Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 5 — Development of the Diffusivity Equation for Liquid Flow

(Skepticism) ... the first step towards truth.
— Diderot (1746)

Topic: Development of the Diffusivity Equation for Liquid Flow

Objectives: (things you should know and/or be able to do)

- Be able to identify all of the components required to develop a governing (partial differential) equation for the flow of fluids in porous media.
  - Mass continuity equation (mass balance on the system):
    - Continuity equation.
  - Equation of motion (force balance):
    - Darcy's law (relates velocity to pressure gradient).
  - Equation of state (relates fluid density with pressure, temperature, and composition):
    - Equation of state for a slightly compressible liquid.
    - Real gas law for compressible gases.
  - Constitutive equation (relates the shear rate and shear stress of the fluid):
    - Unnecessary, we assume Newtonian fluids (viscosity term in Darcy's Law).
  - Energy conservation equation (energy balance on the system):
    - Unnecessary, we assume isothermal flow conditions.

- Mass Continuity Equation: Be able to describe in words and in terms of mathematical expressions, the mass continuity relation for flow through porous media.

  Words:
  \[
  \text{rate of mass flow into the system during the interval, } \Delta t \quad - \quad \text{rate of mass flow out of the system during the interval, } \Delta t = \text{rate of mass accumulation in the system during the interval, } \Delta t
  \]

  Mathematical Expression:
  \[
  \nabla \cdot (\rho \vec{v}) = - \frac{\partial (\rho \phi)}{\partial t}
  \]

- Equation of Motion: Be able to recognize that Darcy's law is a force balance on the system (relates flow velocity and pressure drop).

  Darcy's law: \( \vec{v} = - \frac{k}{\mu} (\nabla p + \rho \vec{g}) \) (vector form)

- Equation of State: Be able to state the definition of compressibility.

  Definition of isothermal compressibility for a fluid: \( c = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \)

- Diffusivity Equations for Liquid Flow ("pressure" form): Be able to develop the "diffusivity" equations for the flow of a slightly compressible liquid in porous media.

  **Gradient-Squared** Case: General form for a slightly compressible liquid.
  \[
  c (\nabla^2 p)^2 + \nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t}
  \]

  **Small and Constant Compressibility** Case: Base relation for all developments in reservoir engineering and well testing.
  \[
  \nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t}
  \]
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 5 — Development of the Diffusivity Equation for Liquid Flow

- **Diffusivity Equations for Liquid Flow ("pseudopressure/pseudotime" forms):** Be able to derive the pseudopressure/pseudotime forms of the diffusivity equation for cases where fluid density and viscosity are functions of pressure.

  "Pseudopressure-Time" Form
  \[ \nabla^2 p_p = \frac{\phi \mu c_i}{k} \frac{\partial p_p}{\partial t} \]

  "Pseudopressure-Pseudotime" Form
  \[ \nabla^2 p_p = \frac{\phi}{k} (\mu c_i)_0 \frac{\partial p_p}{\partial t_a} \]

  where the "pseudopressure" function, \( p_p \), is given by:

  \[ p_p = \left[ \frac{\mu B}{k} \right] \int_{p_{\text{base}}}^{p} \frac{k}{\mu B} \, dp \quad \text{or} \quad p_p = (\mu B)_0 \int_{p_{\text{base}}}^{p} \frac{1}{\mu B} \, dp \]

  and the "pseudotime" function, \( t_a \), is given by:

  \[ t_a = (\mu c_i)_0 \int_0^t \frac{1}{\mu(p) c_i(p)} \, dt \]

Lecture Outline:

- Review and discuss the mass continuity concept.
- Discuss the physical concepts required to develop relations for the flow of fluid in porous media (Darcy's law). Be able to draw analogies with the following:
  - Heat flux (Fourier's law)
  - Mass flux (Fick's law)
  - Electric flux (Ohm's law)
  - Fluid mechanics—flow in pipes (Poiseuille's law)
- Introduce the density form of the diffusivity equation for fluid flow in porous media. Discuss its practicality as a flow relation for petroleum engineering applications.
- Derivation of the "diffusivity" equation for the flow of a slightly compressible liquid in porous media (i.e., the "pressure" formulation), for a general flow geometry.
  - Derivative expansions
  - Definitions of rock and fluid compressibility
  - Identification of "non-linear" terms
  - Final, linearized partial differential equation. Discuss applications and limitations.
- Derivation of the "diffusivity" equation in terms of "pseudopressure" and "pseudotime" for any homogeneous fluid (gas or liquid), for a general flow geometry.
  - Density/formation volume factor relations
  - Initial (nonlinear) forms of the gas and liquid diffusivity equations.
  - Introduction of "pseudopressure" concept and derivative expansions
  - Incorporation of pseudopressure concept in place of pressure
  - Identification of "non-linear" terms and introduction of "pseudotime" concept
  - Final, linearized partial differential equation. Discuss applications and limitations.
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 5 — Development of the Diffusivity Equation for Liquid Flow

Reading Assignment:

- Review the attached notes.
  - Derivation of the "diffusivity" equation—slightly compressible liquid case.
  - Derivation of the "diffusivity" equation in terms of "pseudopressure" and "pseudotime" for any homogeneous fluid (gas or liquid).
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 5 — Development of the Diffusivity Equation for Liquid Flow

Exercises: For your own practice/skills building—do NOT turn in!

- In each of these derivations/problems you are to work in complete detail and you must show all work.
  - Derive the "diffusivity" equation for the flow of a slightly compressible liquid in porous media—"pressure" form, general flow geometry.
  - Derive the "diffusivity" equation in terms of "pseudopressure" and "pseudotime" for any homogeneous fluid (gas or liquid)—general flow geometry.
- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- **Problem:**
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?

- **Assumptions and Limitations:**
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?

- **Practical Applications:**
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how could the solutions/results be used in practice?

- **Discussion:**
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.

- **Recommendations/Extensions:**
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Derivation of the Diffusivity Equation for Single-Phase Liquid Flow (in any Flow Geometry)

(from Petroleum Engineering 620 Course Notes — 1997)

Petroleum Engineering 620
Fluid Flow in Reservoirs
Diffusivity Equation for Single Phase Liquid Flow in any Flow Geometry

General Introduction

Our goal is to develop a mathematical expression (i.e., a partial differential equation) which describes the flow of a single fluid in a porous media with respect to time and distance (i.e., the so-called "diffusivity equation"). This relation should make no assumptions about rates or pressures which may be imposed on the system. We should begin by developing results for a general flow geometry, then adapt these results later to a specific flow geometry (e.g., radial, linear, or spherical flow) — as the need arises.

Derivation of the Single-Phase Diffusivity Equation (Density Formulation)

In order to develop a partial differential equation for the flow of fluids in porous media the following physical concepts must be considered:

1. Mass continuity equation (mass balance on the system):
   - Continuity equation.
2. Equation of motion (force balance):
   - Darcy's law (relates velocity to pressure gradient).
3. Equation of state (relates fluid density with pressure, temperature, and composition):
   - Equation of state for a slightly compressible liquid.
   - Real gas law for compressible gases.
4. Constitutive equation (relates the shear rate and shear stress of the fluid):
   - Unnecessary, we assume Newtonian fluids (viscosity term in Darcy's Law).
5. Energy conservation equation (energy balance on the system):
   - Unnecessary, we assume isothermal flow conditions.

The mass continuity equation is described in words as

\[
\begin{bmatrix}
\text{rate of mass flow into the system during the interval, } \Delta t \\
\text{rate of mass flow out of the system during the interval, } \Delta t \\
\end{bmatrix} = \begin{bmatrix}
\text{rate of mass accumulation in the system during the interval, } \Delta t 
\end{bmatrix}
\] (1)

The mathematical form of the mass continuity equation is

\[
\nabla \cdot (\rho \vec{v}) = -\frac{\partial (\phi \rho)}{\partial t}
\] (2)

where
\[
\begin{align*}
\vec{v} & = \text{fluid velocity vector} \\
\rho & = \text{fluid density} \\
\phi & = \text{porosity, fraction of bulk volume} \\
t & = \text{time}
\end{align*}
\]

The equation of motion used for fluid flow in porous media is known as Darcy's law. Although Darcy's law was empirically derived by performing experiments of water flow through sand packs, and while this concept was only partially verified in an analytical sense (using Navier-Stokes theory), we consider Darcy's law accurate for the laminar flow of fluids in porous media. Darcy's law is given (in vector form) as

\[
\vec{v} = -\frac{k}{\mu} (\nabla p + \rho \vec{g})
\] (3)

where
\[
\begin{align*}
\nabla p & = \text{pressure gradient} \\
k & = \text{effective permeability} \\
\mu & = \text{fluid viscosity} \\
\vec{g} & = \text{gravity vector}
\end{align*}
\]
 Combining Eqs. 2 and 3 gives us
\[
\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi \rho)}{\partial t} \nonumber \quad \text{...........................................(4)}
\]

If we assume horizontal flow, the gravity term can be dropped, which reduces Eq. 4 to
\[
\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi \rho)}{\partial t} \nonumber \quad \text{...........................................(5)}
\]

The general \( \nabla \) operator is defined as
\[
\nabla \cdot a = \frac{\partial a}{\partial y} \nabla y
\]

where pressure or pseudopressure will be used as the "y" variable. The \( \nabla \cdot \nabla a \) terms are given by:

Linear Flow:
\[
\nabla \cdot \nabla a = \nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}
\]

Radial Flow:
\[
\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial a}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}
\]

For the developments in this section, we will leave the results in the \( \nabla \) notation (indicating a general flow geometry). In later developments the appropriate operator(s) for a particular flow geometry (e.g., radial flow) will be substituted.

Eq. 5 is the fundamental form of the diffusivity equation in terms of fluid density and it represents the starting point for our developments in terms of pressure, pressure-squared, and pseudopressure. At this point, the compressibility equation-of-state has not been incorporated into the diffusivity equation because this substitution is not yet necessary. In fact, the diffusivity equation can be worked entirely in terms of density—however, this form would not be practical for field applications.

We note that Eq. 5 includes the constitutive equation implicitly via the viscosity term in Darcy's law (Eq. 3). Virtually all well testing conditions are considered isothermal, hence the conservation of energy equation can be neglected.

Assuming that the permeability, \( k \), and the fluid viscosity, \( \mu \), are constant (i.e., the usual "liquid" flow assumptions), Eq. 5 reduces to
\[
\nabla \cdot \left[ \rho \nabla p \right] = \frac{\mu}{k} \frac{\partial (\phi \rho)}{\partial t} \nonumber \quad \text{...........................................(6)}
\]

**Derivation of the Single-Phase Diffusivity Equation (Slightly Compressible Liquid Case)**

If we expand the terms on the left-hand-side of Eq. 6 using the product rule, and we expand the term on the right-hand-side using the chain rule, then we obtain
\[
(\nabla \cdot \rho) \nabla p + \rho \nabla \cdot \nabla p = \frac{\mu}{k} \frac{\partial (\phi \rho)}{\partial t} \nonumber \quad \text{...........................................(7)}
\]
Using the chain rule for the grad terms (\(\nabla\)), the first term on the left-hand-side of Eq. 7 can be written as

\[
(\nabla \cdot \rho) \nabla p = \frac{\partial \rho}{\partial p} \nabla p \cdot \nabla = \frac{\partial \rho}{\partial p} (\nabla p)^2
\]  

(8)

The second term on the left-hand-side of Eq. 7 can also be reduced using the definition of the \(\nabla\) operator. This gives

\[
\rho \nabla \cdot \nabla p = \rho \nabla^2 p
\]  

(9)

Substituting Eqs. 8 and 9 into Eq. 7, we obtain

\[
\frac{\partial p}{\partial p} (\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k} \frac{\partial^2 \phi}{\partial t \partial p}
\]  

(10)

By applying the product rule to the \(\frac{\partial (\phi p)}{\partial p}\) term in Eq. 10 we have

\[
\frac{\partial (\phi p)}{\partial p} = \phi \frac{\partial p}{\partial p} + \rho \frac{\partial \phi}{\partial p}
\]

Factoring out the \(\phi p\) terms gives us

\[
\frac{\partial (\phi p)}{\partial p} = \phi p \left[ \frac{1}{\rho} \frac{\partial p}{\partial p} + \frac{1}{\phi} \frac{\partial \phi}{\partial p} \right]
\]

The definition of fluid compressibility is given as

\[
c = \frac{1}{\rho} \frac{\partial p}{\partial p}
\]  

(11)

and, the definition of pore-volume compressibility is given as

\[
f = \frac{1}{\phi} \frac{\partial p}{\partial \phi}
\]  

(12)

Combining the definition of compressibility with the expansion of \(\frac{\partial (\phi p)}{\partial p}\) we obtain

\[
\frac{\partial (\phi p)}{\partial p} = \phi p (c + f)
\]

The "total" compressibility is defined as \(c_t = c + f\), and the last relation can be written as

\[
\frac{\partial (\phi p)}{\partial p} = \phi p c_t
\]  

(13)

Substituting Eq. 13 into Eq. 10, we obtain

\[
\frac{\partial p}{\partial p} (\nabla p)^2 + \rho \nabla^2 p = \rho \frac{\phi c_t}{k} \frac{\partial p}{\partial t}
\]

Dividing through by the fluid density gives

\[
\frac{1}{\rho} \frac{\partial p}{\partial p} (\nabla p)^2 + \nabla^2 p = \frac{\phi c_t}{k} \frac{\partial p}{\partial t}
\]

Using the definition of fluid compressibility (Eq. 11) we obtain our "final" form

\[
c (\nabla p)^2 + \nabla^2 p = \frac{\phi c_t}{k} \frac{\partial p}{\partial t}
\]  

(14)

The \(c(\nabla p)^2\) term in Eq. 14 is clearly nonlinear, as this term squares the gradient of pressure (which is nonlinear in and of itself) and multiplies this group by the compressibility (which is generally a weak function of pressure for liquids above the bubblepoint.
pressure). Solving Eq. 14 by analytical means is very difficult, and the solution given by Finjord and Aadnoy\textsuperscript{1} (which is a perturbation solution) is not well suited for the practical analysis of well test and/or production data.

However, if we assume a small and constant compressibility (as required by the equation of state for a "slightly" compressible liquid), then $c(\nabla p)^2$ can be neglected. For reference, the equation of state for a "slightly" compressible liquid is given as

$$\rho = \rho_0 \exp[c(p - p_0)]$$

Therefore, if the $c(\nabla p)^2$ term in Eq. 14 is neglected, Eq. 14 relation becomes

$$\nabla^2 p = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t}$$

Eq. 16 is familiar as a typical diffusion-type, partial differential equation and it can be solved analytically for a variety of flow geometries and boundary conditions (all of which assume that the diffusivity term, $(\phi \mu c_t)/k$, is constant).

**Derivation of the Single Phase Diffusivity Equation (Pseudopressure and Pseudotime Case)**

The developments shown in the previous section focused on the derivation of the diffusivity equation for single-phase liquid flow where the liquid was considered to be of small and constant compressibility. Generally, these conditions are only satisfied for black oil fluids above the bubblepoint pressure. For black oil fluids below the bubble point pressure (and for natural gases) Eq. 16 will be an approximation at best, and a complete failure at worst.

Therefore, we must develop a form of the diffusivity equation where a general behavior for $\rho$, $\mu$, and $k$ can be assumed. The, $(\rho k)/\mu$, term in Eq. 5 will be treated as general, with no assumptions on its behavior with respect to pressure. Recalling Eq. 5, we have

$$\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi \rho)}{\partial t}$$

For a black oil fluid, the oil density is given by

$$\rho_o = \frac{1}{B_o} (\rho_{osc} + 0.01357 R_g \gamma_g)$$

where

- $\rho_o$ = Oil density at reservoir conditions, lbm/ft\textsuperscript{3}
- $\rho_{osc}$ = Oil density at stock tank (or standard) conditions, lbm/ft\textsuperscript{3}
- $B_o$ = Oil formation volume factor, RB/STB
- $R_g$ = Gas-oil-ratio, scf/STB (constant for $p > p_b$)
- $\gamma_g$ = Stock tank gas gravity (air=1.0)

For gases, the density is given by the real gas law as

$$\rho_g = \frac{p M}{z R T}$$

where

- $\rho_g$ = Gas density at reservoir conditions, lbm/ft\textsuperscript{3}
- $p$ = Reservoir pressure, psia
- $M$ = Molecular weight of the gas, lbm/lb-mole
- $z$ = $z$-factor at reservoir pressure
- $R$ = Universal gas constant (10.732 [psia-ft\textsuperscript{3}]/[lb-mole-oR])
- $T$ = Reservoir temperature, oR

and the gas formation volume factor is given by
\[ B_g = \frac{P_{sc}}{z_{sc} T_{sc}} \frac{z T}{p} \]  

(19)

Combining Eqs. 18 and 19 we obtain

\[ \rho_g = \frac{P_{sc} M}{z_{sc} R T_{sc}} \frac{1}{B_g} \]

Using the definition of gas density at standard conditions we have

\[ \rho_g = \frac{\rho_{gsc}}{B_g} \]  

(20)

where

\[ \rho_{gsc} = \frac{P_{sc} M}{z_{sc} R T_{sc}} \]

Substituting Eq. 17 into Eq. 5 gives us the general diffusivity equation for a liquid

\[ \nabla \cdot \left[ k_o \frac{k_o}{\mu_o B_o} \nabla p \right] = \frac{\partial}{\partial t} \left[ \phi \right] \]  

(21)

and substitution of Eq. 20 into Eq. 5 gives us the general diffusivity equation for the flow of a single gas phase.

\[ \nabla \cdot \left[ k_g \frac{k_g}{\mu_g B_g} \nabla p \right] = \frac{\partial}{\partial t} \left[ \phi \right] \]  

(22)

We note that the forms given by Eqs. 21 and 22 are identical. Using a set of general parameters \( \mu, k, \) and \( B \), a single form of the diffusivity equation for both single-phase oil and gas flow can be developed. We must also note that the \( k/(\mu B) \) terms in Eqs. 21 and 22 represent non-linearities which must be resolved. In particular, we need to define a function that combines these variables but retains the \( k/(\mu B) \nabla p \) form. This function is called "pseudopressure" and is defined by the behavior of the \( k/(\mu B) \) term.

The normalized pseudopressure function, \( p_p \), is defined as

\[ p_p = \left[ \frac{\mu B}{k} \right] n \int_{P_{base}}^{P} \frac{k}{\mu B} dp \]  

(23)

where \( n \) indicates the normalizing condition—where the normalizing pressure, \( p_n \), is completely arbitrary. Throughout this work, the initial reservoir pressure, \( P_i \), will be used as the normalizing pressure (this form, while arbitrary, has several advantages—particularly in coupling the gas material balance relation with various forms of the gas flow equation).

Using the definition given by Eq. 23, the pseudopressure gradient, \( \nabla p_p \), can be written as

\[ \nabla p_p = \frac{\partial p_p}{\partial p} \nabla p \]  

(24)

Combining Eqs. 23 and 24 gives us

\[ \frac{\partial p_p}{\partial p} = \left[ \frac{\mu B}{k} \right] n \frac{k}{\mu B} \]  

(25)

The partial derivative of the pseudopressure function with respect to time is obtained using the chain rule. This result is

\[ \frac{\partial p_p}{\partial t} = \frac{\partial p_p}{\partial p} \frac{\partial p}{\partial t} \]  

(26)
Combining these developments and using a generic form given by Eqs. 21 and 22, we obtain the following general relation as our starting point

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right]$$

Expanding the right-hand-side of Eq. 27 using the chain rule gives us

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] \frac{\partial p}{\partial t}$$

Using the product rule we have

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \left[ \frac{1}{B} \phi \frac{\partial B}{\partial t} - \frac{\phi}{B} \right] \frac{\partial p}{\partial t}$$

which after some rearranging into a more familiar form gives us

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \frac{\phi}{B} \left[ \frac{1}{\phi} \frac{\partial \phi}{\partial t} - \frac{1}{B} \frac{\partial B}{\partial t} \right] \frac{\partial p}{\partial t}$$

Recalling the definition of fluid compressibility, we have

$$c = \frac{1}{\rho} \frac{\partial p}{\partial \rho} = -\frac{1}{B} \frac{\partial B}{\partial \rho}$$

Recalling the definition of pore volume compressibility, we have

$$c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial \rho}$$

Substituting Eqs. 12 and 29 into Eq. 28, and then using the definition, \(c_t = c + c_f\), in this result, gives us the following

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \phi c_t \frac{\partial p}{\partial t}$$

Substituting Eq. 30 back into Eq. 27 we obtain

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \phi c_t \frac{\partial p}{\partial t}$$

Combining Eqs. 24 and 25 and solving for \(\frac{k}{\mu B} \nabla p\) we have

$$\frac{k}{\mu B} \nabla p = \left[ \frac{k}{\mu B} \right]_{B} \nabla p_p$$

Also, if we combine Eqs. 25 and 26 and solve for \(\frac{\partial p}{\partial t}\) we obtain

$$\frac{\partial p}{\partial t} = \left[ \frac{k}{\mu B} \right] \frac{\mu B}{k} \frac{\partial p_p}{\partial t}$$

Substituting Eqs. 32 and 33 into Eq. 31 yields

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p_p \right] = \frac{\phi u c_t}{k} \frac{\partial p_p}{\partial t}$$

Canceling the \(\frac{k}{\mu B}\) terms we obtain

$$\nabla \cdot \nabla p_p = \frac{\phi u c_t}{k} \frac{\partial p_p}{\partial t}$$

Recalling that \(\nabla \cdot \nabla a = \nabla^2 a\), we obtain the following as our final result

$$\nabla^2 p_p = \frac{\phi u c_t}{k} \frac{\partial p_p}{\partial t}$$
While the left-hand-side of Eq. 34 has no nonlinear terms, the μ, c_t, and k terms on the right-hand-side are not assumed constant and represent non-linearities due to the multiplication of these terms with \( \frac{dp}{dt} \), as well as with each other. In order to develop analytical solutions using Eq. 34, we must either assume \( \frac{μc_t}{k} \) to be constant, or we must develop a linearizing pseudotime that accounts for changes in \( \frac{μc_t}{k} \) with respect to pressure.

For gas reservoirs, we typically assume that the effective permeability, k, is constant, and we also assume that \( μc_t \) is constant during transient drawdown flow conditions.

Because we know that the \( \frac{μc_t}{k} \) term is not really constant, we must then develop another linearizing function (in this case a pseudotime) in order to account for the variation in the \( \frac{μc_t}{k} \) term. Agarwal\textsuperscript{4} proposed the following "intuitive" pseudotime function

\[
t_a = (μc_t)n \int_0^t \frac{1}{μ(p)c_t(p)} \, dt \quad \text{(35)}
\]

Lee and Holditch\textsuperscript{5} verified this function for the analysis of pressure buildup tests and gave analytic criteria for the application of Eq. 35 using wellbore pressures to evaluate μ and c_t (but these results are only valid for the transient pressure buildup case, pseudotime is not used for the analysis or modelling of transient pressure drawdown behavior).

Fraim and Wattenbarger\textsuperscript{6} proposed a slightly different form of the Agarwal\textsuperscript{4} pseudotime function for boundary-dominated (pseudosteady-state) flow. This pseudotime is given by

\[
t_a = (μc_t)n \int_0^t \frac{1}{μ(p)c_t(p)} \, dt \quad \text{(36)}
\]

where the average reservoir pressure, \( \bar{p} \), is used in place of the wellbore pressure, for the evaluation of \( μc_t \). While Eq. 36 is rigorously correct for boundary-dominated flow, simulation studies suggest that Eq. 36 is also correct for modelling gas well performance during transient flow.

Our present objective is to combine the definition of the pseudotime with our general form of the diffusivity equation, Eq. 34. Taking the partial derivative of Eq. 35 with respect to time gives

\[
\frac{∂t_a}{∂t} = \frac{(μc_t)n}{μc_t} \quad \text{(37)}
\]

Applying the chain rule to the \( \frac{∂p}{∂t} \) term in Eq. 34, we have

\[
\frac{∂p}{∂t} = \frac{∂t_a}{∂t} \frac{∂p}{∂t_a}
\]

Combining this result with Eq. 37 gives us

\[
μc_t \frac{∂p}{∂t} = (μc_t)n \frac{∂p}{∂t_a}
\]

Substituting Eq. 34 into Eq. 38, we obtain

\[
\nabla^2 p_p = \frac{φ}{k} (μc_t)n \frac{∂p}{∂t_a} \quad \text{(39)}
\]

where the normalized pseudopressure function, \( p_p \), is given by
\[ p_p = \left[ \frac{\mu B}{k} \right]_n \int_{P_{base}}^P \frac{k}{\mu B} \, dp \] ..................................................(23)

and the normalized pseudotime function, \( t_a \), is

\[ t_a = (\mu c_l)_n \int_0^t \frac{1}{\mu(p)c_l(p)} \, dt \] ..................................................(35)

If we assume that the effective permeability, \( k \), is constant (as is typically done), Eq. 23 reduces to

\[ p_p = (\mu B)_n \int_{P_{base}}^P \frac{1}{\mu B} \, dp \] ..................................................(40)

Finally, we note that the definitions of pseudopressure and pseudotime are valid for the single-phase flow of both gases and liquids. The pseudopressure and pseudotime concepts can be extended to multiphase flow, but these results require knowledge of both the pressure and saturation histories, and, as such, these results are considered to be theoretical, with little (if any) practical applications possible without full-scale reservoir simulation.

Because the pseudopressure/pseudotime diffusivity equation is written in a general form (i.e., the "liquid equivalent" form), we can use the solutions derived for the "slightly compressible liquid" case to model the single-phase flow of both gases and compressible liquids—assuming that the appropriate pseudopressure and pseudotime functions are used for each case.

References
Validation of the Concept of Pseudotime—Solution-Gas-Drive Reservoirs Case


Petroleum Engineering 620
Fluid Flow in Reservoirs
Fig. 4—Correlation of average pseudopressure, constant-pressure production.

Fig. 5—Variation of average compressibility/mobility ratio with time.

Fig. 12—Variation of $\alpha \frac{\partial p}{\partial t}$ vs. distance, constant-rate production.

References — Diffusivity Equation for Liquid Flow:


Petroleum Engineering 620
Fluid Flow in Reservoirs
Calculation of Productivity Factors for Oil-gas-water Systems in the Steady State

By H. H. Evinger* and M. Muskat*

(New York Meeting, February 1942)

A method of calculating productivity factors for oil, gas, and water systems in the steady state is presented as an illustration of the quantitative application of the fundamental data on the flow properties of heterogeneous fluid systems. While the numerical results of the calculations cannot be applied directly to field conditions, because they are based on permeability data for unconsolidated sands and involve specific assumptions regarding the nature of the reservoir fluids, the methods of analysis and interpretation should have significance in considering certain specific field situations when the necessary field and laboratory data become available. Moreover, it is felt that even under the idealized conditions to which the numerical calculations refer they should serve to give at least the orders of magnitude of the effects of gas-oil ratio and connate water on observed productivity factors. Within the indicated limitations, the computations imply that the foregoing factors can explain only a part of the apparent discrepancy between the homogeneous fluid productivity factors and those obtained recently in field measurements on the West Coast.

Introduction

Sufficient laboratory research has been carried out in the last several years on the fundamental features of the flow of heterogeneous fluids through porous media to call for consideration of what can be done with this type of information. It has, of course, already led to a much clearer general understanding of the mechanics of oil production. However, examples of the quantitative application of the results of such laboratory studies are few in the literature. Part of the explanation for this fact lies undoubtedly in the lack of sufficient laboratory data relating to practical fluids and sand systems such as would occur in actual oil reservoirs. However, unless it can be shown that if such data were obtained quantitative applications could be made, serious doubt may be cast upon the desirability of carrying on further extensive experimental research work on fluid flow.

The purpose of this paper is to present one approach to the problem of quantitative application of laboratory data to practical field problems. Because the published results on the basic experiments with three-phase fluid systems in porous media are limited entirely to unconsolidated sands, the numerical applications made here will not pertain directly to any actual field situation. However, they should serve to illustrate what could be done with fundamental laboratory data if they were extended to cover the field of consolidated sands. From this point of view, it is hoped it will serve to encourage further laboratory research on this problem.

The particular application to be treated here relates to the calculation of the productivity factors for oil-gas-water systems in the steady state. This specific problem seems especially appropriate for discussion in view of the recent publication by Johnston1 of the results of produc-

* Gulf Research and Development Co., Pittsburgh, Pa.

1 References are at the end of the paper.
tivity-factor measurements in California (see also Pyle and Sherborne\cite{3}). These measurements indicated that the productivity factors as determined in the field are generally very much smaller, by factors of 50 to 100, than those calculated on the radial homogeneous fluid flow formula. As Johnston points out, it is to be expected that the homogeneous fluid formula will predict productivity factors higher than those that will be observed in practical field tests where the fluid system is multiphase. It is stated, in fact, that the apparent discrepancy involving the factors of 50 to 100 must arise from the heterogeneous nature of the flow system in the actual field reservoirs and the quality of the well completion. It is obviously pertinent to inquire how much of the apparent discrepancy can be attributed to the multiphase characteristics of the flow systems. The treatment to follow represents the results of such an inquiry. Admittedly its significance is limited because it has of necessity been based upon data obtained from unconsolidated sands.\cite{3} Moreover, the fluid system assumed in the numerical part of the work undoubtedly differs to some extent from those involved in the actual field tests. Nevertheless, it is felt that the calculations are useful in indicating the order of magnitude of the effect that may be attributed to the multiphase features of the flow system. In any case they should serve to show how the proper calculations could be carried out if and when the appropriate practical laboratory data on consolidated sands become available.

**Analytical Procedure**

The analytical basis for the following calculations is the generalization of that previously developed and applied in the study of two-phase systems flowing through porous media.\cite{4} For the general case of a three-phase system such as gas, oil and water, the dynamical equations are obtained by writing an equation of continuity for each of the three phases, as follows:

\[
\nabla \cdot \left( \frac{k_v}{\mu_v} \nabla p \right) = f \frac{\partial}{\partial t} (\rho_v/\beta) \quad [1a]
\]

\[
\nabla \cdot \left( \frac{S k_o}{\beta \mu_o} \nabla p + \frac{\gamma k_w}{\mu_w} \nabla p \right) = f \frac{\partial}{\partial t} \left\{ \frac{S \rho_o}{\beta} + \gamma (1 - \rho_o - \rho_w) \right\} \quad [1b]
\]

\[
\nabla \cdot \left( \frac{k_w}{\mu_w} \nabla p \right) = f \frac{\partial \rho_w}{\partial t} \quad [1c]
\]

where \(k, \mu,\) and \(\rho\) denote, respectively, permeability, viscosity, and saturation, and the subscripts \(o, g,\) and \(w\) refer to the oil, gas and water phases; \(p\) denotes the fluid pressure, \(f\) the porosity, \(\gamma\) the gas density and \(S\) the solubility of the gas in the oil expressed as volume of gas dissolved per unit volume of residual oil. \(\beta\) represents the expansion factor of the oil, and for any particular pressure it is equal to the volume occupied by unit volume of residual oil. In setting up these equations, it has been assumed that both the solubility of the gas in the water and the expansion of the water may be neglected.

If the assumption is introduced that the flow takes place under steady-state conditions, which might be regarded as justifiable for short periods of time, the right sides of Eqs. 1a, 1b and 1c become zero and their first integrals may be obtained easily. Equating the left members of these equations to zero, the first integrals for a radial system are given by:

\[
Q_o = \frac{2\pi rh k_o}{\beta \mu_o} \frac{d\rho}{dr} \quad [2a]
\]

\[
Q_o = 2\pi rh (S k_o/\beta \mu_o + \gamma k_w/\mu_w) \frac{d\rho}{dr} \quad [2b]
\]

\[
Q_w = 2\pi rh \frac{k_w}{\mu_w} \frac{d\rho}{dr} \quad [2c]
\]

where \(Q\) represents volume rates of production, the units being cubic centimeters of residual oil per second for the oil phase and of gas (reduced to standard conditions) per second for the gas phase.
Denoting the gas-oil ratio \( Q_g/Q_o \) by \( R \), and the water-oil ratio \( Q_w/Q_o \) by \( W \), it is found from Eqs. 2, by simple division, that:

\[
R = S + \alpha(\phi)k_0/k_o \quad [3a]
\]

and

\[
W = \Phi(\phi)\frac{k_w}{k_o} \quad [3b]
\]

where:

\[
\alpha(\phi) = \frac{\gamma \beta_\phi}{\mu_\phi} \quad \text{and} \quad \Phi(\phi) = \frac{\beta_\phi}{\mu_w} \quad [4]
\]

It should be noted here that \( \alpha \) and \( \Phi \) are properties of the fluid system and are functions of \( \phi \) alone (for isothermal conditions), whereas the permeabilities, \( k_i \), are properties of the porous medium and are determined by the saturations \( \rho_w \) and \( \rho_o \).

Direct integration of Eq. 2a gives:

\[
\frac{Q_o}{2\pi h k} \log \frac{r_o}{r} = \int_0^{r_o} \frac{k_i}{k_0} \frac{1}{\mu_\phi} d\phi \quad [5]
\]

where \( r_o \) is the reservoir radius, \( r_0 \) the effective reservoir radius, and \( k \) the homogeneous fluid permeability. The indefinite integral may be evaluated, as was done for the two-phase system, and the pressure distribution may be determined. However, it will be sufficient for the calculation of the productivity factor to consider only the limiting form of Eq. 5 obtained by putting \( r = r_w, \phi = \phi_w \), and then letting \( \phi_w \) approach \( \phi_o \). Eq. 5 then takes the following form:

\[
\frac{Q_o}{h k \Delta \phi} = \frac{2\pi (k_i/k_0)}{\mu_\phi \log \frac{r_o}{r_w}} = F \quad [6]
\]

where \( F \) is the productivity factor, \((k_i/k_0)\), is the relative permeability to oil corresponding to the sand saturation at the reservoir radius, and \( r_w \) is the well radius. The values of \( \mu_\phi \) and \( \beta \) to be used in numerical calculations are those corresponding to the reservoir pressure. From Eq. 6 it is clear that, so far as the effect of fluid saturation is concerned, the term \((k_i/k_0)\), alone will cause variations in \( F \). For the case in which the pore space is completely filled with oil, this term has the

value unity. If free gas or connate water is present, however, its value will be lowered. This effect is shown in some of the accompanying curves.

Several items should be noted with respect to the definition of the productivity factor as given by Eq. 6. In the first place, it has been defined so as to refer to the limiting slope of the production-rate vs. pressure-differential curve for vanishing pressure differential; that is, at the origin. This special form for the definition has been used here because the theoretical calculations, as made previously for the two-fluid system, and as would be implied by Eq. 5 for the three-fluid system, lead to a nonlinear variation of the production rate with pressure differential. Thus, the productivity factor, as ordinarily defined, which involves division of the production rate by the pressure differential, would not be constant, but would decrease as the production rate increases. It is necessary therefore, from a theoretical point of view at least, to refer the productivity factor to some basic reference point, and the most natural reference basis seems to be the limiting conditions at vanishing pressure differential. From a practical standpoint this means that if there is any pronounced curvature in the curve of observed production rate vs. pressure differential the curve should be extrapolated to the origin and its slope at that point be used in calculating the productivity factor.

Another factor to be noted about the definition of Eq. 6 relates to the reduction of the productivity factor to unit sand thickness, thus removing the effect of the absolute value of the thickness of the producing zone. This corresponds to the specific productivity factor that has already appeared in the literature. Where no difficulties arise from the linear averaging of strata of different effective permeabilities, the over-all productivity factors for a natural system should be given simply by
the product of the $F$ values of Eq. 6 and the effective sand thickness.

The expression of $F$ in terms of a unit permeability has likewise been introduced only to avoid the assumption of specific available the simplifications can be dropped and the appropriate modifications can be made in the calculations. To avoid misinterpretation, explicit values of $F$ will not be presented here, and the numerical values

![Diagram](image)

**Fig. 1.—Solubility, Expansion Factor, and Oil Viscosity for Reservoir Fluids as a Function of Reservoir Pressure (Gauge).**

Expansion factors are equivalent to $\beta - 1$.

numerical values for the permeability. This is not to be construed as implying that the relative permeability curves for the various fluid phases are the same for all sands regardless of the homogeneous fluid permeability. On the contrary, it is to be understood that the only way in which quantitative checks between field and laboratory computations of the productivity factors can be made is by using for the relative oil permeability, $(k_o/k)$, data appropriate to the particular producing pay in question. The reduction of the productivity factor to a unit permeability will then be only a matter of convenience and convention, and, in fact, the use of $Fk$ rather than $F$ for the productivity factor would then probably be more advisable from a practical point of view.

