq. (1) is negligible then we call this "stabilized flow". Through each rate change introduces a new transient that keeps the well from reaching true pseudosteady-state flow, this does not keep the well from exhibiting a pseudosteady-state-like flow regime that we call "stabilized flow". Also, we use the term "stabilized flow" for the time to reach pseudosteady state for a constant rate. This gives us the lower bound for the start of stabilized flow. However, a large rate change will cause the infinite series to dominate eq. (1) and stabilized flow will not exist, though after some time the infinite series will become negligible and stabilized flow will again exist. What this means is that each new transient will eventually die off and stabilized flow will dominate, so there will be alternating periods of both transient and stabilized flow. Therefore, if we neglect the infinite series in eq. (1) (i.e., assume stabilized flow) and generalize the drainage area shape by using the Dits' shape factor, $Q_t$, the following variable-rate approximation can be written for stabilized flow:

$$\frac{\Delta p}{q_m} = 70.6 \frac{B_u}{kh} \ln \frac{4A}{e c_r A^2} \cdot \frac{0.2339 \frac{B}{\phi \mu c_r A^2}}{t_{w}} \cdot \bar{c} \mbox{.........(2)}$$

where

$$\bar{c} = \frac{Q_t}{Q_m} = \frac{\sum_{j=1}^{m} \left( q_j - q_{j-1} \right)}{Q_m} \mbox{.........(3)}$$

Eq. 2 suggests that a graph of $\frac{\Delta p}{q_m}$ vs. $\bar{c}$ will be a straight line of slope

$$\frac{m_{vr}}{q_m} = 0.2339 \frac{B}{\phi \mu c_r A^2} \mbox{.........(4)}$$

and intercept

$$b_{vr} = 70.6 \frac{B_u}{kh} \ln \frac{4A}{e c_r A^2} \mbox{.........(5)}$$

will result. Since we know that transient and stabilized flow alternate for each rate change, we can suggest that for a small rate change stabilized flow will dominate. However for large rate changes the infinite series in Eq. (1) (i.e., transient flow) dominates and Eq. (2) is not valid. This means that transient data will not lie on the straight line predicted by Eq. (2). Therefore, these outlying points should not be used when obtaining the slope and intercept of the graph.

Because Eq. (2) is an approximation and we know that stabilized flow will not be achieved for all rate changes, we must investigate empirically the accuracy and applicability of Eq. (2). Our investigation used the complete solution (Eq. (1)) and a finite-difference reservoir simulator. The finite-difference formulation was fully implicit. The simulator modeled radial, single-phase flow of a single-phase liquid of small and constant compressibility. The simulator was verified by comparison with the analytical solution for transient and pseudosteady-state flow in a bounded circular reservoir at a constant producing rate.

Earlougher developed a method to determine reservoir drainage area for wells with rate histories that are cyclically constant or square wave, as shown in Figure 2. He proved that the slope of a $p_r$ vs. time graph provided an estimate of pore volume. However, he did not give an explicit method for estimating reservoir drainage shape for this case. Earlougher used superposition to generate this data, therefore, to verify our method for the situation be studied we reproduced his examples using a similar approach. The main difference between our examples and Earlougher's is that he simulated a well centered in a square reservoir and we simulated a well in the center of a circular reservoir. This difference does not distort the interpretation or comparison of results.

Table 2 shows the reservoir data for the cases comparing results from the new method with results using Earlougher's method. Figure 3 is a graph of rate vs. time for the three cases that were simulated using Eq. (1). The first case is the square wave rate case, which serves as the basis for the Earlougher analysis technique. The second case is a sinusoidal rate case and though Earlougher did not model this exact case, he did suggest that his method was applicable to seasonal rate changes. This case shows the rate demand schedule for approximately one year. The third and final case in this comparison is for a completely random rate schedule from a single well.
Earlougher gave a completely random rate case in his work, though it could not be realistically analyzed using his method.

The conventional $p_{sw}$ vs. time plot is shown in Figure 4 and we see that only the square wave rate case can realistically be analyzed using Earlougher's method. Figure 5 shows our method with $\Delta p/q_m$ vs. $\bar{t}$. Note that the square wave rate case gives two straight lines of the same slope and two intercepts. Since there is no theory for interpreting the two intercepts we simply took an arithmetic average. The sinusoidal and random rate cases essentially line up on the same trend, which is what we expect since all results are for the same reservoir. Table 3 shows the relative error in slope and intercept for each of these three cases. The slope is directly proportional to pore volume while the intercept is exponentially proportional to the shape factor, $C_a$. Slopes and intercepts were determined from the best straight-line in a least squares sense passing through the linear portion of the data in each case. Table 3 shows our method is more accurate than Earlougher’s for determining slope and intercept, and thus for determining drainage area size and shape from variable-rate production data.

We next investigated the effect of the type of rate decline on the results obtained using our new method. We modeled six cases with our finite-difference simulator. Table 4 gives the reservoir properties used in the simulations. The types of rate decline schedules were constant rate, logarithmic decline, exponential decline, hyperbolic decline, discrete (stair-step) decline, and the average schedule resulting from constant bottom-hole pressure production. Figure 6 shows $p_{sw}$ vs. time for each case while Figure 7 shows $p_{sw}$ vs. $t$ for each of these cases. Only the constant rate case can be analyzed using Figure 7. Figure 8 shows $\Delta p/q_m$ vs. $\bar{t}$. Note that virtually all of the points fall on the same line, and again this is expected because all cases are for the same reservoir data. Error analysis results are given in Table 5. This table shows that the new method had small error in all cases, which suggests that the introduction of transients due to changes in rate has little effect on the stabilized flow solution (Eq. (2)).

