Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2 — Solution of the Radial Flow Diffusivity Equation

A man will do more for his stubbornness than for his religion or his country.
— Edgar Watson Howe (1911)

Topic: Solution of the Radial Flow Diffusivity Equation

Objectives: (things you should know and/or be able to do)

Infinite-Acting Reservoir Case:

• Be able to recognize that the Laplace transform of the dimensionless form of the single-phase radial flow diffusivity equation is the modified Bessel differential equation. Also, be able to write the general solution for this transformed differential equation.

<table>
<thead>
<tr>
<th>Dimensionless Diffusivity Equation</th>
<th>Laplace Transform of Diffusivity Equation</th>
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<tbody>
<tr>
<td>[ \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \bar{p}_D}{\partial r_D} \right] = \frac{\partial^2 \bar{p}_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial \bar{p}_D}{\partial r_D} = \frac{\partial \bar{p}_D}{\partial t_D} ]</td>
<td>[ \frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{d \bar{p}_D}{dr_D} \right] = u \bar{p}_D ]</td>
</tr>
</tbody>
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Transformed Diffusivity Equation: (Modified Bessel Function form)

\[ z^2 \frac{d^2 \bar{p}_D}{dz^2} + z \frac{d \bar{p}_D}{dz} = z^2 \bar{p}_D \quad (z = \sqrt{u} r_D) \]

• General Solution

\[ \bar{p}_D(r_D,u) = A I_0(\sqrt{u} r_D) + B K_0(\sqrt{u} r_D) \]

• Derivative of the General Solution

\[ \frac{d \bar{p}_D}{dr_D} = A \sqrt{u} I_1(\sqrt{u} r_D) - B \sqrt{u} K_1(\sqrt{u} r_D) \]

• Be able to develop the particular solution (in the Laplace domain) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition. Also, be able to use the van Everdingen and Hurst result which "converts" the constant rate case to the constant wellbore pressure case.

• Constant Rate Solution: (infinite-acting reservoir)

\[ \bar{p}_D(r_D,u) = \frac{1}{u} \frac{K_0(\sqrt{u} r_D)}{\sqrt{u} K_1(\sqrt{u})} = \frac{1}{u} K_0(\sqrt{u} r_D) \]

• Constant Rate-Constant Pressure Relation: (from van Everdingen and Hurst)

\[ \bar{q}_D(u) = \frac{1}{u^2} \frac{1}{\bar{p}_D(u)} \]

• Be able to develop the real domain (time) solution (i.e., the Exponential Integral solution) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition—using both the Laplace transform as well as the Boltzmann transform approaches. Also be able to develop the "log-approximation" for this solution.

• Boltzmann Transform of the Diffusivity Equation:

\[ \frac{d^2 \bar{p}_D}{de_D^2} + \left[ 1 + \frac{1}{e_D} \right] \frac{d \bar{p}_D}{de_D} = 0 \quad \text{(infinite-acting reservoir case only)} \]
Infinite-Acting Reservoir Case: (Continued)

- Real domain (time) solutions: (Continued)
  - "Exponential Integral" Solution for the Diffusivity Equation:
    \[ p_D(t_D, r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4 t_D} \right] \]
  - "Log Approximation" Solution for the Diffusivity Equation:
    \[ p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4 t_D}{e^t r_D^2} \right] \]

Laplace Domain Solutions:

- Be able to derive the particular solutions (in Laplace domain) for a well produced at a constant flowrate in a homogeneous reservoir for the following initial condition—subject to the following inner and outer boundary conditions:
  - **Initial Condition** (Uniform Pressure Distribution)
    \[ p_D(r_D, t_D = 0) = 0 \]
  - **Inner Boundary Condition** (Constant Flowrate at the Well)
    \[ \left[ \frac{\partial p_D}{\partial r_D} \right]_{r_D = 1} = -1 \]
  - **Outer Boundary Conditions**
    a. "Infinite-Acting" Reservoir
      \[ p_D(r_D \to \infty, t_D) = 0 \]  (No reservoir boundary)
    b. "Prescribed Flux" at the Boundary
      \[ \left[ \frac{\partial p_D}{\partial r_D} \right]_{r_D = r_{eD}} = q_{D_{ext}}(t_D) \]  (Specified flux across the reservoir boundary)
    c. Constant Pressure Boundary
      \[ p_D(r_{eD}, t_D) = 0 \]  (Constant pressure at the reservoir boundary)

- **Particular Solutions in the Laplace Domain**
  - "Infinite-acting" reservoir behavior: "cylindrical source" solution
    \[ \bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\sqrt{u} r_D)}{\sqrt{u} K_1(\sqrt{u})} \]
  - "Infinite-acting" reservoir behavior: "line source" solution
    \[ \bar{p}_D(r_D, u) = \frac{1}{u} K_0(\sqrt{u} r_D) \]  (where \( \sqrt{u} K_1(\sqrt{u}) \to 1 \); for \( \sqrt{u} \to 0 \))
  - "Infinite-acting" reservoir behavior: "log approximation" solution
    \[ \bar{p}_D(r_D, u) = \frac{1}{u} K_0(\sqrt{u} r_D) = \frac{1}{2u} \ln \left[ \frac{4 \cdot \frac{1}{u}}{e^{2\sqrt{r_D^2}} u} \right] \]  (\( \approx 0.577216 \) ... Euler's Constant)
Laplace Domain Solutions: (Continued)

- Particular solutions in the Laplace domain: (Continued)
  - Bounded circular res. — "no-flow" at the outer boundary (i.e., \( q_{Dex}(tD)=0 \))
    \[
    \bar{p}_D(rD,u) = \frac{1}{u} \frac{K_0(\sqrt{urD}) I_1(\sqrt{urD}) + K_1(\sqrt{urD}) I_0(\sqrt{urD})}{\sqrt{u} K_1(\sqrt{u}) I_1(\sqrt{urD}) - \sqrt{u} I_1(\sqrt{u}) K_1(\sqrt{urD})}
    \]
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    \[
    \bar{p}_D(rD,u) = \frac{1}{u} \frac{K_0(\sqrt{urD}) I_0(\sqrt{urD}) - K_0(\sqrt{urD}) I_0(\sqrt{urD})}{\sqrt{u} K_1(\sqrt{u}) I_0(\sqrt{urD}) + \sqrt{u} I_1(\sqrt{u}) K_0(\sqrt{urD})}
    \]
  - Bounded circular reservoir — "prescribed flux" at the outer boundary
    \[
    \bar{p}_D(rD,u) = \frac{1}{u} \frac{K_0(\sqrt{urD}) I_1(\sqrt{urD}) + K_1(\sqrt{urD}) I_0(\sqrt{urD})}{\sqrt{u} K_1(\sqrt{u}) I_1(\sqrt{urD}) - \sqrt{u} I_1(\sqrt{u}) K_1(\sqrt{urD})}
    + \frac{1}{u} \frac{q_{Dex}(u)}{\sqrt{urD}} \frac{K_0(\sqrt{urD}) \sqrt{u} I_1(\sqrt{u}) + I_0(\sqrt{urD}) \sqrt{u} K_1(\sqrt{u})}{\sqrt{u} K_1(\sqrt{u}) I_1(\sqrt{urD}) - \sqrt{u} I_1(\sqrt{u}) K_1(\sqrt{urD})}
    \]

Real Domain Solutions (via Inversion of the Laplace Domain Solutions):

- Be able to derive the following particular solutions in the real domain using the appropriate Laplace transform solutions for an unfractured well produced at a constant rate in a homogeneous reservoir for the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior (line source solution)
    \[
    p_D(tD,rD) = \frac{1}{2} E_1 \left[ \frac{rD^2}{4tD} \right]
    \]
  - "Infinite-acting" reservoir behavior (the so-called "log approximation," also a line source solution)
    \[
    p_D(tD,rD) = \frac{1}{2} \ln \left[ \frac{4}{e} \frac{tD}{rD^2} \right]
    \]
  - Bounded circular reservoir — "no-flow" at the outer boundary
    \[
    p_D(tD,rD,r_eD) = \frac{1}{2} E_1 \left[ \frac{rD^2}{4tD} \right] - \frac{1}{2} E_1 \left[ \frac{r_eD^2}{4tD} \right] + \frac{2tD}{r_eD} \exp \left[ -\frac{r_eD^2}{4tD} \right] + \left[ \frac{rD^2}{2r_eD^2} - \frac{1}{4} \right] \exp \left[ -\frac{rD^2}{4tD} \right]
    \]
    and its "well testing" derivative function, \( p'_D = d/dtD[p_D(tD,rD)]\) is given by
    \[
    p'_D(tD,rD,r_eD) = \frac{1}{2} \exp \left[ -\frac{rD^2}{4tD} \right] + \frac{2tD}{r_eD} \exp \left[ -\frac{r_eD^2}{4tD} \right] + \left[ \frac{rD^2}{2tD} - \frac{r_eD^2}{8} \right] \exp \left[ -\frac{rD^2}{4tD} \right]
    \]
Laplace Domain Solutions: (Continued)
- Real Domain Solutions (via Inversion of the Laplace Domain Solutions): (Continued)
  - Bounded circular reservoir — "constant pressure" at the outer boundary

\[
p_D(t_D, r_D, r_e D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4 t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_e D^2}{4 t_D} \right] + \frac{1}{8 t_D} (r_e D^2 - r_D^2) \exp \left[ \frac{-r_e D^2}{4 t_D} \right]
\]

and its "well testing" derivative function, \( p'_D = \frac{d}{dt_D} p_D(r_D, t_D) \) is given by

\[
p'_D(t_D, r_D, r_e D) = \frac{1}{2} \exp \left[ \frac{-r_D^2}{4 t_D} \right] - \frac{1}{2} \exp \left[ \frac{-r_e D^2}{4 t_D} \right] + \frac{1}{8 t_D} (r_e D^2 - r_D^2) \exp \left[ \frac{-r_e D^2}{4 t_D} \right] - 1 \exp \left[ \frac{-r_D^2}{4 t_D} \right]
\]
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2a — Solutions of the Radial Flow Diffusivity Equation Using the Laplace Transform for a Well
Produced at a Constant Rate in an Infinite-Acting Reservoir

A wise man will make more opportunities than he finds.

— Francis Bacon (1625)

**Topic:** Solutions of the Radial Flow Diffusivity Equation Using the Laplace Transform for a Well Produced at a Constant Rate in an Infinite-Acting Reservoir

**Objectives:** (things you should know and/or be able to do)

- Be able to recognize that the Laplace transform of the dimensionless form of the single-phase radial flow diffusivity equation is the modified Bessel differential equation. Also, be able to write the general solution for this transformed differential equation.

  \[
  \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \bar{p}_D}{\partial r_D} \right] = \frac{\partial^2 \bar{p}_D}{\partial z^2} + \frac{1}{r_D} \frac{\partial \bar{p}_D}{\partial r_D} = \frac{\partial \bar{p}_D}{\partial r_D}
  \]

  **Transformed Diffusivity Equation:** (Modified Bessel Function form)

  \[
  z^2 \frac{d\bar{p}_D}{dz^2} + z \frac{d\bar{p}_D}{dz} = z^2 \bar{p}_D \quad (z = \nu r_D)
  \]

- **General Solution**

  \[
  \bar{p}_D(r_D, u) = A I_0(\nu r_D) + B K_0(\nu r_D)
  \]

- **Derivative of the General Solution**

  \[
  \frac{d\bar{p}_D}{dr_D} = A \nu I_1(\nu r_D) - B \nu K_1(\nu r_D)
  \]

- Be able to develop the particular solution (in the Laplace domain) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition. Also, be able to use the van Everdingen and Hurst result which "converts" the constant rate case to the constant wellbore pressure case.