Again it is to be emphasized that these various simplifying factors of the definition of Eq. 6 can be justified only because the calculations as a whole are intended to serve essentially as an illustration of the method of calculating productivity factors. It is to be understood, however, that when the necessary data for specific cases are will be limited to those of $k_o/k$ which give the ratio of the productivity factor for the multiphase system to that for the equivalent homogeneous fluid system.

**Experimental Data**

The properties of the oil used in these calculations are the same as those employed in the previous work on the low-pressure, two-phase system. Fig. 1 gives the variation with pressure of the gas solubility $S$, oil expansion $\beta$, and oil viscosity $\mu_o$. For the gas, the viscosity was assumed to be constant at 0.01 c.p., and its density was taken to be proportional to the absolute pressure. The permeability-saturation data are shown graphically in Figs. 2, 3, 4 and 5. These curves were obtained, by means of a great deal of smoothing and interpolation, from the data of Leverett and Lewis. As already indicated, the porous media employed in their experiments consisted of unconsolidated sands, and inasmuch as no similar work has been published on consolidated sand there exists an uncertainty as to the extent to which the final calculations might be influenced by differences in
the properties of the sand. Another factor that tends to limit the usefulness of these data is the scarcity of measurements for Wyckoff and Botset, because the data for this case were much more complete than those of Leverett and Lewis. For

![Diagrams](image)

**Fig. 2.—Variation of** $k_g/k_o$ **with Oil Saturation** ($p_o$) **for Fixed Values of Water Saturation** ($p_w$).

$k_g =$ gas permeability; $k_o =$ oil permeability.

**Fig. 3.—Variation of** $k_w/k_o$ **with Oil Saturation** ($p_o$) **for Fixed Values of Water Saturation** ($p_w$).

$k_w =$ water permeability; $k_o =$ oil permeability.

values of water saturation between 0 and 30 per cent. This is a rather important range because it includes the cases in which the connate-water content may be sufficient to decrease the oil permeability considerably and yet be too low to result in the production of water.

In Fig. 2 are plotted curves of the ratio of gas to oil permeability, $k_g/k_o$, vs. the oil saturation $p_o$ for constant values of the water saturation $p_w$. Eq. 3 shows that for fixed reservoir pressure the gas-oil ratio varies linearly with $k_g/k_o$. The curve for $p_w = 0$ was taken from the work of fixed water saturation $p_w$, $k_g/k_o$, decreases with increasing oil saturation, which, of course, is to be expected, since the gas permeability should decrease and the oil permeability increase as $p_o$ becomes larger. The effect of increasing the water saturation for fixed oil saturation is to decrease the ratio $k_g/k_o$. Probably this is chiefly because $k_g$ itself decreases on account of the necessarily decreasing values of the gas saturation. Fig. 3 shows similar curves for the ratio of water to oil permeabilities, $k_w/k_o$, which is pro-
portional to the water-oil ratio of the production. The general characteristics of these can be explained by considerations similar to those applied to Fig. 2.

Fig. 4 shows the variation of relative permeability to oil, \( k_o/k \), with the ratio of oil for \( W \), and 297 to 2000 cu. ft. per barrel* oil for \( R \), the figure of 297 representing the saturation value at the assumed reservoir pressure of 725 lb. per sq. in. From the assumed values of \( R \) and \( W \), \( k_o/k \) and \( k_w/k_o \) were determined by mean s

\[ k = \text{homogeneous-fluid permeability}; \quad k_o = \text{gas permeability}; \quad k_w = \text{oil permeability}. \quad k_o/k \text{ is equivalent to ratio between productivity factor for actual flow system and that for corresponding homogeneous-fluid system}. \]

\( k_o/k \) for fixed values of \( \rho_w \). Again, Wyckoff and Botset's data were used for the case \( \rho_w = 0 \). The curves of Fig. 5 were obtained by cross plotting the curves of Fig. 4, the odd values of \( k_o/k \) corresponding to the values of gas-oil ratio \( R \) assumed in the subsequent calculations.

**Numerical Application**

The first step in the application of these data was to determine the reservoir saturations \( \rho_o \) and \( \rho_w \), required to yield specified values of gas-oil ratio \( R \) and water-oil ratio \( W \). The range of numerical values employed was 0 to 5 bbl. water per barrel

\[ k = \text{homogeneous-fluid permeability}; \quad k_o = \text{gas permeability}; \quad k_w = \text{oil permeability}; \quad k_o/k = \text{(productivity factor for actual system)}/(\text{productivity factor for corresponding homogeneous-fluid system)}. \]

of Eqs. 3a and 3b. Then, using the resulting values of \( k_o/k \) and \( k_w/k_o \), it was possible, by the use of the curves of Figs. 2 and 3, to make a graphical determination of \( \rho_w \) and \( \rho_o \). With \( \rho_w \) known, the relative permeability to oil, \( k_o/k \), corresponding to the selected value of \( R \), was determined from the appropriate curve of Fig. 5.

It is thus possible, if the properties of the gas-oil mixture are known, to determine

* Gas-oil ratios higher than the solution ratio are used here only for illustrative purposes. In practice such a situation would imply that the original pressure was considerably higher than 725 lb. per sq. in. and that the sand has already been appreciably depleted.
the values of \( \rho_o \) and \( \rho_w \) (and also \( \rho_t \), since \( \rho_o + \rho_t + \rho_w = 1 \)), which will yield specified values of \( R \) and \( W \). The permeabilities for each phase can then be found from the permeability-saturation data. It should be noted here that the numerical values of permeability and saturation found in this manner refer to conditions at \( r_t \), the reservoir boundary. If it were desired, the radial distribution of these quantities could be found as a function of the pressure differential in a manner similar to that employed for the two-phase system. For the purpose of the limiting productivity-factor calculation, however, this is unnecessary.

The numerical results of the procedure described above are presented in graphical form in Figs. 6 to 9. Fig. 6 gives the relation between the limiting reservoir values of \( \rho_o \) and \( \rho_w \) for fixed values of the gas-oil ratio. For the case \( R = 297 \), all of the gas is carried in solution in the oil, at the reservoir radius, and the curve merely expresses the relationship \( \rho_o + \rho_w = 1 \). The fact that all of the curves are approximately of the form \( \rho_o + \rho_w + c = 1 \), where \( c \) is a constant, shows that the gas saturation \( \rho_g \) is determined essentially by the gas-oil ratio \( R \). The curves of Fig. 7 are obtained directly from those of Fig. 6, and show the relationship between the gas-oil ratio and oil saturation for fixed values of water saturation. As is to be expected, \( R \) increases rapidly for decreasing values of \( \rho_o \). As \( \rho_o \) decreases with a fixed value of \( \rho_w \), \( \rho_o \) must necessarily increase. This results in an increase in \( k_o \) and a decrease in \( k_w \). Eq. 36 shows that this leads to an increase in the value of \( R \).

In Fig. 8 the relationship is plotted between the water-oil ratio \( W \) and \( \rho_w \) for fixed values of \( R \). Eq. 56 shows that \( W \) is determined essentially by the ratio \( k_w/k_o \). In the discussion of Fig. 6 it was shown that a constant value of \( R \) implies only a small variation in the gas saturation \( \rho_o \) so that an increase in \( \rho_w \) for fixed \( R \) necessitates a decrease in \( \rho_o \). The ratio \( k_w/k_o \), and hence \( W \), will therefore increase with \( \rho_w \) not only because of the increase in \( k_w \) but also because of the decrease
in \( k_s \). In Fig. 9 are shown curves of \( W \) vs. \( \rho_o/\rho_w \) for fixed values of \( R \). Here, as is to be expected, increasing values of \( \rho_o/\rho_w \) give rise to a decrease in water-oil ratio. One feature of these curves to be noted is that for a particular value of \( \rho_o/\rho_w \), \( W \) is relatively insensitive to changes in \( R \). Thus, when \( R \) is increased by a factor of nearly seven, the maximum change in \( W \), for the range covered by the curves, is less than a factor of three.

**Application to Production Problems**

The foregoing analysis has shown that with the gas-oil ratio and water-oil ratio given, it is possible, in principle, to calculate the reservoir saturations \( \rho_o \) and hence the permeabilities and productivity factor. Conversely, if one knew the reservoir saturations, the gas-oil ratio, water-oil ratio and the productivity factor could be found. The determination of the reservoir saturations is one of the central problems of core analysis, of course. In virgin fields the determination of the interstitial water from cores in the oil-water zone should, by difference, give reliable values for the oil saturation. For depleted reservoirs, however, even more indirect methods must be used in the determination of the oil saturation, so that a greater uncertainty may be involved in the final calculation of the productivity factor.

If we suppose that the reservoir saturations \( \rho_o \) and \( \rho_w \) have been determined, the corresponding value of \( k_o/k \) may then be found from curves such as those of Fig. 2. Curves such as those of Fig. 4 will then provide the relative permeability to the oil, \( k_o/k \). Since the composite effect upon the gas and water phases on the oil flow is included in the value of \( k_o/k \), Fig. 4 should then give the productivity factors of the three-phase system relative to that for a homogeneous fluid system.

It is of interest to note that the numerical results corresponding to the preceding figures would suffice to explain only a part of the large differences reported by Pyle and Sherborne and Johnston between the observed values of the produc-
tivity factor and those calculated on the basis of the homogeneous fluid permeability. In fact, even for cases in which \( \rho_o \) is sufficiently large to give rise to a 10 per cent water production the relative permeability to oil \( k_o/k \) still exceeds 0.1. On the other hand, for some of the cases cited by Pyle and Sherborne and by Johnston this ratio would be less than 0.01, notwithstanding the fact that the wells were producing only negligible amounts of water. It is, of course, true, as has been previously emphasized, that the numerical calculations reported here refer to a specific fluid system with a fixed reservoir pressure of 725 lb. And, what is more important, the permeability data relate only to the unconsolidated sands studied by Leverett and Lewis.\(^3\) Undoubtedly, the numerical results of the calculations would be different if the corresponding data for consolidated sands had been used, and, in fact, they would be likely to vary with each particular type of consolidated sand. However, in view of the low values of the relative oil permeability already used in the calculations, even with the unconsolidated sand, as shown in Fig. 4, it seems very doubtful whether the whole of the residual discrepancy could be bridged merely by corrections in the assumed permeability curves.

The alternative to attributing the discrepancy to approximations in the theoretical calculations appears to lie in the physical significance of the field observations. This alternative, however, is also beset with difficulties. Thus the failure to reach strictly steady states in the field tests should have led to apparent productivity factors greater than those calculated, rather than lower. The use here of a limiting productivity factor, for zero pressure differential, in contrast to the conventional usage, in which the production rate is divided by the nonvanishing pressure differential, would tend to make the field results low. However, this difference could account at most for a factor of 2 in the discrepancy. One is thus forced to raise the question of mudding or plugging of the producing sand faces during the well completion as the reason for the abnormally low productivity factors observed in the field tests. This, of course, is admittedly too broad an explanation to be especially convincing. For one might have expected more exceptions with high productivity factors, corresponding to really clean sand faces, among all the wells studied. On the other hand, there is a considerable spread—corresponding to a factor of 5 to 10—in the data plotted by Johnston, which is suggestive of an erratic element in the data, such as would be expected of mudding effects.

The implications of these calculations that the field results as reported by Johnston cannot be entirely explained in terms of the heterogeneous character of the oil and gas seems to be strongly supported by the data recently reported by Horner, Lewis and Stekoll.\(^6\) These authors find that the ratios between the productivity factor as calculated by the homogeneous fluid formula and those observed are in general even less than 2 and occasionally agree with the value unity within experimental errors. On the basis of these results, it would seem that under field conditions the heterogeneous character of the flow may affect the productivity factor far less, rather than more, than would be indicated by the present calculations. While it is true that many of the cases cited by Horner, Lewis and Stekoll refer to virgin conditions with practically no free gas voidage within the sand, even those where free gas was present required no more than a factor of 2 as a correction to bring the observed results in line with those obtained under virgin conditions. In the latter, of course, the major factor reducing the permeability to the oil was the connate water. Hence, at least for the wells studied by these authors,
the connate water did not appear to cut down the productivity factor to less than half of the ideal homogeneous fluid value. To achieve a satisfactory reconciliation between these divergent results, it is therefore clear that further work needs to be done with respect to the gathering of both laboratory and field data.

Aside from the bearing of the foregoing calculations on the problem of productivity factors, other interesting conclusions may be drawn. Thus Fig. 7 shows the rapid increase of the gas-oil ratio with decreasing oil saturation. This does not mean a corresponding increase in gas-oil ratio with increasing cumulative oil production, since Fig. 7 is based on the assumption that the reservoir pressure is kept fixed at 725 lb. On the other hand, it does indicate the relatively low efficiency of direct gas drive in repressuring as a secondary-recovery operation.

A very high sensitivity of the water production—water-oil ratio—to the water saturation may be inferred from Fig. 8. This is at least qualitatively in accord with the field experience that the transition period between the flow of clean oil and complete drowning out of the oil by water is often of very short duration. Of course, the basic permeability saturation data of Fig. 4 directly indicate this type of behavior.

Acknowledgment

The authors wish to thank Dr. Paul D. Foote, Executive Vice President, Gulf Research and Development Co., for permission to publish this paper.

References

1. N. Johnston: Oil and Gas Jnl. (May 22, 1941) 40, 76.
The Calculated Performance of Solution-Gas-Drive Reservoirs

J. S. LEVINE*  
MEMBERS AIME  
SHELL DEVELOPMENT CO.  
HOUSTON, TEX.

ABSTRACT

Several methods are available for calculating the performance of solution-gas-drive reservoirs from the PVT properties of the oil and from the relative permeability and other properties of the formation. These methods require a number of simplifying assumptions. The present method of computation has made use of a high-speed computer to solve simultaneously the nonlinear partial differential equations that describe two-phase flow by solution-gas drive in order to calculate the performance of a reservoir. Some of the results obtained by the nonlinear partial differential equation solution are compared with those obtained with an approximate method, which has been called the semi-steady-state solution. The pressure and saturation profiles from the wellbore to outer boundary calculated by the two methods are compared for one constant-terminal-rate case and two constant-terminal-pressure cases. The agreement in these profiles, as well as in the values of average reservoir pressure and cumulative recovery, leads to the conclusion that, for most engineering calculations, the semi-steady-state method will give a reasonable approximation to the numerical solution of the differential equations describing solution-gas drive.

An unfavorable (as regards ultimate oil production) set of relative permeability curves was used in the calculations in the belief that the effect of the parameters which were studied would be emphasized to a greater degree. Furthermore, the reservoir was assumed to be completely homogeneous, and these results should not be considered applicable to any other type of reservoir. Gravity effects are not considered.

The absolute permeability was varied from 25 to 0.5 md. At an economic limit of 2 B/D, the recovery for a 25-md reservoir is about 1.8 times as great as that for a 0.5-md reservoir. The effect of permeability on the producing gas-oil ratio is minor. Once PVT properties of the oil and the relative permeability properties of the reservoir are fixed, the producing gas-oil ratio is found to be a function of the fraction of oil recovered.

Well spacings of 10, 40 and 80 acres were considered. For the assumed homogeneous-reservoir properties, the effect of spacing on recovery at an economic limit of 2 B/D was very slight.

Certain dimensionless groups can be used to extend the results to other fields having different permeabilities, spacings, reservoir thicknesses, well radius and porosities, so long as the PVT and relative permeability properties are similar to those used in this paper.

INTRODUCTION

An important aspect of reservoir engineering is the prediction of the performance of the reservoir based on the limited information normally available early in the life of a field. Usually, soon after a field has been discovered, it is necessary to decide upon the spacing to be employed and the production program to be used for most efficient utilization of reservoir energy. Where these decisions have not been dictated by legal or political considerations, they have frequently been based on experience factors which indicate that two fields with similar characteristics will probably have similar performances. To the extent that the industry has been successful in economically exploiting oil reservoirs, this rule-of-thumb method has been found to have merit.

A large amount of theory has been developed with which it is possible to predict the performance of a reservoir by using certain known properties of the oil and the formation. Because of the difficulties of the mathematics involved for a drainage area having square or rectangular boundaries, the problem usually has been simplified and idealized by assuming each drainage area to have radial symmetry and the field to be represented by a number of these drainage areas. Muskat1 was one of the first to formulate the theory for two-phase flow and to solve the equations for a few cases by numerical integration, using the relatively slow means of computing available at that time. His equations describing

---

1References given at end of paper.

SOCIETY OF PETROLEUM ENGINEERS JOURNAL
multphase flow in the presence of gravity are called "exact" equations in this paper, even though it is recognized they require gross assumptions regarding phase behavior and relative permeability concepts. Muskat later simplified the two nonlinear, second-order differential equations to one nonlinear, first-order differential equation giving an approximation that could be solved easily by numerical methods. This gave a means of determining the pressure and saturation of the reservoir at the external boundary and of calculating an "average" performance of the reservoir. Calculations showing the effect of reservoir fluid and rock characteristics on production histories were made by Muskat and Taylor in 1946. In 1955, Arps and Roberts used the simplified equation of Muskat to calculate the performance of solution-gas-drive reservoirs for a large number of different relative permeability ratios and solution-gas-oil ratios. The curves presented in their paper gave a rapid means of determining the relation between recovery and average pressure in a reservoir if the average \( k_r/k_o \), solution-gas-oil ratio and oil gravity were known.

The development of the very high-speed computer with large storage capacity made possible a numerical solution of the "exact" equations for unsteady-state two-phase flow. West, Garvin and Sheldon were the first ones to do this by replacing the derivatives by finite differences. Their method was programmed for the IBM 701 computer, and the results for one set of conditions were presented for both a radial and a linear flow system. The present report will show the results obtained by a variation of the West method used with a program written for the IBM 704 computer. Some details of this method will be discussed later. In addition, the results of the exact method (that is, the computational results of the finite-difference equations representing the exact equations of Ref. 1) will also be compared with the semisteady-state results for three cases.

PURPOSE OF THE WORK

This work was undertaken for three primary purposes: (1) to compare the solution-gas-drive results obtained by the exact method with those obtained by an approximate method in order to determine whether the latter is suitable for engineering purposes; (2) to compare a constant-terminal-rate with a constant-terminal-pressure method of production; and (3) to determine the effect of absolute permeability and spacing on the recovery at economic limit from homogeneous solution-gas-drive reservoirs.

These objectives have been reached only to the extent that a single set of PVT and relative permeability data has been used. In a sense, therefore, the work to be discussed is exploratory, and the objectives will not be attained entirely until other cases are considered.

RESERVOIR CHARACTERISTICS AND ASSUMPTIONS

The PVT properties of the oil and gas are shown in Fig. 1. Fig. 2 gives the gas and oil relative permeability relationship. It is apparent to one acquainted with various types of relative permeability curves that the ones shown are "unfavorable"; that is, they will result in a low ultimate oil production. These curves were chosen intentionally, with the idea that such relative permeability data might emphasize the effect of the various parameters.

It was assumed that the reservoir was at the bubble-point pressure and had the following additional characteristics: \( \phi = 0.139 \), \( S_{wc} = 0.177 \), \( b = 23.5 \) ft, \( r_m = 1/3 \) ft and \( p_b = 2,060 \) psig.

Three different external boundary distances were used – 372, 745 and 1,053 ft – corresponding to an equivalent circular area of 10, 40 and 80 acres, respectively. The reservoir was assumed to be horizontal, completely homogeneous and not to be affected by gravity. Vertical differences in pressure due to hydrostatic gradient were neglected. The relative permeability values indicated by Fig. 2 were assumed to be independent of the absolute permeability of the formation. At the critical gas saturation of 2 per cent (corresponding to a total liquid saturation of 98 per cent), the relative permeability to gas was taken to be zero. There was no free gas initially in the reservoir.

Throughout this report, the term "economic limit" will refer to an arbitrarily chosen minimum production rate of 2 BOPD. Thus, the oil recovery at

FIG. 1 – PVT PROPERTIES OF THE OIL AND GAS.
economic limit will mean the cumulative recovery when the production rate has reached 2 B/D. This cumulative recovery will be expressed either in barrels or as a per cent of the initial oil in place.

The program for solution of the exact equations on the IBM 704 allows one to specify either the wellbore rate or pressure as a function of time. In addition, one other condition must be specified, namely, the minimum pressure to which the producing well can go. For all the calculations of this report, the minimum well pressure was assumed to be 50 psig. In those cases where the specified rate could be maintained long enough to obtain a cumulative recovery of at least 3 per cent of the initial oil in place, the problem has been called a "constant-terminal-rate" case. Where the specified rate could not be maintained for this amount of production, the problem has been called a "constant-terminal-pressure" case. Obviously, all problems automatically became constant-terminal-pressure cases when the wellbore pressure reached 50 psig.

FIG. 2 - GAS AND OIL RELATIVE PERMEABILITY PROPERTIES OF THE RESERVOIR.

METHODS OF SOLUTION

The preceding section discussed the properties of the reservoir rock and the crude in it, as well as the shape and orientation of the reservoir itself. It also mentioned that gravity effects were neglected. Two different things are meant by this: (1) buoyancy effects resulting from density difference were not considered; and (2) the effect of the varying hydrostatic head on the pressure at the various elevations throughout the thickness of the formation was neglected. If this last effect were not neglected, then at any fixed distance from the well the pressure would be lowest at the top of the formation and more gas would be evolved at the top of the formation than at any point directly below it. This would necessitate determining the pressure and saturation not only as a function of the distance from the well, but also as a function of the vertical position within the reservoir. This cannot be done with available methods. In neglecting both gravity effects, in fact, we are essentially considering an infinitesimally thin reservoir. Of course, results based on this thin-reservoir concept are applied later to practical cases where variations in hydrostatic head may be as large as 100 psi (and higher), but usually are less. Also, when a nominal back-pressure of, say 50 psig is put on a formation, the actual back-pressure will depend on the position along the producing interval. These nominal backpressures are usually referred to the midpoint of the producing interval. Although not strictly so, the numerical results based on the thin-reservoir concept will refer to the horizontal plane through the midpoint of the producing interval. The pressure and saturation are then only a function of the distance from the well, and the system of differential equations describing solution-gas-drive reservoirs can be written as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r_k k_{ro}}{B_o P_o} \frac{\partial P}{\partial r} \right) = -\frac{\partial}{\partial t} \left( \frac{S_o}{B_o} \right), \ldots \quad (1)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \frac{k_{ro} R_o}{\mu_o B_o} + \frac{k_{rg}}{B_g P_g} \right) \frac{\partial P}{\partial r} \right] = -\frac{\partial}{\partial t} \left( \frac{1 - S_{wa} - S_o}{B_g} + \frac{S_o R_o}{B_o} \right), \ldots \quad (2)
\]

At the outer boundary, both pressure and saturation gradients vanish, whereas at the well the production rate or the pressure can be specified. In addition, the initial pressure and saturation distribution are specified.

This system of differential equations and its associated conditions have been solved by two different methods, both numerical. In one method, Eqs. 1 and 2 are represented by finite-difference equations after the substitution

\[
\alpha = \ln \frac{r_w}{r}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\]
has been made. The range of \( a \) is divided into \( M \) equal increments, as shown schematically in Fig. 3. (\( M = 30 \) for most of the calculations in this report.) The pressures are obtained at the midpoint of the cells, whereas the saturations are obtained at the cell boundaries. Whenever saturations are needed at the midpoints of the cells, these are obtained from the arithmetic average of the saturations at the adjoining cell interfaces. The resultant equations are, in general,

\[
\begin{align*}
\left[ (q_o)_{2m+2} - (q_o)_{2m} \right]_{t+T/2} &= \frac{2nb \phi z_{2m+1} \sinh \Delta \alpha}{5.615(t/2)} S_o B_o \left( \frac{S_o}{B_o} \right)_{t+T/2} - \left( \frac{S_o}{B_o} \right)_{t} \\
&= 0 \ldots \ldots (4) \\
\left[ (q_g + R_0 q_o)_{2m+2} - (q_g + R_0 q_o)_{2m} \right]_{t+T/2} &= \frac{2nb \phi z_{2m+1} \sinh \Delta \alpha}{5.615(t/2)} \left( \frac{S_o - S_o}{B_o} + \frac{R_0 S_o}{B_o} \right)_{t+T/2} - \left( \frac{S_o - S_o}{B_o} + \frac{R_0 S_o}{B_o} \right)_{t} \\
&= 0 \ldots \ldots (5)
\end{align*}
\]

These equations are modified near the inner and outer boundaries to incorporate the proper boundary conditions and are then solved on an IBM 704, by procedures similar to those reported by West, Garvin and Sheldon,\(^5\) to give pressure and saturations at the time increment \( T/2 \). The pressures and saturations corresponding to the end of the time step are obtained from those at the beginning and midpoint of the time step by linear extrapolation.

The program is designed to double the time interval of each succeeding time step, providing this gives convergence. There are also provisions for reducing the time interval to any level required to give a convergent series of iterations. A maximum time interval could also be specified. About 40 time steps were usually required to complete a problem.

It should be pointed out that significant cumulative errors can result from approximating the derivatives by finite differences. However, cumulative material balances run at each time step indicated small errors, with representative errors at the end of the run being 1 bbl of oil and 800 cu ft of gas.

The factor 5.615 appearing in Eqs. 4 and 5 and the factor \( 1.127 \times 10^{-3} \) appearing in Eqs. 6 and 7 result from the facts that input data for the problem are expressed in practical engineering units. That is, data required for the system of Eqs. 4 through 7 are given in units of barrels per day, feet, centipoise, millidarcies and pounds per square inch. In fact, except in the basic Eqs. 1 and 2, practical units are used throughout this report.

It was stated earlier in this report that one of the purposes of the work was to compare the numerical solution of Eqs. 1 and 2 with the solution obtained by an approximate method which will be called the semisteady-state solution. The present authors wish to point out that the basic derivation of the approximate method was undertaken many years ago by A. F. van Everdingen, E. H. Timmerman and G. Stewart of Shell Oil Co., but the results have never been publicly circulated. More recently, the Koninklijk Shell Laboratorium, The Netherlands, have used the assumptions proposed by van Everdingen, Timmerman and Stewart to derive an equation which could be integrated very simply on a medium-speed computer.

The approximation proposed by van Everdingen, et al., consisted in assuming that, at any one instant of time, both the rate of decline of stock oil in place and the total gas-oil ratio are everywhere constant. With the assumption that the rate of decline of stock oil in place is everywhere constant, the right-hand side of Eq. 1 becomes equal to the production rate divided by the reservoir pore volume. Workers at the Koninklijk Shell Laboratorium have pointed out that the resultant expression can be integrated directly to give

\[
\int_{PD} P_D \left[ \frac{887 q_o}{4n k h p_b} \left( \frac{T_D - T_D e}{t_D} \right)^2 \right] \frac{r_D}{r_D e} dp = \frac{887 q_o}{4n k h p_b} \left( \frac{T_D - T_D e}{t_D} \right)^2 \frac{r_D}{r_D e} \ldots \ldots (8)
\]

when the integrand is evaluated at constant GOR conditions. The pressure and saturation relationship which results in a constant total GOR everywhere is the same as given by Muskat and is called the differential material balance. Results of

![FIG. 3—SPACE POSITIONS AT WHICH PRESSURES AND SATURATIONS ARE CALCULATED IN THE EXACT SOLUTION.](image)

SEPTEMBER 1961
this single integration of Eq. 8 will yield pressures and saturations as a function of $r_d$, and further manipulation of the results will yield production-history curves.

### NUMERICAL RESULTS

**COMPARISON OF THE EXACT AND APPROXIMATE METHODS**

One of the purposes of this study was to compare results obtained by the exact and the approximate (or semisteady-state) methods of solution, discussed previously. For this comparison, three cases were calculated by the exact method on the IBM 704, and some of the same data were used to obtain a solution by the approximate method. Table 1 shows the results for a constant-terminal-rate case in a reservoir with 40-acre spacing and 25-md permeability. The external boundary pressures and production rates used were the same as those calculated by the IBM 704, and the pressure and saturation distribution, average pressure and percent recovery were calculated for several periods in the production history. It can be seen that the results agree quite well. The average reservoir pressures are always within a few pounds per square inch, although this result is not surprising since the outer boundary pressures were assumed to be the same and the average pressure is always nearly equal to the outer boundary pressure. The recoveries are also in very good agreement by both methods. The biggest difference is in the producing gas-oil ratios, which are too low by the approximate method when the reservoir is producing at the assumed constant rate and too high when the rate can no longer be maintained and the problem becomes essentially a constant-terminal-pressure case. In every case, the constant gas-oil ratio used for the approximate method is that obtained from the differential material balance for the pressure and saturation at the outer boundary. This present case and the other two to be discussed later all indicate that, later in the life of a field, the actual gas-oil ratio is more favorable (lower) than would be anticipated from the differential material balance at the same average reservoir pressure. The higher gas-oil ratios are the result of lower oil saturations calculated by the approximate method than those obtained by the exact method. This can be seen in Fig. 4, which shows the pressure and saturation distributions throughout the reservoir obtained by both methods. It is evident that in this case the semisteady-state method gives too high a pressure near the wellbore and, therefore, a somewhat higher pressure distribution throughout the reservoir at any given instant. For the corresponding times, the oil saturations throughout the reservoir are first slightly higher by the approximate method and then slightly lower. However, as indicated in Table 1 and Fig. 4, the general results obtained by the two methods are in reasonable agreement. It should be pointed out that the use of a log scale to plot the radial distances for the pressure and saturation distributions shown in Fig. 4 (also Figs. 5 and 6) essentially overemphasizes that portion near the producing well, which comprises a very small percentage of the drainage.

### TABLE I — COMPARISON OF THE EXACT AND APPROXIMATE METHODS FOR THE SAME PRODUCTION RATE AND EXTERNAL BOUNDARY PRESSURE

<table>
<thead>
<tr>
<th>Curves (Fig. 4)</th>
<th>$p_0$ (psia)</th>
<th>$q_0$ (B/D)</th>
<th>$\bar{p}$ (psia)</th>
<th>$H_N$ (per cent)</th>
<th>$R$ (cu ft/bbl)</th>
<th>$P_{iw}$ (psia)</th>
<th>$\bar{p}$ (psia)</th>
<th>$H_N$ (per cent)</th>
<th>$R$ (cu ft/bbl)</th>
<th>$P_{iw}$ (psia)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a = a'</td>
<td>1,770</td>
<td>50.0</td>
<td>1,751</td>
<td>1,751</td>
<td>3.34</td>
<td>3.36</td>
<td>2,340</td>
<td>2,350</td>
<td>1,273</td>
<td>1,304</td>
</tr>
<tr>
<td>b = b'</td>
<td>1,605</td>
<td>50.0</td>
<td>1,581</td>
<td>1,581</td>
<td>4.28</td>
<td>4.23</td>
<td>4,570</td>
<td>4,150</td>
<td>865</td>
<td>940</td>
</tr>
<tr>
<td>c = c'</td>
<td>1,485</td>
<td>50.0</td>
<td>1,453</td>
<td>1,457</td>
<td>4.75</td>
<td>4.67</td>
<td>6,610</td>
<td>5,840</td>
<td>480</td>
<td>610</td>
</tr>
<tr>
<td>d = d'</td>
<td>1,400</td>
<td>50.0</td>
<td>1,368</td>
<td>1,368</td>
<td>4.99</td>
<td>4.92</td>
<td>7,680</td>
<td>7,220</td>
<td>153</td>
<td>277</td>
</tr>
<tr>
<td>e = e'</td>
<td>1,115</td>
<td>26.8</td>
<td>1,086</td>
<td>1,088</td>
<td>5.84</td>
<td>5.62</td>
<td>12,790</td>
<td>13,150</td>
<td>65</td>
<td>119</td>
</tr>
<tr>
<td>f = f'</td>
<td>730</td>
<td>10.1</td>
<td>711</td>
<td>712</td>
<td>6.24</td>
<td>6.26</td>
<td>21,110</td>
<td>22,250</td>
<td>65</td>
<td>65</td>
</tr>
<tr>
<td>g = g'</td>
<td>240</td>
<td>1.14**</td>
<td>233</td>
<td>233</td>
<td>6.93</td>
<td>6.91</td>
<td>27,730</td>
<td>30,210</td>
<td>65</td>
<td>65</td>
</tr>
</tbody>
</table>

* $q_0 = 9.71$ B/D by the approximate method.
** $q_0 = 1.02$ B/D by the approximate method.
FIG. 5—PRESSURE AND SATURATION PROFILES FOR A CONSTANT-TERMINAL-PRESSURE CASE, $k = 25$ MD.

Table 2 gives a comparison of the results for two constant-terminal-pressure cases for reservoirs of 80-acre spacing and of 25-md and 2.5-md permeability, respectively. In both cases, the approximate solutions were calculated by using the same external boundary pressures as were obtained by the exact method, as well as the same constant terminal pressure of 65 psia. Shown in the table are the average reservoir pressures, gas-oil ratios, use cent recoveries and production rates. As in the case discussed previously, agreement of the various values is very good. The largest discrepancy of about 10 per cent is in the production rate calcu-

<table>
<thead>
<tr>
<th>(Curves)</th>
<th>$p_e$ (psia)</th>
<th>$p_w$ (psia)</th>
<th>$P_{o}$ (psia)</th>
<th>$N_{oD}$ (per cent)</th>
<th>$R$ (cu ft/bbl)</th>
<th>$q_o$ (b/D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h-h'$</td>
<td>1,757</td>
<td>65</td>
<td>1,708</td>
<td>1,712</td>
<td>3.24</td>
<td>2,600</td>
</tr>
<tr>
<td>$l-l'$</td>
<td>1,143</td>
<td>65</td>
<td>1,377</td>
<td>1,378</td>
<td>4.78</td>
<td>6,250</td>
</tr>
<tr>
<td>$k-k'$</td>
<td>1,077</td>
<td>65</td>
<td>1,054</td>
<td>1,050</td>
<td>5.57</td>
<td>12,830</td>
</tr>
<tr>
<td>$s-l'$</td>
<td>529</td>
<td>65</td>
<td>519</td>
<td>517</td>
<td>6.38</td>
<td>24,280</td>
</tr>
</tbody>
</table>

A = 80 Acres, $k = 25$ md

$A = 80$ Acres, $k = 2.5$ md (Fig. 6)

FIG. 6—PRESSURE AND SATURATION PROFILES FOR A CONSTANT-TERMINAL-PRESSURE CASE, $k = 2.5$ MD.

lated for the 25-md reservoir at a value of $p_e = 1,757$ psia. As the outer boundary pressures decrease, however, the production rates agree more closely. For the 2.5-md case, the agreement is good in all instances. The pressure and saturation distributions for these two cases are shown in Figs. 5 and 6, respectively. The pressure distributions in both cases are almost identical but, of course, were assumed to be the same at the inner and outer boundaries. The slight differences in saturation distributions are the result of the assumptions made for the semisteady-state method.

Fig. 7 shows a typical set of curves of gas-oil ratio as a function of radius for the 2.5-md reser-
FIG. 7—GAS-OIL RATIO PROFILES FOR A CONSTANT-TERMINAL-PRESSURE CASE, \( k = 2.5 \text{ MD} \).

The curves are for the same instant as the corresponding curves in Fig. 6. It is perhaps surprising that, although initially the gas-oil ratio at the wellbore is higher than at the outer boundary, this situation soon reverses, and during much of the producing life, the gas-oil ratio is less at the producing well than at the boundary. This was true in all the cases calculated by the exact method. The maximum variation in gas-oil ratio from outer boundary to wellbore was found to be about 10 per cent, so the assumption of a constant gas-oil ratio used in the approximate method of solution is not too radical. It should be emphasized that these results were obtained for a particular set of PVT and relative permeability data and may not be applicable if the PVT properties are drastically changed.