**APPLICATION OF METHOD**

We suggest the following procedure to obtain the best results from our variable-rate test analysis technique.

1. Measure or estimate both bottom-hole pressures and flowrates as functions of time.
2. Calculate $\Delta p/q_m$ and the plotting function, $\bar{t}$ (Eq. (3)).
3. Plot $\Delta p/q_m$ vs. $\bar{t}$ on Cartesian coordinate graph paper.
4. Determine the slope, $m_{vr}$, and the intercept, $b_{vr}$, of the best straight line on the graph. A least-squares fit gives more accurate results than "eyeball" fits.
5. Estimate reservoir size from the slope, $m_{vr}$, of the graph using Eq. (4).
6. Estimate the reservoir shape factor, $C_a$, from the intercept, $b_{vr}$, of the graph and Eq. (3). Skin factor and reservoir permeability must be known (from a pressure buildup test, for example) to make this estimate.

Table 1 is a summary of plotting and analysis techniques for reservoir limits tests. Figure 1 shows the type of graph for each case. In each case, the slope provides an estimate of drainage area size, $A$, and the intercept provides an estimate of shape factor, $C_a$. A numerical value of shape factor allows us to estimate drainage area by using a table of shape factors and shapes, such as the one presented by Dietz. The new method for analyzing variable-rate production includes the other methods as special cases, similar results will be obtained whether the new general method or the older methods for special cases are used.

**EXAMPLE — Random Rate Production**

In this example we simulated a random rate decline in a bounded circular reservoir with the analytical solution. This is the same case shown earlier in Figures 3, 4 and 5. Also, the pertinent reservoir data is given in Table 2. The production data is given in Table 6.

Using least squares on the straight-line portion of Figure 9, the following slope and intercept were obtained,

$\bar{m}_{vr} = 1.269 \times 10^{-9}$ psi/STB/D/ft
$\bar{b}_{vr} = 2.542 \times 10^{-1}$ psi/STB/D

The reservoir drainage area, $A$, is estimated from Eq. 4,

$\bar{m}_{vr} = 0.2339 \frac{B}{\phi h c L A}$

therefore,

$A = 0.2339 \frac{B}{\phi h c L m_{vr}}$

$= 0.2339 \left( \frac{1.0}{(0.15)(100)(5\times10^{-6})(1.269\times10^{-5})} \right)$

$= 2.458 \times 10^8$ ft$^2$

$= 5643$ acres
The reservoir shape factor, $C_A$, is estimated from Eq. 5,

$$b_{vr} = 70.6 \frac{Bh}{k} \ln \frac{4A}{e'^2 C_A W}$$

therefore,

$$C_A = \frac{4A}{b_{vr} k h e'^2 \exp \left( \frac{4A}{e'^2 C_A W} \right)}$$

$$= \frac{4 \times (2.458 \times 10^8)}{(1.781)(0.5)} \exp \left( \frac{2.542 \times 10^{-1}}{100}(100)(100) \right)$$

$$C_A = 33.46$$

Our estimate for the reservoir shape factor, $C_A$, is slightly high, this is because if we exponentiate a small error in the intercept, $b_{vr}$, it becomes a large error in the reservoir shape factor, $C_A$. Therefore, a better comparison would be that of the input and calculated intercepts. This is shown in the summary table below.

**Summary of Results**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Input</th>
<th>This Work</th>
<th>Z Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>5760</td>
<td>5643</td>
<td>-2.30 x 10^0</td>
</tr>
<tr>
<td>$C$</td>
<td>31.62</td>
<td>33.46</td>
<td>5.83 x 10^1</td>
</tr>
<tr>
<td>$b_{vr}$</td>
<td>2.553 x 10^{-1}</td>
<td>2.542 x 10^{-1}</td>
<td>-4.27 x 10^{-1}</td>
</tr>
</tbody>
</table>

**SUMMARY AND CONCLUSIONS**

We have developed a method of determining drainage area size and shape for wells with variable-rate production histories. This method is an approximation that gives very good results for cases where the rate changes are small. However, large rate changes only dominate the stabilized flow solution (Eq. (2)) until the transient introduced by that rate change becomes negligible. Therefore, the new method should be considered accurate for any production period so long as the reservoir outer boundary is being felt by the pressure response.

The condition of the reservoir can also prescribe the applicability of the new method in that if any of the assumptions concerning the reservoir are violated, the method may not work. Specifically, if water influx, solution gas evolution, multi-phase flow, or reservoir heterogeneities exist then this method should not be used.