- **Constant Rate Solution:** (infinite-acting reservoir)

  \[
  \bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\nu r_D)}{\nu \nu K_1(\nu \nu)} \approx \frac{1}{u} K_0(\nu r_D)
  \]

- **Constant Rate-Constant Pressure Relation:** (from van Everdingen and Hurst)

  \[
  \bar{q}_D(u) = \frac{1}{u^2} \frac{1}{\bar{p}_D(u)}
  \]

- Be able to develop the real domain (time) solution (i.e., the Exponential Integral solution) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition—using both the Laplace transform as well as the Boltzmann transform approaches. Also be able to develop the "log-approximation" for this solution.

- **Boltzmann Transform of the Diffusivity Equation:**

  \[
  \frac{d^2 \bar{p}_D}{de_D^2} + \left[ 1 + \frac{1}{\nu e_D} \right] \frac{d\bar{p}_D}{de_D} = 0 \quad \text{(infinite-acting reservoir case only)}
  \]
Objectives: (Continued)

- "Exponential Integral" Solution for the Diffusivity Equation:
  \[ p_D(t_D, r_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) \]

- "Log Approximation" Solution for the Diffusivity Equation:
  \[ p_D(t_D, r_D) = \frac{1}{2} \ln \left( \frac{4 \cdot t_D}{e^\frac{r_D^2}{4}} \right) \]

Lecture Outline:

- Review of the Dimensionless Forms of the Single-Phase Radial Flow Diffusivity Equation and the Initial and Boundary Conditions
- Boltzmann Transform Solution
- Laplace Transform Solution
  - Laplace transform of the partial differential equation, application of the initial condition
  - Recast the Laplace transformed differential equation into an ordinary differential equation for a single independent variable. Note that the resulting form is the modified Bessel differential equation, and then give the general solution.
  - Differentiate the general solution with respect to the dimensionless radius, \( r_D \), for application of the inner boundary condition
  - Develop the particular solution, which requires application of both the inner and outer boundary conditions.
  - Attempt to invert the particular solution, give rationale for simplifications and discuss further approximations.

Reading Assignment:

- Review attached notes.
  - Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case:
    - Real domain solution via the Boltzmann transform.
    - Real domain solution via inversion of the Laplace transform solution.
  - Boltzmann Transform Identities

Exercises: For your own practice/skills building—do NOT turn in!

From the attached notes you are to rederive the following, and show all details.

- Derive the Boltzmann transform solution for the infinite-acting reservoir case.
- Derive the Laplace transform solutions for the infinite-acting reservoir case.
  - "Cylindrical source" solution (just show the result—you do not have to derive)
  - "Line source" solution (E1 formulation)
  - "Line source" solution (log approximation)
- Plot the \( p_{wD}(t_D) \) solutions for \( 1 \leq t_D \leq 1 \times 10^5 \), on semilog (x-axis) and log-log scales.
Solution of the Dimensionless Radial Flow Diffusivity Equation:

• Infinite-Acting Reservoir Case: Boltzmann Transform and the Laplace Transform Approach

• Laplace Transform Solutions

• Real Domain Solutions via Inversion of the Laplace Transform Solutions

Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Solution via the Boltzmann Transform
- Boltzmann Transform Identities
- Real Domain Solution via Inversion of the Laplace Transform Solution

Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Solution via the Boltzmann Transform

Radial Flow Solution for an Infinite-Acting Homogeneous Reservoir: Boltzmann Transform Approach

This method has been demonstrated by a variety of authors—the approach we choose was presented by J.L. Johnston in the 2nd edition of the Lee Well Testing text.

The basic partial differential equation is given in dimensionless form as

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial t}
\]

(1)

where

\[ \phi = \frac{r}{r^3} \quad (2) \quad \phi = \phi_e \frac{k}{8\mu} \left( \frac{r_e}{r} - 1 \right) \quad (3) \quad \phi = \phi_{te} \frac{k}{8\mu} \left( \frac{r_e}{r} - 1 \right) \quad (4) \]

where \( \phi_{te} \) and \( \phi_e \) are given by

<table>
<thead>
<tr>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{de} )</td>
<td>1</td>
<td>3.557 x 10^-4</td>
</tr>
<tr>
<td>( \beta_{de} )</td>
<td>2\pi</td>
<td>7.081 x 10^-3</td>
</tr>
</tbody>
</table>

The "initial" condition is given as

\[ \phi(0, t=0) = 0 \] (uniform pressure distribution) (6)

The constant rate inner boundary condition is

\[ \left. \frac{\partial \phi}{\partial r} \right|_{r=r_e} = -1 \] (constant flow rate at the well) (6)

The "infinite-acting" outer boundary condition is given by

\[ \phi(\infty, t) = 0 \] (7)

Rewriting Eq. 1 we have

\[ \frac{1}{r_e} \frac{\partial}{\partial r} \left[ r_e \frac{\partial \phi}{\partial r} \right] = \frac{\partial \phi}{\partial t} \]

(8)

The Boltzmann transform variable, \( \epsilon \), is defined as

\[ \epsilon = \frac{r_e}{t_{de}} \]

(9)

where for our problem we have

\[ a = 1/4, \quad b = 2, \quad c = -1 \]

which yields

\[ \epsilon = \frac{r^2}{t_{de}} \]

(10)
Expanding the $\frac{d^2 y}{dx^2}$ term we have
\[ \frac{d}{dx} \left[ \frac{dy}{dp} \right] + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} = \frac{dy}{dp} \] \hspace{1cm} (11)

Applying the chain rule,
\[ \frac{dy}{dx} = \frac{dy}{dp} \frac{dp}{dx} \]

which combined with Eq. 11 gives
\[ \frac{dy}{dx} \left[ \frac{d^2 y}{dp^2} \right] + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} = \frac{dy}{dx} \frac{dy}{dp} \]

Expanding,
\[ \frac{dy}{dp} \left[ \frac{d}{dp} \left( \frac{dy}{dp} \right) \frac{dy}{dp} + \frac{dy}{dp} \frac{d^2 y}{dp^2} \right] + \left[ \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

Isolating terms,
\[ \left[ \frac{dy}{dp} \right]^2 \frac{d^2 y}{dp^2} + \left[ \frac{dy}{dp} \frac{d}{dp} \left( \frac{dy}{dp} \right) + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

Dividing through by $(\frac{dy}{dp})^2$ gives
\[ \frac{d^2 y}{dp^2} + \frac{1}{(\frac{d}{dp})^2} \left[ \frac{dy}{dp} \frac{d}{dp} \left( \frac{dy}{dp} \right) + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

Reducing the $\frac{d}{dp}$ term we have
\[ \frac{d^2 y}{dp^2} + \frac{1}{(\frac{d}{dp})^2} \left[ \frac{dy}{dp} \frac{d}{dp} \left( \frac{dy}{dp} \right) + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

which can be further reduced to
\[ \frac{d^2 y}{dp^2} + \frac{1}{(\frac{d}{dp})^2} \left[ \frac{dy}{dp} \frac{d}{dp} \left( \frac{dy}{dp} \right) + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

Completing the factorization of the $(\frac{dy}{dp})^2$ gives
\[ \frac{d^2 y}{dp^2} + \frac{1}{(\frac{d}{dp})^2} \left[ \frac{dy}{dp} \frac{d}{dp} \left( \frac{dy}{dp} \right) + \frac{1}{\frac{d}{dx}} \frac{dy}{dp} \frac{dp}{dx} \right] \frac{dy}{dp} = 0 \]

Using Eq. 10 we take the following derivatives
\[ \frac{dy}{dp} = \frac{d}{dp} \left( \frac{d^2 y}{dp^2} \right) = \frac{d}{dp} \left( \frac{1}{t_D} \right) = -\frac{1}{t_D} \frac{d^2 y}{dp^2} = -\frac{1}{t_D} \]

\[ \frac{dy}{dp} = \frac{d}{dp} \left( \frac{d^2 y}{dp^2} \right) = \frac{d}{dp} \left( \frac{1}{t_D} \right) = -\frac{1}{t_D} \frac{d^2 y}{dp^2} = -\frac{1}{t_D} \]

\[ \frac{d^2 y}{dp^2} = \frac{d}{dp} \left[ \frac{d}{dp} \left( \frac{d^2 y}{dp^2} \right) \right] = \frac{d}{dp} \left( \frac{1}{t_D} \right) = -\frac{1}{t_D} \frac{d^2 y}{dp^2} = -\frac{1}{t_D} \]

\[ \frac{d^2 y}{dp^2} = \frac{d}{dp} \left( \frac{d^2 y}{dp^2} \right) = \frac{d}{dp} \left( \frac{1}{t_D} \right) = -\frac{1}{t_D} \frac{d^2 y}{dp^2} = -\frac{1}{t_D} \]

\[ \frac{d^2 y}{dp^2} = \frac{d}{dp} \left( \frac{d^2 y}{dp^2} \right) = \frac{d}{dp} \left( \frac{1}{t_D} \right) = -\frac{1}{t_D} \frac{d^2 y}{dp^2} = -\frac{1}{t_D} \]
Substituting Eqs. 13-15 into Eq. 12 gives
\[ \frac{d^2 \rho}{d \rho^2} + \left[ \frac{1}{(2 \varepsilon_D / \rho)^2} + \frac{1}{\rho} \left( \frac{1}{\varepsilon_D / \rho} \right)^2 \right] \frac{d \rho}{d \rho} = 0 \]

reducing to
\[ \frac{d^2 \rho}{d \rho^2} + \left[ \frac{1}{2 \varepsilon_D} + \frac{1}{\varepsilon_D} + 1 \right] \frac{d \rho}{d \rho} = 0 \]

or finally we have
\[ \frac{d^2 \rho}{d \rho^2} + \left[ 1 + \frac{1}{\varepsilon_D} \right] \frac{d \rho}{d \rho} = 0 \]  (16)

Where Eq. 16 is our "Boltzmann" transformed differential equation.

We must now establish the initial and boundary conditions in terms of the Boltzmann transform variable, \( \varepsilon_D \). Recalling the initial condition, Eq. 5, we have
\[ \rho_0 (r_D, \varepsilon_D = 0) = 0 \]  (5)

where for \( r_D = 0 \), \( \varepsilon_D = 0 \), which gives
\[ \rho_0 (\varepsilon_0 = 0) = 0 \]  (12)

Recalling the outer boundary condition, Eq. 7, we have
\[ \rho_0 (r_D = \infty, \varepsilon_D) = 0 \]  (7)

or as \( r_D \to \infty \); \( \varepsilon_D \to \infty \), which yields
\[ \rho_0 (\varepsilon_D \to \infty) = 0 \]  (18)

Where Eqs. 17 and 18 are the same, which illustrates that the Boltzmann transform "collapses" 2 conditions into 1. Combining this observation with the inner boundary condition, we have 2 "boundary" conditions. Coupling this observation with the fact that Eq. 16 is only a function of the Boltzmann variable, \( \varepsilon_D \), we can solve Eq. 16 uniquely. Note that the "collapsing" of the initial and outer boundary conditions must occur or the Boltzmann transform is technically invalid.