EFFECTS OF PERMEABILITY AND SPACING

Some of the results obtained by the exact method of solution on the IBM 704 are shown in Table 3. It can be seen that the effect of spacing on the economic recovery becomes greater as the permeability decreases. For permeabilities of 25 and 2.5 md, the recovery efficiency is decreased from 6.75 to 6.64 per cent and from 5.80 to 5.74 per cent, respectively (slightly over 1 per cent of the total), when the spacing is increased from 10 to 80 acres. However, for a permeability of 0.5 md the recovery efficiency is decreased from 3.85 to 3.53 per cent (about 7 per cent of the total) for the same increase in spacing. The effect of permeability on the economic recovery is much more pronounced. At the economic limit of 2 B/D, the per cent recovery for a 25-md reservoir is about 1.8 times as great as that for a 0.5-md reservoir. This difference in recovery for different permeabilities is, except for spacing effects, entirely the result of having to stop production when a minimum is reached; it should not be construed as a difference in the production mechanism. The calculated values indicate that the maximum production for the same spacing,

<table>
<thead>
<tr>
<th>Spacing (acres)</th>
<th>Constant Terminal Pressure</th>
<th>Economic Limit ( N_{PD} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>25.0</td>
<td>10,600</td>
</tr>
<tr>
<td>40</td>
<td>25.0</td>
<td>42,200</td>
</tr>
<tr>
<td>80</td>
<td>25.0</td>
<td>84,100</td>
</tr>
<tr>
<td>10</td>
<td>2.50</td>
<td>9,100</td>
</tr>
<tr>
<td>80</td>
<td>2.50</td>
<td>72,000</td>
</tr>
<tr>
<td>10</td>
<td>0.50</td>
<td>6,030</td>
</tr>
<tr>
<td>80</td>
<td>0.50</td>
<td>44,200</td>
</tr>
</tbody>
</table>

without regard to economic rates, would be independent of the absolute permeability. This trend can be seen in Fig. 8, where the production rate has been plotted against per cent cumulative oil produced for reservoirs of 25-, 2.5- and 0.5-md permeabilities. If the production had been continued to very low rates (<0.1 B/D), all the curves would have come together at about the same recovery efficiency of around 7.3 per cent. It is evident that, if the economic limit of production had been raised to about 5.5 B/D, no oil would have been produced from the 0.5-md reservoir because its maximum production rate at initial reservoir pressure was only 5.3 B/D.

FIG. 8—OIL-PRODUCTION RATE AS A FUNCTION OF CUMULATIVE OIL PRODUCED.
Fig. 8 also shows that, for a constant terminal pressure and at a given permeability, the closer-spaced wells will produce at a higher rate than the wider-spaced wells at the same per cent recovery. This means that the closer the spacing, the faster a well will drain a given fraction of oil from its drainage area. The time required to reach the economic limit is also shown in Table 3. It can be seen that, for a given permeability, the time is approximately proportional to the spacing. Based on the associated cumulative production and rate performance, one could choose a spacing to optimize profits.

Only one case was run in which the effect of rate of production on the economic recovery was indicated. Fig. 8 shows the curve of rate vs per cent cumulative oil produced for a well which initially produced at a constant terminal rate of 200 B/D, which was four times greater than the constant 50-B/D production of the other wells for the 25-md reservoir. After approximately 1 per cent of the initial stock tank oil in place had been produced at 200 B/D, the well pressure reached the limiting 50-psi gauge; thereafter, the well produced as a constant-terminal-pressure case. It can be seen that at the economic limit the production from this well was 6.64 per cent, compared to 6.71 per cent from an identical well whose initial rate was one-fourth as great. However, Table 3 shows that the time to reach economic limit was 2,680 days, compared to 3,360 days. Thus, if the well produces at approximately the maximum attainable rate at all times, the operating life is decreased about 25 per cent, while the economic recovery is decreased only about 1 per cent of the total recovery.

Fig. 9 shows a plot of the average recovery efficiency at the economic limit as a function of permeability. This curve shows that a formation having a \( k_b \) product of about 6 md-ft (0.215 md × 23.5 ft) would not be able to produce any oil at a rate of 2 B/D. Similarly, extension of the curve to higher permeabilities would indicate that, no matter how great the permeability, the maximum recovery efficiency would be about 7.3 per cent at the economic limit. (This recovery efficiency compares with a value of 7.32 per cent obtained by differential material balance at a pressure of 50 psi.) Shown also in Fig. 9 is the average gas-recovery efficiency (defined as standard cubic feet of produced gas divided by standard cubic feet of gas initially in place) as a function of oil-recovery efficiency. For the 0.5-md reservoir about 10 per cent of the original gas has been produced at the economic limit, while for the 25-md reservoir about 68 per cent of the original gas has been produced. In any economic evaluation, this difference in produced gas must be considered.

**GAS-OIL RATIO AND CUMULATIVE OIL PRODUCED**

Fig. 10 is a plot of the producing gas-oil ratio for various per cent recoveries for all the runs listed in Table 3. No attempt has been made to distinguish between the various cases, but it is obvious that at the same recovery the effects of permeability and spacing on the producing gas-oil ratio are minor. Thus, once the PVT properties of the oil and the relative permeability properties of the reservoir are fixed, the producing gas-oil ratio will be essentially independent of the spacing and of the absolute permeability. The peak occurs at a recovery of about 6.75 per cent and starts to decline after that in the normal way.

**EXTENSION OF RESULTS BY DIMENSIONLESS GROUPS**

It was emphasized earlier that the results obtained in this study were for a particular set of fluid and reservoir properties. Therefore, it will be of interest to determine whether the range of applicability of the results can be extended by the use of some dimensionless groups pertinent to depletion-type reservoirs.

The theory of dimensionally scaled models of petroleum reservoirs has been adequately discussed in papers by Rapoport\(^6\) and Geertsma, \( et \ al.\)^\(^7\) Proper
dimensionless expressions for the time, oil-production rate, and cumulative oil production can be written as follows.

\[ I_d = \frac{1.266 \times 10^{-2} (p_i - p_w) kt}{\mu_b \delta (r_e^2 - r_o^2)} \]  \hspace{1cm} (9)

\[ q_{oD} = \frac{141.2 \mu_b \delta \phi_i}{k b (p_i - p_w)} \]  \hspace{1cm} (10)

\[ N_p = \frac{B_{oi}}{S_{oi}} t_0 \int_0^{t_0} q_{oD} dt \]  \hspace{1cm} (11)

These equations have been used to calculate the values of the dimensionless quantities for all the problems discussed in this report. Fig. 11 shows a plot of dimensionless rate as a function of dimensionless time. The parameters that have been varied are the permeability, which ranged from 0.5 to 25 md, and the external radius, which ranged from 372 to 1,053 ft. It can be seen that the points lie essentially along two curves which can be classified according to whether the problem approximated a constant-terminal-pressure case (lower curve) or a constant-terminal-rate case (upper curve). It was mentioned earlier that, although only one problem was actually a constant-terminal-pressure case, several others approximated it in that the producing well pressure dropped to the minimum permissible value in a relatively short time after oil production started. The duration of this transient effect for those problems intended as constant-terminal-rate cases determined whether the points fell closer to the upper or to the lower curve of Fig. 11. Thus, it might be said that the constant-terminal-pressure curve is a limiting curve giving the maximum dimensionless rate at any dimensionless time, and that all rates less than this maximum will produce a series of dimensionless curves which must fall above the limiting curve. Those runs which behaved as constant-terminal-rate cases did so until a dimensionless time of about 50 was reached.

For constant-terminal-rate cases, the dimensionless rates determined from Eq. 10 will vary with the wellbore pressure. For constant-terminal-pressure cases, the dimensionless rates will vary proportionally to the actual production rate, while the pressure drop has the highest possible value. Because the pressure drop is the highest possible, the dimensionless production rate for the constant-terminal-pressure case is less than that for the constant-terminal-rate case.

The curves of Fig. 11 give a means of determining the oil-producing rate at any time for other permeabilities, spacings, sand thicknesses, well radii and porosities. One need only calculate the values of real time and rate from the dimensionless values, using Eqs. 9 and 10. Thus, the results of this report can be extended to other reservoirs having similar PVT and relative permeability characteristics.

Fig. 12 is a plot of dimensionless cumulative oil production as a function of dimensionless oil rate. Here again, there is a spread in a portion of the curve, depending on whether the early production was at constant terminal pressure or at constant

![Graph showing dimensionless oil rate as a function of dimensionless time.](image)

**FIG. 11 — DIMENSIONLESS OIL RATE AS A FUNCTION OF DIMENSIONLESS TIME.**
CONCLUSIONS

Based on the results discussed previously for a completely homogeneous reservoir and for the range of conditions as given, the following conclusions can be made.

1. The methods described in this report provide a useful means for calculating the performance of solution-gas-drive reservoirs. To solve the exact equations governing two-phase flow, a high-speed computing machine with large storage capacity is required. By comparison, the semisteady-state method does not require such high-speed computers and can be calculated by means of an IBM 650 or equivalent. For the ranges covered in this report, it has been shown that for engineering calculations the semisteady-state method will give a satisfactory approximation to the correct solution.

2. At an arbitrarily chosen economic limit of 2 B/D, the per cent recovery for a 25-md reservoir is 1.8 times as great as that for a 0.5-md reservoir. This is not the result of a difference in production mechanism but, rather, is the result of arbitrarily cutting oil production when a minimum rate is reached. The lower this economic rate, the more nearly will the recovery be independent of permeability.

3. Producing gas-oil ratio is essentially independent of both spacing and permeability at the same value of fractional recovery.

4. The effect of well spacing on recovery at an economic limit of 2 B/D is not significant for permeabilities above 2.5 md (corresponding to a \( k_b = 60 \) md-ft). For a 0.5-md permeability (= 12 md-ft), the recovery is decreased about 7 per cent for a change in spacing from 10 to 80 acres.

5. For the PVT and relative permeability properties given in this report, certain dimensionless groups can be used to extend a given set of results to other fields having different permeabilities, spacings, reservoir thicknesses, well radii and porosities. The effect of whether early production is at constant terminal rate or constant terminal pressure is important; however, for dimensionless rates less than 0.01, the cumulative dimensionless production is independent of initial rate.

NOMENCLATURE

\[ A = \text{area, acres} \]
\[ a = \ln \frac{r_w}{r} \]
\[ B_g = \text{gas formation volume factor, bbl/cu ft} \]
\[ B_o = \text{oil formation volume factor, bbl/bbl} \]
\[ B_{oi} = \text{initial oil formation volume factor} \]
\[ \phi = \text{porosity, fraction} \]
\[ h = \text{thickness of reservoir, ft} \]
\[ k = \text{absolute permeability, md} \]
\[ k_{rg} = \text{relative permeability to gas, fraction} \]
\[ k_{ro} = \text{relative permeability to oil, fraction} \]
\[ M = \text{total number of cells into which reservoir is divided} \]
\[ m = \text{cell number} \]
\[ N = \text{initial oil in place, bbl} \]
\[ N_p = \text{cumulative oil produced, bbl} \]
\[ N_{pD} = \text{cumulative oil produced, fraction or per cent,} \]
\[ \rho = \text{pressure, psi} \]
\[ \rho_D = \text{pressure, dimensionless,} \]
\[ r = \text{radius, ft} \]
\[ r_D = \text{radius, dimensionless,} \]
\[ r_e = \text{external boundary radius, ft} \]
\[ S_e = \text{gas saturation, fraction} \]
\[ S_o = \text{oil saturation, fraction} \]
\[ S_{oi} = \text{initial oil saturation, fraction} \]
$S_{TL}$ = total liquid saturation, fraction
$S_{wc}$ = critical water saturation, fraction
$t$ = time, days
$t_D$ = time, dimensionless
$\tau$ = time increment, days
$\mu_g$ = gas viscosity, cp
$\mu_o$ = oil viscosity, cp
$\mu_{ob}$ = oil viscosity at bubble point, cp

REFERENCES

***
Methods for Predicting Gas Well Performance

D. G. RUSSELL
J. H. GOODRICH
MEMBERS AIME
G. E. PERRY
J. F. BRUSKOTTER
MEMBER AIME

SHELL DEVELOPMENT CO.*
HOUSTON, TX.
SHELL OIL CO.
BAKERSFIELD, CALIF.
HOUSTON, TX.
SHELL DEVELOPMENT CO.
HOUSTON, TX.

ABSTRACT

The depletion performance of gas wells has been investigated by mathematical simulation techniques. The gas well model which was studied consisted of a single well located in the center of a bounded, cylindrical, homogenous reservoir. Dependency of gas compressibility and viscosity on pressure was considered in studies of well performance at both constant mass flow rate and constant flowing pressure conditions. To carry out the investigation, the nonlinear, second-order, partial differential equation which describes Darcy flow of a nonideal gas through porous media was solved numerically.

Some of the previous investigations of gas well performance have been of limited general use because assumptions were introduced to simplify either the gas properties or the basic differential equation. Other studies have been rigorous in these respects but have presented a very limited set of calculated results. The present study was attempted to present a rigorous theoretical model and sufficient numerical results to permit meaningful conclusions to be drawn.

It was found that all terms must be retained in the partial differential equation to make accurate predictions. The neglect of higher-order terms, e.g., terms of the order of the "gradient squared", leads to serious material balance errors at large times and to conservative estimates of well performance. The higher the gas flow rate and/or the lower the permeability-thickness product of the formation, the more pronounced are these deviations. For example, in a well draining 640 acres in a 25-md-ft formation (8,120 MMcf gas in place) at a constant rate of 993 Mcf/D, the rigorous solution predicts a bottom-hole pressure decline from 4,000 to 1,000 psia in 8.7 years. If higher-order terms are neglected in the differential equation, this decline in pressure is predicted to occur in 5.3 years.

With the results of the numerical solution of the differential equation as a basis, simple, easy-to-use approximations for predicting gas well performance for Darcy flow conditions have been developed. These simple approximations are based on the familiar equations for flow of a single, slightly compressible fluid. The approximate methods possess a high degree of accuracy and enable the prediction of long-term gas well performance to be made quickly and accurately without the use of a digital computer. Both transient and stable flow period approximations were developed.

INTRODUCTION

In recent years income from the sale of natural gas and associated products has represented an ever-increasing fraction of the industry's total revenue from operations. To meet the surge in demand for natural gas, the industry has depended heavily upon established reserves and has actively pursued development of new reserves. The search has progressively led to reservoirs which yesterday were too tight and/or deep to yield the desired return on invested capital. More than ever before, evaluation accuracy is now required to forecast the criteria upon which engineering recommendations and management decisions are based.

Considerable effort has been expended by both research and operations personnel on the development and application of methods for analyzing and predicting the performance of gas wells. Fundamentally, the problem is the familiar one of extracting data during the drilling, testing and early production life of a well and applying these data within an accurate simulation model to predict long-term behavior. During the past 30 or more years a voluminous literature dealing specifically with gas field problems has been generated. A recent book lists a comprehensive bibliography of published material through 1959. Over 1,200 references are cited. Since then 39 additional articles on natural gas technology have been published in Transactions volumes of the Society of Petroleum Engineers of AIME.

Most existing theory for predicting gas well performance requires that one or more idealizations (e.g., steady-state flow, ideal gas of constant viscosity, small and constant compressibility and constant-viscosity fluid) be applied. Although existing theory may apply directly or be adapted by various artifices to describe specific gas well and reservoir behavior, no widely applicable method is available, and existing methods appear to be subject to appreciable error unless better limits of applicability are defined.

The objectives of this paper are (1) to present numerical solutions to the partial differential equation describing gas flow under conditions of general interest in gas well performance prediction work, (2) to present solutions which possess a high degree of accuracy and stabilized flow periods of a well producing at constant rate or constant pressure, and (3) to develop and present simplified relationships which can be used as high-order approximations to the exact numerical results for fast and accurate predictions of gas well performance at the operating level. Combined, these objectives are designed gen-

*References given at end of paper.
eraly to furnish a firm basis for prediction of long-term behavior of gas wells and, specifically, to eliminate the technical barrier that has maintained the historic distinction between good gas wells and tight gas wells.

These objectives have been accomplished through formulation and solution of a mathematical analog of the reservoir situation of interest.

THE PHYSICAL PROBLEM

To formulate a mathematical analog of gas flow in the reservoir, we assume single-phase, horizontal, isothermal flow of gas at velocities for which Darcy’s law is applicable. We assume a completely penetrating well located in the center of a circular, cylindrical, homogeneous and isotropic reservoir. Flow in this system is therefore radially symmetrical. If, in addition, we assume the porosity and permeability of the reservoir rock to be independent of pressure, then by combining the equation of continuity, Darcy’s law and the isothermal equation of state, we can write the following equation to describe the pressure behavior within the reservoir:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{M_p}{zRT} \left( -\frac{k}{\mu} \frac{\partial P}{\partial r} \right) \right] = -\phi \frac{\partial}{\partial t} \left( \frac{M_p}{zRT} \right) \]  

(1)

To make Eq. 1 easier to handle numerically, the following dimensionless variables and functions are introduced:

\[ x = \ln \left( \frac{r}{r_o} \right), \quad t_o = \frac{kt}{\phi \mu c r_o^2}, \quad c = \frac{1}{p} - \frac{1}{z} \frac{\partial}{\partial p}, \]

\[ f(p) = \frac{\mu c \xi z r_o}{p}, \quad \xi = \frac{z}{p} \frac{\partial}{\partial p} \]

When these variables are introduced into Eq. 1, the following transformed equation results:

\[ \frac{\partial P}{\partial x} = e^\xi f(p) \frac{\partial P}{\partial t} \]  

(2)

Formulation of the problem in terms of the pseudopressure variable \( P \) was suggested by Prats.7

Since the gas well being considered produces from a bounded drainage volume which is circular in shape, there can be no flow of gas across the drainage boundary of the well. Therefore, the condition which is imposed at the drainage boundary can be expressed mathematically as:

\[ \frac{\partial P}{\partial x} \bigg|_{r=r_o} = 0 \]  

(3)

Eq. 3 simply expresses that, to satisfy the no-flow condition at the outer boundary, the flow velocity and consequently the pressure gradient in Darcy’s law must vanish on this surface.

At the wellbore, Eq. 2 has been solved for both the case of a constant volumetric flow rate at surface temperature and pressure and the condition of a constant bottom-hole flowing pressure. In the constant rate case the requirement of a constant volumetric flow rate at standard conditions is equivalent to the requirement of a constant mass flow rate at the wellbore. The mass flow rate can be expressed as:

\[ q_m = -\frac{2\pi k kh M_p}{\mu c z RT} \frac{\partial P}{\partial x} \bigg|_{r=r_o} \]  

For convenience, a reduced flow rate is introduced at this point and is related to the mass flow rate by the relationship:

\[ q_e = \frac{q_m \mu c z RT}{2\pi kh M_p} \]  

These equations are combined to yield:

\[ q_e = \frac{\partial P}{\partial x} \bigg|_{r=r_o} \]  

(4)

which represents the condition which must be satisfied at the wellbore during the constant rate portion of the well’s performance history.

In the case of constant flowing pressure at the well, it is required that the pressure be fixed and equal to some arbitrarily chosen value \( p_w \), which is less than the original reservoir pressure. This is expressed mathematically in terms of the transformed pressure variable as:

\[ P = p_w \]  

at \( x = 0 \)  

(5)

In the case of production at constant flowing pressure, the main interest is the mass flow rate of gas produced from the well during the constant pressure portion of the performance history. This is determined by the relationship:

\[ q_e = -\frac{2\pi k kh M_p}{\mu c z RT} \frac{\partial P}{\partial x} \bigg|_{r=r_o} \]  

(6)

To complete the mathematical formulation, there must be an initial condition imposed. In all cases, this initial condition is: pressure everywhere in the reservoir is uniform and equal to \( p_w \). This is expressed mathematically in terms of transformed pressure as:

\[ P = 0 \]  

at \( t_o = 0 \)  

(7)

To obtain the pressure and production performance history of a gas well, Eqs. 2, 3, 4 and 7 and Eqs. 2, 3, 5 and 7 must be solved simultaneously for the cases of constant rate performance and constant pressure performance, respectively. These nonlinear systems of equations are very difficult, if not impossible, to solve analytically; therefore, it was necessary to employ numerical finite difference methods to obtain mathematical and numerical solutions. The procedures used and the accuracy of the solutions are discussed in Appendix A. We believe that a high order of numerical accuracy was achieved.

RESULTS

The numerical results presented in the paper, unless otherwise stated, are based on the gas deviation and viscosity data shown on Figs. 1 and 2. Two sets of gas data are used. The first (labeled Gas A) represents a typical gas of 0.7 gravity at a reservoir temperature of 226°F. Viscosity and deviation data for this gas were prepared with standard correlations which are available in the gas engineering literature. The other set of data (called Anadarko Gas) is for a typical natural gas produced in the Anadarko Basin region of western Oklahoma. In this case, gas gravity is 0.675 at a reservoir temperature of 140°F. The viscosity and gas deviation behavior for both gases demonstrates the need for inclusion of nonideal behavior and variable viscosity in the problem formulation and solution.

Various solutions for radial flow of ideal and nonideal gases were obtained for comparison with previously published solutions and with a simplified solution obtained earlier in our work.

COMPARISON WITH PREVIOUS WORK

Carter7 presented solutions to constant rate-radial flow of nonideal gas. His investigation was limited to transient flow behavior and was focused on short flow test analysis. The results of our work are compared with Carter’s results
for two cases (Fig. 3). Good agreement is observed for both cases. Minor divergences are believed due to the difference in pressure reference used to calculate gas viscosity and deviation and to the different methods used to establish gas compressibility. Carter used the following relationship:

\[ \frac{1}{\bar{p}} = \frac{1}{\rho} \]

\[ \bar{p} = \left( \frac{\rho_1^3 + \rho_2^3}{2} \right)^{\frac{1}{3}} \]

and \( \rho_1 \) and \( \rho_2 \) were referred to \( \bar{p} \). The \( \rho_c \) parameter is \( \rho_{\text{eq}} \) for laminar flow.

Bruce and co-workers obtained a comprehensive set of numerical data on the behavior of an ideal gas of constant viscosity by means of a finite difference solution of the gas flow equation. To compare the results of our method with theirs, the following equalities between parameters from the two studies must be observed:

\[ Q_c = \frac{2q_1 t_0}{\rho_c (\tau_0 - 1)} \quad f(p) = \frac{p_1}{p} \]

Comparative data for two rates, in the form of reduced pressure vs fraction of original gas produced, are shown on Fig. 4. The rates employed for these cases differ by a factor of 5. Virtually exact agreement was obtained between the method of this paper and the earlier work of Bruce et al. Additional investigations have been made concerning unsteady-state flow behavior of ideal and nonideal gases. Because of basic differences in mathematical formulation, a comparison with results of these earlier studies was not attempted.

EFFECTS OF NEGLECTING \((\nabla \rho)^2\) TERM

Also shown on Fig. 4 are results of a numerical solution to Eq. 1 in the case that terms of the order \((\nabla \rho)^2\) are considered negligible (derivation of this simplified gas flow equation is in Appendix B). Good agreement is shown for the low rate case, but considerable divergence results at the higher rate. This same effect is observed for the nonideal gas case (Fig. 5) which depicts comparison data for the simplified and more rigorous numerical solutions. PVT properties of Gas A are used in both solutions, with results calculated from an initial pressure of 4,000 psia to a limiting pressure of 1,000 psia. The simplified numerical results show a rapid drop-off in flow pressure, down to the limiting pressure, at \( t_0 = 5.2 \times 10^6 \). The results from the more exact numerical treatment do not show this rapid drop-off. Rather, a smooth, almost linear pressure history is obtained. The pressure history is not entirely linear, however, which indicates that nonideal gas behavior has an effect on the numerical results.

Fig. 6 shows pressure profiles at \( t_0 = 5.1 \times 10^6 \) for the case of Fig. 5. The differences in pressure profiles are a direct result of basic differences in the mathematical models used. On the basis of the simplified model, one might postulate that the rapid decline in pressure is attributable
to the fact that maintenance of a constant mass flow rate at the well for declining pressure requires a constantly increasing volumetric flow rate. It is true that a more rapid decline in pressure is obtained near the end of the constant rate life, even with the more rigorous mathematical model. However, the almost discontinuous drop in pressure is not obtained.

Fig. 7 shows the complete producing history of the case used in Figs. 5 and 6. When wellbore pressure declined to 1,000 psia this pressure was maintained, whereas the producing rate declined. It can be seen that the prediction of the simplified model is quite conservative. In addition, it was not possible to maintain good material balance agreement at large times in the simplified solution.

COMPOSITE PERFORMANCE CALCULATIONS

In general, the production performance of a gas well can be summarized in a chronological manner:

1. After production is begun at constant rate, a period of rapid well pressure decline occurs during which time transient conditions of flow are present in the reservoir. This transient period is entirely analogous to the familiar transient production period from an oil well.

2. After the transient period at constant producing rate, a period of stabilized performance occurs. During this period, reservoir pressure declines fairly uniformly everywhere in the reservoir, and the productivity index of the well exhibits a very slow decline. This stabilized flow period is analogous to the period of semisteady-state production from an oil well. The lower the limiting pressure for constant rate flow, the higher the rate of wellbore pressure decline with time during the later portion of the constant rate period.

3. After wellbore pressure declines to a limiting value the producing rate begins to decline as the well produces against a constant pressure. Production rate decline during this period appears to be hyperbolic in character.

To illustrate the composite performance characteristics for typical reservoir conditions which might be encountered, some computer runs are presented with typical Anadarko Basin gas well and gas characteristics. These calculations were made for a drainage radius equivalent to 640-acre drainage, an effective wellbore radius of 10 ft and the Anadarko Basin gas properties shown on Figs. 1 and 2. An initial pressure of 2,400 psia and a limiting wellbore pressure of 1,000 psia were assumed. Plots of wellbore pressure, average reservoir pressure and producing rate vs dimensionless time for initial reduced flow rates of 100 and 25 are shown on Figs. 8 and 9. The initial reduced rates for each case are expressed in equivalent Mcf/D/ md-ft on the figures. For the high rate case (Fig. 8) there is a progressive divergence between wellbore pressure and the average reservoir pressure during the constant rate portion of the production history. This divergence leads to a decline in productivity index of the well during the constant rate portion of its history. In the lower rate case of Fig. 9, however, the productivity index is, for all practical purposes, constant. In this case most of the production during the well life is obtained during the constant rate period.

To illustrate the effect of a lower limiting wellbore...
pressure, the case of Fig. 8 was repeated with a terminal pressure of 300 psia (Fig. 10). The effect of the lower terminal pressure is to extend the producing period at constant rate by about 50 per cent. Inspection of the pressure history from 1,000 to 300 psia indicates that the rate of pressure decline at the wellbore steadily increases during this period. This decline in pressure is attributable to the fact that maintenance of a constant mass flow rate for declining pressures requires a constantly increasing volumetric flow rate. This brings about higher pressure gradients and lower pressures at the well.

To simulate the installation of compressor facilities and their effect on the flowing performance of a gas well, calculations were made for cases in which the limiting wellbore pressure is reduced to a second level (Fig. 11). Anadarko Basin gas properties and parameters, as used on Figs. 8 through 10, have been assumed. The reduced producing rate during the constant rate period was set at \( q_a = 50 \) (19.8 Mcf/D/mb-ft), and the first limiting well pressure assumed was 1,000 psia. The reduced rate was allowed to decline to a value of 3 (1.98 Mcf/D/mb-ft), at which time the limiting wellbore pressure was reduced to 300 psia. The production rate was increased to the initial level \( q_a = 50 \) when wellbore pressure was reduced. This caused a short period of constant rate production before decline again set in. In this case reduction of the wellbore pressure caused an additional amount of gas to be produced which was roughly equivalent to the amount produced during the declining rate portion of the life prior to the terminal pressure reduction. From calculations of this type, accurate assessment of the economics of compressor facilities can be made.

Having developed the mathematical model, a numerical method of solution and a realistic production scheme, comprehensive and dependable predictions of long-term gas well behavior can be prepared. However, it is not practical to obtain numerical solutions on a large computer for every gas well. This is particularly true when the quality of input data is questionable and a suite of computer runs are required to establish economic probabilities. In order that gas well performance predictions can be made with a minimum of time and expense at the operating level, dependable approximate solutions have been derived and are presented in the next section.

**APPROXIMATE METHODS FOR PREDICTING GAS WELL PERFORMANCE**

By comparison with the numerical results, we have found that the familiar equation describing the flow of a fluid of constant compressibility and viscosity can be adapted to describe the behavior of a gas well. This equation is:

\[
p_r - p_0 = \frac{q_a}{2kkT} \left[ P(t_a) + S \right], \quad \ldots \ldots \quad (8)
\]

in which \( P(t) \) is the unit function of van Everdingen and Hurst, and \( S \) is the skin factor.

The gas flow rate is converted from surface conditions to reservoir conditions by the isothermal equation of state for nonideal gas:

\[
q_a = q_r \frac{p_r}{P_r} \left[ \frac{Z(p_r) T_a}{P_r} \right], \quad \ldots \ldots \quad (9)
\]

Viscosity of the gas is also pressure-dependent and is expressed as \( \mu(p_r) \).

Methods have been developed to obtain unique solutions of Eq. 8 to describe both the transient and stabilized pressure behavior of a gas well producing at constant rate and at constant pressure. For both transient and stabilized flow conditions, it was necessary to specify the pressure \( p_{r*} \) which appears in Eq. 9 and which controls the variation of viscosity. This was accomplished by trial-and-error.

**TRANSIENT APPROXIMATION**

For transient flow, \( p_{r*} \) was found to be an average pressure \( (p_1 + p_2)/2 \). The \( P(t_a) \) function for transient conditions can be expressed as:

\[
P(t_a) = \frac{1}{2} \ln \left( t_o + 0.4045 \right), \quad \ldots \ldots \quad (10)
\]

If we substitute Eqs. 9 and 10 and replace \( p_{r*} \) with \( p_{r*} \).

**JANUARY, 1966**

103
Eq. 8 becomes:

\[ \rho_v - \rho_{sr} = \frac{25,152 \rho_v T q_{sr} \mu(p_{sr})Z(p_{sr})}{T_v kh \rho_{sr}} \left[ \frac{1}{2} \ln \frac{0.006336 k}{\phi \mu_c \rho_{hr}} + 0.4045 + S \right] \]  \( \text{(11)} \)

This equation is expressed in practical units (md, days, psia, ft, and feet).

It should be noted that the skin factor \( S \) and wellbore radius \( r_w \) are related to the effective wellbore radius \( r_w' \) by:

\[ r_w = r_w' e^{a'} \]  \( \text{(12)} \)

and that \( r_w' \) can be substituted in the \( P(t) \) approximation. Therefore, the skin term can be eliminated if desired.

To achieve a unique solution for the condition of constant flow rate, incremental pressures \( p_w \) are assumed and the corresponding real times are calculated with Eq. 11 rearranged in the form:

\[ \log t = \frac{T_v kh}{57,915 \rho_v T q_{sr} \mu(p_{sr})Z(p_{sr})} \left[ \frac{p_v^2 - p_{sr}^2}{0.006336 k} \phi \mu_c \rho_{hr} r_w' e^{a'} + 0.351 + 0.875 \right] \]  \( \text{(13)} \)

Since all parameters in the expression are constant except \( \rho_v, z, t \) and (depending on whether the rate or flowing pressure is constant) either \( p_w \) or \( q_{sr} \), Eq. 13 can be reduced to more convenient forms.

**For constant rate solution**, assume \( p_w \) increments and solve directly for time, using:

\[ \log t = C_i \frac{p_v^2 - p_{sr}^2}{q_{sr} \mu(p_{sr})Z(p_{sr})} - C_i \]  \( \text{(14)} \)

where

\[ C_i = \frac{T_v kh}{57,915 \rho_v T q_{sr} \mu(p_{sr})Z(p_{sr})} \left[ \frac{0.006336 k}{\phi \mu_c \rho_{hr} r_w' e^{a'}} + 0.351 + 0.875 \right], \]

and \( G_s = q_{sr} \).

In low permeability, large drainage area cases where transient flow is often effective for a period of time longer than a year, constant pressure solutions may be required. For this condition cumulative production \( G_p \) can be determined from numerical integration of the rate and time relationship. Average reservoir pressure can be found at any time by material balance in the form:

\[ \bar{p} = \frac{p_v}{z(\bar{p}) (1 - \delta)} \]  \( \text{(15)} \)

where

\[ \delta = \frac{G_p}{G_s}, \]

**For constant pressure solution**, assume \( q_{sr} \) increments and solve directly for time, using:

\[ \log t = C_i \frac{q_{sr}}{C_i} \]  \( \text{(16)} \)

where \( C_i \) and \( C_i \) are defined above, and

\[ C_i = \frac{T_v kh}{57,915 \rho_v T q_{sr} \mu(p_{sr})Z(p_{sr})} \left[ \frac{0.006336 k}{\phi \mu_c \rho_{hr} r_w' e^{a'}} + 0.351 + 0.875 \right]. \]

For transient flow solutions, it is also possible to assume a time \( t \) at which to evaluate \( P(t) \), estimate \( p_w \), and solve for \( p_w \) by an iterative process. This procedure should be used with caution, however, because two or more \( p_w \) values may satisfy the equation.

The excellent agreement between results of the approximate solution and the numerical solution to the differential equation will be illustrated after development of the approximation method for stabilized flow.

**STABILIZED FLOW APPROXIMATION**

From the equations which describe flow of a single, slightly compressible fluid in a bounded, circular reservoir, we know that for semisteady-state conditions \( P(t) \) can be expressed as:

\[ P(t) = \frac{2k}{\phi \mu_c r_w^2} \ln \frac{r_w}{\rho_{hr}} + \frac{3}{4} \]  \( \text{(17)} \)

Combination of this relationship with Eq. 8 yields:

\[ \bar{p} - \bar{p}_w = \frac{q_{sr} \mu}{2 \pi k h} \ln \frac{r_w}{\rho_{hr}} + \frac{3}{4} \]  \( \text{(18)} \)

This result is developed specifically for constant flow rate behavior but has also proved applicable in describing flow at constant pressure conditions.

By combining Eqs. 9 and 18, expressing \( p_w \) as \( p \) and converting to practical units, we obtain:

\[ \bar{p} - \bar{p}_w = \frac{25,152 \rho_v T q_{sr} \mu(p_{sr})Z(p)}{T_v kh \rho_{hr} r_w^2} \ln \frac{r_w}{\rho_{hr}} + \frac{3}{4} + S \]  \( \text{(19)} \)

Again by trial-and-error, Eq. 19 was found to be a good approximation to stabilized flow conditions when

\[ \bar{p} = \bar{p}_w + \frac{p_w}{2} \]  \( \text{(20)} \)

Eq. 19 is nonlinear in the unknown \( p_w \) under conditions of constant flow rate and must be solved iteratively. The material balance of Eq. 13 is used to establish the relationship between cumulative gas produced \( G_p \) and time \( t \).

**For constant rate solution**, assume \( p \) increments and solve for \( p_w \) by iteration, using:

\[ p_w^2 = p_v^2 - C_i q_{sr} \mu(p_{sr}) \]  \( \text{(20)} \)

where

\[ C_i = \frac{115,830 \rho_v T}{T_v kh} \left[ \log \frac{r_w}{r_w'} - 0.326 + 0.4345 \right], \]

and

\[ \delta = 1 - \frac{p_v}{p_{sr}}, \]  \( \text{and} \)

\[ t = \frac{G_p \delta}{q_{sr}}. \]

During the constant pressure period, flow rate can be solved directly at incremental values of assumed \( p \). The material balance Eq. 15 must be used to find \( G_p \). The relationship between \( G_p \) and time is found by numerical integration.