**NOMENCLATURE**

$A$ = Reservoir drainage area, ft$^2$ (m$^2$)

$b_{cp}$ = $70.6 \frac{Bh}{k} \ln \frac{4A}{e'^2 C_A W}$

intercept of log $q_m$ vs. $t$ graph for

constant pressure case, STB/Day (std m$^3$/d)

$b_{cr}$ = $70.6 \frac{Bh}{k} \ln \frac{4A}{e'^2 C_A W}$, intercept of $P_{cr}$ vs. $t$ graph for constant rate case, psi (kPa)

$b_{vr} = 70.6 \frac{Bh}{k} \ln \frac{4A}{e'^2 C_A W}$, intercept of $(\delta P/q_m)$ vs. $t$ graph for general variable-rate case, psi/STB/D (kPa/std m$^3$/d)

$B$ = Liquid formation volume factor, RB/STB (res m$^3$/std m$^3$)

$C_A$ = Reservoir shape factor, dimensionless

$c_f$ = Porosity compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c_o$ = Gas compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c_o$ = Oil compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c$ = $c_f + c_o + c_w$, total compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c_w$ = Water compressibility, psi$^{-1}$ (kPa$^{-1}$)

$h$ = Net pay thickness, ft (m)

$J$ = $q/\bar{q}$, productivity index, STB/D/psi

$q = P - P_{wf}$ (std m$^3$/d/kPa)

$J_0$ = Zero order Bessel function of the first kind

$J_1$ = First order Bessel function of the first kind

$k$ = Effective formation permeability, md

$m_{cp}$ = $-0.001439 \frac{k}{\phi \mu_c A \ln \frac{4A}{e'^2 C_A W}}$

slope of log $q_m$ vs. $t$ graph for constant pressure case, cycle/hour (cycle/hr)

$m_{cr}$ = $-0.2339 \frac{\phi h c_A}{\phi h c_A}$, slope of $P_{cr}$ vs. $t$ graph for constant rate case, psi/hour (kPa/hr)

$m_{gw}$ = $-0.2339 \frac{\phi h c_A}{\phi h c_A}$, slope of $P_{wf}$ vs. $t$ graph for square wave rate case, psi/hour (kPa/hr)

$m_{vr}$ = $0.2339 \frac{\phi h c_A}{\phi h c_A}$, slope of $(\delta P/q_m)$ vs. $t$ graph, psi/STB/D/hr (kPa/std m$^3$/D/hr)

$\Delta p$ = $P_i - P_{wf}$, pressure drop, psi (kPa)
\[ \bar{p} = \text{Average reservoir pressure for outer boundary of reservoir, psi (kPa)} \]

\[ p_D = \frac{k h}{141.2 \ q \ \mu} (p_i - p_{wf}), \text{ dimensionless pressure} \]

\[ p_i = \text{Original formation pressure, psi (kPa)} \]

\[ p_{wf} = \text{Flowing bottom-hole pressure, psi (kPa)} \]

\[ p_r = \text{Pressure at radius } r \text{ from the center of the well, psi (kPa)} \]

\[ \bar{p}_r = \text{Average reservoir pressure for radius } r \text{ from the center of the well, psi (kPa)} \]

\[ q = \text{Liquid flowrate, STB/D (std m}^3/d) \]

\[ \bar{q} = \text{Average liquid flowrate, for square wave rate case, STB/D (std m}^3/d) \]

\[ q_m = \text{Liquid flowrate at time } t, \text{ variable-rate case, STB/D (std m}^3/d) \]

\[ Q_m = \text{Cumulative liquid production at time } t, \text{ variable-rate case, STB/D (std m}^3/d) \]

\[ r = \text{Distance from center of the well, ft (m)} \]

\[ r_D = \frac{r}{r_e}, \text{ dimensionless distance} \]

\[ r_e = \text{Drainging radius of the well, ft (m)} \]

\[ r_w = \text{Wellbore radius, ft (m)} \]

\[ r_w' = r_w^2 \phi, \text{ effective wellbore radius, ft (m)} \]

\[ S = \text{Skin factor, dimensionless} \]

\[ S_g = \text{Gas saturation, fraction} \]

\[ S_o = \text{Oil saturation, fraction} \]

\[ S_w = \text{Water saturation, fraction} \]

\[ t = \text{Flowing time, hr} \]

\[ t' = \text{Dummy variable of integration, hr} \]

\[ \bar{\tau} = \frac{\int q(t') dt'}{q_m}, \text{ plotting function for variable-rate tests, hr} \]

\[ t_D = 0.0002637 \frac{k t}{\phi e_r}, \text{ dimensionless time} \]

\[ t_{DA} = 0.0002637 \frac{k t}{\phi e_r A}, \text{ dimensionless time} \]

\[ V_r = r(t^2 - r_w^2) \phi, \text{ radial pore volume, ft}^3 (m^3) \]

\[ X_n = \text{Root of first order Bessel function of the first kind (i.e., } J_1 (X_n) = 0) \]

\[ \gamma = 0.577216, \text{ Euler's constant} \]

\[ \mu = \text{Liquid viscosity, cp (Pa·s)} \]

\[ \phi = \text{Porosity, Fraction} \]

**Integer Subscripts**

\[ j = \text{Rate counter} \]

\[ m = \text{Number of rates up to time } t \]

\[ n = \text{Infinite series counter in Muskat's equation} \]

**Other Subscripts**

\[ c_p = \text{Constant pressure case} \]

\[ c_r = \text{Constant rate case} \]

\[ s_w = \text{Square wave rate case} \]

\[ v_r = \text{General variable-rate case} \]

**REFERENCES**


**APPENDIX**

In this appendix, we will derive the general variable-rate equation for stabilized flow in a bounded circular reservoir. We will then use the average reservoir pressure approach presented by Dietz and Templer-Lietz to derive a variable-rate equation for stabilized flow in any bounded reservoir.