Recalling the constant rate inner boundary condition, Eq. 6,
\[ \left[ \frac{\rho_0}{d \rho_0} \right]_{\rho_D=1} = -1 \quad \text{or} \quad \left[ \rho_0 \frac{d \rho_0}{d \rho_D} \right]_{\rho_D=1} = -1 \quad (\text{line source condition}) \]  (6)

or
\[ \left[ \rho_0 \frac{d \varepsilon_D}{d \rho_0} \frac{d \rho}{d \varepsilon_D} \right]_{\rho_D=0} = \left[ \rho_0 \left( \frac{2 \varepsilon_D}{\rho_D} \right) \frac{d \rho}{d \varepsilon_D} \right]_{\rho_D=0} = 2 \left[ \rho_0 \frac{d \varepsilon_D}{d \rho_0} \rho_D \right]_{\rho_D=0} = -1 \]
which can be rearranged to yield
\[
\left[ \frac{e_0}{d} \frac{d^2}{e_0 - \varepsilon_0} \right] = -\frac{1}{z}
\]  

(19)

Making the following variable of substitution
\[
v = \frac{d\varepsilon_0}{d\varepsilon_0}
\]

(20)

Substituting Eq. 20 into Eq. 16, and noting the use of ordinary derivatives
\[
\frac{dv}{d\varepsilon_0} + \left[ 1 + \frac{1}{\varepsilon_0} \right] v = 0
\]

\[
\frac{1}{v} \frac{dv}{d\varepsilon_0} = -\left[ 1 + \frac{1}{\varepsilon_0} \right]
\]

\[
\frac{d\varepsilon_0}{d\varepsilon_0} = -\varepsilon_0 - \frac{1}{\varepsilon_0}
\]

Integrating
\[
\ln(v) = -\varepsilon_0 - \ln(\varepsilon_0) + \beta
\]

\[
\beta = \text{constant of integration}
\]

Exponentiating
\[
v = \exp \left[ -\varepsilon_0 - \ln(\varepsilon_0) + \beta \right]
\]

or
\[
v = \exp \left[ -\varepsilon_0 \right] \exp \left[ -\ln(\varepsilon_0) \right] \exp \left[ \beta \right]
\]

which reduces to
\[
v = \frac{\alpha}{\varepsilon_0} \exp \left[ -\varepsilon_0 \right]
\]

(21)

where \(\alpha = \exp[\beta]\), i.e., the constant of integration. Recalling Eq. 20 and combining gives
\[
\frac{d\varepsilon_0}{d\varepsilon_0} = \frac{\alpha}{\varepsilon_0} \exp \left[ -\varepsilon_0 \right]
\]

(22)

Multiplying through by \(\varepsilon_0\) gives
\[
\frac{d\varepsilon_0}{d\varepsilon_0} = \alpha \exp \left[ -\varepsilon_0 \right]
\]

(23)

Substitution of Eq. 23 into Eq. 19 gives
\[
\alpha \lim_{\varepsilon_0 \to \varepsilon_0} \left[ \exp[-\varepsilon_0] \right] = -\frac{1}{2}
\]

or
\[
\alpha = -\frac{1}{2}
\]

(24)

Substitution of Eq. 24 into Eq. 22 gives
\[
\frac{d\varepsilon_0}{d\varepsilon_0} = -\frac{1}{2} \exp \left[ -\varepsilon_0 \right]
\]

(25)
Separating and integrating Eq. 25 gives
\[ \int_{\rho_0=0}^{\rho_0} d\rho_0 = -\frac{1}{z} \int_{t_0=0}^{t_0} e^{-\frac{t_0}{4t_0}} \, dt_0 \]
where we note that \( \rho_0 = 0 \) at \( t_0 = \infty \) is the initial outer boundary condition. Completing the integration and reversing the limits we have
\[ \rho_0 = \frac{1}{z} \int_{t_0=\frac{\rho_0^2}{4t_0}}^{\infty} \frac{1}{y} e^{-\frac{y}{y}} \, dy \quad \text{(26)} \]
We note that the integral in Eq. 26 is the exponential integral, \( E_i(x) \), which is given by
\[ E_i(x) = \int_x^{\infty} \frac{1}{y} e^{-\frac{y}{y}} \, dy \quad \text{(27)} \]
Combining Eqs. 26 and 27 gives our final result
\[ \rho_0(\rho_0, t_0) = \frac{1}{z} E_i \left( \frac{\rho_0^2}{4t_0} \right) \quad \text{(28)} \]
Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

• Boltzmann Transform Identities

Definition of Boltzmann Variable for Transient Radial Flow

Recall that the Boltzmann transform variable, $v_0$, is defined as

$$v_0 = a^b r_0^c$$

But what are the $a, b, c$ constants? How do we define these constants?

Starting with the original form of the radial flow diffusivity equation we have

$$\frac{k}{k_0} \left[ \frac{v_{r0}}{v_{T0}} \right] + \frac{1}{r_0} \frac{v_{r0}}{v_{T0}} = \frac{v_{r0}}{v_{T0}}$$

Combining Eqs. 1 and 2 we have (without details)

$$\frac{k^2}{k_0^2} + \left[ \frac{1}{(v_{r0}/v_{T0})^2} \frac{k^2}{k_0^2} + \frac{1}{r_0} \frac{k}{k_0} \right] - \frac{1}{(v_{r0}/v_{T0})^2} \frac{k}{k_0} = 0$$

Taking $v_{r0}/v_{T0}, v_{T0}/v_{r0}, v_{T0}^2/v_{r0}^2$ we have

$$\frac{k}{k_0} = a \left( r_0^b v_0^c \right) = a r_0^b v_0^c \frac{v_{r0}}{v_{T0}} = a r_0^b v_0^c \frac{v_{r0}}{v_{r0}} = a r_0^b v_0^c \frac{v_{r0}}{v_{r0}}$$

or

$$\frac{k}{k_0} = c v_0$$

or

$$\frac{k}{k_0} = \frac{v_0}{v_0}$$

or

$$\frac{k^2}{k_0^2} = \frac{k^2}{k_0^2}$$

which gives

$$\frac{k^2}{k_0^2} = a r_0^b v_0^c \left( r_0^b v_0^c \right)^{-2} = a r_0^b v_0^c \left( r_0^b v_0^c \right)^{-2}$$

or

$$\frac{k^2}{k_0^2} = b(b-1) v_0^2$$

Recalling the multiplier term in Eq. 5 gives

$$\frac{1}{(v_{r0}/v_{T0})^2} \frac{k^2}{k_0^2} + \frac{1}{r_0} \frac{k}{k_0} \left[ \frac{1}{v_{T0}^2} \frac{k}{k_0} \right] - \frac{1}{(v_{r0}/v_{T0})^2} \frac{k}{k_0} = 0$$

or

$$\frac{1}{(v_{r0}/v_{T0})^2} \frac{k^2}{k_0^2} + \frac{1}{r_0} \frac{k}{k_0} \left[ \frac{1}{v_{T0}^2} \frac{k}{k_0} \right] - \frac{1}{(v_{r0}/v_{T0})^2} \frac{k}{k_0} = 0$$
Substituting Eqs. 4-6 into Eq. 7 we have
\[
\frac{1}{(b e_b r_p)^2} \frac{b(b-1) \epsilon_0}{r_p^2} + \frac{1}{b^2 (b e_b / r_p)} \frac{1}{(b e_b r_p)^2} \frac{\epsilon}{\epsilon_0} \]

or
\[
\frac{1}{(b^2 e_b^2 / r_p)} \frac{b(b-1) \epsilon_0}{r_p^2} + \frac{1}{b^2 (b e_b / r_p)} \frac{1}{(b^2 e_b^2 / r_p^2)} \frac{\epsilon}{\epsilon_0} \]

reducing to
\[
\frac{b-1}{b} \frac{1}{\epsilon_0} + \frac{1}{b} \frac{1}{\epsilon_0} - \frac{c}{b^2} \frac{\epsilon_0}{\epsilon_0} \frac{1}{\epsilon_0} \]  \hspace{1cm} (8)

Eq. 8 still remains in terms of \( r_p \) and \( t_p \), rather than only in terms of \( \epsilon_0 \). In particular, the last term is of interest due to the \( r_p \) and \( t_p \) terms - and how do we "convert" these terms into an \( \epsilon_0 \) term?

Looking at the last term in Eq. 8 we have
\[
-c \frac{r_p^2}{b^2 t_p} \frac{1}{\epsilon_0} = -c \frac{r_p^2}{b^2 t_p} \frac{1}{\epsilon_0} = -c \frac{1}{b^2} \frac{r_p^2}{t_p^{-1}} \]

We will attempt to determine \( a, b, \) and \( c \) by setting the entire term equal to 1, then eliminating the \( r_p \) and \( t_p \) terms by establishing the constants \( b \) and \( c \). Systematically we have
\[
\frac{r_p^2}{b} = 1 \text{ if } b = 2 \text{ and } \frac{t_p^{-1}}{c} = 1 \text{ if } c = -1 \]

Setting the entire term equal to unity we obtain
\[
-c \frac{1}{ab^2} \frac{r_p^2}{t_p^{-1}} \frac{1}{\epsilon_0} = 1 \]  \hspace{1cm} (9)

Assuming the following conditions
\[
b = 2 \]  \hspace{1cm} (10)
\[
c = -1 \]  \hspace{1cm} (11)

and substituting Eqs. 10 and 11 into Eq. 9
\[
-(-1) \frac{1}{a(b)^2} \frac{r_p^2}{t_p^{-1}} \frac{1}{\epsilon_0} = 1 \]

or
\[
a = \frac{1}{4} \]  \hspace{1cm} (12)

And the final form of Eq. 8 is given by
\[
\frac{2-1}{z} \frac{1}{\epsilon_0} + \frac{1}{z} \frac{1}{\epsilon_0} + 1 = \left[ \frac{1}{\epsilon_0} + 1 \right] \]  \hspace{1cm} (13)
Making the final equality of Eqs. 7 and 13 we have

\[
\frac{1}{(\text{de}_0/\text{dr}_0)^2} \frac{\text{d} \text{e}_0}{\text{d} \text{r}_0} + \frac{1}{\text{v}_0} \frac{1}{(\text{de}_0/\text{dr}_0)^2} \frac{\text{d} \text{e}_0}{\text{d} \text{r}_0} = 1 + \frac{1}{\text{e}_0}
\]  \hspace{1cm} (14)

Combining Eqs. 3 and 14 gives

\[
\frac{\text{d} \text{e}_0}{\text{d} \text{e}_0} + \left[ 1 + \frac{1}{\text{e}_0} \right] \frac{\text{d} \text{v}_0}{\text{d} \text{e}_0} = 0
\]  \hspace{1cm} (15)

where the following definitions are used:

\[
\text{e}_0 = a \text{v}_0^b + c
\]  \hspace{1cm} (1)

\[
a = \frac{1}{4}
\]  \hspace{1cm} (12)

\[
b = 2
\]  \hspace{1cm} (10)

\[
c = -1
\]  \hspace{1cm} (11)