**For constant pressure solution**, assume \( q_{sr} \) increments and solve directly for \( q_{sr} \), using:

\[ q_{sr} = \frac{\bar{p} - \bar{p}_w}{C_i(p)Z(p)} \]  \( \text{(21)} \)

where

\[ G_p = \frac{G_{sr}}{p_v Z(p)} \left[ 1 - \frac{\bar{p}_w}{p_v \bar{p}_w} \right], \]

and

\[ t = t_0 + \sum_{k=1}^n \frac{G_p(p_k) - G_p(p_{k-1})}{k q_{sr}(p_k) + q_{sr}(p_{k-1})}. \]

The approximate method of solution of stabilized gas flow has also been found to duplicate effectively the computer numerical solutions. The transient and stabilized approximation methods can be combined to give accurate long-term, composite predictions of gas well performance.
In field application of the approximate performance prediction techniques a key parameter is the value for the permeability-thickness product. Experience in low-permeability reservoirs has shown that the apparent permeability, as determined by pressure build-up, often displays a general decline over the life of the well. Several factors, including liquid condensation in the formation, reservoir compaction and fracture healing, probably contribute to this effect. Use of permeability values determined from short, transient well tests will likely lead to overly optimistic performance predictions. Experience in field application of the techniques of this paper has shown that permeability values obtained from extended pressure drawdown tests yield the best performance predictions.

COMPOSITE PERFORMANCE CALCULATIONS: APPROXIMATE METHODS

Fig. 12 shows the comparison between results of the numerical and approximate solutions for transient flow. These solutions are for reduced flow rates of 10, 25, 50 and 100, with gas properties and parameters shown on the figure. It can be seen that almost identical results are derived by the two methods until some time after the transient pressure drop has reached the outer boundary.

The pressure disturbance first reaches the outer boundary at \( t_r \approx 0.1r_o^2 \). Comparison of results of the two methods, however, shows that the transient approximation is normally accurate to \( t_r \approx 0.2r_o^2 \). Stabilized flow was observed early in our investigation to begin at \( t_r \approx 0.3r_o^2 \). However, by comparison of computer results for the two methods of this paper, we found that reasonable accuracy is attained by assuming the start of stabilized flow at \( t_r = 0.2r_o^2 \).

Results with the assumption that transient flow ends and stabilized flow begins at \( t_r \approx 0.2r_o^2 \) are shown for the cases of Fig. 12 on Fig. 13. Only minor deviations from a smooth transition have been observed for numerous constant rate and constant pressure solutions by the approximate method over pressure ranges up to 10,000 psia.

Figs. 14 and 15 compare the composite performance predictions by the approximate methods with the computed solutions for the \( q_e = 100 \) case of Figs. 8 and 10, respectively. Almost perfect agreement is obtained for all phases of the calculation. The trend toward increasing decline rate in pressure during the constant rate period of Fig. 14 is accentuated on Fig. 15, where the limiting pressure is reduced to 300 psia.

From these results, it is concluded that the approximate solutions are very accurate and flexible and can be used with a high degree of confidence in making long-term performance predictions for gas wells. Although a digital computer was used in all analyses in this paper, the method is developed for easy application with a desk calculator or slide rule. Some sacrifice in accuracy is made when large pressure or rate increments are used in hand calculations; however, limits of accuracy can be ascertained by determining the general shape of pressure and rate decline curves for field conditions of interest.

Similarities can be observed between Eq. 19 and the long-used empirically derived relationship \( q_* = C(G^2 - p_*)^{-\frac{3}{4}} \). Eq. 19, when rearranged, is

\[
q_* = \frac{T_o kh}{25,135 \mu_o \mu_r} \frac{1}{2 \mu_o (\rho \rho_r) \left( \ln \frac{r_e}{r_i} - \frac{3}{4} + S \right)}
\]

Since Eq. 19 is known to fit Darcy flow gas well perform-

---

**Fig. 13**—Comparison of results during transition from transient to stabilized flow.

**Fig. 14**—Comparison of composite performance predictions.
nce, we have established for stabilized flow that

\[ n = 1, \]

\[ C = \frac{T \mu h}{25,152 p_r T} \frac{1}{2 \mu(p)z(p)} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} + S \right) \]

The fact that the coefficient \( C \) is a function of pressure suggests that plots of field data of the form \( \log q_n \) vs \( \log (p' - p) \) will always yield non-unity \( n \) values. Performance prediction methods based on the \( n \) and \( C \) values derived from this type of plot or variations thereof will generally give erroneous results. This is particularly true for thin, tight reservoirs when substantial pressure drawdown occurs, even during short flow tests. In fact, except for high permeability, small drainage volume cases, normal potential testing is conducted under transient conditions with \( C \) being a function of time as well as pressure.

DECLINE CURVE ANALYSIS

Further inspection of composite performance curves reveals that under stabilized flow conditions, and within certain limits, the production rate decline is hyperbolic during the constant pressure period. This observation is demonstrated by Fig. 16, which is a repeat of data and conditions used in Fig. 11. On Fig. 16 the production forecast for hyperbolic decline is compared with the approximate solution.

For both constant pressure periods the hyperbolic rate decline is analyzed and extended by conventional means, with one year's decline history being solved for the exponent and decline factor. Slide rule accuracy is used. For both cases the exponent is calculated to be 2.0. Decline factors are 0.127 and 0.1025, respectively. Hyperbolic decline during the 1,000-psia constant pressure period becomes progressively optimistic after about five years. The cumulative gas production forecast for this 12-year period is about 5.8 per cent optimistic. For the 300-psia terminal pressure period the two methods are more compatible. In this case the hyperbolic extension is only 1.7 per cent optimistic, with an estimated 1,343 MMcf recovery over 17.7 years. The production rate at the end of this period is about 6 per cent too high.

The limit of application for the hyperbolic rate decline is believed to be a function of the variation of gas properties with pressure. In all cases investigated where the product \( \mu(p)z(p) \) is a smoothly declining function with pressure, the hyperbolic exponent is 2.0, and the approximation is quite accurate.

DISCUSSION AND CONCLUSIONS

The second-order nonlinear partial differential equation that describes flow of natural gas through a porous medium has been solved numerically. Results obtained are believed to be theoretically and numerically correct. An approximate method for prediction of gas well performance was successful in duplicating the results of the rigorous numerical techniques. Development of the rigorous and approximate solutions, largely through application of a high-speed digital computer, eliminates the necessity for the computer in routine gas well performance prediction work.

The approximate solutions provide a quick and accurate method of predicting long-term performance of gas wells at the operating level. In addition, the transient flow solution constitutes an improved tool for evaluating basic reservoir parameters and completion techniques, particularly in tight reservoirs where older methods are often insufficient.

The method presented in this paper, when used within the framework of the assumptions made in the derivation, should yield dependable forecasts of gas well production and pressure performance.

NOMENCLATURE*

\[ A_i = \text{time-space coefficient} = \frac{\Delta x \Delta z}{\Delta t} \]

\[ c = \text{gas compressibility factor} = \frac{1}{p} - \frac{1}{z} \frac{\partial p}{\partial z} \]

\[ f(p) = \text{pressure-dependent function} \]

\[ f(p) = \text{pseudo-pressure variable function} \]

\[ G = \text{total initial gas in place in reservoir, Mcf} \]

\[ G_i = \text{cumulative gas produced, Mcf} \]

\[ h = \text{net pay thickness, ft} \]

\[ i = \text{iteration counter} \]

\[ j = \text{space increment index} \]

\[ k = \text{time increment index} \]

\[ k = \text{permeability, md} \]

\[ p = \text{pressure, psia} \]

\[ \bar{p} = \text{average reservoir pressure, psia} \]

*Units for the symbols are those used in application of information contained in this paper. Units employed in the basic derivation and finite difference equations are Darcy units.

Fig. 15—Comparison of Composite Performance Predictions.

Fig. 16—Comparison of Composite Performance Predictions by Approximate and Decline Analysis Methods.
\( \tilde{p} = \text{mean pressure} = (\tilde{p} + \tilde{p}_0)/2 \)

\( P = \text{pseudopressure variable} \)

\( F(t) = \text{unit function of van Everdingen and Hurst} \)

\( q = \text{volumetric gas production rate, Mcf/D} \)

\( q_a = \text{mass rate of gas production} \)

\( q_r = \text{reduced mass flow rate parameter} \)

\( r = \text{radial distance, ft} \)

\( r = \text{dimensionless radius} = r/r_w \)

\( S = \text{skin factor, dimensionless} \)

\( S_w = \text{water saturation, fraction} \)

\( t = \text{time, days} \)

\( t_s = \text{dimensionless time} = kt/\mu_c r_w^2 \)

\( T = \text{temperature, °R} \)

\( \nu = \text{flow velocity} \)

\( x = \text{dimensionless distance variable} = \ln r/r_w \)

\( z = \text{gas deviation factor, dimensionless} \)

\( \delta = \text{fraction of produced gas originally in place} \)

\( \mu = \text{gas viscosity, cp} \)

\( \rho = \text{density} \)

\( \phi = \text{gas-filled porosity, fraction} \)

**APPENDIX A**

**NUMERICAL SOLUTION BY FINITE DIFFERENCE METHODS**

**FINITE DIFFERENCE FORMULATION**

The Crank-Nicolson procedure, incorporating an iterative-type solution is used to solve the nonlinear differential equation:

\[
\frac{\partial P}{\partial x} = e^{\nu} f(P) \frac{\partial P}{\partial t},
\]

in which \( P \) is the pseudopressure variable. The difference formulation is centered at the \((k + \frac{1}{2})\Delta t\) time level and at space point \( j\Delta x \) (Fig. 17). At the time level \( k + \frac{1}{2} \):

\[
\frac{\partial P}{\partial x} \bigg|_{j\Delta x + \frac{1}{2} \Delta x} = \frac{1}{2} \left[ \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta x^2} \right. \left. + \frac{P_{i+1,j+1} - 2P_{i,j+1} + P_{i-1,j+1}}{\Delta x^2} \right],
\]

and the time derivative of \( P \) at point \( x \) is approximated by:

\[
\frac{\partial P}{\partial t} \bigg|_{j\Delta x + \frac{1}{2} \Delta x} = \frac{P_{i+1,j} - P_{i,j}}{\Delta t}.
\]

The value of \( x \) in \( e^\nu \) is \((-1)\) \( j \Delta x \), where \( j=1, 2, \ldots, n \).

For simplification, \( P \) is designated at the unknown time level \( k + 1 \) by the symbol \( P^{*} \) and introduces the parameter:

\[
A_j = \frac{\Delta x^2 e^{\nu} z_{i\Delta x + \frac{1}{2} \Delta x}}{\Delta t}.
\]

Using this notation and substituting the expressions for the derivatives into the partial differential equation, we obtain:

\[
P_{i+1,j+1} - 2[1 + A_j f(P_{i+1,j})]P_{i+1,j} - P_{i-1,j} = 2[1 - A_j f(P_{i+1,j})]P_{i,j} - P_{i+1,j} - P_{i,j}.
\]

An equation of this type can be written at each space point in the finite difference grid at each time level, with \( P(P_{i+1,j}) \) evaluated at each space point. The pseudopressure \( P_{i+1,j+1} \) is approximated as a linear function of time by:

\[
P_{i+1,j} = P_{i,j} + \frac{\Delta t}{2} \frac{\partial P}{\partial t} \bigg|_{j\Delta x + \frac{1}{2} \Delta x}.
\]

Substituting from the basic differential equation and writing Eq. A5 in finite difference form, we obtain:

\[
P_{i+1,j} = P_{i,j} + \frac{e^{\nu} z_{i\Delta x + \frac{1}{2} \Delta x}}{\Delta x^2} \left[ f(P_{i,j}) P_{i+1,j} + \frac{P_{i+1,j+1} - 2P_{i,j+1} + P_{i-1,j+1}}{2} \right]
\]

which can be used in Eq. A4 to evaluate \( f(P_{i+1,j}) \).

**OUTER BOUNDARY**

The pseudopressure variable is \( P_{i} \) at the outer boundary, where \( \partial P/\partial x = 0 \). The grid point next to the bound-

**REFERENCES**


ary is so imagined that \( P_{n+1} = P_{n-1} \), and Eq. A4 becomes:

\[
P_{n+1} - [1 + A f(P_{n+1})] P_n - P_{n-1} = 0 \quad \ldots \ldots \quad (A7)
\]

By imaging the difference expression Eq. A6, \( P_{n+1} \) is obtained.

INNER BOUNDARY:

CONSTANT RATE CASE

The inner boundary condition is a constant mass rate of production at the wellbore corresponding to constant surface rate of production at standard temperature and pressure. This is expressed as:

\[
\frac{\partial P}{\partial x} \bigg|_{x=0} = -q_{a} \quad \ldots \ldots \ldots \ldots \ldots \quad (A8)
\]

In finite difference form, at time level \( k + 1 \), the expression is:

\[
3P_n - 4P_{n+1} + P_{n+2} = 2q_{a} \Delta x \quad \ldots \ldots \quad (A9)
\]

A series of \( n \) equations (at the \( n \) space points in the grid) are reduced to the form of a tridiagonal matrix of order \( n = 1 \), in which the first equation is:

\[
6[1 + A f(P_{n+1})] - 4P_n + 2P_{n+1} = 2q_{a} \Delta x
\]

Inversion of this matrix is accomplished by an algorithm that has been described in detail elsewhere in the literature. This evaluates \( P_{n+1} \) through \( P_{n-1} \), and \( P_n \) is solved by Eq. A9.

Since \( f(P_{n+1}) \) is an implicit function of \( P_{n+1} \), it is reevaluated by averaging known and calculated pseudopressures, and the matrix is again solved to obtain the next solution. Iteration is continued until

\[
\left| \frac{P_{n+1} - P_{n+1,0}}{P_{n+1,0}} \right| \leq 10^{-4}
\]

INNER BOUNDARY—

CONSTANT PRESSURE CASE

For the constant terminal pressure case the boundary condition at the well is also Eq. A9, \( 3P_n - 4P_{n+1} + P_{n+2} = 2q_{a} \Delta x \), in which \( P_{n+1} \) is known. Again, a tridiagonal matrix solution is used to determine \( P_n \) at all other grid points, and \( q_{a} \) is calculated from the above equation. To ensure accuracy, the solution is iterated to the same precision as the constant rate case by testing the change in summation of all pseudopressures in the grid at \( k + 1/2 \).

ACCURACY OF THE NUMERICAL SOLUTION

The numerical solutions were subjected to a number of tests to determine their accuracy. The first of these tests was a comparison of the analytical solutions describing flow of a liquid of constant compressibility and viscosity with results for similar conditions generated from the computer program. The excellent agreement obtained for both constant rate and constant pressure cases indicated no numerical errors.

Perhaps the most reassuring evidence that the numerical solutions possess a high degree of accuracy is the excellent agreement noted in all cases between the average pressure obtained from integration of the numerically determined pressure profiles and that determined by material balance. In all cases which were calculated, the deviation of this numerically integrated average pressure from that determined by material balance was never more than from 3 to 5 psi (absolute pressure values in these cases were 1,500 to 4,000 psi). In addition, the fact that these small deviations seemed to be distributed randomly in time about 0 indicated no progressive growth of any calculation error. The very small errors noted are probably the result of round-off and truncation errors in the computer which was employed.

**APPENDIX B**

**DERIVATION OF THE GAS FLOW EQUATION NEGLECTING HIGHER-ORDER TERMS**

As described in Eq. 1 of the text, the differential equation for radial flow of a nonideal gas is given by:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r M_p \left( -\frac{k}{\mu} \frac{\partial P}{\partial r} \right) \right] = -\phi \frac{\partial}{\partial r} \left( \frac{M_p}{z RT} \right) \quad (B1)
\]

If we assume isothermal flow of a gas of constant composition, \( M/R \) remains constant. Further, if we consider a medium of constant permeability and define viscosity as a function of pressure, Eq. B1 can be rearranged and simplified to:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{p}{\mu} \frac{\partial p}{\partial r} \right) = \frac{\phi}{k} \frac{\partial}{\partial r} \left( \frac{p}{z} \right) \quad \ldots \ldots \quad (B2)
\]

The expression on the left can be written as:

\[
\frac{p}{\mu z} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial p}{\partial r} \frac{\partial}{\partial r} \left( \frac{p}{\mu z} \right)
\]

is of the order \((\nabla p)^2\) and is therefore considered negligible when compared with

\[
\frac{p}{\mu z} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right)
\]

We can then write

\[
\frac{p}{\mu z} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{\phi}{k} \frac{\partial}{\partial r} \left( \frac{p}{z} \right) \quad \ldots \ldots \quad (B3)
\]

After expansion of the derivative expression \( (\partial/\partial r)(p/z) \) and rearrangement, Eq. B3 becomes:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = \frac{\phi c \mu}{k} \frac{\partial p}{\partial t} \quad \ldots \ldots \quad (B4)
\]

in which

\[
c = \frac{1}{p} \frac{1}{0} \frac{\partial z}{\partial p},
\]

and \( \mu = \mu(p) \). To simplify Eq. B4, we introduce the variables

\[
x = \ln \frac{r}{r_o} \quad \text{or} \quad r = r_o e^x
\]

\[
t_o = \frac{kt}{\mu c_i r_o^2}
\]

and let

\[
f(p) = \frac{\mu c}{\mu c_i}
\]

These can be substituted into Eq. B4 to obtain the equation

\[
\frac{\partial p}{\partial t} = \phi^2 f(p) \frac{\partial p}{\partial t_o}, \quad \ldots \ldots \quad (B5)
\]

which is identical with Eq. 2 of the text, expressed in pressure rather than in the pseudopressure variable.
The Flow of Real Gases Through Porous Media

R. AI-HUSSAINY  
JUNIOR MEMBER AIME  
H. J. RAMEY, JR.  
P. B. CRAWFORD  
MEMBERS AIME

TEXAS A&M U.  
COLLEGE STATION, TEX.

ABSTRACT

The effect of variations of pressure-dependent viscosity and gas law deviation factor on the flow of real gases through porous media has been considered. A rigorous gas flow equation was developed which is a second order, non-linear partial differential equation with variable coefficients. This equation was reduced by a change of variable to a form similar to the diffusivity equation, but with potential-dependent diffusivity. The change of variable can be used as a new pseudo-pressure for gas flow which replaces pressure or pressure-squared as currently applied to gas flow.

Substitution of the real gas pseudo-pressure has a number of important consequences. First, second degree pressure gradient terms which have commonly been neglected under the assumption that the pressure gradient is small everywhere in the flow system, are rigorously handled. Omission of second degree terms leads to serious errors in estimated pressure distributions for tight formations. Second, flow equations in terms of the real gas pseudo-pressure do not contain viscosity or gas law deviation factors, and thus avoid the need for selection of an average pressure to evaluate physical properties. Third, the real gas pseudo-pressure can be determined numerically in terms of pseudo-reduced pressures and temperatures from existing physical property correlations to provide generally useful information. The real gas pseudo-pressure was determined by numerical integration and is presented in both tabular and graphical form in this paper. Finally, production of real gas can be correlated in terms of the real gas pseudo-pressure and shown to be similar to liquid flow as described by diffusivity equation solutions.

Applications of the real gas pseudo-pressure to radial flow systems under transient, steady-state or approximate pseudo-steady-state injection or production have been considered. Superposition of the linearized real gas flow solutions to generate variable rate performance was investigated and found satisfactory. This provides justification for pressure build-up testing. It is believed that the concept of the real gas pseudo-pressure will lead to improved interpretation of results of current gas well testing procedures, both steady and unsteady-state in nature, and improved forecasting of gas production.

INTRODUCTION

In recent years a considerable effort has been directed to the theory of isothermal flow of gases through porous media. The present state of knowledge is far from being fully developed. The difficulty lies in the non-linearity of partial differential equations which describe both real and ideal gas flow. Solutions which are available are approximate analytical solutions, graphical solutions, analogue solutions and numerical solutions.

The earliest attempt to solve this problem involved the method of secessions of steady states proposed by Muskat. Approximate analytical solutions were obtained by linearizing the flow equation for ideal gas to yield a diffusivity-type equation. Such solutions, though widely used and easy to apply to engineering problems, are of limited value because of idealized assumptions and restrictions imposed upon the flow equation. The validity of linearized equations and the conditions under which their solutions apply have not been fully discussed in the literature. Approximate solutions are those of Heatherington et al., MacRoberts and Janicek and Katz. A graphical solution of the linearized equation was given by Cornell and Katz. Also, by using the mean value of the time derivative in the flow equation, Rowan and Clegg gave several simple approximate solutions. All the solutions were obtained assuming small pressure gradients and constant gas properties. Variation of gas properties with pressure has been neglected because of analytic difficulties, even in approximate analytic solutions.

Green and Wilts used an electrical network for simulating one-dimensional flow of an ideal gas. Numerical methods using finite difference equations and digital computing techniques have been used extensively for solving both ideal and real gas equations. Aronofsky and Jenkins gave numerical solutions for linear and radial gas flow. Douglas et al. gave a solution for a square drainage area. Aronofsky included the effect of slippage on ideal gas flow. The most important contribution to the theory of flow of ideal gases through porous media was the conclusion reached by Aronofsky and Jenkins that solutions for the liquid flow case could be used to generate approximate solutions for constant rate production of ideal gases.

An equation describing the flow of real gases has been solved for special cases by a number of investigators using numerical methods. Aronofsky and Ferris considered linear flow, while Aronofsky and Porter considered radial gas flow. Gas properties were permitted to vary as linear functions of pressure. Recently, Carter proposed an empirical correlation by which gas well behavior can be estimated from solutions of the diffusivity equation using instantaneous values of pressure-dependent gas

*References given at end of paper.

JOURNAL OF PETROLEUM TECHNOLOGY
properties evaluated at an average pressure also defined empirically. Carter gave a limited number of numerical solutions as a basis, and suggested some relations which might give a better correlation. However, the proposed relations were not evaluated in the mentioned work. Solutions have been presented by Eilerts et al. for flow of gas-condensate fluids in linear and radial systems.

It has been observed that as the gas flow velocity increases, departure from Darcy’s law occurs. Such flow is termed non-Darcy, or turbulent flow. Flow is transitional, and not truly turbulent. A gas flow equation including a quadratic velocity term to account for turbulence near the producing well has been solved by Swift and Kiep and Tek et al. for ideal gases. Eilerts et al. and Carter also included non-Darcy flow in their models for real gases. An approximate solution including non-Darcy flow has been presented by Rowan and Clegg.

Two other calculational procedures appear in the works of Roberts and Kidder for solving the one-dimensional flow equation for an ideal gas. Roberts used a stepwise forward integration in time by joining a sequence of solutions for linearized differential equations. Kidder, applying perturbation technique and using the well-known Boltzmann transformation in the theory of diffusion, gave an exact analytic solution for gas flow in a semi-infinite porous medium. Kidder’s solution is very similar to a more general one reported by Polubarinova-Kochina on the movement of ground water.

In summary, only a limited number of solutions for flow of real gases are available, and these are not of general utility. Furthermore, methods of analyzing gas reservoir performance in current use are generally based on solutions for the flow of ideal gases under the assumption of small pressure gradients. These methods fail to describe the behavior of low permeability and high pressure reservoirs.

FLOW OF REAL GASES

The following concerning the flow of real gases through porous media is drawn from an analogy with the theory of heat conduction in solids. Variation of gas physical properties with the pressure correspond to that of temperature-dependent properties in the theory of heat conduction.

The mechanism of fluid flow through a porous medium is governed by the physical properties of the matrix, geometry of flow, PVT properties of the fluid and pressure distribution within the flow system. In deriving the flow equations and establishing the solutions, the following assumptions are made. The medium is homogeneous, the flowing gas is of constant composition and the flow is laminar and isothermal. Assumption of laminar flow can be removed, but will be used to simplify the presentation.

The principle of conservation of mass for isothermal fluid flow through a porous medium is expressed by the well-known continuity equation:

\[ \nabla \cdot \rho \mathbf{v} = -\frac{\partial \rho}{\partial t} \]

The velocity vector in Eq. 1 is given by Darcy’s law for laminar flow as:

\[ \mathbf{v} = -\frac{k(p)}{\mu(p)} \nabla p \]

Substituting Eq. 2 in Eq. 1 yields:

\[ \nabla \left[ \rho \frac{k(p)}{\mu(p)} \nabla p \right] = \phi \frac{\partial p}{\partial t} \]

For real gases:

\[ \rho = \frac{M}{RT} \frac{p}{z(p)} \]

Density can be eliminated from Eq. 3 to yield:

\[ \nabla \left[ \frac{k(p)}{\mu(p) z(p)} p \nabla p \right] = \phi \frac{\partial}{\partial t} \left[ \frac{p}{z(p)} \right] \]

Eq. 5 is one form of the fundamental non-linear partial differential equation describing isothermal flow of real gases through porous media.

The pressure-dependent permeability for gas was expressed by Klinkenberg as:

\[ k(p) = k_0 \left( 1 + \frac{b}{p} \right) \]

where \( k_0 \) is effective permeability to liquids; and \( b \) = slope of a linear plot of \( k(p) \) vs \( 1/p \).

However, the dependency of permeability on pressure is usually negligible for pressure conditions associated with gas reservoirs, as pointed out by Aronofsky. In a subsequent report, Aronofsky and Ferris indicated that variations of gas properties with pressure are more important than variations of permeability with pressure. Therefore, liquid permeability can be used for gas flow, and the following equation is correct for all practical purposes:

\[ \nabla \left[ \frac{\rho}{\mu(p) z(p)} \nabla p \right] = \phi \frac{\partial}{\partial t} \left[ \frac{p}{z(p)} \right] \]

Eq. 7 can be expanded to many different forms. For example, Eq. 7 can be rearranged to point out explicitly the real gas diffusivity

\[ \frac{k}{\phi(p) \sigma_f(p)} \]

Since

\[ p \nabla p = \frac{1}{2} \nabla p^2 \]

Eq. 7 becomes, after some rearrangement:

\[ \nabla^2 p' = d \left[ \ln \mu(p) z(p) \right] \frac{\partial}{\partial p} \left( \frac{p}{\mu(p) z(p)} \right) \frac{\partial}{\partial t} \left[ \frac{p}{z(p)} \right] \]

From the definition of the isothermal compressibility of gas:

\[ c_f(p) = \frac{1}{\rho} \frac{d \rho}{d p} = \frac{z(p)}{p} \frac{d}{d p} \left( \frac{p}{z(p)} \right) = \frac{1}{p} - \frac{1}{z(p)} \frac{d z(p)}{d p} \]

Thus:

\[ \frac{\partial}{\partial t} \left[ \frac{p}{z(p)} \right] = \frac{pc_f(p)}{z(p)} \frac{\partial p}{\partial t} \]

Combining Eqs. 9 and 11:

*Permeability can be considered an important function of pressure for a wet condensate gas as used by Eilerts. This case can be handled, as will be shown later in this paper.*
\[ \nabla \rho^* = \frac{d(\mu p z(p))}{dp} \frac{\nabla \rho^*}{\partial t} = \frac{\phi \rho(z(p)) \partial p^*}{k} \frac{\partial \rho^*}{\partial t}. \] (12)

If it is assumed that viscosity and gas law deviation factors change slowly with pressure change, the pressure differential of \(1/\mu p z(p)\) becomes negligible. On the other hand, the assumption that pressure gradients are small will permit omission of terms of order \( \nabla \rho^* \). In either event, Eq. 12 can be simplified to:

\[ \nabla \rho^* = \frac{\phi \rho(z(p)) \partial p^*}{k} \frac{\partial \rho^*}{\partial t}. \] (13)

Eq. 13 is similar in form to the diffusivity equation. However, the diffusivity is a function of pressure, even for a perfect gas. In this form, the close analogy with liquid flow found by Jenkins and Aronofsky is emphasized. However, the assumption that pressure gradients are small everywhere in the flow system cannot be justified in many important cases. The assumption of small pressure gradients is implicit in all of the pressure build-up and drawdown methods currently in use which are based upon ideal gas flow solutions or liquid flow analogies. We return, then, to the rigorous Eq. 7.

Eq. 7 can be transformed to a form similar to that of Eq. 13 without assuming small pressure gradients, by making a scale change in pressure. Define a new pseudo-pressure \(m(p)\) as follows:

\[ m(p) = 2 \int_{P_*}^{p} \frac{\rho}{\mu(p)z(p)} dp \] (14)

where \(P_*\) is a low base pressure. The variable \(m(p)\) has the dimensions of pressure-squared per centipoise. Since \(\mu(p)\) and \(z(p)\) are functions of pressure alone for isothermal flow, this is a unique definition of \(m(p)\). It follows that:

\[ \frac{\partial m(p)}{\partial t} = \frac{\partial m(p)}{\partial \rho} \frac{\partial \rho}{\partial t} = \left( \frac{2p}{\mu(p)z(p)} \right) \frac{\partial \rho}{\partial t}, \] (15)

and

\[ \frac{\partial m(p)}{\partial z} = \left( \frac{2p}{\mu(p)z(p)} \right) \frac{\partial \rho}{\partial z}. \] (16)

with similar expressions for \( \frac{\partial m(p)}{\partial y} \) and \( \frac{\partial m(p)}{\partial x} \).

Therefore, Eq. 7 can be rewritten in terms of the variable \(m(p)\) using the definition of \(z(p)\) given by Eq. 10 as:

\[ \nabla \cdot \nabla m(p) = \frac{\phi \rho(z(p)) \partial m(p)}{k} \frac{\partial \rho}{\partial t} \] (17)

or

\[ \nabla m(p) = \frac{\phi \rho(z(p)) \partial m(p)}{k} \frac{\partial \rho}{\partial t}. \] (18)

Comparison of Eqs. 13 and 18 shows that the form of the diffusivity equation is preserved in terms of the new variable \(m(p)\). However, Eq. 18 is still non-linear because diffusivity is a function of potential. The gas law deviation factor \(z(p)\) does not appear in the equation, but is involved in \(m(p)\) and \(z(p)\). Eq. 18 does not involve the assumption of small pressure gradients, nor that of slow variation of \(z(p)\).

The importance of Eq. 18 deserves emphasis. It is a fundamental partial differential equation which describes the flow of real gases. To the authors' knowledge, this equation has not been presented previously in connection with gas flow. Equations of this type have been called quasi-linear flow equations.\(^{23}\) The real importance lies in the extreme utility of this form of the equation. As will be shown, the form of the equation suggests a powerful engineering approach to the flow of real gases.

To solve Eq. 18, it is necessary to convert the usual initial and boundary conditions into terms of the new pseudo-pressure \(m(p)\). Important considerations are as follows.

The gas mass flux is:

\[ \frac{q}{A} = \frac{\phi \rho(z(p))}{2RT} \frac{\partial \rho}{\partial t} \] (19)

In terms of \(m(p)\), the mass flux is:

\[ \frac{q}{A} = -\frac{M}{2RT} \frac{\partial m(p)}{\partial t}. \] (20)

The usual boundary conditions are either specification of pressure or the gas flux across bounding surfaces. When pressure is fixed, \(m(p)\) can be determined from Eq. 14. If flux is specified, the boundary conditions can be determined from Eq. 20. If the outer boundary is impermeable, then:

\[ \frac{\partial m(p)}{\partial n} = 0, \] (21)

where \(n\) is the direction normal to the boundary.

Steady-state flow occurs when pressure distribution and fluid velocity are independent of time. Eq. 18 reduces to:

\[ \nabla \cdot \nabla m(p) = 0 \] (22)

which is Laplace's equation. Thus, previous solutions of the Laplace equation can be used if \(m(p)\) is used as the potential.

Steady-state flow can rarely be obtained in reality because gas wells usually produce gas from a limited, finite reservoir or drainage volume. There can be no flow across the outer boundary. Thus, pressure must decline as production continues. True steady-state would require pressure to remain constant at the outer boundary, which implies flow across the outer boundary. Production of a bounded reservoir at constant production is an important problem, which will be considered later in this paper.

### REAL GAS PSEUDO-PRESSURE

To obtain generally useful solutions for Eq. 18, the proper physical properties for natural gases must be specified. Fortunately, all required physical properties have been correlated as functions of pseudo-reduced pressures and temperatures for many gases met in field work. It should be emphasized that the concept of the real gas pseudo-pressure is not limited to use of specific gas property correlations. Pseudo-reduced pressure and temperature are defined, respectively, as:

\[ \rho_p^* = \frac{\rho}{\rho_{cr}} \] (23)

and

\[ T_p^* = \frac{T}{T_{cr}} \] (24)

where \(\rho_{cr}\) is the pseudo-critical pressure and \(T_{cr}\) is the pseudo-critical temperature. Real gas law deviation factors \(z(p)\) have been presented by Standing and Katz.\(^{24}\) Vis-

\[ \text{JOURNAL OF PETROLEUM TECHNOLOGY} \]
Compressibilities of natural gases have been correlated by Carr et al.\textsuperscript{9} as the ratio of viscosity at any pressure to that at one atmosphere. Thus:

$$\frac{\mu(p)}{\mu_1} = f(p_r, T_r)$$

(25)

The integral can be evaluated generally from reduced properties correlations.

**EVALUATION OF REAL GAS PSEUDO-PRESSURE**

To establish the relationship between $p_r$ and $m(p)$, the integral must be evaluated numerically for various isotherms. The lower limit of the integration $(p_r)_0$ can be set arbitrarily. A value of 0.20 was chosen. Selected isotherms from pseudo-reduced temperatures of 1.05 to 3.0 were used.

Fig. 1 presents the argument of the integral in Eq. 27 vs pseudo-reduced pressure for various pseudo-reduced temperatures. The dashed line represents the ideal gas case with both viscosity ratio and gas law deviation factor equal to unity. The magnitude of gross variations of gas properties with pressure and temperature is apparent.

Fig. 2 presents $m(p)$ integrals as functions of pseudo-reduced pressures and temperatures. The integrals were evaluated by means of the Trapezoidal rule using an IBM 709 digital computer. Values of the integrals are also presented in Table 1. Interpolation between the curves or between the values presented in the table can be performed easily.

Use of Fig. 2 or Table 1 is limited to gases containing small amounts of contaminants for which changes in viscosity and gas law deviation factor can be handled by appropriate changes in the pseudo-critical properties, as suggested by Carr et al.\textsuperscript{9} However, useful charts can be prepared for gases containing large amounts of contaminants if complete properties are known. See Robinson.
et al." for density data for gases containing large amounts of contaminants.

In general, it is useful to prepare a chart of \( m(p) \) in units of psi-kg/ft\(^3\) centipoise vs pressure in psi for any given reservoir to aid engineering use of the ideal gas pseudo-pressure. The \( m(p) \) can be computed readily for any specific gas and reservoir temperature if density and viscosities are known as functions of pressure. The integration can be performed using the Trapezoidal rule or graphical integration. More sophisticated integrations are usually not required.

The \( m(p) \) values in Fig. 2 and Table 1 are presented as a convenience because it is necessary to assume many gases do follow the existing correlations because lack of specific data. It is emphasized that the concept of the real gas potential is general and is not limited to use of the \( m(p) \) values presented herein. If viscosity and density data are available for a specific gas, it should be used in preference to Fig. 2 and Table 1 to prepare \( m(p) \) plots for the specific gas.

**TRANSIENT FLOW**

**CONSTANT-RATE PRODUCTION**

As has been described in the introduction of this paper, Eq. 7 has been solved for specific flow cases under appropriate boundary and initial conditions by a number of authors using finite difference solutions. We seek a general solution which can be used for engineering purposes without the aid of a digital computer. Eq. 18 and the work of Aronofsky and Jenkins\(^*\) provide a basis for an approach. For radial flow of ideal gas, the continuity equation leads to:

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi c_d(p) \rho}{k} \frac{\partial \rho}{\partial t}, \quad \ldots \ldots (28)
\]

where \( c_d(p) \) for an ideal gas is the reciprocal of the pressure. Several features of Eq. 28 are noteworthy. First,
the second degree pressure gradient term \( (\partial p/\partial r)^2 \) does not appear for an ideal gas. Second, Eq. 28 has the form of the diffusivity equation, but the diffusivity is proportional to pressure. Viscosity is a function of temperature, but not of pressure for an ideal gas. Aronofsky and Jenkins found that for constant rate production of an ideal gas from a closed radial system, the pressure at the producing well could be correlated as a function of a dimensionless time based on a compressibility evaluated at the initial pressure. The correlation was slightly sensitive to the production rate, but not sensitive enough to affect engineering accuracy.

Aronofsky and Jenkins demonstrated that production of ideal gas from a closed radial system could be approximated very closely from the solutions for transient liquid flow of van Everdingen and Hurst. Matthews later pointed out the application of this conclusion to pressure build-up analysis for gas wells as a liquid case analog.