When Duhamel's theorem is applied to the constant rate solution for a continuously changing flow rate, the result is

\[ P_1 - P_R = 141.2 \frac{BU}{kh} \int_0^t q(t') \frac{dp(t-t')}{dt} dt'. \tag{A-1} \]

Applying the convolution theorem to Eq. (A-1) gives

\[ P_1 - P_R = 141.2 \frac{BU}{kh} \int_0^t q(t') \frac{dp(t-t')}{dt} dt'. \tag{A-2} \]

To model discrete rate changes, we discretize Eq. (A-2) the result is

\[ P_1 - P_R = 141.2 \frac{BU}{kh} \int_{q_1}^{q_2} \int_{t_1}^{t_2} q(t') \frac{dp(t-t')}{dt} dt'. \tag{A-3} \]

\[ \int_{q_1}^{q_2} \int_{t_1}^{t_2} q(t') \frac{dp(t-t')}{dt} dt'. \tag{A-4} \]

Eq. (A-4) is a general equation which matches arbitrary rate changes in a producing well. Specific flow regimes such as the transient or pseudosteady-state can be modeled with the appropriate \( \frac{dp(t)}{dt} \) function in Eq. (A-4). At this point we will develop a relation for variable-rate flow in a bounded reservoir, using Eq. (A-4). This requires that we know \( \frac{dp(t)}{dt} \) for the specific reservoir geometry modeled by Matthews, Brons, and Hazebrook and Earlougher, et al. Method of determining \( \frac{dp(t)}{dt} \) for various reservoir geometries. However, the most convenient geometry is a well centered in a bounded circular reservoir. Muskat's give this solution as

\[ p_D \left( r_c, \frac{t}{D} \right) = -\ln \frac{r_D}{r_c} = -\frac{3}{2} + \frac{2}{r_D} + S + 2n \tau_D \]

\[ n = 1 \] is significant root of \( J_0 \left( \frac{r_c}{r_D} \right) \exp \left( -x_n^2 \frac{\pi}{n} \right) \cdots (A-5) \]

where

\[ \tau_D = r/r_e \] (A-6)

\[ X_n \] are the positive roots of \( J_1 \left( \frac{r_c}{r_D} \right) = 0 \] (A-7)

\[ \tau_D = 0.0002637 \left( \frac{kt}{\phi \mu c_A} \right) \] (A-8)

\[ A = \pi \frac{r_e}{x_n^2} \] (A-9)

Combining Eqs. A-5, A-6, A-8, and A-9, and letting \( r = r_w \) gives

\[ p_D \left( r_w, t \right) = \ln \frac{r_w}{r_e} - \frac{3}{4} + \frac{r_w}{2r_e^2} + S \]

\[ + 2 \pi (0.0002637) \left( \frac{kt}{\phi \mu c_A} \right) \] (A-10)

Combining Eqs. A-4 and A-10 gives

\[ \Delta p = P_1 - P_w = 141.2 \frac{BU}{kh} \int_q \ln \frac{r_c}{r_w} - \frac{3}{4} + \]
\[
\frac{r_w^2}{2r_e^2} \left[ \frac{\ln \frac{r_w}{r}}{2} + S \right] + 2\pi (0.0002637) \frac{k}{\phi \mu c_v A} Q_m
\]

\[
-2 \sum_{j=1}^{m} (q_j - q_{j-1}) \prod_{n=1}^{m} \frac{J_0 \left( \frac{r_w}{r_e} \right)}{X_n^2} \frac{z_j^2}{X_n^2} (X_n)
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-11)}
\]

where

\[
Q_m = \int_{t_{j-1}}^{t_j} r_q(t') \, dt' = \sum_{j=1}^{m} q_j(t_j - t_{j-1}) \] 

\[
\text{(A-12)}
\]

We will now develop a plotting function from Eq. (A-11). Following the Odeh and Jones derivation of the plotting function for variable-rate transient flow, we divide both sides of Eq. (A-11) by the final flow rate \( q_m \) Letting \( \bar{t} = \frac{t_j}{q_m} \) yields

\[
\frac{\Delta p}{q_m} = \frac{141.2}{q_m} \frac{B}{kh} \left[ \ln \frac{r_w}{r} \right] + \frac{\frac{r_w}{2}}{2 \bar{t}} + S \]

\[
+ 0.2339 \frac{B}{\phi \mu c_v A} \bar{t}
\]

\[
- 282.4 \frac{B}{kh} \sum_{j=1}^{m} \left( \frac{q_j}{q_m} - \frac{q_{j-1}}{q_m} \right) \prod_{n=1}^{m} \frac{J_0 \left( \frac{r_w}{r_e} \right)}{X_n^2} \frac{z_j^2}{X_n^2} (X_n)
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-13)}
\]

For a usable plotting function to be developed, the infinite series in Eq. (A-13) must be negligible. This is true for the constant rate case at pseudo-steady state and was shown to be approximately true for the variable-rate case at steady reservoir flow conditions using simulated examples in this paper. Therefore, we can approximate Eq. (A-13) by neglecting the infinite series.

\[
\frac{\Delta p}{q_m} = \frac{141.2}{q_m} \frac{B}{kh} \left[ \ln \frac{r_w}{r} \right] + \frac{\frac{r_w}{2}}{2 \bar{t}} + S \]

\[
+ 0.2339 \frac{B}{\phi \mu c_v A} \bar{t}
\]

\[
\text{(A-14)}
\]

Eq. (A-14) is the variable-rate approximation for stabilized flow for a well centered in a bounded circular reservoir.