Substituting Eqs. 10-12 into Eq. 1, we have

\[
\text{e}_0 = \frac{\text{v}_0^2}{4 \text{e}_0}
\]  \hspace{1cm} (16)

which is the basis for the application of the Boltzmann transform to the radial flow diffusivity equation.
Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Real Domain Solution via Inversion of the Laplace Transform Solution

Radial Flow Solution for an Infinite-Acting Homogeneous Reservoir
Laplace Transform Approach

The basic partial differential equation (i.e., the diffusivity equation) is given in dimensionless form by

\[ \frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial t} \]  
\[ \text{or} \quad \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d\phi}{d\rho} \right] = \frac{d\phi}{dt} \]  

(1) or (2)

where

\[ \rho_e = \frac{r}{r_w} \]  
\[ \phi_e = \frac{\phi_c}{k} \left( \frac{r_w^2}{r^2} - 1 \right) \]  
\[ t_0 = \frac{r_c^2}{k\mu c} \]  

(3) (4) (5)

where \( r_w \) and \( r_c \) are given by

<table>
<thead>
<tr>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.637 \times 10^{-4}</td>
<td>3.557 \times 10^{-6}</td>
</tr>
<tr>
<td>2\pi</td>
<td>7.081 \times 10^{-3}</td>
<td>5.356 \times 10^{-4}</td>
</tr>
</tbody>
</table>

The initial condition is given as

\[ \phi_e (\rho, t_0 = 0) = 0 \]  

(uniform pressure distribution)  

(6)

The constant rate inner boundary condition is

\[ \left[ \frac{\rho \phi_e}{\phi_c} \right]_{\rho = 1} = -1 \]  

(constant flow rate at the well)  

(7)

The "infinite-acting" outer boundary condition is given by

\[ \phi_e (\rho = \infty, t) = 0 \]  

(8)

Laplace Transform Formulation: \( \tilde{\phi} = \mathcal{L} \{ \phi_e (\rho, t) \} \)

Taking the Laplace transform of Eq.2 gives

\[ \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d\tilde{\phi}}{d\rho} \right] = s\tilde{\phi} - \tilde{\phi} (t_0 = 0) \]  
\[ \left[ \frac{\rho \phi_e}{\phi_c} \right]_{\rho = 1} = \left[ \frac{\phi_e (\rho, t)}{\phi_c} \right]_{\rho = 1} \]  

(9)

We recognize immediately from Eq.6 that \( \tilde{\phi} (t_0 = 0) = 0 \), combining Eqs. 6 and 9 we obtain

\[ \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d\tilde{\phi}}{d\rho} \right] = s\tilde{\phi} \]  

(10)

Taking the Laplace transform of the inner boundary condition gives

\[ \left[ \frac{\rho \phi_e}{\phi_c} \right]_{\rho = 1} = -\frac{1}{s} \]  

(11)

Taking the Laplace transform of the outer boundary condition gives

\[ \tilde{\phi} (\rho = \infty, s) = 0 \]  

(12)
multiplying through Eq 10 by \( \rho_0^2 \) we have
\[
\frac{\rho_0}{1} \frac{d}{d\rho} \left[ \frac{\rho_0}{\rho_0} \frac{d\rho_0}{d\rho} \right] = 5 \rho_0^2 \rho_0
\]  
(13)

we will define a variable of substitution, \( \bar{z} \), as
\[
\bar{z} = \sqrt{\rho_0}
\]  
(14)
or
\[
\rho_0 = \bar{z}/\sqrt{\bar{z}}
\]  
(15)

Applying the chain rule on the \( d(1)/d\rho \) terms in Eq.13 we have
\[
\frac{\rho_0}{\sqrt{\rho_0}} \frac{d}{d\bar{z}} \left[ \frac{\rho_0}{\sqrt{\rho_0}} \frac{d\rho_0}{d\bar{z}} \right] = 5 \rho_0^2 \rho_0
\]  
(16)

where,
\[
\frac{d\bar{z}}{d\rho} = \frac{1}{\sqrt{\rho}} (\sqrt{\rho_0} \rho_0) = \sqrt{\bar{z}}
\]  
(17)

Substituting Eqs.15 and 17 into Eq.16 we have
\[
\frac{\bar{z}}{\sqrt{\bar{z}}} \frac{d}{d\bar{z}} \left[ \frac{\bar{z}}{\sqrt{\bar{z}}} \frac{d\rho_0}{d\bar{z}} \right] = \bar{z} \rho_0
\]  
(18)

Cancelling the \( \sqrt{\bar{z}} \) terms on the left-hand-side we have
\[
\bar{z} \frac{d}{d\bar{z}} \left[ \frac{\bar{z} d\rho_0}{d\bar{z}} \right] = \bar{z} \rho_0
\]  
(18)

Expanding the left-hand-side terms we have
\[
\bar{z} \frac{d}{d\bar{z}} \left[ \frac{\bar{z} d\rho_0}{d\bar{z}} \right] = \bar{z} \rho_0
\]  
(18)

From Abramowitz & Stegun, Handbook of Mathematical Functions (p.374, Eq. 9.6.1) the modified Bessel differential equation is given by
\[
\bar{z}^2 \frac{d^2w}{d\bar{z}^2} + \bar{z} \frac{dw}{d\bar{z}} = (\bar{z}^2 + \nu^2) w
\]  
(19)

The general solution of Eq. 20 is given by
\[
w = A J_\nu(\bar{z}) + B K_\nu(\bar{z})
\]  
(20)

where \( J_\nu(z) \) and \( K_\nu(z) \) are the modified Bessel functions of the first and second kinds, respectively. Comparing Eqs. 19 and 20 we find that \( \nu=0 \), which yields the general solution of our diffusivity equation. The general solution is given by
\[
\frac{\rho_0}{\sqrt{\rho_0}} = A J_0(\sqrt{\rho_0}) + B K_0(\sqrt{\rho_0})
\]  
(21)

or using \( \bar{z} = \sqrt{\rho_0} \) in Eq.22 we have
\[
\rho_0(\bar{z}) = A J_0(\bar{z}) + B K_0(\bar{z})
\]  
(22)
In order to develop the "particular" solution (i.e., solve for the A and B parameters), we need to determine \( \frac{dP}{dr_0} \). Using the chain rule we have

\[
\frac{dP}{dr_0} = \frac{dP}{d\zeta} \cdot \frac{d\zeta}{dr_0}
\]

Substituting Eq. 17 into Eq. 24 we obtain

\[
\frac{dP}{dr_0} = \sqrt{5} \frac{dP}{d\zeta}
\]

and the \( \frac{dP}{d\zeta} \) term is given by

\[
\frac{dP}{d\zeta} = A \frac{dI_0(\zeta)}{d\zeta} + B \frac{dk_0(\zeta)}{d\zeta}
\]

From Abramowitz and Stegun, *Handbook of Mathematical Functions*, we have

\[
\frac{dI_0(\zeta)}{d\zeta} = I_1(\zeta) \quad \text{(Eq. 9.1.27, p. 376)}
\]

\[
\frac{dk_0(\zeta)}{d\zeta} = -k_1(\zeta) \quad \text{(Eq. 9.1.27, p. 376)}
\]

Substituting Eqs. 27 and 28 into Eq. 26 we have

\[
\frac{dP}{d\zeta} = A \frac{dI_0(\zeta)}{d\zeta} - B \frac{dk_0(\zeta)}{d\zeta}
\]

Combining Eqs. 25 and 29 and using \( \zeta = \sqrt{5} r_0 \) we obtain

\[
\frac{dP}{dr_0} = A \sqrt{5} I_1(\sqrt{5} r_0) - B \sqrt{5} k_1(\sqrt{5} r_0)
\]

Summarizing our efforts so far we have

Laplace Transform of Inner Boundary Condition

\[
\left[ r_0 \frac{dP}{dr_0} \right]_{r_0=1} = -\frac{1}{s}
\]

Laplace Transform of Outer Boundary Condition

\[
P_0(\rho \rightarrow \infty, s) = 0
\]

General Solution in Laplace Domain

\[
P_0(\rho, s) = A I_0(\sqrt{5} \rho) + B k_0(\sqrt{5} \rho)
\]

Radial Derivative of the General Solution in Laplace Domain

\[
\frac{dP_0}{dr_0} = A \sqrt{5} I_1(\sqrt{5} \rho) - B \sqrt{5} k_1(\sqrt{5} \rho)
\]

or

\[
\rho \frac{dP_0}{dr_0} = A \sqrt{5} I_1(\sqrt{5} \rho) - B \sqrt{5} k_1(\sqrt{5} \rho)
\]
Combining Eqs. 11 and 31 for the inner boundary condition at \( r = 1 \), we have
\[
A \sqrt{5} I_1(\sqrt{5} r) - B \sqrt{5} k_1(\sqrt{5} r) = -\frac{1}{5}
\]
(32)

Combining Eqs. 12 and 23 for the outer boundary condition, we obtain
\[
\lim_{r \to 0} \left[ A I_0(\sqrt{5} r) + B k_0(\sqrt{5} r) \right] = 0
\]
(33)

The conventional approach would be to solve Eqs. 32 and 33 simultaneously to determine the \( A \) and \( B \) parameters; however, the \( r \to 0 \) condition simplifies matters somewhat. In particular, we consider the behavior of the following terms:

\[
\lim_{z \to \infty} I_0(z) = 0
\]
and
\[
\lim_{z \to \infty} k_0(z) = 0
\]

considering the behavior illustrated above, if \( \lim_{z \to \infty} I_0(z) = 0 \) and \( \lim_{z \to \infty} k_0(z) = 0 \), then from Eq. 33 we find that \( A = 0 \) and by Eq. 33 \( B \) is indeterminate, i.e., \( B(0) = 0 \). Therefore, \( B \) is determined from the inner boundary condition. Using \( A = 0 \) and solving Eq. 32 for \( B \) we have

\[
B \sqrt{5} k_1(\sqrt{5} r) = -\frac{1}{5}
\]

or
\[
B = \frac{1}{5} \frac{1}{\sqrt{5} k_1(\sqrt{5} r)}
\]
(34)

and of course
\[
A = 0
\]
(35)

Substituting Eqs. 34 and 35 into the general solution (Eq. 23) we obtain the particular solution which is given as
\[
\bar{P}(r, \xi) = \frac{1}{5} \frac{k_0(\sqrt{5} r)}{\sqrt{5} k_1(\sqrt{5} \xi)}
\]
(36)

Eq. 36 is the so-called "cylindrical" source solution. Unfortunately, the quotient of \( k_0(\sqrt{5} r) / (\sqrt{5} k_1(\sqrt{5} \xi)) \) cannot be inverted directly except by Residue methods as illustrated by van Everdingen and Hurst.
The van Everdingen and Hurst results are given as

\[ p_0(t_0, t_p) = \frac{1}{\pi} \int_0^\infty \frac{1 - \exp(-u^2 t_p^2)}{u^2 [I_0(\lambda_0) - Y_1(\lambda_0)]} \, du \]  

(37)

and for the wellbore solution (i.e., \( v_0 = 1 \)) we have

\[ p_0(t_0, t_p) = 4 \frac{1}{\pi^2} \int_0^\infty \frac{1 - \exp(-u^2 t_p^2)}{u^2 [I_0(\lambda_0) + Y_1(\lambda_0)]} \, du \]  

(38)

However, neither Eq. 37 nor Eq. 38 is computationally efficient -- and for general applications we recommend numerical inversion of Eq. 36.