For radial flow of a real gas, Eq. 18 becomes:

\[
\frac{\partial m(p)}{\partial t} + \frac{1}{r} \frac{\partial m(p)}{\partial r} = \frac{\phi p(p)c_r(p)}{k} \frac{\partial m(p)}{\partial r} . \tag{29}
\]

The close analogy between Eqs. 28 and 29 suggests that the real gas pseudo-pressure \( m(p) \) should correlate as a function of a dimensionless time based on viscosity and compressibility evaluated at the initial pressure, if the variation of the viscosity-compressibility product with \( m(p) \) for a real gas is similar to the variation of compressibility for an ideal gas (1/p) with pressure squared. Fig. 3 shows the comparison.

In view of the close resemblance between \( m(p) \) vs \( m(p) \) for the real gas, and \( p^4 \) vs \( p^2 \) for the ideal gas, it is reasonable to expect solutions for the flow of real gases to correlate as functions of a dimensionless time based on initial values of viscosity and compressibility. That is, let:

\[
t_\phi = \frac{kt}{\phi(m_c),r_\phi} . \tag{30}
\]

Further, define a dimensionless real gas pseudo-pressure drop \( m_\phi(t_\phi,r_\phi) \):

\[
m_\phi(t_\phi,r_\phi) = \pi k h \frac{\Delta m(r,t)}{q_i r_\phi^2} , \tag{31}
\]

where \( r_\phi = r/r_\phi \). The dimensionless real gas pseudo-pressure drop is thus analogous to the van Everdingen-Hurst dimensionless pressure drop \( p_\phi(t_\phi) \).

Fig. 4 shows the comparison between \( p_\phi(t_\phi) \) and \( m_\phi(t_\phi) \) for a closed, radial reservoir produced at constant rate.

**Fig. 4—\( p_\phi(t_\phi) \) and \( m_\phi(t_\phi) \) vs \( t_\phi \) for a closed, radial reservoir produced at constant rate.**

![Graph showing \( p_\phi(t_\phi) \) and \( m_\phi(t_\phi) \) vs \( t_\phi \)](image)

**Fig. 5—\( m_\phi(t_\phi) \) and \( p_\phi(t_\phi) \) vs \( t_\phi \) for radial flow of liquid, ideal gas, and real gases.**

![Graph showing \( m_\phi(t_\phi) \) and \( p_\phi(t_\phi) \) vs \( t_\phi \) for liquid, ideal gas, and real gases](image)

**Fig. 3—Pressure-dependent diffusivity terms vs potential for ideal and real gas flow.**

![Graph showing pressure-dependent diffusivity terms for ideal and real gas flow](image)
(Fig. 5). Also shown on Fig. 5 are the Aronofsky-Jenkins' ideal gas flow results. It is clear that both the ideal and real gas cases lead to dimensionless pressure drops which are lower than the liquid case—and which are flow-rate dependent. Another important difference is illustrated by the case \( Q = 0.05 \). The ideal gas line terminates at the point where the well pressure is zero. The real gas solutions terminate at a well pressure of 10 per cent of the initial pressure. Although not shown on Fig. 5, the production times for the real gas cases to reach a limiting production pressure are about two and a half times those required for the ideal gas flow cases. Clearly, production forecasts based on the ideal gas solutions will be far too conservative.

Another important observation can be made from Fig. 5 by comparing the real gas solutions for natural gas and condensate for a flow rate \( Q \) of 0.05. Although the natural gas line is close to the liquid case, the condensate line is far below the liquid case line. The terminal producing pressure is reached earlier for the condensate line than for the natural gas line. This indicates the importance of gas property variations upon the results. That is, no single set of \( m_t(\alpha) \) correlations could be expected to apply for all real gas production times. It is also clear from Fig. 5 that the real gas results tend to approach the liquid case results as flow rate decreases, and at small production times.

Aronofsky and Jenkins introduced the concept of a transient drainage radius \( r_0 \). This term should not be confused with the dimensionless radial coordinate \( r_n \). From the Aronofsky-Jenkins definition of the transient drainage radius, we write for real gas flow:

\[
\ln \frac{r_0}{r_n} = \frac{1}{q_n p_r T} \left[ m'(p) - m'(p_0) \right] = m_0(p_0) - m_0(p)
\]  

(32)

The Eilerts et al. results can also be correlated as transient drainage radius vs. dimensionless time. The results are presented in Fig. 6, and agree with the Aronofsky-Jenkins results and the liquid flow results almost exactly. Actually, the correlation of the real gas flow solutions in terms of the transient drainage radius (Fig. 6) is a much better correlation than the correlation in terms of \( m_t(\alpha) \) (Figs. 4 and 5). The drainage radius correlation is excellent for all values of production time. Thus, Eq. 32 provides the most useful engineering approach to the transient flow of real gases. As recommended by Jenkins and Aronofsky for ideal gas flow, the transient drainage radius for real gas flow can be found from:

\[
\ln \frac{r_0}{r_n} = p_t(\alpha) - 2p_0 \left( \frac{r_0}{r_n} \right)^3
\]  

(33)

and the \( m'(p) \) can be found from the materials balance:

\[
\left( \frac{p}{\mu_c} \right) - \left( \frac{\mu}{\mu_c} \right) = \frac{T p_r q_m T_r}{z h r_n} = \frac{1}{2} \int m(p) \, dp
\]  

\[
(\mu c) dm(p) = \frac{\mu_c}{2} \left[ m(p) - m(p_0) \right]
\]  

(34)

Eqs. 32 through 34 are not strictly a solution to Eq. 18. They represent an excellent engineering approximation which applies for a wide range of conditions. The method appears to be every bit as good as the Jenkins-Aronofsky result for ideal gas flow.

Fig. 6 shows that at long production times \( r_n \) takes the constant value 0.472 \( r_n \). This is similar to the Aronofsky-Jenkins finding for ideal gas. Substitution of long-time values for \( p_t(\alpha) \) in Eq. 33 also leads to this conclusion. Thus, Eq. 32 becomes similar in form to the liquid case pseudo-steady-state equation at times long enough that the outer boundary effect is controlling. The fact that \( r_n \) eventually becomes constant at 0.472 \( r_n \) does not mean the physical drainage radius stabilizes about half-way out in the reservoir. The entire reservoir volume is being drained, as can be seen by inspection of any of the Eilerts et al. production figures.

The Eilerts et al. solutions have provided an excellent set of information to test the linearization of the real gas flow solutions for production. Eilerts et al. specified that the effective permeability was a function of pressure (assuming pressure drop would result in condensation and reduction of effective permeability). Effective permeability can thus be taken within the \( m(p) \) integral. Correlations in Figs. 4 through 6 do include a pressure-dependent permeability. Thus, if an approximation of the effect of pressure drop upon liquid condensation and reduction in permeability near the wellbore can be made, the performance can be estimated from:

\[
\ln \frac{r_0}{r_n} = \frac{z h T_n}{q_n p_r T} \left[ m'(p) - m'(p_0) \right]
\]  

(35)

where

\[
m'(p) = 2 \int_p^{p_m} \frac{k p d p}{
\mu c}
\]  

(36)

and \( k \) is a known function of pressure.

The usefulness of considering \( k \) as a function of pressure to handle condensate flow might be open to question. Nevertheless, it is clearly indicated that variation of \( k \) as a function of pressure can be included in the real gas pseudo-pressure.

Correlation of the Eilerts et al. data presented previously involves calculation of \( m(p) \) and determination of relationships between the Eilerts et al. nomenclature and that used in this paper. (Necessary relationships are in the Appendix).

Eilerts et al. also determined performance with a steady-state, non-Darcy flow region near the producing well. As a result, a steady-state skin effect can also be introduced to yield the following approximation for the radial flow of real gases during production:

\[
\frac{z h T_n}{q_n p_r T} \left[ m(p) - m(p_0) \right] = \ln \frac{r_0}{r_n} + s + D q_m
\]  

(37)

![FIG. 6-JEIKNS-ARONOFSKY DRAINFAGE RADIUS VS. R, FOR A CLOSED, RADIAL RESERVOIR PRODUCED AT CONSTANT RATE.](image-url)
where \( s \) is the skin effect and \( D \) is the non-Darcy flow coefficient.

**CONSTANT RATE INJECTION**

All of the preceding discussion of real and ideal gas transient flow deals with production only. Injection results, as was clearly shown by Aronofsky and Jenkins for radial ideal gas flow, cannot be linearized in as simple a fashion. Aronofsky and Jenkins correlated injection well pressures for radial flow of an ideal gas as functions of a dimensionless time based on gas compressibility evaluated at the initial formation pressure before injection. The dimensionless pressure rise at a given dimensionless time was generally greater than that for a liquid case, and increased with injection rate. Aronofsky and Jenkins showed that injection case results were very close to the liquid case for low injection rates. Although injection is of practical importance in itself, the major utility of injection case correlations is in application of the principle of superposition to generate variable rate production cases, including the important pressure build-up case.

Superposition, as it has been applied in gas well testing, requires that dimensionless times for both injection and production be based on the same gas physical property evaluation. Although superposition could be based on different dimensionless times for injection and production, the added complexity of such a scheme does not appear justified. Thus, an obvious question is: will injection solutions correlate closer to the liquid case if dimensionless times are based on physical properties evaluated at a pressure above the initial, low formation pressure?

We rule out the scheme of using a point evaluation at the existing injection pressure because this would yield a result not usable for forecasting. That is, it would be necessary to know the injection pressure-time history before it could be calculated. An obvious possibility is to evaluate physical properties at the final, elevated injection pressure, or in the case of superposition applied to reservoir production or build-up, at the initial formation pressure before production was started. This idea is fundamentally the basis for all gas well pressure build-up applications currently in use.

In brief, correlations for injection based on an elevated pressure are no better (or worse) than those based on physical properties evaluated at the initial, low formation pressure. This is true for both the ideal and real gas flow cases. Fig. 7 presents the dimensionless real gas potential rise for the Eilerts et al. injection case (their Fig. 8) correlated vs dimensionless times based on both the initial, low formation pressure and the final injection pressure. The dashed line presents the liquid flow solution. Two facts are apparent: the slopes of the correlations are similar, and correlations based on final injection pressure are no worse than those based on initial, low formation pressure. From the Jenkins-Aronofsky studies of ideal gas flow, we can also conclude that the difference between the injection case correlations and the liquid case become smaller as injection rate decreases; in any case, the differences aren't large.

Fig. 7 can lead to another idea. Correlation based on a dimensionless time evaluated with physical properties about half-way between the extremes might be quite good. This idea follows immediately from the theorem of the mean. That is, if flowing fluid physical properties vary monotonically with potential, the proper result is limited by those evaluated at the extreme values of physical properties. Friedmann proved that results must lie between those evaluated at the extremes of physical properties whether physical properties are monotonic functions or not. The injection problem has been the subject of much investigation in the fields of heat transmission and ground-water movement (Friedmann Storm and Polubarinova-Kochina). As has been shown by these authors, it cannot always be assumed that evaluation at an average property will yield good results. Sometimes the answer will vary from one extreme to the other.

**SUPERPOSITION OF LINEARIZED SOLUTIONS**

Superposition is rigorously correct only for linear partial differential equations. Nevertheless, the extremely close check between the linearized real gas solutions correlated on the basis of the \( m_s(t_0) \), as given by Eq. 31, and a \( t_2 \) given by Eq. 30, and the liquid flow solutions of van Everdingen and Hurst, indicates the possibility that superposition might be quite good for matching an increasing rate production schedule. An increasing rate

![Figure 7](image-url)

**FIG. 7—\( P_p(t_0) \) AND \( m_s(t_0) \) VS \( t_2 \) FOR INJECTION OF A REAL GAS (CORRELATION OF EILERTS ET AL. DATA).**

![Figure 8](image-url)

**FIG. 8—\( P_p(t_0) \) AND \( m_s(t_0) \) VS \( t_0 \) FOR CONSTANT RATE PRODUCTION WITH WELLBORE STORAGE.**
schedule would require superposition of positive incremental rates. However, the real gas flow solutions do depend slightly upon production rate. Thus, the only way that the application of the principle of superposition (as an acceptable approximation) to real gas flow can be established is by comparison with finite-difference solutions of variable-rate, real gas flow problems.

Such a comparison can be made for an increasing production rate schedule from data for real gas flow published by Carter. Carter studied the effect of wellbore storage on gas production. For his solutions, it was assumed that the surface flow rate was held constant, but 0.02965 Mcf was withdrawn from the wellbore per psi pressure drop in the wellbore. This resulted in the sand face flow rate increasing as a function of time toward the constant surface flow rate. This case is almost exactly analogous to the wellbore unloading case presented by van Everdingen and Hurst in their Eq. VIII-II. The wellbore storage constant $C$ for Carter's solutions can be determined from Eq. 6 presented by Ramey. The value of $C$ for Carter's solutions does vary slightly with pressure, but a value of 300 is quite good. Fig. 8 presents a comparison between the $m_d(t_0)$ obtained from Carter's solutions, both with and without wellbore storage, and the van Everdingen-Hurst $p_d(t)$ solutions for the liquid flow case. As can be seen, the comparison with constant rate liquid flow without storage is excellent. This was previously shown for the EiUuTa et al. solutions. Of more interest, the comparison between the liquid flow case with wellbore storage and Carter's two solutions with wellbore storage are also excellent. This establishes that superposition of the linearized real gas flow solutions for an increasing flow rate should be a very good approximation—at least before outer boundary effects are controlling.

Although superposition in an increasing production rate schedule appears quite good, it is not apparent that a decreasing rate schedule is susceptible to superposition. This results because the dimensionless real gas injection pressure increases do not correlate with the liquid case as well as do production data. Even for transient injection of an ideal gas, the resulting dimensionless pressure rise appears to depend upon injection rate, but does approach the liquid case solution as injection rate decreases. The fact that injection results do approach the liquid case as injection rate approaches zero suggests that superposition of small positive incremental rates (injections) would be feasible. Again, the possibility can only be checked by comparison with finite-difference solutions.

Fortunately, both Carter and Dykstra have presented finite-difference solutions for decreasing flow-rate production. Dykstra's data provide an excellent set for comparison of finite-difference solutions with superposition of the linearized solutions. Fig. 9 presents a comparison of Dykstra's computed flowing pressures with those obtained by superposition of linearized real gas flow solutions. The line is Dykstra's result, while points represent results of superposition using only four or five incremental rate changes to represent a rapidly changing flow rate. The flow rate is shown by the dashed line. For the example shown, the permeability was 0.25 md, thickness was 179 ft, initial pressure 6,150 psi and flow rate declined quadratically as a function of time from 6,556 to 2,500 Mcf/D by 50 days' producing time. Superposition was accomplished using dimensionless times based on the initial pressure and the $m_d(t_0)$ taken equal to the liquid case $p(t_0)$ values. The maximum difference between Dykstra's result and those computed by super-

position was 20 psi out of a drawdown of 2,150 psi—a difference of 0.9 per cent. The 50-day production period was long enough that initial rate changes were influenced by the outer boundary. Thus, we conclude that superposition can be used to reproduce variable-rate drawdown data with acceptable accuracy.

The previous remarks concerning superposition of incremental rate increases are, of course, directly applicable to pressure build-up testing. Although insufficient comparisons between finite-difference build-up solutions and superposition solutions for the real gas flow case have been made to completely explore this problem, it does appear that build-up theory can be used with good accuracy. An interesting test of pressure build-up can be made by comparison of Dykstra's solutions with superposition solutions. Because Dykstra's cases involved a variable-rate production period, permeability was low and pressure gradients high, it is believed that a fairly extreme test results. Fig. 10 presents the build-up following the drawdown of Fig. 9. As can be seen, the

---

**Fig. 9—Comparison of Finite Difference and Superposition Flowing Pressures for a Decreasing Production Rate.**

---

**Fig. 10—Comparison of Finite Difference and Superposition Build-Up Pressures for a Real Gas.**
superposition result yields a similar build-up curve of identical slope, but about 60 psi below Dykstra’s finite-difference solutions. Again, the percentage difference is small; the final static pressure is about 1.1 per cent too low. It appears that superposition of the real gas flow linearization will always yield a pressure build-up static pressure that is too low, but as good or better than results of current methods. Furthermore, field application would be to the field measured data—the real solution—which would tend to correct for this error. We conclude that pressure-build-up analysis based on superposition can be done for real gas flow with acceptable accuracy, but that further study of pressure build-up for real gas flow is desirable.

STEADY-STATE AND PSEUDO-STEADY-STATE FLOW

Radial gas flow at constant production rate will be considered. A horizontal homogeneous porous medium of constant thickness \( h \) with impermeable upper and lower boundary, and a well of radius \( r_w \) located in the center of a radial reservoir, constitutes the flow system. The outer radius \( r_e \) represents either the real boundary or the radius of drainage. Two cases will be considered: (1) constant pressure at \( r_e \), and (2) no flow across \( r_e \).

CONSTANT PRESSURE AT OUTER BOUNDARY

The steady-state equation for a real gas in axisymmetrical coordinates can be written from Eq. 22 as:

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dm(p)}{dr} \right] = 0
\]  

(38)

The boundary conditions for two concentric cylinders of radii \( r_w \) and \( r_e \) are:

\[
r = r_w : m(p) = m(p_w) \quad \quad \quad \quad (39)
\]

\[
r = r_e : m(p) = m(p_e) \quad \quad \quad \quad (40)
\]

Integrating Eq. 38 and using the boundary conditions, the steady-state pressure distribution in the system is:

\[
m(p_w) - m(p_e) = \frac{q_w p_w T}{\pi h T_w} \left( \ln \frac{r_e}{r_w} \right) \quad \quad \quad \quad (41)
\]

Eq. 41 can be evaluated for \( p = p_e \) at \( r = r_e \) and rearranged to provide an equation analogous to the normal radial flow equation:

\[
q_w = \frac{\pi h T_w [m(p_e) - m(p_w)]}{T_p \ln \frac{r_e}{r_w}}
\]  

(42)

Both Eqs. 41 and 42 are in darcy, or cgs units. Thus, the \( m(p) \) have the units of sq atm/cp.

NO FLOW ACROSS OUTER BOUNDARY

As was shown previously by Eq. 32 and Fig. 6 at long times, a flow equation for the closed outer boundary, constant mass rate production, radial flow case can be written:

\[
\ln \frac{r_e}{r_w} = \ln \frac{0.472 \frac{r_w}{r_e}}{r_w} = \frac{\pi h T_w [m(p) - m(p_w)]}{q_w p_w T}
\]  

(43)

Since the \( m(p) \) values were determined from a materials balance, the \( p \) argument represents the average pressure which would yield the proper average density, or the static pressure following a complete pressure build-up. It is not a volumetric average pressure. Eq. 43 coupled with the normal material balance for a bounded drainage volume provides a useful means to couple production rate and gas recovery.

In the case of liquid flow, an equation similar to Eq. 43 can be derived using the concept of pseudo-steady-state flow. That is, a condition is eventually reached for constant liquid production when the rate of pressure decline becomes constant everywhere in the reservoir. This condition is expressed mathematically by setting the Laplacian of the pressure equal to a constant (other than zero). Although it can be shown that the Laplacian of pressure-squared for an ideal gas, or the Laplacian of the real gas pseudo-pressure cannot be equal to a constant rigorously, a flow condition similar to pseudo-steady-state does appear to exist for both ideal and real gas flow, for all practical purposes. The existence of such a condition is suggested by Eq. 43. Fig. 11 presents an interesting inspection of the pressure behavior during the period that Eq. 43 applies for one of the Eilerts et al. cases. Also shown is the \( p_h(t_o) \) for comparison with the liquid case. As was seen previously in Fig. 5, the \( m_0(t_o) \) does not change at a constant rate during this period. Although it matches the liquid case solution at early times, eventually the \( m_0(t_o) \) drops below the liquid case solution. The most interesting feature of Fig. 11, however, is that the \( m_0(r_e, t_o) \) for all radial locations is essentially parallel. Thus, the \( m(p) \) profile is essentially independent of time. This condition can be described approximately by setting the Laplacian of \( m(p) \) equal to a constant. As shown in Refs. 39 and 42, this leads to an equation similar to Eq. 32, but in terms of an average \( m(p) \) rather than \( m(p) \). Although it can be shown that these two averages tend to be equivalent for practical ranges of conditions, it does not appear worthwhile to show the development here. In any event, Eq. 32 describes the long-time flow behavior of closed radial systems with remarkable accuracy.

Another consequence of inspection of Fig. 11 is that the \( m(p) \) distribution can be obtained readily. For ex-

![Fig. 11—m_0(r_e, t_o) vs t_o for constant rate production of a real gas from a closed, radial reservoir.](image-url)
ample, the following equation also describes flow reasonably well:

$$\ln \left( \frac{0.606 \cdot f}{r_{w}} \right) = \frac{z k h T_{L}}{q_{c} r_{w} T} [m(p_{L}) - m(p_{c})] . \quad (44)$$

### DISCUSSION AND CONCLUSIONS

The purpose of the preceding was to describe fundamental considerations which can be used successfully to analyze the flow of real gases. The concept of the real gas pseudo-pressure promises a considerable simplification and improvement in all phases of gas well testing analysis and gas reservoir calculations. Such applications will be described in useful engineering form in a companion paper.

Several remarks concerning the real gas pseudo-pressure are in order. No claim of originality can be made for the substitution we have called the real gas pseudo-pressure. Carslaw and Jaeger reviewed application of a similar transformation which was used in solution of heat conduction problems as early as 1894 and the early 1930’s. Recently, McMorris pointed out the utility of this sort of transformation in heat conduction problems. There have been numerous mentions of the use of a transformation similar to the $m(p)$ function in connection with flow through porous media. In 1949, Muskat used the same transformation in a discussion of the theory of potentiometric models. In 1953, Leibenzon used the transformation, and Russian authors refer to it as the Leibenzon transformation. In 1951, Fay and Prats discussed use of a similar transformation in connection with transient liquid flow. In 1955, Atkinson and Crawford evaluated numerically a similar function but with constant viscosity. In 1962, Carter used a gas mobility term $M(p)$, which was defined as:

$$M(p) = \frac{k h p}{T_{mz}}.$$

Clearly, the $m(p)$ function is proportional to the pressure integral of Carter’s $M(p)$. In 1963, Hurst et al. used a similar integral, but with constant viscosity. To our knowledge, however, this paper represents the first application of the real gas pseudo-pressure to linearization of transient real gas flow. Perhaps the most surprising fact is that the realization of the utility of this concept has been so long in coming.

In the original draft of this paper and the companion paper, we called the $m(p)$ function the real gas potential. It was stated in those papers that the $m(p)$ transformation was not a true potential. Carslaw and Jaeger termed a similar substitution in heat conduction an effective potential, while Muskat termed the transformation a potential as a matter of convenience. We feel that the $m(p)$ transformation will be an important function in gas reservoir engineering, and it is important that the function be given a suitable name. If we were to name the transformation as Russian authors have, we would call it the Muskat transformation. In the belief that the name should be reasonably descriptive and brief, the term real gas pseudo-pressure was finally selected. This name was originally suggested to us by M. Prats, with Shell Development Co.

It appears that the following conclusions are justified. The transformation called the real gas pseudo-pressure in this paper reduces a rigorous partial differential equation for the flow of real gas in an ideal system to a form similar to the diffusivity equation, but with potential-dependent diffusivity. Because the variation of the diffusivity of real gas with pressure was similar to that of an ideal gas, it was possible to correlate finite difference solutions for the ideal radial production of real gas from a bounded system with the liquid flow solutions of van Everdingen and Hurst, and the ideal gas solutions of Aronofsky and Jenkins. This correlation avoids the assumption of small pressure gradients in the reservoir and offers generally useful solutions for the radial flow of real gas.

An investigation of the injection of real gas into a bounded radial system also gave a reasonable correlation—not as good a correlation as production data. The correlation was as good as, or better than, the correlation of ideal gas flow results made by Aronofsky and Jenkins.

An investigation of the possibility of superposition of the linearized results indicated that superposition can be used as an acceptable engineering approximation to generate variable rate flow of real gases in a radial system. Pressure build-up for real gas flow was thus justified for the first time. (No justification for pressure build-up for the non-linear problem of ideal gas flow has yet been presented.)

Accurate and simple equations can be written to describe unsteady flow of real gases which properly consider variation of gas physical properties.

### NOMENCLATURE

- $\nabla$ = grad
- $\nabla \cdot$ = divergence
- $\nabla \times$ = Laplacian operator

- $A$ = area, sq cm
- $b$ = slope of a straight line in a plot of $k(p)$ vs $1/p$
- $c_e(p)$ = real gas compressibility defined by Eq. 10
- $h$ = thickness, cm
- $k(p)$ = effective permeability, darcies
- $M$ = molecular weight
- $m(p)$ = real gas pseudo-pressure defined by Eq. 14
- $p$ = pressure, atm
- $q$ = flow rate, cm$^2$/sec
- $r$ = radius, cm
- $R$ = gas constant
- $t$ = time, sec
- $T$ = temperature, °K
- $v$ = velocity, cm/sec
- $V$ = pore volume, cm$^3$

- $x, y, z$ = direction notation

- $z(p)$ = gas deviation factor, a function of pressure at constant temperature

- $\rho$ = density, gm/cm$^3$

- $\mu(p)$ = real gas viscosity, a function of pressure at constant temperature, cp

- $\mu_r$ = viscosity at atmospheric pressure, cp

- $n$ = normal distance scale

- $\phi$ = hydrocarbon porosity, fraction

### SUBSCRIPTS

- $e$ = external boundary
- $I$ = liquid
- $pc$ = pseudo-critical
- $r$ = radius
- $sc$ = standard conditions
- $w$ = internal boundary, the well
ACKNOWLEDGMENTS

The authors wish to acknowledge financial support of the Texas A&M U., the Texas Engineering Experiment Station of Texas A&M and the Texas Petroleum Research Committee. This paper represents a composite of research effort conducted over a period of time by several agencies. Creation of the $m(p)$ function and a more recent evaluation of the function were described by Al-Hussainy. The authors also wish to acknowledge the encouragement in the course of this study by R. L. Whiting. Portions of this work were done by Al-Hussainy in partial fulfillment of graduate degree requirements in petroleum engineering at Texas A&M U. Finally, and sincerely, the authors wish to acknowledge the numerous helpful suggestions made by the reviewers of the original draft of this paper.

REFERENCES

APPENDIX

CORRELATION OF EILERTS ET AL.* SOLUTIONS

Eilerts et al.* solved the following equation numerically (in their nomenclature):
\[
\frac{\partial}{\partial U} \left[ W(P) \frac{\partial P}{\partial U} \right] = e^\phi \frac{\partial}{\partial P} \left[ \frac{P}{Z(P)} \right] , \ldots \quad (A-1)
\]

where
\[
p = p, P, K(p) = k(p)K(P), z(p) = z(p)Z(P), \mu(p) = \mu(p) \quad \mu(p), w(p) = \frac{k(p)}{\mu(p)z(p)}
\]

and
\[
w(p) = w(p)W(P) , \ldots \quad (A-2)
\]

Dimensionless time is defined as:
\[
H = \frac{p, k(p)}{2 \mu(p)r_0} \quad t \quad , \ldots \quad (A-3)
\]

and the dimensionless radius:
\[
U = 1n\frac{r}{r_o} , \ldots \quad (A-4)
\]

In terms of the \( m(p) \) function, and using the dimensionless variables Eq. A-2, the flow equation takes the form:
\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial m(P)}{\partial r} \right] = \frac{\phi(p, z)p, z(P)C(p)}{\mu(p, z)} \frac{\partial m(P)}{\partial t} , \ldots \quad (A-5)
\]

where
\[
m(P) = 2 \int_0^P PW(P)dP , \ldots \quad (A-6)
\]

Let the coefficient on the right side of Eq. A-5 be evaluated at the initial conditions, and define:
\[
r_o = \frac{r}{r_o} , \ldots \quad (A-7)
\]

\[
t_o = \frac{k(p, z)r_o}{\mu(p, z)} = \frac{2H}{C(z(p))} \left( \frac{r_o}{r_o} \right) , \ldots \quad (A-8)
\]

Notice that \( \mu(p) \) and \( K(p) \) are equal to one at the initial \( P \). Hence, Eq. A-5 takes the form:
\[
1 \frac{1}{r_o} \frac{\partial}{\partial r_o} \left[ r_o \frac{\partial m(P)}{\partial r_o} \right] = \frac{\partial m(P)}{\partial t} , \ldots \quad (A-9)
\]

The flow rate at the producing face as given by Eilerts et al.* is:
\[
Q = 2W(P) \frac{\partial P}{\partial U} = 2 \frac{\partial m(P)}{\partial U} , \ldots \quad (A-10)
\]

and the closed boundary:
\[
\frac{\partial m(P)}{\partial U} = 0 , \ldots \quad (A-11)
\]

Thus, in terms of the dimensionless real gas pseudo-pressure drop:
\[
m_P (r_o, t_o) = \frac{2}{Q} \Delta m(P) , \ldots \quad (A-12)
\]

The \( m(P) \) for the Eilerts et al. natural gas and condensate fluid are shown in Fig. 12. The large difference between physical properties of the two fluids is apparent.

***
"REAL GAS PSEUDO-TIME" – A NEW FUNCTION FOR PRESSURE BUILDUP ANALYSIS OF MF GAS WELLS

by Ram G. Agarwal, Member SPE-AIME,
Amoco Production Co.

ABSTRACT

A new time function has been defined which considers variations of gas viscosity and compressibility as a function of pressure, which in turn is a function of time. This function appears to be similar to the real gas pseudo-pressure, m(p) of Al-Hussainy et al., which takes into account the variations of gas viscosity and z-factor as a function of pressure. However, this is an approximate function as opposed to m(p). This time function will be referred to in this paper as the real gas pseudo-time, t_g(p). This function has aided in post-treatment pressure buildup analysis of fractured (including MF) gas wells by type curve analysis. Results of computer simulated pressure buildup analysis indicate that the use of t_g(p) provides satisfactory values of computed fracture lengths in fractured gas wells.

In this paper the real gas pseudo-time is described and its application is demonstrated by means of example problems. Although the discussion in this paper is limited to pressure buildup analysis of vertically fractured gas wells, the utility of this function is not meant to be restricted to such wells only.

INTRODUCTION

In recent years, type curve analysis methods have become well known in the petroleum industry for analyzing both pressure drawdown and buildup data in oil and gas wells. These methods are meant to be used in conjunction with the conventional methods whenever possible. Exceptions appear to be MF gas wells with finite flow capacity fractures where conventional methods are not readily applicable and, at least to date, only type curve methods appear practical to determine fracture length and fracture flow capacity. Although the majority of published type curves, including those for MF wells, are based on the pressure drawdown solutions for liquid systems, they can be used in an approximate fashion to analyze pressure data from real gas wells. The first requirement is that the dimensionless pressure and time variables are appropriately defined for gas wells. For example, to use the liquid system type curves for an MF gas well, dimensionless variables are defined as follows:

\[ P_{WB} = \frac{k_f h \Delta m(p)}{1424 q_i} \]  

(1)

(In SI units, the numerical constant is 1.28x10\(^{-3}\))

Dimensionless pressure, for a gas well, may also be expressed in terms of \( \Delta(p^g) \) or \( \Delta p \).

Dimensionless time,

\[ t_{DF} = \frac{2.634 \times 10^{-4}}{\phi \mu c_t} \times_f x_f^2 \]  

(2)

(In SI units, the numerical constant is 3.6x10\(^{-9}\))

The definition of dimensionless fracture capacity remains the same.

\[ F_{CD} = \frac{k_f^{1/2}}{k_x} \]  

(3)

Note that in Eq. (1), the real gas pseudo-pressure, m(p) of Al-Hussainy et al. has been used to take into account the variations of gas viscosity and z-factor as a function of pressure. In Eq. (2), viscosity-compressibility (\( \mu c_t \)) is shown to be evaluated at the initial reservoir pressure.

In analyzing pressure drawdown data from real gas wells using a liquid system type curve, it is recommended that the real gas pseudo-pressure is
used and $\Delta m(p)$ to be used in Eq. (1) is defined as follows:

$$\Delta m(p) = m(p_f) - m(p_{f,p})$$  \hspace{1cm} (4)

It has been also established that in analyzing pressure drawdown data from gas wells by type curve matching, reasonable answers are obtained if the $(\mu_p)$ product in Eq. 2 is evaluated at the initial reservoir pressure.1

In analyzing pressure buildup data using drawdown type curves, the additional restriction which should be imposed is that the producing time, $t_p$ prior to shut-in is significantly greater than the shut-in time, $\Delta t$ that is $(t_p + \Delta t)/\Delta t = 1).$ This should apply to both oil and gas wells. In the case of gas well buildup analysis, $\Delta m(p)$ is defined as follows:

$$\Delta m(p) = m(p_{\Delta t}) - m(p_{\Delta t=0})$$  \hspace{1cm} (5)

However, in using type curves for analyzing pressure buildup data especially from fractured gas wells (even if the effect of producing time is insignificant), it is not clear as to which pressure level the $(\mu_p)$ product in the dimensionless term should be evaluated. The question arises whether the $(\mu_p)$ product should be evaluated at (1) the initial reservoir pressure, (2) the final flowing pressure prior to shut-in, or (3) some average pressure. Results of this study indicate that the use of any one of the above three pressure levels is less than satisfactory. In general, the use of the initial reservoir pressure resulted in a fracture length value greater than the actual; the use of the final flowing pressure provided a computed fracture length value which was smaller than the actual; and the use of an average pressure provided a fracture length value which was different than that used in the computer model.

To overcome the above difficulty, a new time function has been studied. This has aided us in post-treatment pressure buildup analysis of fractured (including MF) gas wells. This function considers variations of gas viscosity and compressibility as a function of pressure, which in turn is a function of time. This function is named real gas pseudo-time, $t(p)$, in this paper. This function is analogous to the real gas pseudo-pressure, $m(p)$ of Al-Hussainy et al.4,5 which includes the effects of pressure dependent gas viscosity and $z$-factor. It should be emphasized that $t(p)$ is an approximate function as opposed to $m(p).$ However, it provides reasonable values of fracture lengths in pressure buildup analysis of vertically fractured gas wells and should be most useful for MF gas wells.

The discussion in this paper will deal with the applicability and limitations of liquid system drawdown type curves in analyzing pressure buildup data from gas wells. However, type curves for only vertically fractured wells will be considered.

The discussion will also include a description of the new time function, its computational procedure and application by means of example problems.

**TYPE CURVES FOR VERTICALLY FRACTURED WELLS**

During the past few years, type curves have appeared in the petroleum literature which can be used to analyze pressure data from vertically fractured wells. Gringarten et al.6 presented type curves for infinite flow capacity fractures. Since their type curves could not be used for MF wells with finite capacity fractures, Cinco et al.7 and Agarwal et al.2 published new sets of type curves (finite fracture flow capacity) for MF wells. Common wellbore rate type curves of Gringarten et al.6 and Agarwal et al.2 are shown in Figs. 1 and 2, respectively. Since all these type curves are based on the pressure drawdown data in liquid systems, it may appear that the same are not used to analyze pressure buildup data in oil and gas wells. Results of this study indicate that the above type curves are not used to analyze type curves for vertically fractured wells with infinite flow capacity. For $x_k/\rho_p = \infty$. Dimensionless pressure rise $P_0^*$ has been plotted as a function of dimensionless shut-in time, $D_0$. A family of type curves is shown with dimensionless producing time, $D_x$, as a parameter. This clearly shows the limitations of drawdown type curves for analyzing pressure buildup data collected after small producing times. The effect of small producing time will not be considered in the subject study. However, a future paper is planned to cover this aspect.

**a) Small producing time**

If the producing time, $t_p$, prior to shut-in is relatively short such that it does affect the pressure transients due to the subsequent buildup, drawdown type curves should not be used. In this case, pressure buildup type curves need to be generated to include the effect of producing time. This aspect of producing time and its effect on type curve analysis has been discussed recently by Raghavan.7 Fig. 3, taken from his paper, presents build-up type curves for a vertically fractured well with infinite flow capacity fracture (for $x_k/\rho_p = \infty$). Dimensionless pressure rise $P_0^*$ has been plotted as a function of dimensionless shut-in time, $D_0$. A family of type curves is shown with dimensionless shut-in time, $D_x$, as a parameter. This clearly shows the limitations of drawdown type curves for analyzing pressure buildup data collected after small producing times. The effect of small producing time will not be considered in the subject study. However, a future paper is planned to cover this aspect.