We will now use Dieltz's approach for constant rate flow in a bounded reservoir of general shape. This requires \( \frac{\partial p}{\partial t} \) for both the constant rate and the variable-rate cases. If we combine Eqs. (A-5), (A-6), (A-8), (A-9), and the definition of dimensionless pressure we obtain the following equation relating pressure and time for constant rate flow:

\[
P_r = P_i - 141.2 \frac{B}{kh} \left[ \ln \frac{r_w}{r} - \frac{3}{4} + \frac{r_w^2}{2r_e^2} + S \right]
\]

\[
+ 2\pi (0.0002637) \frac{k}{\phi \mu c_v A} - 2 \sum_{j=1}^{m} \frac{J_0 \left( \frac{r_w}{r_e} \right)}{X_n^2} \frac{z_j^2}{X_n^2} (X_n)
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-15)}
\]

Solving Eq. (A-11) for \( p_r \) (i.e., \( r_w = r \)) gives an equation relating pressure and time for variable-rate flow:

\[
P_r = P_i - 141.2 \frac{B}{kh} \left[ q_m \left[ \ln \frac{r_w}{r} - \frac{3}{4} + \frac{r_w^2}{2r_e^2} + S \right]
\]

\[
+ 2\pi (0.0002637) \frac{k}{\phi \mu c_v A} Q_m
\]

\[
- 2 \sum_{j=1}^{m} (q_j - q_{j-1}) \prod_{n=1}^{m} \frac{J_0 \left( \frac{r_w}{r_e} \right)}{X_n^2} \frac{z_j^2}{X_n^2} (X_n)
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-16)}
\]

Taking the time derivative of each term in Eq. (A-15) yields:

\[
\frac{\partial P_r}{\partial t} = -0.2339 \frac{qB}{\phi \mu c_v A} \left[ 1 + \sum_{n=1}^{m} \frac{X_n^2}{J_0^2(X_n)} \right]
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-17)}
\]

\[
\frac{\partial P_r}{\partial t} = -0.2339 \frac{qB}{\phi \mu c_v A} \left[ 1 + \sum_{j=1}^{m} \frac{q_j - q_{j-1}}{q_m} \prod_{n=1}^{m} \frac{J_0 \left( \frac{r_w}{r_e} \right)}{X_n^2} \frac{z_j^2}{X_n^2} (X_n)
\]

\[
\exp(-X_n^2 \pi (0.0002637) \frac{k}{\phi \mu c_v A} (t - t_{j-1})) \] 

\[
\text{(A-18)}
\]

When stabilized flow occurs (i.e., when the bounded reservoir terms dominate the transient terms) the infinite series in Eqs. (A-17) and (A-18) become negligible. Therefore, the two derivatives become

\[
\frac{\partial P_r}{\partial t} = -0.2339 \frac{qB}{\phi \mu c_v A} \] 

\[
\text{(A-19)}
\]
\[
\frac{\partial p}{\partial t} \bigg|_{vr} = -0.2339 \frac{q_m^B}{\phi h c_l A} \quad \text{(A-20)}
\]

Only the rates in Eqns. (A-19) and (A-20) differ. Note that there is no dependence on shape for either equation.

Dietz \(^{11}\) developed an average reservoir pressure relation and a reservoir shape relation for constant rate flow based on Eq. (A-19). We will now derive similar relations for variable-rate flow using Eq. (A-20). The general equation of radial flow or the diffusivity equation was given by Muskat \(^{9}\) as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{\phi \mu c_t}{0.0002637k} \frac{\partial p}{\partial t} \quad \text{(A-21)}
\]

Combining Eqns. (A-20) and (A-21) yields

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = -282.4 \frac{q_m^B \mu r}{kh^2} \quad \text{(A-22)}
\]

Integrating Eq. (A-22) yields

\[
\frac{\partial p}{\partial t} = -141.2 \frac{q_m^B \mu}{kh^2} \left( \frac{1}{r} - \frac{x}{r_e} \right) \quad \text{(A-23)}
\]

Applying the outer boundary condition,

\[
\frac{\partial p}{\partial t} = 0 \text{ at } r = r_e \quad \text{(A-24)}
\]

Combining eqns (A-23) and (A-24)

\[
\frac{\partial p}{\partial t} = 141.2 \frac{q_m^B \mu}{kh^2} \left( \frac{1}{r} - \frac{r}{r_e} \right) \quad \text{(A-25)}
\]

Integrating Eq. (A-25) yields

\[
p_r = 141.2 \frac{q_m^B \mu}{kh^2} \left( \ln r - \frac{r^2}{r_e^2} \right) + C_2 \quad \text{(A-26)}
\]

Applying the inner boundary condition,

\[
p_r = p_{wf} \text{ at } r = r_w \quad \text{(A-27)}
\]

Combining Eqns. (A-26) and (A-27)

\[
p_r = p_{wf} + 141.2 \frac{q_m^B \mu}{kh^2} \left( \ln \frac{r}{r_w} - \frac{r^2}{r_w^2} \right) \quad \text{(A-28)}
\]

Introducing the average reservoir pressure, \( p_r \), for any radius \( r \),

\[
\frac{1}{V} \frac{\partial}{\partial t} \int_r p_r \text{ dw} = -p_r \quad \text{(A-29)}
\]

where

\[
V_r = \pi \left( r^2 - r_w^2 \right) \phi h \quad \text{(A-30)}
\]

\[
dV_r = 2 \pi r \phi h \text{ dr} \quad \text{(A-31)}
\]