**Line Source Solution:**

Starting with the cylindrical source solution we will develop a line source solution by simplifying the denominator term. Recalling the cylindrical source solution we have

\[ p_0(t_0, s) = \frac{1}{5} \frac{k_0(\sqrt{s} r_p)}{\sqrt{s} k_1(\sqrt{s})} \]  

(36)

Looking at the denominator term we have

\[ \text{denom} = \sqrt{s} k_1(\sqrt{s}) \]

and recalling that the Laplace transform parameter, \( s \), and the dimensionless time function, \( t_0 \), are inversely proportional we have

\[ s \propto \frac{1}{t_0} \]

As a matter of reducing the denominator term we will consider the "large" time (i.e., small \( s \)) behavior of the \( \sqrt{s} k_1(\sqrt{s}) \) term. In particular

\[ \lim_{s \to 0} \sqrt{s} k_1(\sqrt{s}) = 0 \]

From Abramowitz and Stegun, *Handbook of Mathematical Functions*, p. 375, Eq. 9.6.9 (for \( v = 1 \)) we have

\[ k_1(z) = \frac{1}{z} \quad \text{for } z \to 0 \]

Multiplying through by \( z \) we have

\[ zk_1(z) = 1 \quad \text{for } z \to 0 \]

Which for our case is given by

\[ \text{denom} = \sqrt{s} k_1(\sqrt{s}) = 1 \quad \text{for } \sqrt{s} \text{ (or } s \to 0) \]
Applying this behavior to Eq. 36 we have

\[ P_s(r, s) = \frac{1}{5} k_0(\sqrt{s} r) \quad \text{as} \quad s \to 0 \quad \text{(line source solution)} \]  

(59)

However, this \( s \to 0 \) condition is not as restrictive as it might seem. For example, Eq. 59 has been shown to be equal to the cylindrical source solution for \( t_p < 10 \).

But the question is how do we obtain the inverse of Eq. 39? The short answer is to simply use Laplace transform tables, but what if Eq. 39 or its exact form are not listed? Then we can proceed to the method of residues, which will yield an infinite integral or series solution, see van Everdigen and Hurst or Carslaw and Jaeger. Or we can simply use a variety of numerical methods to obtain \( P_s(r, t_p) \) values.

The fact is that Eq. 39 can be inverted analytically to yield a compact (non-infinite series), closed form solution. Our first effort will be a simple table lookup in a relatively obscure reference. Our second effort will also involve a table lookup for part of the inversion, but here we will use a fundamental theorem of the Laplace transform to develop the final result.

---

**Case 1: Inversion of Eq. 39 by a single table lookup**

<table>
<thead>
<tr>
<th>( \hat{f}(s) )</th>
<th>( f(t) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{5} k_0(\sqrt{s} a) )</td>
<td>( \frac{1}{2} e^{\frac{-a^2}{4t}} )</td>
<td>Carslaw and Jaeger; Conduction of Heat in Solids, Table V, Eq. 26, p. 495</td>
</tr>
<tr>
<td>( k_0(\sqrt{s} a) )</td>
<td>( \frac{1}{2t} \exp\left(-\frac{a^2}{4t}\right) )</td>
<td>Abramowitz and Stegun; Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.120, p. 1028 and Roberts and Kaufman; Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304</td>
</tr>
</tbody>
</table>
Combining the Carslaw and Jaeger result for \( \psi_0 k_0 (5 \alpha) \) with the line source solution, Eq. 39, we obtain
\[
P_b (\psi_0, t_D) = \frac{1}{2} \frac{\psi_0}{(\frac{\psi_0^2}{4t_D})}
\]  
(40)

Case 2: Inversion of Eq. 39 by a table lookup and integration
Recalling the integration theorem for Laplace transforms we have
\[
f(t) = \int_0^\infty q(t) dt = \mathcal{L}^{-1}\left\{ \frac{1}{s} \hat{q}(s) \right\}
\]  
(41)

Combining Eqs. 39 and 41 we obtain
\[
P_b (\psi_0, t_D) = \int_0^{t_D} \frac{dp}{dt_D} dt_D
\]  
(42)
or
\[
P_b (\psi_0, t_D) = \int_0^{t_D} \frac{dl}{\xi} \frac{1}{l} \left\{ \frac{1}{s} \hat{k}_0 (\sqrt{\xi} \psi_0) \right\} dt_D
\]  
(43)

But what is \( \mathcal{L}^{-1}\{ \frac{1}{s} \hat{k}_0 (\sqrt{\xi} \psi_0) \} \)? From the table on the previous page we find
\[
\mathcal{L}^{-1}\{ \frac{1}{s} \hat{k}_0 (\sqrt{\xi} \psi_0) \} = \frac{1}{2} \frac{\exp\left(\frac{-\psi_0^2}{4t_D}\right)}{4t_D} \frac{d\psi_0}{dt_D}
\]  
(44)

Substituting Eq. 44 into Eq. 42 we have
\[
P_b (\psi_0, t_D) = \frac{1}{2} \int_0^{t_D} \frac{1}{\xi} \exp\left(-\frac{\psi_0^2}{4t}\right) dt (\xi \text{ is a dummy variable})
\]  
(45)

Introducing a variable of substitution for Eq. 45 we have
\[
m = \frac{\psi_0^2}{4t} \quad (46) \quad \frac{dm}{dt} = -\frac{\psi_0^2}{4t^2} \frac{1}{\xi} \quad \text{or} \quad \frac{dt}{t} = -\frac{1}{m} dm \quad (47)
\]

where the limits are
at \( t_D = 0 \); \( m = \infty \)
at \( t_D = t_0 \); \( m = \frac{\psi_0^2}{4t_0} \)

These substitutions yield
\[
P_b (\psi_0, t_D) = \frac{-1}{2} \int_0^{\frac{\psi_0^2}{4t_0}} \frac{1}{m} \left( \frac{1}{m} \right) dm
\]
or reversing the limits we have
\[
P_b (\psi_0, t_D) = \frac{1}{2} \int_0^{\frac{\psi_0^2}{4t_0}} \frac{1}{m} e^{-m} dm
\]  
(48)
As with the Boltzmann solution, we note that the integral in Eq. 48 is the exponential integral, $E_i(x)$, which is given in Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 5.1.1, p. 238. This relation is

$$E_i(x) = \int_x^\infty \frac{1}{t} e^{-t} \, dt$$  (49)

Combining Eqs. 48 and 49, we obtain

$$p_0 (\delta, t_0) = \frac{1}{2} E_i \left( \frac{v_0^2}{4t_0} \right)$$  (50)

The "Log" Approximation: Real Domain Approach

The application of Eq. 50 is often hampered by the tedious computational nature of the exponential integral function, $E_i(x)$. Consider the infinite series form of $E_i(x)$ given by

$$E_i(x) = -x - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n n!}$$  (51)

where for $x \leq 0.01$ we can neglect the infinite series term, which gives

$$E_i(x) = -x - \ln x \quad \text{for } x \leq 0.01$$  (52)

Substituting Eq. 52 into Eq. 50, we obtain

$$p_0 (\delta, t_0) = \frac{1}{2} \ln \left( \frac{4t_0}{v_0^2} \right) \quad \text{for} \quad \frac{v_0^2}{4t_0} \leq 25 \quad \text{or} \quad \frac{4t_0}{v_0^2} \geq 100$$  (53)

The "Log" Approximation: Laplace Domain Approach

Recalling the line source solution in the Laplace domain, Eq. 34, we have

$$\tilde{p}_0 (\tilde{s}, \delta) = \frac{1}{s} \tilde{k}_0 (s \delta)$$  (54)

We need an approximation for $k_0(s)$ as $s \to 0$. From Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.6.13, p. 375 we have

$$k_0(s) = -\left[ \ln \left( \frac{1}{2} \right) + s \right] \tilde{S}_0 (s) + \left( \frac{1}{(1+1) \frac{1}{2}} \right) \left[ \frac{1}{4} \tilde{S}_0 \left( \frac{1}{2} \right) \right] + \cdots$$

where we note that the $s^2$ terms diminish as $s \to 0$ so we are left with

$$k_0(s) \approx -\left[ \ln \left( \frac{e^s}{2} \right) \right] \tilde{S}_0 (s)$$
which can also be written as
\[ k_0(z) \propto \ln\left(\frac{e^z}{2} \frac{1}{z} \right) I_0(z) \]
or multiplying and dividing by \( z \) we have
\[ k_0(z) \propto \frac{1}{z} \ln\left(\frac{z}{e^z} \frac{1}{2} \right) I_0(z) \]
using the rules of logarithms we have \( \ln(x) = \ln(x^a) \), which gives
\[ k_0(z) \propto \frac{1}{z} \ln\left(\frac{4}{e^{2z}} \frac{1}{z^2} \right) I_0(z) \quad \text{as } z \to 0 \quad (54) \]
But what is the behavior of \( I_0(z) \) as \( z \to 0 \)? Using the series representation from Abramowitz and Stegun, *Handbook of Mathematical Functions*, Eq. 9.6.12, p. 375, we have
\[ I_0(z) = 1 + \frac{1}{(1)!^2} \left[ \frac{1}{4} z^2 \right] + \frac{1}{(2)!^2} \left[ \frac{1}{4} z^2 \right]^2 + \frac{1}{(3)!^2} \left[ \frac{1}{4} z^2 \right]^3 + \ldots + \]
where the behavior as \( z \to 0 \) is
\[ I_0(z) \approx 1 \quad \text{as } z \to 0 \quad (55) \]
Combining Eqs. 54 and 55 we have
\[ k_0(z) \propto \frac{1}{z} \ln\left(\frac{4}{e^{2z}} \frac{1}{z^2} \right) \quad \text{as } z \to 0 \quad (56) \]
Substituting Eq. 56 into Eq. 59 gives
\[ \tilde{f}_0(r_0, s) = \frac{1}{2s} \ln\left(\frac{4}{e^{2s}} \frac{1}{r_0^2 s} \right) \quad \text{as } s \to 0 \quad (large \ r_0) \quad (57) \]
Rearranging we have
\[ \tilde{f}_0(r_0, s) = \frac{1}{2s} \left[ -\frac{1}{s} \ln(1/s) + \frac{1}{s} \ln\left(\frac{4}{e^{2s}} \frac{1}{r_0^2 s} \right) \right] \]
or
\[ \tilde{f}_0(r_0, s) = \frac{1}{2s} \left[ -\frac{1}{s} \ln(s) + \frac{1}{s} \ln\left(\frac{4}{e^{2s}} \frac{1}{r_0^2 s} \right) \right] \quad (58) \]
The inverse Laplace transform of the \( \ln(s) \) term is given by
\[
\begin{array}{ccc}
\tilde{f}(s) & f(t) & \text{Reference} \\
-\frac{1}{s} \ln(s) & \ln(t) + \gamma & \text{Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.98, p. 1027} \\
\quad & \ln(e^{\gamma t}) & \text{and} \\
\quad & & \text{Roberts and Kaufman: Table of Laplace Transforms, Section 7, Eq. 9.1, p. 258}
\end{array}
\]
Term-by-term inversion of Eq. 58 gives

\[ \eta_b(r_0, t_0) = \frac{1}{2} \ln \left( \frac{4}{\pi} \frac{1}{r_0^2} \right) + \frac{1}{2} \ln \left( \frac{4}{\pi} \frac{1}{r_0^2} \right) \]