**b) Long producing time**

If the producing time prior to shut-in is significantly long (that is $(t_p + \Delta t)/\Delta t = 1$) so that it does not affect the transients due to pressure buildup, drawdown type curves may be used to analyze pressure buildup data. This should be obvious from Fig. 3. The basis for this is depicted schematically in Fig. 4 in which the pressure behavior is shown during both constant rate drawdown and pressure buildup periods. According to the pressure transient theory $p_0$ during the shut-in drawdown should be equal to that during pressure buildup (for $t=\Delta t$) provided they are defined as follows:
(6a) \[(\Delta p)_{\text{drawdown}} = p_i - p_w \]
= (initial reservoir pressure)
- (wellbore flowing pressure)

(6b) \[(\Delta p)_{\text{buildup}} = \frac{p_{\Delta t} + p_t}{p} \]
= (shut-in pressure)
- (wellbore flowing pressure in absence of shut-in)

However, since \(p_t + \Delta t\) is not readily available, \((\Delta p)_{\text{buildup}}\) is normally defined as equal to \((p_{\Delta t} - p_{\Delta t = 0})\).

Thus there is a difference between the \((\Delta p)_{\text{drawdown}}\) and the way \((\Delta p)_{\text{buildup}}\) are calculated. This difference is equal to \((p_{\Delta t = 0} - p_t + \Delta t)\) and is shown as the cross hatched area in Fig. 4.

To further investigate this difference, an NRF simulator was used to simulate pressure buildup in an NRF gas well using the liquid system analogy (\(m_c = \text{constant}\)). Reservoir and fracture data are shown in Table 1. The well was allowed to produce at a constant rate for 180 days followed by a pressure buildup test for 14 days. Pressure drawdown and buildup data expressed as \(\Delta m(p)/q\) as function of time (\(t\) or \(\Delta t\)) in days are plotted on coordinate graph paper and are shown in Fig. 5. \(\Delta m(p)\) is defined as follows:

\[(\Delta m(p))_{\text{drawdown}} = m(p_i) - m(p_w) \]
(7a)

and

\[(\Delta m(p))_{\text{buildup}} = m(p_{\Delta t}) - m(p_{\Delta t = 0}) \]
(7b)

Note that there is virtually no difference between the drawdown and buildup data at early times. However, the difference gets bigger as time increases. This difference is due to the way \(\Delta m(p)_{\text{buildup}}\) is calculated. Also shown on Fig. 5 as the cross hatched area is \([m(p_{\Delta t = 0}) - m(p_t + \Delta t)]\), which is equal to the above difference. Thus there is a basic difference between the drawdown and buildup type curves. However, in many cases this difference is not significant. This fact should also be apparent from Fig. 3 where the difference between the drawdown and buildup curve for long producing times is shown to be small. This indicates that in liquid systems, for large producing times, pressure buildup data can be analyzed using drawdown type curves to obtain reasonable answers.

2. Use of Liquid System Type Curves for Gas Wells

Next, we will examine the applicability of liquid system type curves for analyzing pressure drawdown and pressure buildup data obtained from real gas wells.

a) Gas well drawdown data

To use the liquid system type curves for gas well drawdown data, it is recommended that real gas pseudo-pressure, \(m(p)\), is used for the dimensionless pressure, \(p_{\Delta t}\), term and \(m_c\), evaluated at the initial reservoir pressure, is utilized in the dimensionless time, \(\Delta t\), term. Fig. 6 shows the comparison between the drawdown type curve for liquid and real gas systems. Drawdown data were generated using the NRF simulator and reservoir data shown in Table 1 and gas properties data shown in Table 2. Results indicate that the use of a liquid system type curve for gas well drawdown data is reasonable provided that the above mentioned conditions are met.

If one questions the applicability of these type curves for a particular gas reservoir because of unusual gas properties and/or reservoir pressure, as suggested in a recent paper, the curves could be generated using an appropriate \(m_c\) function, temperature, and pressure ranges specific to the reservoir under study. These type curves should then be used for pressure transient analysis of data from that reservoir.

b) Gas well buildup data

To use the drawdown type curves for gas well buildup data, considerations regarding the duration of producing time, as discussed earlier, should also apply. Consequently, in this study it will be assumed that the effect of producing time on pressure buildup data is insignificant. The effect of the \(m_c\) product on pressure buildup data will be mainly considered. As mentioned earlier, for pressure buildup analysis it is not clear as to the pressure level at which the \(m_c\) product in the dimensionless time term should be evaluated.

To study this problem, the NRF simulator was utilized to generate pressure buildup data on an NRF gas well using real gas properties. Reservoir data and gas properties used are shown in Tables 1 and 2. As was done for the liquid case, the well was produced at a constant rate for 180 days followed by a pressure buildup for about 14 days. Pressure drawdown and buildup data expressed as \(\Delta m(p)/q\) as function of time in days are plotted and shown on Fig. 7. Note that this figure is similar to Fig. 5 presented earlier for the liquid case. \(\Delta m(p)\) has been appropriately defined. Also shown on Fig. 7, as the cross hatched area, is \([m(p_{\Delta t = 0}) - m(p_t + \Delta t)]\). Notice that there is a marked difference between the drawdown and buildup type curve. This difference is much greater than that shown by the cross hatched area, mentioned above and discussed earlier.

The big difference in the drawdown and buildup curves for gas wells is due to large variations of \(m_c\) or \(m_c\) product as a function of pressure. For gas wells, \(m_c\) is approximately equal to \(s_{n - 1}/m_c\). Fig. 8 shows the graph of \(m_c\) vs pressure for the simulated gas well buildup case. Note that the variations in \(m_c\) are much larger in the low pressure range (say below 2000 psi) than those in the high pressure range (say above 3000 psi). This indicates that during the pressure buildup, changes in the value of the \(m_c\) product
Real Gas Pseudo-Time, \( t_a(p) \)

Al-Hussainy et al.\(^4\) defined real gas pseudo-pressure,

\[
m(p) = 2 \int_0^p \frac{\mu(p) \zeta(p)}{p_0} \, dp \tag{8}\]

which takes into account the variations of gas viscosity and \( z \)-factor as a function of pressure with \( p_0 \) as a low base pressure.

In this paper an analogous function is defined as follows:

\[
E_a(t) = \int_{t_0}^t \frac{dt}{\mu(t) \zeta(t)} \tag{9}\]

where \( \mu \) and \( \zeta \) are used to denote viscosity and system compressibility as a function of time rather than \( \mu \) and \( c \), which are usually expressed as functions of pressure. If \( E_a(t) \) is redefined as a function of pressure, a new function is obtained as

\[
t_a(p) = \int_p^{p_0} \frac{dt}{\mu(p) \zeta(p)} \, dp \tag{10}\]

where \( \mu \) and \( c \) are functions of pressure. This function is referred to as real gas pseudo-time, \( t_a(p) \), in this paper.

If time and pressure are assumed to vary linearly with each other, over small time increments, Eq. 10 can be approximated as

\[
t_a(p) \approx \sum_{j=1}^n \frac{t_j - t_{j-1}}{(p_j - p_{j-1})} \frac{P_j}{p_j c \zeta(p)} \tag{11}\]

Note that in Eq. 11, \( t \) represents flowing time for a drawdown test and shut-in time for a buildup test. Eq. 11 may be rewritten as

\[
t_a(p) = \sum_{j=1}^n \frac{(t_j - t_{j-1})}{(p_j - p_{j-1})} [(I(p_j) - I(p_{j-1})] \tag{12}\]

where an integral

\[
I(p) = \int_{p_0}^p \frac{dp}{\mu(p) \zeta(p)} \tag{13}\]

can be evaluated beforehand using \( \mu \) and \( c \) as functions of pressure. In Eq. (13), \( p_0 \) is a low base pressure and \( p \) is the maximum pressure of interest. The above integral, expressed in graphical or tabular form, can be used in conjunction with Eq. 12 to compute real gas pseudo time, \( t_a(p) \). Since \( c \) rather than \( c \) is normally available as a function of pressure, the following relationships may be utilized.

\[
c \approx S_g c_g + S_o c_o + S_w c_w + c_f \tag{14}\]

For a gas well, Eq. 14 is usually approximated as

\[
c \approx S_g c_g \tag{15}\]

Going back to Eq. 11, it should be noted that during the time interval, \( \Delta t = t_j - t_{j-1} \), and the pressure change, \( \Delta p = p_j - p_{j-1} \), the viscosity-compressibility product \((pc)_{t_j} \) is defined by

\[
\frac{1}{(pc)_{t_j}} = \frac{P_j}{\Delta p_j} \frac{dp}{\mu(p) c \zeta(p)} \tag{16}\]

where, \( j = 1, 2, \ldots, n \).
This definition in Eq. 11 gives

\[ t_a(p) \approx \sum_{j=1}^{n} \frac{\Delta t_j}{(\mu c_p)_{j-1}} \]  

(17)

Eq. 17 clearly indicates that the units in \( t_a(p) \) consist not only of time but a combination of time, viscosity, and compressibility.

**COMPUTATION OF REAL GAS PSEUDO-TIME, \( t_a(p) \)**

Since Eq. 12 for \( t_a(p) \) contains an integral, \( I(p) \), given by Eq. 13, computation can be performed using either graphical or tabular data. Simply, it can be accomplished by means of a computer or a desk calculator. Trapezoidal or Simpson's rule can be used. Integration can be performed by reading mid-point values of \( (1/\mu c_p) \) from the table or graph and multiplying by \( \Delta p \). Computations of \( I(p) \) utilizing gas properties in Table 2 are outlined in Table 3. Fig. 9 shows a graph of \( (1/\mu c_p) \) and \( I(p) \) as a function of pressure. Although not shown, it is also useful to prepare a similar graph for real gas pseudo-pressure, \( m(p) \). Thus \( I(p) \) and \( m(p) \) curves prepared for the gas in a specific reservoir can be used as master graphs for future wells in that reservoir. The \( I(p) \) curve is used in conjunction with Eq. 12 to convert real times to corresponding pseudo-times for the specific application.

**CERTAIN USEFUL ASPECTS OF \( t_a(p) \)**

Before discussing the application, let us consider certain aspects of real gas pseudo-time, \( t_a(p) \):

**Definition of dimensionless time**

If \( t_a(p) \) is used to express the dimensionless time term, \( t_{Dax} \), then

\[ t_{Dax} = \frac{2.634 \times 10^{-k}}{\phi x} t_a(p) \]  

(18)

Since the viscosity-compressibility product is already included in \( t_a(p) \), it does not appear in Eq. 18. To express an analogy between Eqs. 18 and 2, the above equation can be multiplied and divided by \( (\mu c_p) \) evaluated at the initial reservoir pressure. This provides

\[ t_{Dax} = \frac{2.634 \times 10^{-6}}{\phi x} [(\mu c_p)_{i-1} \cdot t_a(p)] \]  

(19)

It should be noted that in Eqs. 18 and 19, a general definition of dimensionless time, \( t_{Dax} \), has been used. Accordingly, it may represent dimensionless drawdown time or dimensionless buildup time depending on whether \( t_a(p) \) in Eq. 12 has been calculated using the flowing time, \( t_f \), or shut-in time, \( t_i \).

**Correspondence between flowing and shut-in times**

Eq. 19 appears very similar to Eq. 2 where time \( t \) has been replaced by \( [(\mu c_p)_{i-1} \cdot t_a(p)] \). This suggests a correspondence between the real time and the pseudo-time. This also implies some correspondence between the flowing time for a drawdown test and the shut-in time for a buildup test as well be shown next.

**APPLICATION OF REAL GAS PSEUDO-TIME, \( t_a(p) \)**

To illustrate the application of real gas pseudo-time, let us consider the pressure drawdown and buildup data, shown in Fig. 7, for the gas well...
case. These data have been replotted in Fig. 10 and are shown as $\Delta \ln(p)/q$ as a function of time ($t$ or $\Delta t$) in days. In this figure, solid line with circles represent the drawdown data whereas the solid line with triangles is for the buildup data. As mentioned earlier, there is a considerable difference between the two curves. These buildup data, being so much on the right side of the drawdown curve, imply that the use of $(\mu_c)_{t}$ at the initial reservoir pressure will result in computed fracture length which is much greater than the actual. This aspect will be investigated later by means of type curve analysis.

To use the concept of real gas pseudo-time, Eq. 12 was used in conjunction with Fig. 9 to convert shut-in time, $\Delta t$, to $t_1(p)$. Eq. 21 was utilized to express these data in terms of equivalent flow time, $t_0$ or $(\mu_c)_{t}$. It enabled us to compare drawdown and buildup data on an equivalent basis. Pressure buildup data plotted as a function of $(\mu_c)_{t}$, $t(p)$ are shown as the dotted line with triangles. Note that the result was to move the buildup data (shown as triangles) horizontally from the solid line on the right to the dotted line on the left. Also, the modified buildup curve came very close to the drawdown curve. This figure also indicates that real shut-in times, $\Delta t$, are equivalent to only about 60% of the equivalent flow times, $t$. For example, the real shut-in time of 6 days is equal to only about 3.75 days of the equivalent flow time. It is possible to plot shut-in time, $\Delta t$ as a function of equivalent flow times, $(\mu_c)_{t}$. This is shown in Fig. 11. The solid line represents the gas case and the dotted line is for liquid case. This figure clearly shows that shut-in times for liquid case are equal to the equivalent flow times, whereas they are much less for the gas case. This indicates that a graph similar to Fig. 11 should also prove useful in the design of a pressure buildup test on an HGF gas well. For example, if a pressure buildup test is required to be run for an equivalent flow time of 6 days to obtain the desired information, it may be necessary to run the test for about 10 days, which is almost twice as long.

Let us next consider the effect of $(\mu_c)_{t}$ product evaluated at different pressure levels on the type curve analysis of gas well buildup data. The values of computed fracture lengths will be compared against the actual total fracture length of 1000 feet used in the simulator. Pressure drawdown and buildup data presented in the preceding example will be utilized for this purpose. Fig. 12 shows a semilog graph of pressure buildup and drawdown data expressed in dimensionless quantities. Dimensionless pressure, $p_{D}$, has been plotted as a function of dimensionless time, $D_{t}$. For drawdown data, $(\mu_c)_{t}$ product in the dimensionless time was evaluated at the initial reservoir pressure. This curve will be considered as the reference type curve.

Data curves for buildup data have been plotted using the $(\mu_c)_{t}$ product at the initial reservoir pressure, $p_{D}$ and the final value of flowing pressure prior to shut-in, $p_{D}$. Since the data curve using $(\mu_c)_{t}$ is on the right-hand side of the drawdown type curve, matching will provide computed fracture length which is greater than the actual. On the other hand, the data curve using $(\mu_c)_{t}$ at $p_{D}=0$, being on the left-hand side of the reference type curve, provided a computed fracture length which is smaller than the actual. A third data curve is also shown which utilized Eq. 18 to incorporate the concept of real gas pseudo-time function. Results of curve matching, using the new time function, gave results which are close to the actual fracture length.

Table 4 provides a comparison of fracture lengths, computed by type curve analysis, using the $(\mu_c)_{t}$ product in the dimensionless time term at various pressure levels. Results of four sets of simulated gas well buildup data are shown, where both the value of fracture length and the level of final flowing pressure $p_{D} = 0$ were varied. Inspection of Table 4 reveals that results are affected by the $(\mu_c)_{t}$ product used and the level of the final flowing pressure at the instant of shut-in. The use of $(\mu_c)_{t}$ provides values of computed fracture length which are too optimistic. The effect is further exaggerated at a lower value of the flowing pressure. The use of the $(\mu_c)_{t}$ product at $p_{D} = 0$ provides a low but reasonable value if $p_{D}$ is relatively high, otherwise it provides pessimistic values of fracture length. The use of real gas pseudo-time provided computed fracture length values similar to those entered in the simulator. Based on a number of computer runs, it appears that the concept of real gas pseudo-time function is useful in analyzing post-treatment buildup data from fractured (including HGF) gas wells.

Steps Used in Applying Real Gas Pseudo-Time Function for Type Curve Matching

The following step-by-step procedure should be useful in applying the concept of real gas pseudo-time to gas well buildup data for type curve matching purposes.

Step 1
Prepare a table of gas properties as shown in Table 2. Compute real gas pseudo-pressure, $m(p)$ and integral, $I(p)$ as a function of pressure and plot them on coordinate graph paper.

Step 2
Tabulate pressure buildup data, $p_{D}$ vs $\Delta t$.

Using the above figure, convert $p_{D}$ to $m(p_{D})$ and compute $\Delta (m(p)) = m(p_{D}) - m(p_{D} = 0)$.

Step 3
Using the figure for $I(p)$ and Eq. 12 convert $\Delta t$ to $(\mu_c)_{t}$. It should be noted that $(\mu_c)_{t}$ already contains the $(\mu_c)_{t}$ product.

Step 4
Plot $\Delta (m(p))$ vs $(\mu_c)_{t}$ on a tracing paper utilizing the appropriate type curve. Type curve matching should be done in the usual manner. For an HGF well, if formation flow capacity is known a priori, the vertical position of the data plot may be fixed on the y-axis of the type curve. Otherwise the matching should be done by sliding the tracing paper parallel to both x and y axes.
Step 5

Once a match is obtained and a match point is selected, the fracture length is calculated as

\[ x_f = \left( \frac{2.63 \times 10^{-4} \Delta t_{\text{r}}}{\phi} \right) \left( \frac{\tau_s(p)}{[\Delta x_f]_H} \right) \]  

(22)

In regard to the \( \phi \) term in the above equation, the following should be pointed out:

If the \( (\mu_c) \) product in the real gas pseudo-time, \( \tau_s(p) \), is based on the system compressibility, \( c_s = S_s c_S + s c_S + s c_S + c_e \), then \( \phi \) should be the total porosity in the system.

If gas compressibility, \( c_g \), has been used instead of \( c_e \), then \( \phi \) should be replaced by hydrocarbon porosity, \( \phi^g \).

Once the value of \( x_f \) is determined, the fracture flow capacity can be determined by Eq. 3 as

\[ (k_s)_{ol} = (F_{CD})(k_w) \text{ md-ft} \]  

(23)

CONCLUDING REMARKS

As a result of this study, the following remarks appear warranted:

1. A new time function [real gas pseudo-time, \( \tau_s(p) \)] has been developed which has aided in post-treatment pressure buildup analysis of fractured (including NWF) gas wells.

2. This function is analogous to the real gas pseudo-pressure, \( m(p) \), of Al-Hussainy et al. Although it is not a rigorous function, it provides excellent engineering answers for vertically fractured gas well buildup analysis.

3. There is a basic difference between the drawdown type curve and the buildup type curve because of the different \( \Delta(m(p)) \) or \( \Delta p \) are calculated for pressure drawdown and buildup. However, drawdown type curves may be used for buildup data provided the producing time prior to shut-in is long (that is \( t_p + \Delta t)/\Delta t = 1 \)).

4. For small producing time prior to shut-in, buildup data should not be analyzed by drawdown type curves.

5. To use liquid system type curves for gas wells, the following points should be noted:

a) For analyzing gas well drawdown data, the use of liquid system type curves appears reasonable provided that the real gas pseudo-pressure is used in the dimensionless pressure term and the \( (\mu_c) \) product in the dimensionless time is evaluated at the initial reservoir pressure. However in certain cases, because of unusual gas properties or reservoir pressure, type curves should be generated using the appropriate \( (\mu_c) \) function, temperature, and pressure ranges and then used for the specific application.

b) For gas well buildup data, the use of \( (\mu_c) \) product at the initial reservoir pressure provides optimistic values for fracture lengths whereas the \( (\mu_c) \) at the final flowing pressure provides pessimistic values for fracture length. The use of the real gas pseudo-time provides satisfactory values of fracture length.

6. Due to the effect of variations of \( (\mu_c) \) on gas well buildup data, it may be necessary to run a buildup test twice as long as it is normally run. This aspect should be considered in the design of pressure buildup tests on NWF gas wells.

7. Although the discussion in this paper is limited to pressure buildup analysis of vertically fractured gas wells, the utility of the real gas pseudo-time is not meant to be restricted to such wells only. For example, this function was also found very useful for gas wells in analyzing wellbore storage data, linear flow data, etc., to name a few.

NOMENCLATURE

\[ c_f = \text{formation compressibility, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ c_g = \text{gas compressibility, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ c_o = \text{oil compressibility, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ c_s = \text{system compressibility, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ c_t = \text{system compressibility as a function of time, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ c_w = \text{water compressibility, psi}^{-1} \text{ (Pa}^{-1}) \]

\[ F_{CD} = \text{dimensionless fracture flow capacity (see Eq. 3)} \]

\[ h = \text{formation thickness, ft (m)} \]

\[ I(p) = \text{integral in Eq. 13, psi}^2 \text{/cp (Pa}^2 \text{/s)} \]

\[ k = \text{formation permeability, md (10}^{-3} \text{um}^2 \text{)} \]

\[ k_f = \text{fracture permeability, md (10}^{-3} \text{um}^2 \text{)} \]

\[ m(p) = \text{real gas pseudo-pressure, psi}^2 \text{/cp (MPa}^2 \text{/pa}\cdot\text{s)} \]

\[ \Delta(m(p)) = \text{difference in real gas pseudo-pressures psi}^2 \text{/cp (MPa}^2 \text{/pa}\cdot\text{s)} \]

\[ p = \text{pressure, psi (MPa)} \]

\[ P_i = \text{initial pressure, psi (MPa)} \]

\[ P_{\Delta t} = \text{shut-in pressure, psi (MPa)} \]
\( P_{\Delta t=0} \) = shut-in pressure at the instant of shut-in, psi (MPa)
\( \tilde{P}_{D_s} \) = dimensionless shut-in pressure rise (see Ref. 7)
\( P_wD \) = dimensionless pressure or pressure drop (see Eq. 1)
\( P_{wf} \) = wellbore flowing pressure, psi (MPa)
\( \Delta p \) = pressure change, psi (MPa)
\( \Delta p_j \) = pressure change, \( p_j - p_{j-1} \), psi (MPa) (see Eq. 11)
\( \Delta(p^2) \) = difference in squares of pressures, psi² (MPa²)
\( q \) = flow rate, MCF/D ("standard" m³/d)
\( S \) = saturation, fraction
\( S_g \) = gas saturation, fraction
\( S_o \) = oil saturation, fraction
\( S_w \) = water saturation, fraction
\( t \) = flowing time, hours
\( t_a(p) \) = real gas pseudo-time, hours-psi/cp (hours-Pa/Pa·s) (see Eq. 11)
\( [t_a(p)]_M \) = real gas pseudo-time for match point, hours-psi/cp (hours-Pa/Pa·s)
\( F_a(t) \) = a function defined by Eq. 9
\( t^{Dx_f} \) = dimensionless time based on \( x_f \) (see Eq. 2)
\( \Delta t^{Dx_f} \) = dimensionless shut-in time (see Ref. 7)
\( t^{Dax_f} \) = dimensionless time based on \( t_a(p) \) (see Eqs. 18 and 19)
\( [t^{Dax_f}]_M \) = dimensionless time based on \( t_a(p) \) for match point (see Eq. 22)
\( t_p \) = producing time prior to buildup, hours
\( \Delta t \) = shut-in time, hours
\( \Delta t_j \) = time interval, \( t_j - t_{j-1} \), hours
\( T \) = reservoir temperature, °R (K)
\( x_e \) = distance from well to the reservoir boundary, ft (m)
\( x_f \) = fracture half length, ft (m)
\( z \) = real gas deviation factor
\( \mu \) = viscosity, cp (Pa·s)
\( \tilde{\mu} \) = viscosity as a function of time, cp (Pa·s)
\( \mu c_L \) = viscosity-compressibility product, cp/psi(Pa·s/Pa)
\( \mu c_L_i \) = viscosity-compressibility product at initial reservoir pressure, cp/psi(Pa·s/Pa)
\( \mu c_L_j \) = viscosity-compressibility product in a given interval, cp/psi(Pa·s/Pa) (see Eq. 16)
\( \Delta \) = delta or difference
\( \phi \) = formation porosity, fraction
\( \Sigma \) = sigma or summation

Subscripts
\( cD \) = dimensionless flow capacity
\( D \) = dimensionless
\( D_{x_f} \) = dimensionless based on \( x_f \) and time
\( D_{ax_f} \) = dimensionless based on \( x_f \) and pseudo-time
\( a \) = apparent or pseudo
\( e \) = external boundary
\( f \) = fracture or formation
\( g \) = gas
\( i \) = initial
\( j \) = index for summation
\( M \) = match point values
\( n \) = index for summation
\( o \) = oil
\( p \) = producing
\( t \) = total
\( w \) = water
\( wf \) = wellbore flowing

REFERENCES


---

**TABLE 1**

Reservoir and Fracture Data for Simulated MHF Well

**Reservoir Data**

- Reservoir pressure, $p_i$ 5000 psi (34.5 MPa)
- Reservoir temperature, $T$ 720°F (400°C)
- Formation thickness, $h$ 50 ft (15.2 m)
- Formation permeability, $k$ $0.1$ md ($9.9 	imes 10^{-6}$ μm$^2$)
- Formation porosity, $\phi$ 0.07 fraction
- Initial gas saturation, $S_g$ 0.50 fraction
- Production Rate, $q$ 500 Mcf/D (14,158 m$^3$/D)

**Fracture Data**

- Total fracture length, $2x_f$ 1000 ft (305 m)
- Fracture flow capacity, $k_{fw}$ 50 md-ft ($15 \times 10^3$ μm$^2$m)
- Dimensionless fracture capacity, $F_{CD}$ 10
### TABLE 2

Gas Properties for Simulated MF Gas Well

<table>
<thead>
<tr>
<th>Pressure (psl)</th>
<th>Viscosity (cp)</th>
<th>z-factor (fraction)</th>
<th>( c_g ) (psi(^{-1}))</th>
<th>( m(p) ) (psi(^2/)cp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>--</td>
<td>1.000</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>600</td>
<td>0.0147</td>
<td>0.971</td>
<td>( 170.6 \times 10^{-5} )</td>
<td>( 2.5 \times 10^{7} )</td>
</tr>
<tr>
<td>1200</td>
<td>0.0155</td>
<td>0.951</td>
<td>( 86.0 \times 10^{-5} )</td>
<td>( 10.0 \times 10^{7} )</td>
</tr>
<tr>
<td>1800</td>
<td>0.0166</td>
<td>0.940</td>
<td>( 56.0 \times 10^{-5} )</td>
<td>( 21.8 \times 10^{7} )</td>
</tr>
<tr>
<td>2400</td>
<td>0.0180</td>
<td>0.939</td>
<td>( 40.9 \times 10^{-5} )</td>
<td>( 37.3 \times 10^{7} )</td>
</tr>
<tr>
<td>3000</td>
<td>0.0197</td>
<td>0.947</td>
<td>( 31.1 \times 10^{-5} )</td>
<td>( 55.6 \times 10^{7} )</td>
</tr>
<tr>
<td>3600</td>
<td>0.0216</td>
<td>0.964</td>
<td>( 24.3 \times 10^{-5} )</td>
<td>( 75.6 \times 10^{7} )</td>
</tr>
<tr>
<td>4200</td>
<td>0.0236</td>
<td>0.986</td>
<td>( 19.5 \times 10^{-5} )</td>
<td>( 96.9 \times 10^{7} )</td>
</tr>
<tr>
<td>4800</td>
<td>0.0255</td>
<td>1.014</td>
<td>( 16.0 \times 10^{-5} )</td>
<td>( 118.9 \times 10^{7} )</td>
</tr>
<tr>
<td>5400</td>
<td>0.0275</td>
<td>1.045</td>
<td>( 13.3 \times 10^{-5} )</td>
<td>( 141.3 \times 10^{7} )</td>
</tr>
</tbody>
</table>

### TABLE 3

Computation of \( \frac{1}{\mu c_t} \) and \( \frac{1}{\mu p} = \int_0^{\frac{g_0}{\mu p c_t}} \frac{da}{\mu p c_t} \) Using Gas Properties

<table>
<thead>
<tr>
<th>Pressure (psl)</th>
<th>( \frac{1}{\mu c_t} ) (cpsi)</th>
<th>( \frac{1}{\mu p c_t} ) (cpsi)</th>
<th>( \Delta p ) (psi)</th>
<th>( \frac{1}{\Delta p} ) (cpsi)</th>
<th>( \frac{1}{\mu p c_t} ) (cpsi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>600</td>
<td>( 12.54 \times 10^{-6} )</td>
<td>( 7.98 \times 10^{-6} )</td>
<td>( 3.99 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 2.39 \times 10^{7} )</td>
</tr>
<tr>
<td>1200</td>
<td>( 6.67 \times 10^{-6} )</td>
<td>( 15.00 \times 10^{-6} )</td>
<td>( 11.49 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 9.28 \times 10^{7} )</td>
</tr>
<tr>
<td>1800</td>
<td>( 4.70 \times 10^{-6} )</td>
<td>( 21.29 \times 10^{-6} )</td>
<td>( 18.13 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 20.17 \times 10^{7} )</td>
</tr>
<tr>
<td>2400</td>
<td>( 3.68 \times 10^{-6} )</td>
<td>( 27.17 \times 10^{-6} )</td>
<td>( 24.23 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 34.71 \times 10^{7} )</td>
</tr>
<tr>
<td>3000</td>
<td>( 3.06 \times 10^{-6} )</td>
<td>( 32.64 \times 10^{-6} )</td>
<td>( 29.91 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 52.66 \times 10^{7} )</td>
</tr>
<tr>
<td>3600</td>
<td>( 2.62 \times 10^{-6} )</td>
<td>( 38.14 \times 10^{-6} )</td>
<td>( 35.37 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 73.82 \times 10^{7} )</td>
</tr>
<tr>
<td>4200</td>
<td>( 2.30 \times 10^{-6} )</td>
<td>( 43.46 \times 10^{-6} )</td>
<td>( 40.78 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 98.35 \times 10^{7} )</td>
</tr>
<tr>
<td>4800</td>
<td>( 2.04 \times 10^{-6} )</td>
<td>( 49.02 \times 10^{-6} )</td>
<td>( 46.24 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 126.09 \times 10^{7} )</td>
</tr>
<tr>
<td>5400</td>
<td>( 1.83 \times 10^{-6} )</td>
<td>( 54.68 \times 10^{-6} )</td>
<td>( 51.85 \times 10^{-6} )</td>
<td>( 600 \times 10^{7} )</td>
<td>( 157.20 \times 10^{7} )</td>
</tr>
</tbody>
</table>

* \( \frac{1}{\mu c_t} = S_{\mu c_t} \)
** \( \frac{1}{\mu p} \) vs. pressure is shown in Fig. 9

Remarks: Real gas pseudo pressure, \( \frac{1}{\mu p} \) can be computed using Eq. 12 and \( \frac{1}{\mu p} \) in conjunction with the desired pressure vs. time data.

### TABLE 4

Comparison of Computed Fracture Lengths from Type Curve Analysis of Gas Well Buildup Data Using P\( \frac{1}{\mu p} \) Product at Various Pressure Levels

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Flow Rate (DICPD)</th>
<th>Initial Pressure (psl)</th>
<th>Final Flowing Pressure (psl)</th>
<th>Simulator</th>
<th>( \frac{1}{\mu c_t} )</th>
<th>( \frac{1}{\mu p} ) at ( \Delta t = 0 )</th>
<th>Total Fracture Length (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>500</td>
<td>5000</td>
<td>2374</td>
<td>1000</td>
<td>1370</td>
<td>955</td>
<td>1025</td>
</tr>
<tr>
<td>1B</td>
<td>650</td>
<td>5000</td>
<td>1332</td>
<td>1000</td>
<td>1500</td>
<td>965</td>
<td>995</td>
</tr>
<tr>
<td>2A</td>
<td>500</td>
<td>5000</td>
<td>2248</td>
<td>1500</td>
<td>2020</td>
<td>1125</td>
<td>1515</td>
</tr>
<tr>
<td>2B</td>
<td>650</td>
<td>5000</td>
<td>1163</td>
<td>1500</td>
<td>2656</td>
<td>1292</td>
<td>1515</td>
</tr>
</tbody>
</table>
Effects of the Quadratic Gradient Term in Steady-State and Semisteady-State Solutions for Reservoir Pressure

J. Finjord, SPE, and B.S. Aadnoy, SPE, Rogaland Regional C.

Summary. Constant-rate analytical solutions of the one-phase radial-flow equation in two dimensions, including effects of the quadratic gradient term, are derived for an oil reservoir with constant diffusivity and compressibility. The combinations of compressibility contributions to the various terms are analyzed. It is shown that the standard condition allowing the quadratic gradient term to be neglected ($\phi_0 < 1$) is incorrect; pressure, $p$, should be replaced by a function of the production rate and other reservoir parameters except absolute pressure.

The effect of the quadratic gradient term, for which quantitative expressions are given for steady-state and semisteady-state flow, may amount to several percent of the pressure drawdown at the wellbore.

Introduction

Considerable emphasis has recently been placed on nonlinear effects in reservoir flow. In particular, a whole literature on effects of a variable diffusivity has appeared. One exception is effects of the quadratic gradient term, which always appears if integral transforms of pressure are not introduced. Virtually the only work on this subject addresses the conditions for linearizing the flow equation by neglecting the term. No rigorous results for the quadratic gradient term's influence on the solution in terms of pressure have been presented.

In an era of increasing sophistication of the prediction methods for reservoir flow, the effects of the quadratic gradient term also deserve quantitative consideration. We hope to contribute here toward this goal by solving analytically the one-phase-flow equation for two-dimensional (2D) flow with the quadratic gradient term included, which allows us to calculate the effect on the solution directly.

The problem was solved in principle for the case of a constant compressibility by Mysak, who showed that the Darcy-continuity equation would be linearized (except for the nonlinear diffusivity) if it were expressed consistently in terms of density instead of pressure. It can indeed be checked that the transformations introduced here are equivalent to the introduction of density. In Ref. 2, however, the emphasis was not on nonlinear contributions to the solutions in terms of pressure, which were therefore never derived explicitly.

The different ways that the various terms in the equation depend on contributions to the total compressibility are also clarified. Conditions are noted for which the compressibility difference has consequences for 2D simulators.

Nonlinear Terms in the 2D Radial-Flow Equation

Assumptions. We study the case of isothermal radial flow of a single, slightly compressible phase toward one well in a 2D horizontal reservoir, where effects of gravity and inhomogeneities can be neglected. Reservoir height, $h$, porosity, $\phi$, viscosity, $\mu$, and total permeability, $k$, can be functions of pressure, but are not assumed to depend on position and time in any other way. Wellbore storage effects and skin effects are not considered.

The flow equation is obtained by introducing Darcy's law into the equation expressing mass conservation. Textbook versions usually assume several of the parameters—$h$, $\phi$, $\mu$, and $k$—to be constants. This approach obscures one point made here, so we use the more general approach of Ref. 5.

Allowing the reservoir height to depend on pressure essentially corresponds to allowing for a bulk-volume variability with pressure. The bulk volume is not allowed to expand radially in our approach. From a solid mechanics point of view, this corresponds to a plane-stress condition.

One way of partially accounting for pressure dependence in reservoir parameters is to introduce integral transforms of the pressure: e.g., what is done when the pseudopressure representation for gas flow is formulated. To compare with the standard procedure for analytical treatment of oil flow, however, we keep pressure, $p$, as the dependent variable.

Quadratic Gradient Term. The basic flow equation in 2D is

$$\nabla \cdot \left[ (\rho(p)(\mu(p)) \nabla p \right] = (\partial / \partial t) (\rho(p)(\mu(p)) \phi(p)). \quad (1)$$

where $\rho$ = fluid density. For circular symmetry, it can be rewritten

$$c_p \rho \mu(p) (c_p + c_h + c_d) \frac{\partial \phi}{\partial t} = \phi(p) \mu(p) (c_p + c_h + c_d) \frac{\partial \rho}{k(p)} \quad (2)$$

The generalized isothermal compressibilities, $c_p$, are given generally by

$$c_p = (1 + \alpha(p)[\partial \rho(p)/\partial p], \quad \alpha(p,h,\phi,\mu,\kappa). \quad (3)$$

They are defined for constant reservoir overburden pressure, $p^*$ (a plane-stress condition). Here, we consider all compressibilities, $c_p$, to be constants.