Combining Eqns. (A-29) and (A-31) and solving for the average reservoir pressure, \( p_r \),

\[
p_r = \frac{1}{n} \frac{1}{\pi \left( r^2 - r_w^2 \right) \phi h} \frac{\int r \text{ dr}}{V_r}
\]

\[
+ 2 \pi h \left( 141.2 \right) \frac{q_m^B \mu}{kh^2} \left( \ln \frac{r}{r_w} - \frac{r^2}{r_w^2} \right) + \frac{2}{8r_e^2} \quad \text{(A-32)}
\]

Completing the integration in Eq. (A-32),

\[
p_r = p_{wf} + \frac{2(141.2) (q_m^B \mu)}{kh^2} \left( \frac{r}{2} \ln \frac{r}{r_w} - \frac{r^2}{2r_w^2} \right) - \frac{(r^2 - r_w^2)}{4} \quad \text{(A-33)}
\]

Simplifying Eq. (A-33),

\[
p_r = p_{wf} + 141.2 \frac{q_m^B \mu}{kh^2} \left( \frac{r}{2} \ln \frac{r}{r_w} - \frac{1}{4} \right) \quad \text{(A-34)}
\]

If we let \( r = r_e \) and note that \( r_e^2 \gg r_w^2 \), Eq. (A-34) becomes

\[
p = p_{wf} + 141.2 \frac{q_m^B \mu}{kh^2} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right) \quad \text{(A-35)}
\]

Eq. (A-35) gives us a relation for the average reservoir pressure, \( p \), during stabilized flow. Dietz \(^{11}\) gave the following relation for the average reservoir pressure, \( p \), for constant rate pseudo-steady-state flow

\[
\bar{p} = p_{wf} + 141.2 \frac{q_m^B \mu}{kh^2} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right) \quad \text{(A-36)}
\]

Note that if we solve for the productivity index, \( J \), using Eqns. (A-35) and (A-36) we obtain the following equality

\[
J = \frac{q_m^B \mu}{\bar{p} - p_{wf}} = \frac{q_m^B \mu}{141.28q_m^B \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right)} \quad \text{(A-37)}
\]

Eq. (A-37) proves that the productivity indices for constant rate pseudo-steady-state and variable-rate stabilized flow are equal. Now we will use Eq. (A-35) and the approximate
variable-rate transient flow relation to develop a general reservoir shape factor, \( C_A \). The approximate variable-rate transient solution is

\[
P_{wf} = P_i - 70.6 \frac{q_B u}{kh} \ln \left( \frac{\alpha}{e \tau_D} \right) \quad \text{(A-38)}
\]

Combining Eqns. (A-35) and (A-38) for \( p_i - p \) yields

\[
P_i - p = 70.6 \frac{q_B u}{kh} \ln \left( \frac{C_A \tau_D}{e} \right) \quad \text{(A-39)}
\]

Where the reservoir shape factor, \( C_A \), is defined for a circular reservoir as

\[
C_A = \frac{4 \pi r_e^{3/2}}{e^2} \quad \text{(A-40)}
\]

When \( C_A \tau_D \leq 1, P_i - \bar{p} = 0, \) so \( P_i = \bar{p} \) up to this time. Therefore, for \( P_i = \bar{p} > 1/C_A \). This means that, for \( r = 1/C_A \), \( p \) can be predicted using Eq. (A-35). Also, since the constant rate and variable-rate solutions are essentially the same then the shape factors predicted for constant rate bare also valid for variable-rate flow. Therefore, we can generalize Eq. (A-14) for other shapes by using the Deitz shape factor, \( C_A' \).

If we neglect \( r_e^2/2r_e \) in Eq. (A-14) and use the effective wellbore radius to model the skin effect, we obtain,

\[
\frac{\Delta p}{q_m} = 70.6 \frac{B u}{kh} \ln \left( \frac{r_e^2}{r_w^2} \right) + 0.2339 \frac{B}{\phi h c_e} \frac{\bar{r}}{C_A} \quad \text{(A-41)}
\]

Now we will include the reservoir shape factor, \( C_A \), relation (eq. (A-40)) in the argument of the natural logarithm:

\[
\frac{\Delta p}{q_m} = 70.6 \frac{B u}{kh} \ln \left( \frac{r_e^2}{r_w^2} \right) \frac{C_A}{2} + 0.2339 \frac{B}{\phi h c_e} \frac{\bar{r}}{C_A} \quad \text{(A-42)}
\]

Eq. (A-42) is the general stabilized flow equation which serves as the basis for our analysis technique. If we plot \( p/q_m \) vs. \( \bar{r} \) on Cartesian coordinate graph paper, Eq. (A-42) dictates the following slope and intercept.

\[
a_{\bar{r}} = 0.2339 \frac{B}{\phi h c_e} \quad \text{A-43}
\]

\[
b_{\bar{r}} = 70.6 \frac{B u}{kh} \ln \left( \frac{r_e^2}{r_w^2} \right) \frac{C_A}{2} \quad \text{A-44}
\]

Solving Eq. (A-43) for the reservoir drainage area, \( A \), yields

\[
A = 0.2339 \frac{B}{\phi h c_e} \quad \text{A-43a}
\]

Solving Eq. (A-44) for the reservoir shape factor, \( C_A' \), yields

\[
C_A' = \frac{2.246 A}{b_{\bar{r}}} \frac{1}{\frac{B}{\phi h c_e}} \frac{1}{r_w^2} \quad \text{A-44a}
\]

Eq. (A-35) can also be put in a general form by use of the shape factor, \( C_A' \).