Collecting terms we have

\[ \eta_b(r_0, t_0) = \frac{1}{2} \ln \left( \frac{4}{\pi} \frac{t_0}{r_0^2} \right) \]  

(59)

where Eqs. 58 and 59 are exactly equivalent.

we can also establish a direct inversion formula for logarithmic functions by comparison of Eqs. 57 and 59. Recalling Eq. 57 we have

\[ \eta_B(r_0, s) = \frac{1}{2s} \ln \left( \frac{4}{\pi} \frac{1}{r_0^2} \frac{1}{s} \right) \]  

(57)

by the Laplace transform parameter, s, and rearranging the terms inside the logarithm we have

\[ s \eta_B(r_0, s) = \frac{1}{2} \ln \left( \frac{4}{\pi} \frac{1}{r_0^2} \frac{1}{s} \right) \]  

(60)

Equating the right-hand-sides of Eqs. 59 and 60 we obtain

\[ t_0 = \frac{1}{e^{\frac{r_0}{s}}} \]

or more importantly, we can solve for s in terms of t_0 as

\[ s = \frac{1}{e^{t_0}} \]  

(61)

We also establish that

\[ \eta_B(r_0, t_0) \approx s \eta_B(r_0, s) \bigg|_{s = \frac{1}{e^{t_0}}} \]  

(62)

where Eq. 62 is called the "Schaperly" inversion formula for logarithms. This result may be of use for modelling and data analysis, but is only strictly valid for radial flow — so applications may be limited.
**Summary of Results:**

**Cylindrical Domain Solutions**

<table>
<thead>
<tr>
<th>case</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ general solution</td>
<td>$\bar{\rho}(\rho, s) = A I_0(\sqrt{s} \rho) + B k_0(\sqrt{s} \rho)$</td>
</tr>
<tr>
<td>$\rho(d\rho/d\rho_z)$ general solution</td>
<td>$r_0 d\bar{\rho}(\rho, s) = A \sqrt{s} \rho I_1(\sqrt{s} \rho) + B \sqrt{s} \rho k_1(\sqrt{s} \rho)$</td>
</tr>
</tbody>
</table>

**Cylindrical Source Solution**

<table>
<thead>
<tr>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\rho}(\rho, s) = \frac{1}{2} \frac{k_0(\sqrt{s} \rho)}{\sqrt{s} k_1(\sqrt{s})}$</td>
</tr>
</tbody>
</table>

**Line Source Solution**

<table>
<thead>
<tr>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\rho}(\rho, s) = \frac{1}{2} k_0(\sqrt{s} \rho) \approx \frac{1}{2s} \ln(\frac{4}{e^{1/2} \rho^2 / s})$</td>
</tr>
</tbody>
</table>

**Real Domain Solutions**

<table>
<thead>
<tr>
<th>case</th>
<th>solution</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical Source Solution</td>
<td>$\rho(\rho_0, t_0) = e^{-1} \left[ \frac{1}{2} \frac{k_0(\sqrt{s} \rho_0)}{\sqrt{s} k_1(\sqrt{s})} \right]$</td>
<td>not invertable to closed form $t_0 &gt; 10$</td>
</tr>
<tr>
<td>Line Source Solution</td>
<td>$\frac{d\rho(\rho_0, t_0)}{dt_0} = \frac{1}{2} \frac{e^{\rho_0^2}}{2 \rho_0^2}$</td>
<td>$t_0 &gt; 10$</td>
</tr>
<tr>
<td>Derivative of Line Source Solution</td>
<td>$\frac{d\rho(\rho_0, t_0)}{dt_0} = \frac{1}{2} \ln(\frac{4}{e^{1/2} \rho_0^2})$</td>
<td>$t_0^{3/2} &gt; 25$</td>
</tr>
<tr>
<td>Log Approximation</td>
<td>$\rho(\rho_0, t_0) \approx \frac{1}{2} \ln(\frac{4}{e^{1/2} \rho_0^2})$</td>
<td>only for radial flow</td>
</tr>
<tr>
<td>&quot;Schaperly&quot; Inversion Formula</td>
<td>$\rho(\rho_0, t_0) \approx \frac{1}{2} \rho(\rho_0, s) \bigg</td>
<td>_{s = \frac{1}{e^{1/2} \rho_0}}$</td>
</tr>
</tbody>
</table>
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2b — Laplace Transform Solutions
of the Radial Flow Diffusivity Equation for a Bounded Circular Reservoir:
Infinite and Finite-Acting Reservoir Cases

It is bad to be oppressed by a minority, but it
is worse to be oppressed by a majority.
— Lord Acton (1907)

**Topic:** Laplace Transform Solutions of the Radial Flow Diffusivity Equation for a
Bounded Circular Reservoir: Infinite and Finite-Acting Reservoir Cases

**Objectives:** (things you should know and/or be able to do)
- Be able to derive the particular solutions (in Laplace domain) for a well produced at a
  constant flowrate in a homogeneous reservoir for the following initial condition, subject
  to the following inner and outer boundary conditions:
  
  **Initial Condition** (Uniform Pressure Distribution)
  \[ p_D(r_D,t_D=0) = 0 \]

  **Inner Boundary Condition** (Constant Flowrate at the Well)
  \[ r_D \left( \frac{\partial p_D}{\partial r_D} \right)_{r_D=1} = -1 \]

  **Outer Boundary Conditions**
  a. "Infinite-Acting" Reservoir
  \[ p_D(r_D \to \infty, t_D) = 0 \] (No reservoir boundary)
  b. "Prescribed Flux" at the Boundary
  \[ r_D \left( \frac{\partial p_D}{\partial r_D} \right)_{r_D=r_{e D}} = q_{D_{ex}}(t_D) \] (Specified flux across the reservoir boundary)
  c. Constant Pressure Boundary
  \[ p_D(r_{e D}, t_D) = 0 \] (Constant pressure at the reservoir boundary)

  **Particular Solutions in the Laplace Domain**
  - "Infinite-acting" reservoir behavior: "cylindrical source" solution
    \[ \overline{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\nu u r_D)}{\sqrt{u}} K_1(\sqrt{u}) \]
  - "Infinite-acting" reservoir behavior: "line source" solution
    \[ \overline{p}_D(r_D, u) = \frac{1}{u} K_0(\nu u r_D) \] (where \( \sqrt{u} K_1(\sqrt{u}) \to 1 \) for \( \sqrt{u} \to 0 \))
  - "Infinite-acting" reservoir behavior: "log approximation" solution
    \[ \overline{p}_D(r_D, u) = \frac{1}{u} K_0(\nu u r_D) = \frac{1}{2u} \ln \left[ \frac{4}{e^{2\gamma}} \frac{1}{r_D^2} \right] \] \( (\gamma=0.577216 \ldots \text{ Euler's Constant}) \)
  - Bounded circular res. — "no-flow" at the outer boundary (i.e., \( q_{D_{ex}}(t_D)=0 \))
    \[ \overline{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\nu u r_D) I_1(\nu u r_D) + K_1(\nu u r_D) I_0(\nu u r_D)}{\sqrt{u} K_1(\sqrt{u}) I_1(\nu u r_D) - \sqrt{u} I_1(\sqrt{u}) K_1(\nu u r_D)} \]
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    \[ \overline{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\nu u r_D) I_0(\nu u r_D) - K_0(\nu u r_D) I_0(\nu u r_D)}{\sqrt{u} K_1(\sqrt{u}) I_0(\nu u r_D) + \sqrt{u} I_1(\sqrt{u}) K_0(\nu u r_D)} \]
Objectives: (Continued)

- Particular Solutions in the Laplace Domain (Continued)
  - Bounded circular reservoir — "prescribed flux" at the outer boundary

\[
\bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\bar{v} r_D) I_1(\bar{v} r_D)}{\bar{v} K_1(\bar{v} u)} + \frac{1}{u} q_{Dext}(u) \frac{K_0(\bar{v} r_D) V_1(\bar{v} u) + I_0(\bar{v} r_D) V_1(\bar{v} u)}{\bar{v} K_1(\bar{v} u) (\bar{v} r_D)}
\]

Lecture Outline: (Continued)

- General Approach to Laplace Transform Solutions:

  - Develop Bessel's modified differential equation from the Laplace transform of the diffusivity equation.

  **Dimensionless Diffusivity Equation**

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \bar{p}_D}{\partial r_D} \right] = \frac{\partial \bar{p}_D}{\partial r_D} + \frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{d \bar{p}_D}{dr_D} \right] = u \bar{p}_D
\]

  **Transformed Diffusivity Equation**: (Modified Bessel Function form)

\[
z^2 \frac{d^2 \bar{p}_D}{dz^2} + z \frac{d \bar{p}_D}{dz} = z^2 \bar{p}_D \quad \text{where} \quad z = \bar{v} r_D.
\]

- Write the appropriate general solution and take the derivative with respect to \( r_D \) of the general solution. These results are:

  - **General solution**: (Bessel's modified differential equation)

\[
\bar{p}_D(r_D, u) = A I_0(\bar{v} r_D) + B K_0(\bar{v} r_D)
\]

  - **Spatial (Radial) Derivative of the General Solution**:

\[
\left[ r_D \frac{d \bar{p}_D}{dr_D} \right]_{r_D} = A \bar{v} r_D I_1(\bar{v} r_D) - B \bar{v} r_D K_1(\bar{v} r_D)
\]

- Boundary Conditions in the Laplace Domain

  - **Inner Boundary Condition** (Constant Flowrate at the Well)

\[
\left[ r_D \frac{d \bar{p}_D}{dr_D} \right]_{r_D=1} = \frac{1}{u}
\]

  - **Outer Boundary Conditions**

    a. "Infinite-Acting" Reservoir

\[
\bar{p}_D(r_D \to \infty, u) = 0 \quad \text{(No reservoir boundary)}
\]

    b. "No Flow" at the Boundary

\[
\left[ r_D \frac{d \bar{p}_D}{dr_D} \right]_{r_D=r_e D} = 0 \quad \text{(No flow at the reservoir boundary)}
\]
Lecture Outline: (Continued)
- General Approach to Laplace Transform Solutions: (Continued)
  c. Constant Pressure Boundary
  \[ \bar{p}_D(r_D, u) = 0 \quad (\text{Constant pressure at the reservoir boundary}) \]
  d. "Prescribed Flux" at the Boundary
  \[ \left[ r_D \frac{d \bar{p}_D}{dr_D} \right]_{r_D=r_D} = \mathcal{L} \left[ q_{D_{\text{ext}}}(t_D) \right] = \bar{q}_{D_{\text{ext}}}(u) \quad (\text{Specified flux at boundary}) \]

- Establish the first "equation" by equating the inner boundary condition (constant rate at the well) and the radial derivative of the general solution.
- Establish the second "equation" by equating the desired outer boundary condition ("no-flow," "constant pressure," or "prescribed flux") and either the general solution or its radial derivative, as appropriate.
- Using the two equations/two unknowns approach, solve for the particular solution in the Laplace domain (i.e., the \( A \) and \( B \) parameters) and reduce to the most fundamental algebraic form.

Reading Assignment:
- Review attached notes.
- Solution of the Dimensionless Radial Flow Diffusivity Equation:
  - Laplace transform solutions.

Exercises: For your own practice/skills building—do NOT turn in!