An interesting point to be made is that if $c_d$ is nonzero, a nonlinearity is by definition introduced at the right side of Eq. 2, because the porosity, $\phi$, must vary with $p$. Likewise, if $c_h$ and/or $c_d$ are nonzero, nonlinearities appear at the right side. A variable porosity should imply variations in permeability, which, according to the Carmen-Kozeniy law, could be quite violent; we do not pursue that question further here.) If one's aim is to study effects of the nonlinear terms as small corrections to the linearized solutions of Eq. 2, then contributions from the quadratic gradient term on the left side and from the pressure dependence of the diffusivity can be studied separately by the use of lowest-order perturbation theory.

The theme of this paper concerns the solutions of Eq. 2 when the quadratic gradient term is kept, assuming constant diffusivity. (Formally, this corresponds to the limit where $c_d$ and $c_h$ are much larger than the other compressibilities; see Eq. 10.) Notably, the combination of compressibilities in its coefficient is not the same as in the diffusivity, which warrants a check of relative-compressibility magnitude together with the search for an analytical solution.

Contributions to Compressibility. The PV compressibility, $c_{vp}$, is defined by

$$c_{vp} = (1/V_p)(\partial V_p/\partial p),$$

$$= c_d + c_h \phi V_p, \quad V_p = (1-\phi) V_b, \quad (4)$$

where $V_p$, $V_r$, and $V_b$ are pore, grain, and bulk volumes, respectively, the last being proportional to $h$. This justifies the rule$^{2-4}$ that the compressibility to be used should be the sum of the fluid
compressibility and the PV compressibility, according to Eqs. 2 and 4.

On the other hand, by noticing that

\[
\left. \frac{1}{V_f} \left( \frac{\partial V_f}{\partial p} \right) \right|_{p} = \left. \frac{1}{V_p} \left( \frac{\partial V_p}{\partial p} \right) \right|_{p} \]

\[
(5)
\]

(the grain compressibility should be negligible compared with other contributions to the total compressibility), we find by insertion for \( V_f \) and \( V_p \) and partial derivatives that

\[
c_o = (1 - \phi)c_v; \quad \text{........................................... (6)}
\]

i.e., for standard values of the porosity, the major contribution to the PV compressibility comes from \( c_o \). The standard numerical example,

\[
c_o = 10 \times 10^{-6} \text{ psi}^{-1}, \quad c_v = 6 \times 10^{-6} \text{ psi}^{-1}, \quad \phi = 0.15, \ldots \]

leads then to

\[
c_p + c_h = 0.2(c_o + c_v); \quad \text{........................................... (7)}
\]

where Eq. 16 is for constant sandface rate. Because the right side of Eq. 9 disappears in this case, the constancy assumption for \( \mu \) and \( k \) does not really apply, except in the specific value of \( c \) (see Eq. 10). The rate and reservoir parameters entering Eq. 11 can be uniquely defined to refer to \( r_{f} = r_w \), where \( r_w \) = well radius.

For the semi-steady state, the equivalent of Eq. 14 is derived from the overall material balance. For constant compressibilities, the result in terms of volume-average quantities when we consider nonzero compressibilities is found to be

\[
\frac{\phi}{\phi_{sh}} \frac{\partial p_D}{\partial t_D} = \frac{\phi}{\phi_{sh}} \frac{r_w^2}{2 - r_w^2} \frac{r_{f}^2}{\rho_f} \left( \frac{\partial p_D}{\partial t_D} \right) \frac{r_w^2}{r_w^2 - r_f^2} \left( 1 - c_f \phi_{sh} (\frac{\partial p_D}{\partial t_D}) + \ldots \right); \quad \text{........................................... (17)}
\]

The absorption of the \( c_k (\frac{\partial p_D}{\partial t_D}) \) term on the right side of Eq. 17 into the rate (which appears in the constant \( K \)) does not make the problem less general: variations in \( c_k (\frac{\partial p_D}{\partial t_D}) \) with time will be of order \((cK)^2\), and \( c_k \) does not change the quadratic gradient term. Likewise, \( \phi \) can be treated as a constant when the influence of the quadratic gradient term to the lowest order in \( cK \) is finally sought.

At least in the infinite-acting period, the influence of a variable \( \phi \) can be described asymptotically as a constant skin contribution. (Such a result is also known for gas flow.9) The condition in Eq. 17 applies most reliably in the case where \( c_p \gg c_h \).

Our final conditions of solution, obtained by replacing the time derivative in Eq. 17 by a local partial time derivative, are

\[
\frac{\phi}{\phi_{sh}} \frac{\partial p_D}{\partial t_D} = \frac{r_w^2}{2 - r_w^2} \frac{r_f^2}{\rho_f} \left( 1 - c_f \phi_{sh} (\frac{\partial p_D}{\partial t_D}) + \ldots \right); \quad \text{........................................... (18)}
\]

\[
p_D(t_D = r_f, t_D = 0) = 0, \quad \text{........................................... (19)}
\]

\[
r_D(\frac{\partial p_D}{\partial t_D})|_{t_D = r_f} = 0 \quad \text{........................................... (20)}
\]

where Eq. 18 is for a constant volumetric sandface rate at flowing bottomhole pressure. The solution we seek is asymptotic in time, because Eq. 18 corresponds to the long-time limit of the van Everdingen-Hurst solution9 for \( c_k = 0 \). With nonzero compressibilities, one might expect \( \frac{\partial p_D}{\partial t_D} \) to have an \( r_f \) dependence. We later estimate the magnitude of the effect of such a possibly neglected dependence to lowest order in \( cK \), assuming that the \( r_f \) dependence can be equivalently expressed as a series in powers of dimensionless pressure.

The van Everdingen-Hurst solution9 is based on linear operations—e.g., the Laplace transform. Therefore, that general solution and technique is of no use in the nonlinear problem we are treating: only the asymptotically stable result expressed by Eq. 18 should be considered. The infinite-acting-period case is treated in a companion paper.10

Steady-State Solution. Combining Eqs. 9 and 14, one makes the substitution

\[
r_D(\frac{\partial p_D}{\partial t_D}) = \frac{1}{(r_D)}; \quad \text{........................................... (21)}
\]

The resultant problem in \( s \) is linear, and together with Eqs. 15 and 16, the full solution is easily found to be

\[
p_D(s) = \frac{1}{cK} \ln \frac{r_f}{r_w}; \quad \text{........................................... (22)}
\]

where \( r_w = r_D \). The traditional linearized result. For a given production rate and given reservoir parameters, the pressure difference between the well and the influx periphery is less than what is predicted by the linearized equation.

SPE Formation Evaluation, September 1989
Semisteady-State Solution. Eqs. 9 and 18 taken together result in a Riccati nonlinear equation for \( r_D \delta p_D / \delta r_D \), which cannot be reduced to the Bernoulli type. It is remarkable that an exact, explicit, and simple analytical solution can still be found. It is shown in the Appendix that the solution to Eq. 9 with the conditions of Eqs. 18 through 20 is

\[
p_{SSD}(r_D, t_d) = \left( \frac{r_D}{r_w} \right)^{1/2} \left( \frac{r_D}{r_w} \right)^{-1} - \frac{1}{1 + \sqrt{\varepsilon}} \left( \frac{r_D}{r_w} \right)^{1/2} \left( \frac{r_D}{r_w} \right)^{-1}
\]

where \( \varepsilon = \left( \frac{2}{cK_0} \right)^{1/2} \).

For small values of \( cK \) (see the Appendix), we expand the Bessel functions to get

\[
q_{SSD} = \frac{r_D}{r_w} \left[ \ln \left( \frac{r_D}{r_w} \right) - \frac{1}{2} \right] + 0(c^2K^3/2) + \frac{1}{cK}\left( \frac{r_D}{r_w} \right)^{1/2} \left( \frac{r_D}{r_w} \right)^{-1}
\]

where \( \varepsilon = \left( \frac{2}{cK_0} \right)^{1/2} \).

Physical Relevance. Steady State. In terms of constant compressibilities, the pressure-dependent parameters become exponential functions of pressure. Applying Eq. 11, and assuming for brevity \( c_k = c_0 = 0 \), we then get for the ratio of mass flow rates at a given radius and at sandface, independent of the type of flow,

\[
q_{SSD} - q_{SS} = \frac{r_D}{r_w} \left[ \ln \left( \frac{r_D}{r_w} \right) - \frac{1}{2} \right] + 0(c^2K^3/2) + \frac{1}{cK}\left( \frac{r_D}{r_w} \right)^{1/2} \left( \frac{r_D}{r_w} \right)^{-1}
\]

Noticing that the average pressure, \( \bar{p} \), is close to \( p_D \), and using the zero-order result (Eq. 27) in the exponential of Eq. 28, we find that the effective correction factor in the pressure difference between the well and the periphery is roughly the same as in the steady-state case, Eq. 22.

Semisteady State. With Eqs. 12, 24, and 29,

\[
q_{SSD} = \frac{r_D}{r_w} \left[ \ln \left( \frac{r_D}{r_w} \right) - \frac{1}{2} \right] + 0(c^2K^3/2) + \frac{1}{cK}\left( \frac{r_D}{r_w} \right)^{1/2} \left( \frac{r_D}{r_w} \right)^{-1}
\]

where \( \varepsilon = \left( \frac{2}{cK_0} \right)^{1/2} \).

By the result for \( \bar{p} \), the exponential contained in the definition of \( cK \) cancels exactly to lowest nontrivial order in \( cK \). For a small well radius,

\[
\frac{q_{SSD}}{q_{SS}} = \left( \frac{1-r_w^2}{r_w^2} \right) \left( 1-\frac{r_D^2}{r_w^2} \right) \ln \left( \frac{r_D}{r_w} \right) - \frac{1}{2} \left( \frac{r_D^2}{r_w^2} \right) - \frac{1}{2} \left( \frac{r_D^2}{r_w^2} \right) + \frac{1}{cK}\left( \frac{r_D^2}{r_w^2} \right) \left( \frac{r_D^2}{r_w^2} \right)
\]

Except at large radii, where in the semisteady state, the mass flow rate diminishes anyway, the \( 0(c^2K) \) contribution is attenuated by a multiplicative factor \( r_D^2/r_w^2 \). Close to the well, the correction is very small.

To estimate how an \( r_D \) dependence at the right side of Eq. 18 would manifest itself, one could replace \( \bar{p} \) in the exponent by the local value \( p \) before expanding the exponential to the first order in \( cK \). Inserting the resultant Eq. 18 into Eq. 9, and neglecting the quadratic gradient term (a perturbative procedure), one would then arrive at the solution

\[
\frac{p_D(r_D,t_d)}{p_D} = \frac{cK}{cK_0} \left[ \frac{cK_0}{cK} \right]^{-1} \left[ 1 + \frac{r_D^2}{r_w^2} \right] + \frac{1}{cK}\left( \frac{r_D^2}{r_w^2} \right)
\]

when the replacement

\[
\sigma = \frac{cK}{cK_0}
\]

had been done (also in \( \varepsilon \) and \( \eta \)) and the definition of \( \bar{p} \) was used once again. \( I_0, I_1, K_0, \) and \( K_1 \) are modified Bessel functions, for which series expansions were used in the second step of Eq. 35. The \( r_D \) dependence introduced into Eq. 18 has transmuted itself into an attenuation factor in the \( 0(cK) \) term, which for small \( r_D \) becomes a very small correction to the constant exponential factor originating from Eq. 18. Eqs. 34 and 35, taken together, indicate that the approximation originally introduced by Eq. 18 is a sound one, in particular for small \( r_D \).

A perturbative expansion of the solution \( p_D \) of Eq. 9 in powers of \( cK \) can be done. Keeping all terms in Eq. 9, and allowing for an \( r_D \) variation in \( \delta p_D/\delta r_D \) by replacing the exponential in Eq. 18 by a power series in \( cK \), one can find the successive terms in \( p_D \) after insertion in Eq. 9 by a straightforward procedure (not to be reproduced here). To orders \( cK \) and \( (cK)^2 \) in \( p_D \), it has been checked explicitly that the effects of a variable right side of Eq. 18 manifest themselves in \( p_D \) only for small \( r_D \). Values whose order of magnitude is not much smaller than unity. For \( r_D \), these contributions vanish, and the orders \( cK \) and \( (cK)^2 \) in \( p_D \) are governed by the quadratic gradient term for such radii.

Therefore, we conjecture that Eq. 24 gives the exact result for \( p_{SSD} - p_D \) in the limit \( r_w < r_D \), to all orders in \( cK \).

Magnitude of the Correction

It became evident from Eq. 9 that the usual condition \( cK \approx 1 \) given for neglecting the quadratic gradient term, \( cK < 1 \), is not the correct one. Instead, the relative magnitude of the correction to linearized behavior of the solution is given to lowest order by the \( cK \)-dependent terms in Eqs. 22 and 28. Its magnitude at the sandface, where it modifies the total pressure drop from drainage radius to wellbore, is

\[
\frac{1}{cK} \frac{r_w}{r_D} = 0.017 \left( \frac{q_{ss}}{1,000 B/D} \right) \left( \frac{\mu}{0.5 cp} \right) \left( \frac{10 md}{5 m} \right) \left( \frac{5 \text{ m}}{h} \right)
\]

SPE Formation Evaluation, September 1989
The number scaling in \( \rho_d / \rho_w \) corresponds to the "canonical" example of a 40-acre [16.2-ha] drainage area and a 3-in. [7.6-cm] wellbore radius. The unmodified pressure drop, for \( \omega \rightarrow \rho_d \), is

\[
K \ln \left( \frac{\rho_d \rho_w}{\rho_w} \right) = 3.400 \text{ psi} \left( \frac{10^{-5}}{1000 \text{ B/D}} \right) \left( \frac{10^{-3}}{0.5 \text{ cp}} \right) \left( 10 \text{ md} \right) \left( \frac{5 \text{ m}}{7.9} \right)
\]

for both steady-state and semisteady-state conditions. Essentially, the product of \( c_p + c_v \) and the rate-dependent pressure drop determines the relative magnitude of the correction. For a small pressure drawdown, the condition for linearization by deletion is less severe than the one given by Ref. 1, as long as one’s concern is the total pressure drop at the wellbore.

For deep wells, where large drawdowns can take place before the bubblepoint is reached, Eqs. 37 and 38 show that corrections at the wellbore can amount to several percent of the pressure drawdown. As an example, an unmodified drawdown of 5,000 psi [34.5 MPa], with \( c = 10^{-5} \) psi \(^{-1} \) [\( 14.5 \times 10^{-6} \) kPa \(^{-1} \)], gives a correction of 2.5% and an absolute correction in wellbore pressure of 125 psi [862 kPa], which would clearly be significant if the drawdown should result in a final well pressure close to the bubblepoint.

The correction reduces the wellbore-pressure drawdown for a given rate. It may therefore partly cancel the effect of an eventual non-Darcy flow effect, \(^4\) and it should be considered when non-Darcy effects in oil flow are estimated.

Other Comments

1. Consider a 2D simulator with a constant block height independent of pressure \(^3\) (\( c_v = 0 \)) for the case of one flowing (oil) phase with constant saturation. It follows from Eqs. 2 and 6 that a contradiction occurs: the simulator does not distinguish clearly between \( c_v \) and \( c_p \). This may be of limited relevance for practical simulator design and for history matching. Note, however, that there may be a basic discrepancy between laboratory values for \( c_v \) and the "rock-compressibility" value for which best agreement between simulation and reality is obtained.

2. A skin factor is usually defined \(^4\) as an additive contribution to the dimensionless pressure. The nonlinear effect studied here is like skin in that its influence in the solution for pressure is largest close to the wellbore. Thus, care must be taken so that effects of the quadratic gradient term are not falsely interpreted as (rate-dependent) skin effects.

3. Two flow cases studied here apply after the transient effects have died out. For the transient case, it has been demonstrated \(^12\) that wellbore-storage effects die asymptotically. So even if the considerations here have been formulated for the limit of zero dimensionless wellbore-storage constant, the results apply without limitation in the long-time limit. For the steady-state case, there is the explicit assumption that all transients have died. The applicability of the semisteady-state solution depends on the ability to maintain a constant volumetric bottomhole flow rate from or to the formation.

Conclusions

Solutions in pressure for one-phase 2D oil flow, including effects of the quadratic gradient term, have been found for flow toward one well with a constant volumetric sandface rate. For steady-state flow with constant compressibilities, the result is exact for all radii; for semisteady flow, the result can be shown to be exact at the sandface (for \( r_w \rightarrow r_d \)) at least to orders \( cK \) and \( cK^2 \).

The solutions modify a standard textbook condition for neglecting the quadratic gradient term. For large pressure drawdowns at the wellbore, the absolute value of the analytical pressure prediction modification can become of order 100 psi [689 kPa], which is particularly significant if the bubblepoint is approached.

Nomenclature

\[ \sigma_{1,2} = \text{integration constants} \]
\[ c = \text{compressibility coefficient of quadratic gradient term} \]
\[ c_v = \text{PV compressibility, psi}^{-1} \text{[kPa}^{-1}] \]
\[ c_s = \text{compressibility, where index} x \text{specifies quantity, psi}^{-1} \text{[kPa}^{-1}] \]
\[ h = \text{formation thickness (reservoir height), ft [m]} \]
\[ I_0, I_1 = \text{modified Bessel functions of first kind, zero and first order} \]
\[ J_0, J_1 = \text{Bessel functions of first kind, zero and first order} \]
\[ k = \text{permeability, total because there is one flowing phase, md} \]
\[ K = \text{constant, dimension of pressure depending on reservoir parameters (Eq. 11)} \]
\[ K_0, K_1 = \text{modified Bessel functions of second kind, zero and first order} \]
\[ p = \text{reservoir pressure (pore pressure), psi [kPa]} \]
\[ \bar{p} = \text{volume-averaged pressure in reservoir, psi [kPa]} \]
\[ p^* = \text{overburden pressure, psi [kPa]} \]
\[ P_D, P_{SD} = \text{dimensionless reservoir pressure; indices} S \text{and} SS \text{denote steady-state and semisteady-state conditions} \]
\[ p_D = \text{volume-averaged dimensionless reservoir pressure} \]
\[ p_i = \text{initial pressure, infinite-acting case, psi [kPa]} \]
\[ p_o = \text{any one of} p_i, p_v, \text{or} p_{ef} (t=0), \text{psi [kPa]} \]
\[ p_{rd} = \text{pressure at drainage radius; constant in steady state, psi [kPa]} \]
\[ p_{rd} (t=0) = \text{pressure at drainage radius at} t=0 \text{in semisteady state, psi [kPa]} \]
\[ p_{sd} = \text{sandface pressure} = p_w \text{ because no skin is assumed, psi [kPa]} \]
\[ p_{SD}, p_{SSD} = \text{dimensionless reservoir pressure, steady state and semisteady state, linearized case (no quadratic gradient term)} \]
\[ p_w = \text{well pressure, psi [kPa]} \]
\[ q = \text{volumetric production rate, B/D [m}^3\text{d]} \]
\[ q_{sd} = \text{volumetric sandface production rate, B/D [m}^3\text{d]} \]
\[ r = \text{radius from center of well} \]
\[ r_d = \text{drainage radius} \]
\[ r_D = \text{dimensionless radius} \]
\[ r_w = \text{well radius} \]
\[ s = \text{inverse value of} r_D \partial p_D / \partial D \text{ (steady state)} \]
\[ t = \text{time} \]
\[ t_D = \text{dimensionless time} \]
\[ V_b = \text{bulk volume} \]
\[ V_r = \text{grain volume} \]
\[ V_p = \text{PV} \]
\[ x = \text{auxiliary variable} \]
\[ \gamma_0, \gamma_1 = \text{Bessel functions of second kind, zero and first order} \]
\[ z = \text{variable proportional to} r_D \text{ (Eqs. 26 and A-5)} \]
\[ Z = \text{function in expression replacing} \Sigma \text{ (Eq. A-4)} \]
\[ \gamma = \text{Euler’s constant, 0.5772} \ldots \]
\[ \eta = \text{value of} z \text{ at} r_D = 1 \]
\[ \kappa = \text{parameter-dependent constant} \text{ (Eq. 27)} \]
\[ 
\mu = \text{viscosity, cp [Pa} \cdot \text{s]} \]
\[ \rho = \text{value of} z \text{ at} r = r_d \]
\[ \rho = \text{fluid density} \]
\[ \rho_v = \text{volume-averaged fluid density} \]
\[ p_{sd}, p_{ssd} = \text{porosity at sandface, the latter for} \Sigma = 0, \text{fraction} \]

Acknowledgments

We thank H. Kilepe, L. Larsen, and S.M. Skjaeveland for discussions of the topics treated in this work.

References


SPE Formation Evaluation, September 1989


**Appendix—Solution of the Semisteady-State Riccati Equation**

In the shorthand

\[ \sigma = 2r_\mu^2 (r_\mu^2 - r^2) e^{-1(r-w)} \]  \hspace{1cm} \text{(A-1)}

and

\[ \Sigma (r_D,p_D) = r_D (\partial \Sigma / \partial r_D), \]  \hspace{1cm} \text{(A-2)}

we get a Riccati nonlinear equation when Eqs. 18 and 19 are combined:

\[ (\partial / \partial r_D) \Sigma = 2K (1/r_D) \Sigma^2 + \sigma D, \]  \hspace{1cm} \text{(A-3)}

By the transformations \( z = (2K)^{1/6} r_D \), \hspace{1cm} \text{(A-4)}

a linear equation results, which is easily solved:

\[ (3/2)(\partial z^2 / \partial \tau) - (1/2)(\partial / \partial \tau) z^2 + Z = 0. \]  \hspace{1cm} \text{(A-5)}

We combine Eqs. A-2, A-4, and the solution of Eq. A-6 to get

\[ p_D = -1/2cK \ln [J_0 (z) + A_1 Y_0 (z)] + A_2, \]  \hspace{1cm} \text{(A-7)}

where \( J_0 \) and \( Y_0 \) are zero-order Bessel functions of the first and second kind, respectively, \( A_1 \) and \( A_2 \) are integration constants. The extra integration constant introduced by the transformation Eq. 4 drops out. Note that \( A_1 \) and \( A_2 \) can, in principle, depend on \( r_D \).

The final solution, stated elsewhere by Eqs. 24 and 25, is then obtained by applying the conditions in Eqs. 18, 19, and 20 to Eq. A-7. In particular, Eq. 24 is obtained with Eq. 20, determining \( A_1 \) according to

\[ \frac{\partial}{\partial z} J_0 (z) = -J_1 (z), \quad \frac{\partial}{\partial z} Y_0 (z) = -Y_1 (z) \]  \hspace{1cm} \text{(A-8)}

and introducing the Wronskian of \( J_0 \) and \( Y_0 \). The maximum value of the argument \( z \) is

\[ z = 2cK [1 - (r_w/r_d)^2]^{1/6} e^{-c(r-w)}, \]  \hspace{1cm} \text{(A-9)}

So, for normal values of the parameters, \( r \neq 1 \). The zeroes of the Bessel functions will then never be encountered for \( r < r_d \), and small-\( z \) expansions\( ^{11} \) can be used.

With Ref. 11, we derive, with \( \gamma \) as Euler's constant,

\[ Y_0 (z) / J_0 (z) = (2/\pi) [\ln (\varepsilon z) + \gamma + 1.4142 + \ldots] \]  \hspace{1cm} \text{(A-10)}

and

\[ J_1 (z) / J_0 (z) = (2/\pi) [\ln (\varepsilon z) + \gamma - (2/3) (1 + 1.78^2 - 1.82^2 + \ldots)]. \]  \hspace{1cm} \text{(A-11)}

Together with the expansions of \( J_0 (z) \) and \( J_1 (z) \), this is what is needed to arrive at the approximate result of Eq. 33.
Boundary-Dominated Flow in Solution-Gas-Drive Reservoirs
Rodolfo G. Camacho-Vera, SPE, and Rajagopal Raghavan, SPE, U. of Tulsa

Summary. The performance of wells in solution-gas-drive reservoirs during the boundary-dominated flow period is examined. Both constant-wellbore-pressure and constant-oil-rate production modes in closed systems are considered. For the constant-wellbore-pressure production mode, Arps' performance-prediction equations are examined, and predictions of future performance are shown to be strong functions of well spacing, well condition, and fluid properties. The parameters $b$ (the decline exponent) and $d$ (the initial decline rate) in the Arps equations are expressed in terms of physical properties. The conditions under which these equations can be used are specified. An empirical procedure to predict production rates is also presented. In the case of constant-oil-rate production, an expression to correlate the pressure distribution in the reservoir is presented. The correlating function permits us to extend the definition of pseudosteady-state flow to solution-gas-drive systems. Its use also allows the simultaneous computation of average properties (pressure and saturation) during boundary-dominated flow from wellbore information.

Introduction
This work documents some theoretical results that are useful for predicting well performance from production data in solution-gas-drive reservoirs during the boundary-dominated flow period. In the process of documenting these results, we also furnish theoretical support for empirical observations that exist on this subject.

Both constant-wellbore-pressure and constant-oil-rate production modes in circular closed systems are considered. The outcomes presented are based on the theoretical results presented in Refs. 1 and 2. Specifically, the results given here follow from our ability to correlate responses of solution-gas-drive systems with the response of a slightly compressible liquid flow during the boundary-dominated flow period for constant-oil-rate and constant-wellbore-pressure production modes.

This work is divided into three parts. First, the theoretical results related to boundary-dominated flow given in Refs. 1 and 2 are outlined to establish a framework for findings presented in this paper. Second, the case of a well flowing at a constant pressure during the boundary-dominated flow period is analyzed. For this case, Fedovich showed that Arps' empirical family of curves can be combined with the slightly compressible liquid flow solution (exponential decline response) to obtain a family of curves that can be used to predict future performance and estimate the reservoir PV. Refs. 1 and 6 report that during the boundary-dominated period, the rate response vs. time does not match a fixed value of the decline exponent, $b$, in the type curve of Ref. 3. An explanation for this observation is presented in Refs. 3 and 7 that $b$ must be less than or equal to unity, but note that if transient data are used, the value of $b$ in the Arps solution can be greater than unity. The development for transient flow presented in Ref. 8 is used to provide a theoretical justification for this observation.

An empirical procedure to predict production rates of wells produced at a constant pressure over short time spans is also presented. Third, the situation when production is held at a constant oil rate is considered. For this case, it is known from Refs. 1, 2, and 9 that the reservoir does not achieve the pseudosteady-state condition; i.e., the derivative of pressure with respect to time is not constant and is not independent of position in the reservoir. In this work, a correlation for the pressure distribution in the reservoir during the boundary-dominated flow period is developed. This correlating function allows an extension of the pseudosteady-state concept to solution-gas-drive reservoirs. Furthermore, this function allows simultaneous computation of the values of average pressure, $P$, and average saturation, S, from wellbore information.

Mathematical Model
A homogeneous closed cylindrical reservoir with a fully penetrating well located at its center is considered. The well is capable of producing at either a constant oil rate or a constant wellbore pressure. An annular region concentric with the wellbore, with a different permeability from the formation is used to include the effect of a skin region. The effects of gravity, capillary pressure, and non-Darcian flow are not considered.

Figs. 1 and 2 show the PVT properties of fluids used in this work. Fig. 3 presents relative permeability data. The data sets shown in Figs. 1 through 3 are identical to the data sets considered in Refs. 1, 2, 8, and 11 and are used here mainly to preserve continuity. The conclusions derived in this work do not depend on the specific data used in the simulations. Table 1 presents information on the range of variables examined.

Background
Following the development in Ref. 1, the dimensionless pseudopressure is defined as

$$P_{pd}(r,0)=\frac{kh}{141.2q_o(0)}\left[\int_0^r \frac{\alpha(p,S_o)\partial p}{\partial r'} dr' + \int_r^0 \frac{\alpha(p,S_o)\partial p}{\partial r} dr\right]. \quad \quad \quad (1)$$

Here, \(\alpha(p,S_o)=k_{rd}S_o/[\mu_o(p)B_o(p)]\), \quad \quad \quad (2)

and $r$=radius corresponding to the position in the reservoir at which $P(r)$=$P_0$. During the boundary-dominated flow period, $F_r$=0.54928$r_e$ (see Fig. 5 of Ref. 2).

Ref. 1 establishes that during boundary-dominated flow, the following results are valid:

$$P_{pd}(r,t)=P_{pd}(0)+\left(\ln r_d^3-\frac{3}{4}\right)+\left(\frac{k}{k_r}-1\right)\left[\frac{1}{4}\frac{r_d^2}{r_d^2} - \frac{r_d^2}{r_d^2}\right]$$

for $1 \leq r_d \leq r_d$, \quad $r_d=r_w$, and

$$P_{pd}(r,t)=P_{pd}(0)+\left(\ln r_d^3-\frac{3}{4}\right)+\left(\frac{k}{k_r}-1\right)\left[\frac{1}{4}\frac{r_d^2}{r_d^2} - \frac{r_d^2}{r_d^2}\right]$$

for $r_d \leq r_d \leq r_d$. Here $r_d=r_d$=dimensionless radius of the skin zone and $P_{pd}(t)$=volumetric average of the pseudopressure. By using the Muskat material-balance equation, we show in Ref. 1 that

$$P_{pd}(t)=\frac{kh}{141.2q_o(0)}\left[\int_0^r \frac{\alpha(p',r)dp'}{\partial r} \right]. \quad \quad \quad (5)$$

**Now at Inst. Mexicano del Petroleo.**
**Now at Texas A&M U.**

SPE Reservoir Engineering, November 1989

503
where $\alpha$ is value of function $\alpha(p, S_o)$ at average conditions $\bar{p}$ and $\bar{S}_o$ and $\Gamma_{AD} = \frac{r^2}{D} \frac{\bar{p}}{A}$, with the dimensionless time, $\bar{t_D}$, defined as

$$\bar{t_D} = \frac{0.006328k}{\phi \mu_o \rho_o \mu_T} \int_0^{\bar{t}} \frac{\bar{q}_d(t') \bar{\lambda}(t') \bar{c}_T(t')}{\bar{c}_T(t')} \, dt'.$$

where $\bar{\lambda}_i$ and $\bar{c}_T$ = system mobility and compressibility, respectively, corresponding to $\bar{p}$ and $\bar{S}_o$. $\bar{\lambda}_i$ and $\bar{c}_T$ are given by

$$\bar{\lambda}_i = \left( \frac{k_{ro}}{\mu_o \rho_o} \right) \left( \bar{p}, \bar{S}_o \right) \left( \bar{p}, \bar{S}_o \right)$$

and

$$\bar{c}_T = -\frac{\bar{S}_o}{B_o(\bar{p})} \left( \frac{dB_o}{dp} \right) + \frac{\bar{S}_o}{B_o(\bar{p})} \left( \frac{dB_o}{dp} \right) + \frac{\bar{S}_o}{B_o(\bar{p})} \left( \frac{dB_o}{dp} \right).$$

For the constant-oil-rate case, Eq. 5 can be simplified as

$$\bar{P}_{PD} = 2 \sqrt{\bar{t}_D} \Gamma_{AD}$$

where $\Gamma_{AD} = \frac{r^2}{D} \bar{p}$ with $\bar{t}_D$ defined as

$$\bar{t}_D = \frac{0.006328k}{\phi \mu_o \rho_o \mu_T} \int_0^{\bar{t}} \frac{\bar{q}_d(t') \bar{\lambda}(t') \bar{c}_T(t')}{\bar{c}_T(t')} \, dt'.$$

Eq. 9 is an extension of the material-balance equation for single-phase liquid flow (production at a constant rate); similarly, Eq. 5 may also be considered a generalization of the material-balance equation for production at a variable rate in solution-gas-drive reservoirs.

For reference purposes, the definition of dimensionless time based on initial system properties is introduced:

$$\bar{t}_D = \frac{0.006328k \rho_{oil} \bar{t}}{\phi \mu_o \rho_o \mu_T} \frac{r^2}{D}.$$

For single-phase liquid flow during the boundary-dominated flow period, the dimensionless flow rate, $q_D$, is given by

$$q_D(\bar{t}_D) = \frac{141.2 q(0) \mu_B}{kh (p_i - p_w)} \frac{1}{D} \exp \left( -\frac{2 \pi \Gamma_{AD}}{D} \right).$$
where $D = \frac{1}{2} \left( \frac{4A}{\pi C_A r_a^2} + 2 \gamma \right)$. .................................. (13)

where $\gamma$ = Euler's constant and $C_A$ = shape factor. It can be shown that the average reservoir pressure and flow rate are related by

$$\bar{p}_D(t_D) = \frac{k h (p_l - p)}{141.2 q(t) B_p} = -D \left[ 1 - \exp \left( \frac{2x t_D}{D} \right) \right].$$ .......................... (14)

Ref. 1 also shows that for solution-gas-drive systems, the following relation is valid:

$$\bar{p}_D(t_{AD}) = -D \left[ 1 - \exp \left( \frac{2x t_{AD}}{D} \right) \right].$$ .......................... (15)

Fig. 4 shows the use of the appropriate dimensionless average pseudopressure, $\bar{p}_D(0)$, and dimensionless time for production at a constant wellbore pressure for both data sets used in this study. The filled-in data points correspond to Data Set 1 with $s = 10$ and $r_D = 8,000$. The open data points correspond to Data Set 2 with $s = -2$ and $r_D = 2,000$. In Fig. 4, the unbroken straight line represents Eq. 5: $\bar{p}_D(t_{AD}) = 2\pi x t_{AD}$. The circular data points represent $\bar{p}_D(t_{AD})$ obtained from simulation runs. An excellent correlation with the liquid solution is obtained for all values of production time. Similar results are obtained for the case of constant-rate production. The triangular points represent $\bar{p}_D$ values plotted as a function of $t_{AD}$. Agreement with Eq. 5 is good for small values of time; however, during the boundary-dominated flow period, the correlation is poor. (Similar results are also observed for constant-rate production.) The unbroken curves in Fig. 4 represent the right side of Eq. 15 for two values of $r_D$. The square data points correspond to $\bar{p}_D$ values obtained from simulation runs and plotted as a function of $t_{AD}$. The results show here suggest that Eq. 15 will predict $\bar{p}_D(t_{AD})$ values with reasonable accuracy until $t_{AD} = 3$. For larger times, Eq. 5 is a better representation of $\bar{p}_D(t_{AD})$.

Considering Eqs. 5 and 15 and expanding the exponential function in the right side of Eq. 15, we obtain

$$t_{AD > t_{AD}}.$$ ........................................ (16)

This inequality explains why the data points in terms of $t_{AD}$ in Fig. 4 fell to the left of the data points in terms of $t_{AD}$ during boundary-dominated flow.

The results in this section form the framework for the findings presented in this work.

**Results**

As explained in the Introduction, the results of this paper are presented in two sections. We start by considering the case of constant-wellbore-pressure production mode. In this section, we establish conditions under which procedures to analyze rate data available in the literature are justified and furnish theoretical support for empirical observations existing on this subject. A procedure to predict production rates of reservoirs produced at a constant pressure is presented.

We then examine the case of production at a constant oil rate and present a correlation for the form of the pressure distribution in the reservoir. This correlation provides an extension of the pseudosteady-state concept to solution-gas-drive reservoirs.

**Constant-Pressure Production Mode. Analysis of Arps' Performance-Prediction Equations.** On the basis of the success in correlating the average pseudopressure in terms of both dimensionless times $t_{AD}$ and $t_{AD'}$, we use these results to examine the assumptions involved in using the Arps' equations for performance predictions.
Differentiating Eqs. 15 and 5 with respect to time, considering the definitions of $\rho_p$ (Eq. 10) and $t_d$ (Eq. 6), we obtain

$$-\frac{1}{q_o} \frac{\partial \phi \alpha}{\partial t} \frac{dx}{q_o} = \frac{2\phi_0}{D} \frac{2\pi 0.06328k}{\phi_0} \frac{\lambda_t}{\phi} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

$$= \exp \left( \frac{2\pi t_d}{\phi D} \frac{2\pi 0.06328k}{\phi} \frac{\lambda_t}{\phi} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d} \right)$$  

(17)

$$\frac{\partial \phi}{\partial t} = \frac{2\pi 0.06328k}{\phi} \frac{\lambda_t}{\phi} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

(18)

respectively, with $t$ given in days.