\[
\bar{p} = v_{wf} + 70.6 \frac{q_B u}{kh} - 1 - \frac{4A}{e^2 C_A' r_w^2} \quad \text{(A-45)}
\]

And finally, we will express Eq. (A-42) in dimensionless variables as

\[
P_D = \frac{1}{2} \ln \left( \frac{4A}{e^2 C_A' r_w^2} \right) + 2 \pi \tau_D \quad \text{(A-46)}
\]

Where \( \bar{r} \) is substituted for \( t \) in \( t_D \). Eq. (A-47) is the same as the constant rate solution presented by Namer and Cobb, but \( \bar{r} \) is substituted for \( t \).
### TABLE 1
SUMMARY OF RESERVOIR LIMITS TEST AND ANALYSIS TECHNIQUES

**TEST METHOD**

**Constant Rate (\(q_{w} \text{ vs. } t\) Graph)**

\[
A = \frac{-0.3339}{\frac{\rho B}{W} \frac{\mu}{\theta}}
\]

\[
C_{A} = \frac{2.264 A}{\exp\left(\frac{k h}{70.6 \mu B} \frac{\mu}{\theta} \right) \frac{\mu}{\theta}^{2} \frac{\mu}{\theta}}
\]

**Constant Pressure (log \(q \text{ vs. } t\) Graph)**

\[
A = -0.1016 \frac{\rho}{\mu} \frac{\mu}{\theta} \exp\left(\frac{k h}{70.6 \mu B} \frac{\mu}{\theta} \right) \frac{\mu}{\theta}^{2} \frac{\mu}{\theta}
\]

\[
C_{A} = \frac{2.264 A}{\exp\left(-0.00159 \frac{k h}{70.6 \mu B} \frac{\mu}{\theta} \right) \frac{\mu}{\theta}^{2} \frac{\mu}{\theta}}
\]

**Square Wave Rate** (\(q_{w} \text{ vs. } t\) Graph)

(Earlougher's Method)

\[
A = -0.3339 \frac{\rho B}{W} \frac{\mu}{\theta}
\]

\[
C_{A} = \text{Not applicable from this method}
\]

**General Variable-Rate (\(q_{w} \text{ vs. } l\) Graph)**

\[
A = \frac{-0.3339}{\frac{\rho}{\mu} \frac{\mu}{\theta} \frac{\mu}{\theta}}
\]

\[
C_{A} = \frac{2.264 A}{\exp\left(\frac{k h}{70.6 \mu B} \frac{\mu}{\theta} \right) \frac{\mu}{\theta}^{2} \frac{\mu}{\theta}}
\]

For all methods, use \(r_{w} = r_{w} e^{-2}\) for a non-zero skin factor.

### TABLE 2
SYSTEM PROPERTIES FOR CASES SIMULATED ANALYTICALLY

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Single Well Centered in a Bounded Circular Reservoir</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, A</td>
<td>5760.0 Acres (9 mi²)</td>
</tr>
<tr>
<td>Reservoir drainage radius, (r_{e})</td>
<td>8937 ft</td>
</tr>
<tr>
<td>Net pay thickness, (h)</td>
<td>100.0 ft</td>
</tr>
<tr>
<td>Wellbore radius, (r_{w})</td>
<td>0.5 ft</td>
</tr>
<tr>
<td>Reservoir permeability, (k)</td>
<td>100.0 md</td>
</tr>
<tr>
<td>Reservoir porosity, (\theta)</td>
<td>0.15</td>
</tr>
<tr>
<td>Total pore volume, (q_{h})</td>
<td>3.76x10⁵ ft³</td>
</tr>
<tr>
<td>Fluid viscosity, (\nu)</td>
<td>2.0 cp</td>
</tr>
<tr>
<td>Total compressibility, (c_{e})</td>
<td>5x10⁻⁶ psi⁻¹</td>
</tr>
<tr>
<td>Initial Pressure, (p_{I})</td>
<td>2000.0 psi</td>
</tr>
<tr>
<td>Formation Volume Factor, (B)</td>
<td>1.0 RB/STB</td>
</tr>
</tbody>
</table>

### TABLE 3
ERROR ANALYSIS FOR CASES SIMULATED ANALYTICALLY

<table>
<thead>
<tr>
<th>Case</th>
<th>New Method Slope Error, (\epsilon)</th>
<th>New Method Intercept Error, (\epsilon)</th>
<th>Earlougher Method Slope Error, (\epsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square wave rate</td>
<td>2.145x10⁰</td>
<td>2.678x10⁰</td>
<td>6.539x10⁰</td>
</tr>
<tr>
<td>Sine Wave rate</td>
<td>4.276x10⁻¹</td>
<td>-1.343x10⁻²</td>
<td>&gt;&gt;100%</td>
</tr>
<tr>
<td>Random rate</td>
<td>2.099x10⁰</td>
<td>-4.370x10⁻¹</td>
<td>8.285x10⁻¹</td>
</tr>
</tbody>
</table>