Derivation of Solutions in the Laplace Domain:
From the attached notes you are to rederive the following, and show all details.
- Starting from the dimensionless diffusivity equation, derive the Laplace transform solutions for a well produced at a constant flowrate (inner boundary condition) in a homogeneous reservoir with the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior
  - Bounded circular reservoir — "no-flow" at the outer boundary
  - Bounded circular reservoir — "constant pressure" at the outer boundary
  - Bounded circular reservoir — "prescribed flux" at the outer boundary
Exercises: For your own practice/skills building—do NOT turn in!

Paper Reviews:

- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- **Problem:**
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?

- **Assumptions and Limitations:**
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?

- **Practical Applications:**
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how could the solutions/results be used in practice?

- **Discussion:**
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.

- **Recommendations/Extensions:**
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Laplace Transform Solutions—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Error in Laplace Transform Solutions—Radial Flow Case (SPE 25479)
Solution of the Dimensionless Radial Flow Diffusivity Equation:

● Laplace Transform Solutions

The fundamental partial differential equation (the diffusivity equation) is given in dimensionless form by:

\[
\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} = \frac{1}{k_D} \left[ \frac{\partial}{\partial r_D} \left( r_D \frac{\partial \theta}{\partial r_D} \right) \right] = \frac{\theta}{t_D}
\]  

(2)

where

\[ r_D = \frac{r}{r_c} \quad (\text{Field Units}) \quad \theta = \frac{k}{k_c} \quad (\text{Darcy Units}) \quad t_D = \frac{r_c^2}{k_c} \quad t \quad (\text{SI Units}) \]

The initial condition is given as

\[ \theta (r_c, t_D = 0) = 0 \quad (\text{uniform pressure distribution}) \]

The constant rate inner boundary condition is

\[ \left[ \frac{\partial \theta}{\partial r_D} \right]_{r_D = 1} = -1 \quad (\text{constant flow rate at the well}) \]

The outer boundary conditions are given by:

a. "Infinite-acting" outer boundary condition

\[ \theta (r_c, t_D) = 0 \]

b. "No-flow" outer boundary condition

\[ \left[ \frac{\partial \theta}{\partial r_D} \right]_{r_D = r_c} = 0 \]

c. "Constant pressure" outer boundary condition

\[ \theta (r_c, t_D) = \theta _{\text{ext}} (t_D) = \text{constant at initial pressure} \]

d. "Specified flux" outer boundary condition

\[ \left[ \frac{\partial \theta}{\partial r_D} \right]_{r_D = r_c} = \frac{q}{k} \theta _{\text{ext}} (t_D) \]

Laplace Transform Formulation: \( \bar{\theta} (s) = \mathcal{L} \{ \theta (r_c, t_D) \} \), \( s = \text{Laplace transform parameter} \)

Taking the Laplace transform of Eq. 2 gives

\[ \frac{1}{\sigma_D} \left[ \frac{\theta}{\sigma_D} \right] = \mathcal{L} \left\{ \theta (r_c, t_D) \right\} = \frac{d \theta (s \sigma_D)}{d \sigma_D} \]

(12)
We recognize from Eq. 6 that \( \psi (r_p, t_p=0) = 0 \), combining Eqs. 6 and 12 we obtain

\[
\frac{1}{r_p} \frac{d}{dr_p} \left[ r_p \frac{d^2 \psi}{dr_p^2} \right] = \mu \frac{\psi}{r_p}
\]  
(13)

Taking the Laplace transform of the inner boundary condition gives

\[
\left[ r_p \frac{d \psi}{dr_p} \right]_{r_p = R_0} = -\frac{1}{\mu}
\]  
(14)

Taking the Laplace transform of the outer boundary conditions,

a. Laplace transform of the "infinite-acting" outer boundary condition

\[
\frac{\psi}{r_p} (r_p = R_0, \mu) = 0
\]  
(15)

b. Laplace transform of the "no-flow" outer boundary condition

\[
\left[ r_p \frac{d \psi}{dr_p} \right]_{r_p = R_p} = 0
\]  
(16)

c. Laplace transform of the "constant pressure" outer boundary condition

\[
\frac{\psi}{r_p} (r_p = R_0, \mu) = \frac{p_{ext}}{\mu} (constant \ at \ initial \ pressure)
\]  
(17)

d. Laplace transform of the "prescribed flux" outer boundary condition

\[
\left[ r_p \frac{d \psi}{dr_p} \right]_{r_p = R_p} = \psi \left( \phi_p (r_p) \right) = \frac{\psi}{r_p} \phi_p (r_p)
\]  
(18)

Multiplying through Eq. 13 by \( r_p^2 \) we have

\[
\frac{1}{r_p} \frac{d}{dr_p} \left[ r_p \frac{d \psi}{dr_p} \right] = \mu r_p \frac{\psi}{r_p}
\]  
(19)

Defining a variable of substitution, \( z \), as follows

or

\[
z = \sqrt{\mu} r_p
\]  
(20)

\[
r_p = \frac{z}{\sqrt{\mu}}
\]  
(21)

Applying the chain rule on the \( d(1/dr_p) \) terms in Eq. 19 we obtain

\[
\frac{r_p}{r_p} \frac{d}{dz} \left[ \frac{d \psi}{r_p} \frac{d}{dz} \right] = \mu \frac{\psi}{r_p} \frac{\psi}{r_p}
\]  
(22)

where

\[
\frac{z}{r_p} = \frac{z}{\sqrt{\mu} r_p} = \sqrt{\mu}
\]  
(23)

Substituting Eqs. 21 and 23 into Eq. 22 we have

\[
\frac{z}{\sqrt{\mu}} \frac{d}{dz} \left[ \frac{z}{\sqrt{\mu}} \frac{d \psi}{dz} \right] = z^2 \frac{\psi}{r_p}
\]
Cancelling the $\sqrt{\nu^2}$ terms on the left-hand-side we obtain
\[
\frac{d}{dz} \left[ \nu \frac{d\phi}{d\nu} \right] = \nu^2 \phi
\]  
(24)

Expanding the left-hand-side terms we have
\[
\nu^2 \frac{d^2\phi}{d\nu^2} + \nu \frac{d\phi}{d\nu} = \nu^2 \phi
\]  
(25)

From Abramowitz and Stegun, Handbook of Mathematical Functions, (p. 374, Eq. 9.6.1), the modified Bessel differential equation is given by
\[
\frac{d^2w}{dz^2} + \frac{z}{d} \frac{dw}{dz} = (z^2 + \nu^2)w
\]  
(26)

The general solution of Eq. 26 is given by
\[
w = A I_\nu(z) + B K_\nu(z)
\]  
(27)

where the functions $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the first and second kinds, respectively. By inspection, our general solution is
\[
\phi_0(z) = A I_\nu(z) + B K_\nu(z)
\]  
(28)

or, substituting $z = \sqrt{\nu^2 r^2}$ (Eq. 20) into Eq. 28 we have
\[
\phi_0(r, \theta) = A I_\nu(\sqrt{\nu^2 r^2}) + B K_\nu(\sqrt{\nu^2 r^2})
\]  
(29)

In order to develop our particular solutions (i.e. to solve for the $A$ and $B$ parameters for each set of boundary conditions), we require the $d\phi_0/dr_0$ term. Using the chain rule we obtain
\[
\frac{d\phi_0}{dr_0} = \frac{\nu z}{det} \frac{d\phi_0}{d\nu}
\]  
(30)

Substituting Eq. 23 into Eq. 30
\[
\frac{d\phi_0}{dr_0} = \frac{\nu z}{det} \frac{d\phi_0}{d\nu}
\]  
(31)

and the $d\phi_0/dz$ term is given by
\[
\frac{d\phi_0}{dz} = A \frac{dI_\nu(z)}{dz} + B \frac{dK_\nu(z)}{dz}
\]  
(32)

From Abramowitz and Stegun, Handbook of Mathematical Functions, we have
\[
\frac{dI_\nu(z)}{dz} = I_{\nu-1}(z) \quad \text{(Eq. 9.6.27, p. 376)}
\]  
(33)

\[
\frac{dK_\nu(z)}{dz} = -K_{\nu-1}(z) \quad \text{(Eq. 9.6.27, p. 376)}
\]  
(34)
Substituting Eqs. 33 and 34 into Eq. 32 we have
\[ \frac{d^2 P}{d \xi^2} = A I_1 (\xi) - B K_1 (\xi) \] (55)

Combining Eqs. 30 and 55, and substituting \( z = \sqrt{\nu} \xi \) (eq. 28) into Eq. 55 we obtain
\[ \frac{d^2 P}{d \xi^2} = A \sqrt{\nu} I_1 (\sqrt{\nu} \xi) - B \sqrt{\nu} K_1 (\sqrt{\nu} \xi) \] (56)

Multiplying through by \( \xi \) gives
\[ \xi \frac{d}{d \xi} \frac{d P}{d \xi} = A \sqrt{\nu} \xi I_1 (\sqrt{\nu} \xi) - B \sqrt{\nu} \xi K_1 (\sqrt{\nu} \xi) \] (57)

Summarizing our efforts so far

General Solution in laplace Domain
\[ P_0 (\xi, \nu) = A I_0 (\sqrt{\nu} \xi) + B K_0 (\sqrt{\nu} \xi) \] (29)

Radial Derivative of the General Solution in Laplace Domain
\[ \left[ \frac{\partial P_0}{\partial \xi} \right] = A \sqrt{\nu} I_0 (\sqrt{\nu} \xi) - B \sqrt{\nu} K_0 (\sqrt{\nu} \xi) \] (57)

Laplace transform of boundary conditions:

Inner boundary condition -
\[ \left[ \frac{\partial P_0}{\partial \xi} \right]_{\xi=1} = 1 \quad (\text{constant rate at well}) \] (14)

Outer boundary conditions -
  a. "infinite-acting" reservoir
\[ P_0 (\xi = \infty, \nu) = 0 \] (15)

  b. "no-flow" outer boundary condition
\[ \left[ \frac{\partial P_0}{\partial \xi} \right]_{\xi = \xi_0} = 0 \] (16)

  c. "constant pressure" outer boundary condition
\[ P_0 (\xi = \xi_0, \nu) = 0 \] (17)

  d. "prescribed flux" outer boundary condition
\[ \left[ \frac{\partial P_0}{\partial \xi} \right]_{\xi = \xi_0} = \bar{P}_{\text{ext}} (\nu) \] (18)

Our goal is to use the boundary conditions to determine the \( A \) and \( B \) parameters. Our first step is to use the constant rate inner boundary condition (Eq. 14) as a starting point then combine this condition with each outer boundary.
condition in order to determine \( A \) and \( B \) for each case.