Because $q_o(t) = \frac{dN_p}{dt}$, Eq. 18 can be written as

$$\frac{dN_p}{dt} = \frac{\phi h}{\phi} \frac{2\pi 0.06328k}{\phi} \frac{\lambda_t}{\phi} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

(19)

where $N_p$ is cumulative oil production. Eq. 19 shows the relative importance that parameters like relative permeability, PV data, and PV have in the prediction of future performance.

Substituting the right sides of Eqs. 15 and 18 for the second and first expressions in the left side of Eq. 17, respectively, and simplifying, we obtain

$$\frac{d}{dt} \ln \frac{q_o}{\phi AD} = \frac{2\pi 0.06328k}{\phi} \frac{\lambda_t}{\phi} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

(20)

Eq. 20 implies that if $\frac{\lambda_t}{\phi}$ is approximately constant with time, then we would obtain a straight line by plotting log $q_o$ vs. time. This observation may also be expected on intuitive grounds, based on single-phase-flow theory.

We can relate Eq. 20 with the decline-curve equations of Arps for hyperbolic decline. Arps' equations are applicable only for boundary-dominated flow. In these expressions, $t$ represents time since the rate was $q_{oi}$. The variables $d_l$ and $b$ in Eqs. 21 and 22 are considered to be constants and represent the nominal rate at which decline takes place and the decline exponent, respectively. For exponential decline $b = 0$, and for harmonic decline $b = 1$; in general, $b$ is in the range $0 < b < 1$.

Considering Eqs. 20 and 21, we can show that $d_l$ is given by

$$d_l = \frac{2\pi 0.06328k}{\phi} \frac{\lambda_t}{\phi}$$  

(21)

Thus, we can deduce from Eq. 23 that the rate will decline in an exponential form only in the case when $\frac{\lambda_t}{\phi}$ is approximately constant. From Eq. 23 it is clear that $d_l$ is dependent on both rock and fluid properties of the reservoir, the physical reservoir dimension, and the wellbore condition (see also Ref. 15).

Our simulation results indicate that for the assumptions in this study, the function $\frac{\lambda_t}{\phi}$ does not remain constant with time. This point is illustrated in Fig. 5, which presents results for both data sets. The variation in $\frac{\lambda_t}{\phi}$ with time is clearly significant. These results are typical of all simulations we conducted and also appear to be typical of results in Ref. 6.

The variable $b$ can be obtained by

$$\frac{d}{dt} \left( \frac{\phi \alpha}{\phi AD} \right) = -b$$  

(24)

From Eq. 20 it can be shown that

$$\frac{d}{dt} \left( \frac{\phi \alpha}{\phi AD} \right) = \frac{D}{2\pi 0.06328k} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

(25)

Thus, combining Eqs. 24 and 25, we obtain

$$b = \frac{D}{2\pi 0.06328k} \frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$$  

(26)

From Eq. 26, we observe that $b$ will be a constant as long as $\frac{\lambda_t}{\phi}$ varies linearly with time.

The observations regarding $d_l$ and $b$, which have not been presented before to our knowledge, demonstrate the assumptions that are inherent in using the Arps relations to analyze data or to predict future performance. Because both $d_l$ and $b$ depend on relative permeability and fluid properties, a simple material-balance equation like Muskat's can be used to study the variation in $\frac{\lambda_t}{\phi}$ for any specific situation to determine the consequences of using the Arps equations. This observation also applies to gas reservoirs. More interesting and important is that these equations clearly indicate that predictions of future performance are strong functions of well spacing, the well condition, and fluid properties, and thus, they also furnish a theoretical support for the concerns expressed in Ref. 18 about the use of the production-rate declines curves for flow-rate predictions. For gas reservoirs, Fraim and Wattenbarger also observed that the rate-vs.-time plot does not match a fixed value of the decline exponent.

Fig. 6 presents a match of data obtained from three simulation runs for Data Set 1 with the Fetkovich type curve. Here $q_{oi}$ and $t_d$ represent dimensionless decline rate and dimensionless decline time, respectively, as defined in Fetkovich (see Eqs. 19 through 22 of Ref. 3). The objective of this plot is to show the behavior of the product $\frac{d}{2\pi 0.06328k}$ in $\frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$. For solution-gas-drive systems, Refs. 3 and 7 suggest that $b$ should be in the range $0.33 < b < 0.67$. The response shown here fits the range $0.4 < b < 0.8$. The match shown here follows the $b=0.7$ curve at early times. At later times, the responses cut across several curves because, for these simulations, $\frac{\lambda_t}{\phi}$ is not a linear function of time. As already mentioned, similar behavior is reported in Refs. 6 and 17.

The fact that the value of $b$ is not constant in most of the theoretical studies reported in the literature deserves comment. First one must consider whether one can obtain a constant value of $b$ for constant-pressure production. We found that $b$ generally is not a constant for constant-pressure production. Second, one must consider the nature of the wellbore pressure response when the production is forced to follow a specific value of $b$ in the Arps equations. Computations suggest that the production mode must be a variable-pressure/variable-rate mode if the rate is to follow a specific $b$ value. In some cases, we noted that to follow a specific value of $b$, the wellbore pressure must increase with time (assuming that the skin zone properties and drainage area remain constant with time). Third, if we assume constant-pressure production, then Eq. 26 indicates that the only possibility to obtain a constant $b$ is for the product $\frac{d}{2\pi 0.06328k}$ in $\frac{dx}{q_o} \frac{1}{t_d} \frac{dx}{t_d}$ to be a constant. Because $D$ includes the skin factor, this reasoning leads to the argument that variable skin factors may account for a constant value of $b$. One obvious possibility to obtain a variable skin factor is non-Darcy flow. Fetkovich also suggested this possibility to us. In Ref. 19 we noted that on the basis of an examination of inflow performance relations, non-Darcy flow may be the norm in solution-gas-drive reservoirs. On the basis of observations given here, it appears that non-Darcy flow may yield a constant value of $b$. This observation does not imply, however, that if field data suggest that $b$ is a constant, this result is a consequence of non-Darcy flow because there are other factors—like changes in skin zone properties and drainage area—that could cause this behavior. The above observations are important in understanding the consequences of using the Arps curves.

Let us now consider the computation of PV for the results shown in Fig. 6. Fetkovich et al. suggest that it is possible to obtain reservoir PV from the match shown in Fig. 6 by the relation

$$V_p = \frac{B_o}{\phi} t \frac{q_o(t)}{\phi_0(t)}$$  

(27)

where $V_p$ is in barrels and $t$ is at time which the initial rate

Substituting appropriate values ($\bar{P} = 5,344.94$ psi [36.85 MPa], $P_{wf} = 3,389$ psi [23.4 MPa], $\bar{S}_i = 0.0336$, $E_i = 1.4466 \times 10^{-5} \text{ psi}^{-1}$ [21 $\times 10^{-3}$ kPa$^{-1}$], $q_{ad} = q_{o} = (t_t)^{1/2}$ $q_{ad} = 220.86$ STB/D [3.11 stock tank m$^3$/d], $t = 100$ days, $I_{AD} = 1.3$) in Eq. 27, we obtain $P_p = 1.039 \times 10^{12}$ bbl [165.2 $\times 10^3$ m$^3$]. In simulations, $P_p$ was assumed to be $1.124 \times 10^{12}$ bbl [178.7 $\times 10^3$ m$^3$]. Although results did not match a curve with a specific value of $b$, it is interesting that for this specific example, a good estimate of the PV is obtained. Note that Eq. 27 indicates that fluid properties should be computed at $\bar{P}$. Justification for this choice follows directly from the use of $P_{ad}$, $I_{AD}$, and $t_{AD}$ as correlating variables.

Application of Pseudotime Concept. Eq. 20 suggests a procedure to take into account variations in $E_i$ and $S_i$. If $b = 0$, solutions to this equation can be used for performance predictions if $P_{ad}$ is used. This observation follows directly from integration of Eq. 20:

$$q_{ad}(t) = q_{ad} \exp(-t_{AD}/D). \tag{28}$$

Here, $P_{ad}$ must be interpreted as a pseudotime since the time that the rate was $q_{ad}$. A similar conclusion for single-phase gas flow is presented in Ref. 18. Note that the exponential decline trend can be approximately warranted by rewriting Eq. 20 as

$$d \ln q_{ad} = t_{AD}/D. \tag{29}$$

Thus, if we use the definition of dimensionless time $t_{AD}$ instead of just $t$ in the semilogarithmic plot of $q_{ad}$ vs. time, we would obtain an approximately straight line for engineering purposes, as mentioned in Ref. 20.

It is important to realize that Eqs. 28 and 29 are approximate. For some simulations conducted, these expressions were very good approximations; in other cases they were inadequate. Fig. 7 demonstrates one case where Eq. 28 does not do as well as one would hope. The circular data points are plots of rate vs. time of $t_{AD}$, and the square data points are results expressed in terms of $t_{AD}$. The unbroken line is the $b = 0$ solution. The response in terms of $t_{AD}$ (or $t_{ad}$) intersect several values of $b$. In this case, however, the application of the pseudotime concept does not resolve matters completely. Eq. 28 is extremely useful, however, if a plot of $q_{ad}$ vs. time intersects several values of $b$; in this case, extrapolations based on a plot of $q_{ad}$ vs. $t$ would result in a much better estimate of flow rate with time.

Application of Decline Curves to Transient Flow. Refs. 3 and 7 note that if transient data are used to compute the value of $b$, then such data will suggest that $b$ is greater than unity. The basis for this observation can be seen in the following development. In Ref. 8, it is shown that during the transient flow period, the skin factor of a well produced at a constant pressure can be obtained by the relations

$$s = \frac{1}{2} \left( \frac{1}{d \ln q_{ad} / d \ln t} + \frac{4I_{AD}}{e^7} \right) \tag{30}$$

where $N_p = $ cumulative oil produced. Eq. 31 is the general form of Eq. 30.

If we combine Eq. 30 with Eq. 24, we obtain

$$b = 2s + \ln(4I_{AD}/e^7). \tag{32}$$

This expression leads to two observations. First, $b$ is a function of time during the transient flow period. Second, if $b$ takes values greater than unity, then Eq. 32 yields

$$s > 0.5[1 - \ln(4I_{AD}/e^7)]. \tag{33}$$

This inequality can be easily satisfied for large reservoirs and large $t_{AD}$; in fact, even for small reservoirs and small $t_{AD}$, the skin factor can in practice satisfy this inequality for most cases. For example, for $t_{AD} = 10^4$, the inequality in Eq. 33 would be satisfied for $s > 5.66$, which is a high negative skin factor. Thus, in most of the cases, $b > 1$ if transient data are used.

A similar conclusion results if Eq. 31 is used. In this case, we obtain

$$b = \frac{1}{2s + \ln(4I_{AD}/e^7)} \left[ 1 - \frac{4I_{AD}}{t_{ad} e^7} \right]. \tag{34}$$

If we assume that $N_p = t_{ad}$, then we obtain

$$b = -\ln(4I_{AD}/e^7)/2s. \tag{35}$$

Eq. 35 again leads to the conclusion that $b$ is a function of time and would be greater than one in most cases when transient data are used to compute $b$. The above development provides theoretical justification for the observations in Ref. 3 that $b \geq 1$ for the Arps procedure to be applicable.

Approximate Determination of Production Rates. The development given above also suggests a procedure to predict flow rates over short time spans without the use of relative permeability data or decline curves. Here, we present a simple procedure to predict flow rates over short time spans that is similar to the inflow performance relationship (IPR) predictions of Standing and Fetkovich.

Numerical computations suggest that for constant-pressure production,

$$\int_{r_{ad}}^{r_{gd}} d r \left( \frac{d \phi}{d \phi_{r_{ad}}} \right) = 0 \tag{36}$$

during the boundary-dominated flow period. Intuitively, at late times, this result could be expected. Differentiating Eq. 3 with
Eq. 37 can also be written as

\[
\frac{dq_d}{dt} = \frac{\alpha_d}{\bar{p}^2} \frac{d\bar{p}}{d\bar{p}} \left( \ln \frac{r_{e0}}{4} + s \right) + \left( \frac{k}{k_s} - 1 \right) \left( \frac{\bar{r}_{e0}^2}{4} - \frac{\bar{r}_{e0}^2}{r_{e0}^2} \right)
\]

where subscripts \( f \) and \( p \) = future and present conditions, respectively.

Note that Eq. 38 resembles Standing's procedure to predict future performance given by

\[
\frac{q_{o,max,f}}{q_{o,max,p}} = \frac{\bar{p}_f^2}{\bar{p}_p^2} \frac{\bar{p}_f}{\bar{p}_p}
\]

where \( q_{o,max} \) is the maximum flow rate (rate corresponding to \( p_{max} = 0 \)).

It should be noted, however, that Eq. 39 can be obtained by assuming that the function \( \alpha(p) \) varies linearly with pressure, but nothing has been assumed regarding the shape of the function \( \alpha(p) \) in the derivation of Eq. 38.

If we now assume that \( \alpha = \theta \bar{p}^2 \), where \( \theta \) is a constant, as suggested by Fetkovich, Eq. 38 yields

\[
\frac{dq_d}{dt} = \bar{p}_f^2 \frac{d\bar{p}}{d\bar{p}} \left( \ln \frac{r_{e0}}{4} + s \right) + \left( \frac{k}{k_s} - 1 \right) \left( \frac{\bar{r}_{e0}^2}{4} - \frac{\bar{r}_{e0}^2}{r_{e0}^2} \right)
\]

Note again that Eq. 40 resembles Fetkovich's procedure to predict future performance given by

\[
\frac{q_{o,max,f}}{q_{o,max,p}} = \bar{p}_f^2 \frac{\bar{p}_f}{\bar{p}_p}
\]

Fig. 8 presents the derivative of the rate data with respect to time vs. average pressure for a case where Data Set 1 is used. The unbroken line represents simulated values of the function \( dq_d/dt \) vs. \( \bar{p} \). The circular and square data points correspond to the rate derivative evaluated from Eqs. 38 and 40, respectively. The computations are done by starting with the simulation value of \( -dq_d/dt \) at \( t_{4D} = 0.1 \). To evaluate the right sides of both Eqs. 38 and 40, simulation values have been used. Close agreement with actual values is obtained with both Eqs. 38 and 40. The results obtained with Eq. 38 were slightly better than those obtained with Eq. 40. Similar results were observed in other simulations.

The main advantage of Eq. 37 is that it gives us an opportunity to predict future production rates. Eq. 37 can be rewritten

\[
\bar{q} = \frac{q_{o0}}{\bar{p}} = \frac{141.2}{kh} \left( \ln \frac{r_{e0}}{4} + s \right) + \left( \frac{k}{k_s} - 1 \right) \left( \frac{\bar{r}_{e0}^2}{4} - \frac{\bar{r}_{e0}^2}{r_{e0}^2} \right)
\]

Integrating Eq. 42 from \( \bar{p}_f \) to \( \bar{p}_p \) and to \( \bar{p}_f \) (with \( \bar{p}_f > \bar{p}_p > \bar{p}_0 \)) and using the assumption \( \alpha = \theta \bar{p}^2 \), we arrive at the result

\[
\frac{\bar{p}_f^2 - \bar{p}_p^2}{\bar{p}_f^2 - \bar{p}_0^2} = \frac{\bar{q}_{o0} - \bar{q}_0}{\bar{q}_{o0} - \bar{q}_{o0}}
\]

Eq. 43 permits us to predict flow rate at a future value of \( \bar{p} \) and in this sense is similar to Eqs. 39 and 41. Once the future value of \( q_{o,max} \) is determined, an IPR curve can be developed from the relations of Vogel or Fetkovich. Note that Eq. 43 requires information at two pressure levels. In this regard, it is similar to the method of Kelkar and Cox and the pivot-point method.

Because Eq. 43 is based on the assumption that \( \alpha(p) = \theta \bar{p}^2 \), we would intuitively expect Eq. 43 to be applicable over short time spans. This limitation also applies to Eqs. 39 and 41 (see Ref. 19). The main advantage of Eq. 43 is that predictions can be made with rate and pressure data alone without the requirement of relative permeability data.

Figs. 9 and 10 show the use of Eq. 43 to predict flow rates. Results of flow rates vs. average reservoir pressure with both data sets are presented. The filled-in data in Fig. 3 correspond to Data Set 1, and the data in Fig. 10 to Data Set 2. The unbroken lines correspond to simulation values. The initial pair of values \( (q_{o0}, \bar{p}_0) \) was taken at the onset of the boundary-dominated flow period \( t_{4D} = 0.078 (\bar{p} = 5,662.59 \text{ psi}) \) and \( t_{4D} = 0.093 (\bar{p} = 1,438.54 \text{ psi}) \) for Data Sets 1 and 2, respectively.
Other details are given in Figs. 9 and 10. Results are normalized by the initial pair of values. The circular data points reflect the predictions made by using Eq. 43 with the second pair of data points taken at $t_{AD}=0,117$ for Data Sets 1 and 2, respectively. The square data points represent results with $q_e$ and $p$ values at the same $t_A$ as the circular points but with the other pair of $q_e$ and $p$ values at a time $t_B$ greater than that for the circular points ($t_{AD_{AB}}=4,4095$ for Data Set 1 and $t_{AD_{AB}}=1,3435$ for Data Set 2). Agreement with the numerical solution is excellent in three of the four cases considered here. For the case corresponding to the circular data points in Fig. 9, agreement is good only for a short time span beyond the second test values ($t_{AD}=0,1243$, $p_{AD}=5,638,89$ psi [38,88 MPa]). Unfortunately, as with other empirical methods, it is not possible to deduce definite conclusions regarding the accuracy of this method. As shown in Fig. 9, if test conditions are different, results may be substantially different. The principal limitation appears to be the assumption $a(b)=\bar{p}^2$. The only observation we can make is that Eq. 43 yields good results in all cases for short time spans; this limitation also applies to the methods given in Refs. 21, 22, 26, and 27. This limitation can be avoided if the method of Ref. 19 is used.

Constant-Oil-Rate Production Mode. In this section, we show that by correlating the pressure distribution in the reservoir during the boundary-dominated flow period, we can extend the pseudosteady-state concept to solution-gas-drive systems.

Eq. 9 can be used to write Eqs. 3 and 4 as

$$p_{PD}(r,t)=2\sqrt{a_{PD}}\ln r_2^2 + k \left( \frac{r_2^2 - 1}{r_2^2} \right) - \ln r_2 - \frac{3}{4} + s$$

for $1 \leq r_{PD} \leq t_{PD}$ and

$$p_{PD}(r,t)=2\sqrt{a_{PD}}\ln r_2 + \frac{1}{2} \ln r_2 - \frac{3}{4} + \ln r_2 - \frac{3}{4} + s$$

for $r_{PD} \leq r_2 \leq r_{PD}$. Eqs. 44 and 45 imply that

$$p_{PD}(r,t)=f\left( \frac{1}{2} \ln r_2^2 + k \frac{r_2^2 - 1}{r_2^2} \right)$$

for $1 \leq r_{PD} \leq r_{PD}$, and

$$p_{PD}(r,t)=f\left( \frac{1}{2} \ln r_2^2 + k \frac{r_2^2 - 1}{r_2^2} \right)$$

for $r_{PD} \leq r_2 \leq r_{PD}$. Because pseudopressure is a function of pressure and saturation, it is possible that equations or both of these variables-pressure and/or saturation-is a function of $X_P$ and $X_D$ in the corresponding range of radial distance. This possibility was explored and it was found that pressure is a function of $X_D$ for $1 \leq r_{PD} \leq r_{PD}$ and $X_P$ for $r_{PD} \leq r_2 \leq r_{PD}$. Saturation, however, is not a unique function of these variables. Fig. 11 illustrates the point that pressure is a function of $X_P$ or $X_D$. In Fig. 11, the pressure ratio $p(r_{PD})/p_1$ has been correlated for $t_{AD}=0,1$. The data points correspond to Data Set 1 with $s=10$, $r_{PD}=4,105$, and $r_{PD}=8,000$. The dimensionless rate $q_{ocD}$ in Fig. 11 is defined as

$$q_{ocD} = 141,2(p_{PD}/p_1)q_e/(kk_{ocD}p_1).$$

For clarity, seven locations within the skin zone and five locations outside the skin region have been considered. The simulation was stopped when an abandonment pressure of 245 psi [1,69 MPa] was reached, which occurred at 450 days ($t_{AD}=0,95$, $t_{AD}=0,7$). The pressure responses form a single curve for the appropriate variable. These results are typical of all simulations conducted in this work. Thus, on the basis of this observation, we can write the following relations:

$$p(r,t)=f\left( \frac{1}{2} \ln r_2^2 + k \frac{r_2^2 - 1}{r_2^2} \right)$$

for $1 \leq r_{PD} \leq r_{PD}$, and

$$p(r,t)=f\left( \frac{1}{2} \ln r_2^2 + k \frac{r_2^2 - 1}{r_2^2} \right)$$

for $r_{PD} \leq r_2 \leq r_{PD}$. Differentiating Eq. 49 with respect to $r$ and $t$ and combining the resulting equations, we obtain the following expression for the pressure gradient:

$$\frac{dp}{dr} = \frac{k}{\phi A \bar{c}_r} \left[ \frac{k}{\phi A \bar{c}_r} \right]$$

for $1 \leq r_2 \leq r_{PD}$. If we differentiate Eq. 9 with respect to time by using the definitions of $p_{PD}$ and $t_{PD}$ for constant rate, we obtain the following expression for $\bar{c}_r$

$$\bar{c}_r = -5,614q_{ocD} \bar{c}_r/(\phi \lambda d \bar{c}_r)$$

Muskat's material-balance equation is given by

$$\frac{d\lambda}{dr} = \frac{-5,614q_{ocD}}{\phi \lambda d \bar{c}_r}$$

where $\beta = S_P/S_D$. In Refs. 1 and 2, it is shown that $\beta(r_{PD})=\bar{p}/\bar{p}_1$, where an overbar indicates volumetric-average values. Denoting $\beta(r_{PD})$ by $\bar{\beta}$, we have

$$\frac{d\lambda}{dr} = \frac{-5,614q_{ocD}}{\phi \lambda d \bar{c}_r}$$

where $q_e$ is production rate. The production rate for $s \neq 0$ is given by

$$q_e = \frac{k_h}{h} \frac{d\lambda}{dr} \frac{1}{s}$$

where $k_h =$ skin-zone permeability. Substituting the right side of Eq. 51 into Eq. 55 for $d\lambda/dr$, neglecting $(r_{PD}/r_2)^s$, and using Eq.
52, we obtain

\[ \alpha(\partial p/\partial t) = \alpha_0(\partial p/\partial t) \]  

\[ \text{(56)} \]

Eq. 56, of course, is also valid for \( s = 0 \). The importance of this expression to solution-gas-drive systems is that it is a generalization of the pseudosteady-state concept at the positions \( r_m \) and \( F \). The question that needs to be addressed at this point is whether the generalization of the pseudosteady-state concept is valid at other positions in the reservoir. To answer this question, let us consider for simplicity the region outside the skin zone. Differentiating Eq. 50 with respect to \( r \) and \( t \) and combining the resulting equations, we obtain

\[ \frac{\partial p}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( -\frac{k}{c_{1g}} \frac{\partial p}{\partial t} \right) \left( 1 - \frac{r^2}{r_m^2} \right) \]  

\[ \text{(57)} \]

The oil rate at any position \( r < r < r_m \) is

\[ q_o(r) = \frac{k h}{141.2} \left( \frac{\partial p}{\partial r} \right) \]  

\[ \text{(58)} \]

Substituting the right side of Eq. 57 into the right side of Eq. 58 using Eq. 52, we obtain

\[ q_o(r) = \frac{\alpha_0}{\alpha} q_o(r_m) \left( 1 - \frac{r^2}{r_m^2} \right) \frac{\partial p/\partial t}{\partial p/\partial t} \]  

\[ \text{(59)} \]

Refs. 1 and 2 show that during the boundary-dominated flow period, the following expression is valid:

\[ q_o(r) = q_o(r_m) \left( 1 - \frac{r^2}{r_m^2} \right) \]  

\[ \text{(60)} \]

Thus, from Eqs. 59 and 60 it follows that

\[ \alpha(\partial p/\partial t) = \alpha(\partial p/\partial t) \]  

\[ \text{(61)} \]

for any position \( r < r_m \). The same result can be obtained for positions inside the skin zone. Thus, Eq. 61 represents a generalization of the pseudosteady-state concept to solution-gas-drive systems for all positions in the reservoir.

Eq. 61 also provides an insight into the behavior of solution-gas-drive systems produced at a constant oil rate. Differentiating Eqs. 3 and 4 with respect to time and considering the definitions of pseudopressure and average pseudopressure given by Eqs. 1 and 5, respectively, we obtain

\[ \frac{\partial p}{\partial t} = \left[ \frac{1}{r} \frac{d}{dr} \left( \alpha(p, S_o) \frac{\partial p}{\partial r} \right) \right] dr = \frac{\partial}{\partial t} \]  

\[ \text{(62)} \]

Comparing Eq. 61 with Eq. 62, we can conclude that the integral term in Eq. 62 can be considered to be negligible when compared with the other terms. This observation has been confirmed by numerical results. In fact, it has been found that the integral in Eq. 62 may even be considered negligible toward the end of the transient flow period. The same result is also true for constant-wellbore-pressure production for the boundary-dominated flow (Eq. 36) and during the transient flow period, as reported by Camacho-V. 1

Fig. 12 is a plot of the variation in \( \partial p/\partial t \) with distance \( r \). The ordinate is normalized in terms of the value of \( \partial p/\partial t \) at \( r = r_m \) i.e., \( \alpha(\partial p/\partial t) \). Our objective was to test the reasonableness of the approximation given by Eq. 61 during the boundary-dominated flow period. Results for the transient period are also presented (filled-in data points). This case represents a severe test of our development because the value of \( s \) is large (\( k h = 8.08 \)). For the case \( t_{id} = 0.089 \), the variation in \( \partial p/\partial t \) is significant, although for small distances \( r < 1,000 \) the variation in \( \partial p/\partial t \) is negligible. This behavior during transient flow can be expected by noting that during this flow period, the pseudopressure (Eq. 1 with \( \beta = \beta \)) may be expressed in terms of the exponential integral function. Thus, if the ratio \( t_{id}/r_{id}^2 \) is big enough for the logarithmic approximation to be valid (\( t_{id} < 10 \)), the derivative of the pseudopressure with respect to time would be independent of \( r \). For the case presented in Fig. 12 at \( r_p = 1,000 \), the ratio \( t_{id}/r_{id}^2 = 17.89 \), which explains the constant behavior of \( \partial p/\partial t \) for \( r < 1,000 \). For the boundary-dominated flow period, the overall variation is less than 25%. Outside the skin zone, the maximum variation is less than 19%. The variation decreases with time.

The validity of the expressions derived above may also be proved by simultaneously solving Eqs. 54 and 56 to compute \( \tilde{p} \) and \( \tilde{S}_o \) and comparing the results with the \( \tilde{p} \) and \( S_o \) values obtained by solving known material-balance equations like

\[ \left( \frac{S_o}{B_o} \right) = \frac{5.615}{\phi h} N_p + \left( \frac{S_o}{B_o} \right) \]  

\[ \text{(63)} \]

and

\[ \left( \frac{S_e + R_s}{B_e} \right) = \frac{5.615}{\phi h} G_o + \left( \frac{S_e + R_s}{B_e} \right) \]  

\[ \text{(64)} \]

where \( N_p \) and \( G_o \) = cumulative oil and gas productions, respectively. Note that the system of equations 63 and 64 is independent of relative permeability data; also these equations are valid for all times.

The data required to solve Eqs. 54 and 56 simultaneously are an initial pair of \( \tilde{p} \) and \( \tilde{S}_o \) values, relative permeability data, an estimate of \( PV \), and the value of the function \( \alpha(r_o) \). Fig. 13 shows computations of average pressure and saturation for Data Set 2. The value of \( r_{id} = 100 \) and the skin factor is zero. The unbroken and dashed lines correspond to simulator values of \( \tilde{S}_o \) and \( \tilde{p} \), respectively. These lines also represent the solution of Eqs. 63 and 64. The circular and square data points correspond to \( \tilde{S}_o \) and \( \tilde{p} \).
respectively, both computed by solving Eqs. 54 and 56 simultaneously with the fourth-order Runge-Kutta procedure. Values of \( p_\text{p} \) and \( \bar{p}_\text{w} \) from simulation runs were used. The maximum relative differences in the values of \( \bar{S}_\text{w} \) and \( \bar{p} \) are 1.1 and 4.5%, respectively. Similar results are obtained in other cases. These results also indicate that the approximation given by Eq. 56 is reasonable. These computations suggest that it is possible to compute \( \bar{p} \) and \( \bar{S}_\text{w} \) from wellbore data (pressure and gas and oil rates).

**Conclusions**

In the past 20 years, it has been shown that decline-curve analysis involves the same basic principles as well-test analysis. Unfortunately, like well-test analysis, extensions to multiphase-flow situations and the development of theoretical foundations for multiphase-flow conditions have been based on analogy and empiricism. Most disconcerting is that it is normally impossible to reproduce Arps’ decline curves with numerical models, although these models reproduce the behavior of wells for other conditions very well. This fact suggests that an important gap exists in our understanding of the physical mechanisms that influence well behavior in solution-gas-drive systems. This study attempted to provide a general framework for studying well performance for multiphase flow in general and solution-gas drive in particular and fills an important gap by highlighting the basic assumptions in the use of Arps’ equations. It is surprising that the basis for the Arps equations has not been examined until now. This rigorous examination permits us to evaluate the consequences of predictions based on the Arps equations and also provides us with an improved analysis procedure for predicting well performance by use of Arps’ exponential decline curve. In addition, the role of various variables—such as well spacing, well condition, and fluid and formation properties (see Conclusions 1 and 2)—on performance predictions have been examined. Besides examining decline-curve analysis, a simple expression for predicting flow rates over short time spans has been presented. The advantage of the method is that relative permeability and fluid-property data are not needed. This empirical procedure is particularly suited for predicting future IPR’s.

1. The variable \( d_1 \) in Arps’ equations depends on the function \( \bar{r}_1/\bar{r}_2 \). This function must be approximately constant with time to obtain an exponential decline in rate.

2. The function \( \tau_1/\tau_2 \) must vary linearly with time for the decline exponent, \( b \), in the Arps equations to be independent of time.

3. Exponential decline behavior can be used to approximate production rate if a pseudotime, \( \bar{t} \), based on \( \bar{r}_1 \) and \( \bar{r}_2 \) is used.

4. If transient data are used with Arps’ decline curves, theoretical justification exists for the conclusion that \( b \) will be a function of time and will take values greater than unity in most of the cases.

5. For constant-oil-rate production, the pressure distribution in the reservoir can be correlated as a function of distance and time.

6. The concept of pseudosteady-state can be extended to solution-gas-drive systems. The product \( \alpha \bar{r} \dot{V}_p/dt \) is approximately independent of distance during boundary-dominated flow.

**Nomenclature**

\( A \) = drainage area, ft\(^2\) [m\(^2\)]

\( b \) = decline exponent in Arps equations

\( B_g \) = gas FVF, RB/scf [res \ m\(^3\)/std m\(^3\)]

\( B_o \) = oil FVF, RB/STB [res \ m\(^3\)/stock-tank m\(^3\)]

\( c_0 \) = total system compressibility at initial conditions, psi\(^{-1}\) [kPa\(^{-1}\)]

\( c_1 \) = total system compressibility at average conditions (Eq. 8), psi\(^{-1}\) [kPa\(^{-1}\)]

\( C_A \) = shape factor

\( d_1 \) = nominal rate at which decline takes place, parameter in Arps equations

\( D \) = dimensionless constant (Eq. 13)

\( G_p \) = cumulative gas production, Msce [std m\(^3\)]

\( h \) = formation thickness, ft [m]

\( k \) = absolute permeability, md

\( k_{rg} \) = relative permeability to gas, fraction

\( k_{ro} \) = relative permeability to oil, fraction

\( n \) = exponent of backpressure curve

\( N_p \) = cumulative oil production, STB [stock-tank m\(^3\)]

\( p \) = pressure, psi [kPa]

\( p_i \) = initial pressure, psi [kPa]

\( P_{dp}(t) \) = dimensionless pseudopressure (Eq. 1)

\( P_{dp}(t) \) = dimensionless volumetric average pseudopressure (Eq. 5)

\( p_{w} \) = wellbore flowing pressure, psi [kPa]

\( \bar{p} \) = average reservoir pressure, psi [kPa]

\( q_{ud} \) = decline curve dimensionless rate defined in Ref. 3

\( q_k \) = gas flow rate, Msce/STB [std m\(^3\)/d]

\( q_o \) = oil flow rate, STB/STB [stock-tank m\(^3\)/d]

\( q_{6d} \) = dimensionless rate (Eq. 48)

\( r \) = distance, ft [m]

\( r_D \) = dimensionless distance

\( r_e \) = external drainage radius, ft [m]

\( r_s \) = radius of altered-permeability zone, ft [m]

\( r_w \) = wellbore radius, ft [m]

\( r_{to} \) = radius where \( p(r_{to})=\bar{p} \), ft [m]

\( R \) = producing GOR, scf/STB [std m\(^3\)/stock-tank m\(^3\)]

\( R_s \) = solution GOR, scf/STB [std m\(^3\)/stock-tank m\(^3\)]

\( s \) = mechanical skin factor

\( S_b \) = gas saturation

\( S_o \) = oil saturation

\( S_{or} \) = residual oil saturation

\( S_{ew} \) = initial and immobile water saturation

\( S_{el} \) = volumetric average of oil saturation

\( S_{e} \) = volumetric average of oil saturation

\( t \) = time, hours or days

\( t_D \) = pseudotime corresponding to \( \bar{r}_D \), hr-psiu/CP

\( t^{*} \) = time at which rate is \( q_{6d} \)

\( t_{ID} \) = decline curve dimensionless time defined in Ref. 3

\( t_{UD} \) = dimensionless time based on \( A \) and initial conditions

\( t_{ID} \) = dimensionless time based on initial conditions (Eq. 11)

\( t_{ID} \) = dimensionless time (Eq. 10)

\( t_{IP} \) = dimensionless time (Eq. 6)

\( V_p \) = pore volume, bbl [m\(^3\)]

\( X_{0d} \) = correlating variable outside skin zone during boundary-dominated flow and constant-rate production (Eq. 47)

\( X_{ID} \) = correlating variable inside skin zone during boundary-dominated flow and constant-rate production (Eq. 46)

\( \alpha \) = function of pressure and saturation (Eq. 2)

\( \beta \) = function of pressure and saturation (Eq. 33)

\( \gamma \) = Euler’s constant, 0.57721...

\( \lambda_{1} \) = total mobility at initial conditions, \( \text{cp}^{-1} \) [in{Pa·s}·s\(^{-1}\)]

\( \lambda_{e} \) = total mobility at average conditions (Eq. 7), \( \text{cp}^{-1} \) [in{Pa·s}·s\(^{-1}\)]

\( \mu_g \) = gas viscosity, cp [mPa·s]

\( \mu_o \) = oil viscosity, cp [mPa·s]

\( \phi \) = porosity

**Subscripts**

\( \bar{D} \) = dimensionless

\( \bar{e} \) = external

\( \bar{i} \) = initial conditions

\( \bar{s} \) = property of skin region or shut-in conditions

\( \bar{w} \) = wellbore

**Acknowledgment**

Computing time was provided by the U. of Tulsa.

**References**


SI Metric Conversion Factors

- bbl × 1.589 873 E−01 = m³
- cp × 1.0* E−03 = Pa·s
- ft × 3.048* E−01 = m
- f³ × 2.831 685 E−02 = m³
- md × 9.869 233 E−04 = µm²
- psi × 6.894 757 E+00 = kPa
- psi−1 × 1.450 377 E−01 = kPa−1
- scf/bbl × 1.801 175 E−01 = std m³/m³

*Conversion factor is exact.

SPE Reservoir Engineering, November 1989