### TABLE 4
SYSTEM PROPERTIES FOR CASES MODELED WITH THE FINITE-DIFFERENCE SIMULATOR

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Single Well Centered in a Bounded Circular Reservoir</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, A</td>
<td>40.0 Acres (9 mi²)</td>
</tr>
<tr>
<td>Reservoir drainage radius, (r_{e})</td>
<td>15.3 x 10⁵ ft³</td>
</tr>
<tr>
<td>Net pay thickness, (h)</td>
<td>30.0 ft</td>
</tr>
<tr>
<td>Wellbore radius, (r_{w})</td>
<td>0.2 ft</td>
</tr>
<tr>
<td>Reservoir permeability, (k)</td>
<td>100.0 md</td>
</tr>
<tr>
<td>Reservoir porosity, (\theta)</td>
<td>0.15</td>
</tr>
<tr>
<td>Total pore volume, (q_{h})</td>
<td>345 ft³</td>
</tr>
<tr>
<td>Fluid viscosity, (\nu)</td>
<td>0.4 cp</td>
</tr>
<tr>
<td>Total compressibility, (c_{e})</td>
<td>5x10⁻⁶ psi⁻¹</td>
</tr>
<tr>
<td>Initial pressure, (p_{I})</td>
<td>4800.0 psi</td>
</tr>
</tbody>
</table>
TABLE 5
ERROR ANALYSIS FOR CASES MODELED WITH THE
FINITE-DIFFERENCE SIMULATOR

<table>
<thead>
<tr>
<th>Case</th>
<th>Slope Error, ( S )</th>
<th>Intercept Error, ( I )</th>
<th>Time</th>
<th>Flowrate</th>
<th>Cum. Prod.</th>
<th>Pressure</th>
<th>( \Delta p_{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Rate</td>
<td>(-3.89\times10^{-4})</td>
<td>(-1.38\times10^{-2})</td>
<td>0.7200e 03</td>
<td>0.1500e 04</td>
<td>0.4500e 05</td>
<td>0.1607e 04</td>
<td>0.7200e 03</td>
</tr>
<tr>
<td>Logarithmic Rate Decline</td>
<td>(-9.218\times10^{-3})</td>
<td>2.000e 10^{-2}</td>
<td>0.1440e 04</td>
<td>0.3000e 03</td>
<td>0.6000e 05</td>
<td>0.1800e 04</td>
<td>0.2893e 03</td>
</tr>
<tr>
<td>Exponential Rate Decline</td>
<td>(-3.232\times10^{-5})</td>
<td>(9.27\times10^{-1})</td>
<td>0.1880e 04</td>
<td>0.3200e 04</td>
<td>0.5900e 06</td>
<td>0.164e 04</td>
<td>0.3180e 04</td>
</tr>
<tr>
<td>Rate Decline</td>
<td>(-3.90\times10^{-3})</td>
<td>(-1.70\times10^{-2})</td>
<td>0.3760e 04</td>
<td>0.3700e 04</td>
<td>0.4935e 06</td>
<td>0.302e 03</td>
<td>0.3158e 04</td>
</tr>
<tr>
<td>Hyperbolic Rate Decline</td>
<td>(-1.697\times10^{-2})</td>
<td>3.263e 10^{-2}</td>
<td>0.7920e 04</td>
<td>0.3900e 04</td>
<td>0.8655e 06</td>
<td>0.343e 03</td>
<td>0.3270e 04</td>
</tr>
<tr>
<td>Constant Pressure</td>
<td>(-1.898\times10^{-4})</td>
<td>2.513e 10^{-1}</td>
<td>0.8640e 04</td>
<td>0.3200e 04</td>
<td>0.8940e 06</td>
<td>0.340e 03</td>
<td>0.3120e 04</td>
</tr>
<tr>
<td>Special Case Results:</td>
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</tr>
<tr>
<td>Constant Rate</td>
<td>(P_{wF} vs. \ t Graph)</td>
<td>5.128x10^{-4}</td>
<td>5.717x10^{-3}</td>
<td>0.1656e 05</td>
<td>0.2200e 04</td>
<td>0.194e 07</td>
<td>0.564e 03</td>
</tr>
<tr>
<td>(Log ( q_{m} ) vs. ( t ) Graph)</td>
<td>-9.953x10^{-1}</td>
<td>-7.221x10^{-1}</td>
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</table>

TABLE 6
SIMULATED PRODUCTION DATA

<table>
<thead>
<tr>
<th>Time</th>
<th>Flowrate</th>
<th>Cum. Prod.</th>
<th>Pressure</th>
<th>( \Delta p_{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(hrs)</td>
<td>(S/3B)</td>
<td>(S/3B)</td>
<td>(psia)</td>
<td>(psia)</td>
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<td>0.7200e 03</td>
<td>0.1500e 04</td>
<td>0.4500e 05</td>
<td>0.1607e 04</td>
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<tr>
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<td>0.3000e 03</td>
<td>0.6000e 05</td>
<td>0.1800e 04</td>
<td>0.2893e 03</td>
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<td>0.5900e 06</td>
<td>0.164e 04</td>
<td>0.3180e 04</td>
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<tr>
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<td>0.4935e 06</td>
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<tr>
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<td>0.3200e 04</td>
<td>0.8940e 06</td>
<td>0.340e 03</td>
<td>0.3120e 04</td>
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<tr>
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<td>0.2200e 04</td>
<td>0.194e 07</td>
<td>0.564e 03</td>
<td>0.2119e 05</td>
</tr>
</tbody>
</table>

Fig. 1—Typical reservoir limits test performance.
Fig. 6—Rate histories for finite-difference simulation cases.

Fig. 7—Pressure drawdown data for finite-difference simulation cases.

Fig. 8—$\Delta p_\text{in}$ vs. $t$ curves for finite-difference simulation cases.

Fig. 9—Example graph of $\Delta p_\text{in}$ vs. $t$ for random rate case simulated analytically.