Starting with the inner boundary condition (Eq. 14) and the derivative of the general solution (Eq. 37) we have
\[
A \sqrt{\nu} I_1(\sqrt{\nu} \mu) - B \sqrt{\nu} K_1(\sqrt{\nu} \mu) = \frac{-1}{\mu}
\]
or
\[
A \sqrt{\nu} I_1(\sqrt{\nu} \mu) - B \sqrt{\nu} K_1(\sqrt{\nu} \mu) = \frac{1}{\mu}
\]

(38)

**Outer Boundary Case 1: Infinite-acting reservoir**

Combining Eqs. 29 and 15 we have
\[
\lim_{\nu \to \infty} \left[ A I_0(\sqrt{\nu} \nu_0) + B K_0(\sqrt{\nu} \nu_0) \right] = 0
\]

(39)

Given that we are taking the limit as \( \nu \to \infty \) we must establish the behavior of \( I_0(x=\infty) \) and \( k_0(x=\infty) \). Considering the behavior of \( I_0(x) \) and \( k_0(x) \) we have

\[
\lim_{x \to \infty} I_0(x) = 0
\]

and

\[
\lim_{x \to \infty} k_0(x) = 0
\]

Since \( I_0(x=\infty)=\infty \), then \( A(\infty) + B(\infty) = 0 \); therefore \( A = 0 \) in order for the solution to be bounded. Setting \( A = 0 \) we solve Eq. 38 for \( B \), which gives
\[
B = \frac{1}{\mu} \frac{1}{\sqrt{\nu} K_1(\sqrt{\nu})}
\]

(40)

and of course
\[
A = 0
\]

(41)

Substituting Eqs. 40 and 41 into the general solution (Eq. 29) we obtain the particular solution for the infinite-acting reservoir case. This result is
\[
\frac{P_0(\nu_0, \mu)}{k_0(\sqrt{\nu_0} \mu)} = \frac{1}{\mu} \frac{k_0(\sqrt{\nu_0} \nu_0)}{\sqrt{\nu} K_1(\sqrt{\nu})}
\]

(42)

Eq. 42 is called the cylindrical source solution.
Unfortunately, Eq. 42 is not readily invertible—therefore we will attempt to reduce Eq. 42 into a more usable form. From Abramowitz and Stegun, Handbook of Mathematical Functions, p. 375, Eq. 9.6.9 (for \( v = 1 \)) we have

\[
k_v(x) = \frac{1}{x}
\]

for \( x \to 0 \)

or, multiplying through by \( x \) we have

\[
xk_v(x) = 1 \quad \text{as} \quad x \to 0
\]

for our case we have

\[
\sqrt{\mu} k_v(\sqrt{\mu}) = 1 \quad \text{for} \quad \sqrt{\mu} \quad \text{(or} \quad \mu \quad \text{)} \to 0
\]

Combining this result with Eq. 42 we obtain

\[
y_v(r, \mu) = \frac{1}{\mu} k_0(\sqrt{\mu} r_0) \quad \text{as} \quad \mu \to 0
\]

Eq. 43 is called the line source solution and can be inverted directly.

Eq. 43 can be reduced further to yield a logarithmic relation that is commonly referred to as the "log approximation." In order to develop this result we require an approximation for \( k_0(x) \) as \( x \to 0 \). From Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.15, p. 375) we have

\[
k_0(x) = -\left[ \ln \left( \frac{x}{2} \right) + y \right] I_0(x) + \frac{1}{(1l)^2} \left[ \frac{1}{4} x^2 \right] + \left( 1 + \frac{1}{2} \right) \frac{1}{(2l)^2} \left[ \frac{1}{4} x^2 \right] + \ldots +
\]

where we note that as \( x \to 0 \), then \( x^2 \to 0 \), which reduces to

\[
k_0(x) \approx \ln \left( \frac{2}{x} \right) I_0(x) \quad \text{as} \quad x \to 0
\]

or multiplying and dividing by \( z \) we have

\[
k_0(\mu) \approx \frac{1}{z} \ln \left( \frac{4}{e^2} \frac{1}{x^2} \right) I_0(x) \quad \text{as} \quad x \to 0
\]

(49)

The behavior of \( I_0(x) \) in the vicinity of \( x \to 0 \) is obtained using the series representation provided in Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.17, p. 375). This expression is

\[
I_0(x) = 1 + \frac{1}{(1l)^2} \left[ \frac{1}{4} x^2 \right] + \frac{1}{(2l)^2} \left[ \frac{1}{4} x^2 \right]^2 + \frac{1}{(3l)^2} \left[ \frac{1}{4} x^2 \right]^3 + \ldots +
\]

where as \( x \to 0 \) we have

\[
I_0(x) = 1 \quad \text{as} \quad x \to 0
\]

(45)
Combining Eqs. 44 and 45
\[ k_0(x) \propto \frac{1}{z} \ln \left( \frac{4}{e^{2\pi} x^2} \right) \quad \text{as } x \to 0 \] (46)

Substituting Eq. 46 into Eq. 43 we obtain
\[ \bar{p}_0(r, \mu) = \frac{1}{2m} \ln \left( \frac{4}{e^{2\gamma} r^2 \mu} \right) \quad \text{as } \mu \to 0 \] (47)

or in a form more amenable to inversion we have
\[ \bar{p}_0(r, \mu) = \frac{-1}{2m} \ln (\mu) + \frac{1}{2m} \ln \left( \frac{4}{e^{2\gamma} r^2 \mu} \right) \] (48)

**Outer Boundary Case 2: No-Flow outer boundary**

Combining Eqs. 37 and 16 we obtain
\[ A \sqrt{v} r_0 I_0 \left( \sqrt{v} r_0 \right) - B \sqrt{v} r_0 k_1 \left( \sqrt{v} r_0 \right) = 0 \] (49)

Solving for the B parameter we obtain
\[ B = \frac{A I_0 \left( \sqrt{v} r_0 \right)}{k_1 \left( \sqrt{v} r_0 \right)} \] (50)

Recalling the inner boundary condition, Eq. 38, we have
\[ A \sqrt{v} I_1 \left( \sqrt{v} r_0 \right) - B \sqrt{v} k_1 \left( \sqrt{v} r_0 \right) = \frac{-1}{\mu} \] (58)

Combining Eqs. 50 and 58 we obtain
\[ A \sqrt{v} I_1 \left( \sqrt{v} r_0 \right) - A \sqrt{v} k_1 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right) = \frac{-1}{\mu} k_1 \left( \sqrt{v} r_0 \right) \]

or solving for A we have
\[ A = \frac{k_1 \left( \sqrt{v} r_0 \right)}{\mu \left[ \sqrt{v} k_1 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right) - \sqrt{v} I_1 \left( \sqrt{v} r_0 \right) k_1 \left( \sqrt{v} r_0 \right) \right]} \] (51)

Substituting Eq. 51 into Eq. 50 we obtain
\[ B = \frac{I_0 \left( \sqrt{v} r_0 \right)}{\mu \left[ \sqrt{v} k_1 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right) - \sqrt{v} I_1 \left( \sqrt{v} r_0 \right) k_1 \left( \sqrt{v} r_0 \right) \right]} \] (52)

Substituting Eqs. 51 and 52 into the general solution, Eq. 29, gives
\[ \bar{p}_0(r, \mu) = \frac{k_0 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right) + k_1 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right)}{\mu \left[ \sqrt{v} k_1 \left( \sqrt{v} r_0 \right) I_0 \left( \sqrt{v} r_0 \right) - \sqrt{v} I_1 \left( \sqrt{v} r_0 \right) k_1 \left( \sqrt{v} r_0 \right) \right]} \] (53)
As before, in the case of an infinite-acting reservoir, we showed \( \sqrt{m} I_1(\sqrt{m}) = 1 \) as \( m \to 0 \).

Similarly from Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.7, p. 375) we have
\[
I_1(x) \approx \frac{1}{2} x
\]
or
\[
 x I_1(x) \approx \frac{1}{2} x^2
\]
where as \( x \to 0 \) we have
\[
x I_1(x) = 0 \quad \text{as} \quad x \to 0
\]
or for our present problem we have
\[
\sqrt{m} I_1(\sqrt{m}) = 0 \quad \text{as} \quad m \to 0
\]
Combining these relations with Eq. 58 we obtain
\[
\frac{\tilde{p}_0(r_0, m)}{\tilde{p}_0(r_0, m)} = \frac{1}{m} k_0(\sqrt{m} r_0) + \frac{1}{m} \frac{k_1(\sqrt{m} r_0)}{I_0(\sqrt{m} r_0)} I_0(\sqrt{m} r_0) \quad (64)
\]

**Outer Boundary Case 3: Constant pressure outer boundary**

Combining Eqs. 29 and 17 we have
\[
A I_0(\sqrt{m} r_0) + B k_0(\sqrt{m} r_0) = 0 \quad (55)
\]
Solving for the \( B \) parameter we obtain
\[
B = -A \frac{I_0(\sqrt{m} r_0)}{k_0(\sqrt{m} r_0)} \quad (56)
\]
Recalling the inner boundary condition, Eq. 38, gives us
\[
A \sqrt{m} I_1(\sqrt{m}) - B \sqrt{m} k_1(\sqrt{m}) = -\frac{1}{m} \quad (58)
\]
Substituting Eq. 56 into Eq. 38 we have
\[
A \sqrt{m} I_1(\sqrt{m}) + A \sqrt{m} k_1(\sqrt{m}) I_0(\sqrt{m} r_0) = -\frac{1}{m} \frac{k_0(\sqrt{m} r_0)}{k_0(\sqrt{m} r_0)} \quad (58)
\]
or
\[
A \left[ \sqrt{m} I_1(\sqrt{m}) k_0(\sqrt{m} r_0) + \sqrt{m} k_1(\sqrt{m}) I_0(\sqrt{m} r_0) \right] = -\frac{1}{m} \frac{k_0(\sqrt{m} r_0)}{k_0(\sqrt{m} r_0)}
\]
or solving for \( A \) we have
\[
A = \frac{-k_0(\sqrt{m} r_0)}{m \left[ \sqrt{m} I_1(\sqrt{m}) k_0(\sqrt{m} r_0) + \sqrt{m} k_1(\sqrt{m}) I_0(\sqrt{m} r_0) \right]} \quad (57)
\]
Substituting Eq. 57 into Eq. 56 gives
\[
B = \frac{I_0(\sqrt{m} r_0)}{m \left[ \sqrt{m} I_1(\sqrt{m}) k_0(\sqrt{m} r_0) + \sqrt{m} k_1(\sqrt{m}) I_0(\sqrt{m} r_0) \right]} \quad (58)
\]
Substituting Eqs. 57 and 58 into the general solution, Eq. 29, we obtain
\[ \bar{\rho}_0(r_0, \mu) = \frac{k_0(\mu r_0) \bar{I}_0(\mu r_0) - k_0(\mu r_{\text{rep}}) \bar{I}_0(\mu r_{\text{rep}})}{m [\bar{I}_0(\mu r_{\text{rep}}) + \sqrt{\lambda} I_1(\mu) k_0(\mu r_{\text{rep}})]} \]  
(57)

As in the previous cases, we want to consider the behavior as \( \mu \to 0 \) (large \( r_0 \)). As before, we have
\[ \sqrt{\lambda} I_1(\mu) = 1 \quad \text{as} \quad \mu \to 0 \]
\[ \sqrt{\lambda} I_0(\mu) = 0 \quad \text{as} \quad \mu \to 0 \]

Combining these relations with Eq. 59, we obtain
\[ \bar{\rho}_0(r_0, \mu) = \frac{1}{m} \frac{k_0(\mu r_0) - k_0(\mu r_{\text{rep}})}{I_0(\mu r_{\text{rep}})} \quad \text{(as} \quad \mu \to 0) \]  
(60)

**Outer Boundary Case 4: Prescribed flux outer boundary**

Combining Eqs. 57 and 58 we obtain
\[ A \sqrt{\lambda} r_{\text{rep}} I_1(\mu r_{\text{rep}}) - B \sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}}) = \bar{\rho}_{\text{ext}} \]
(61)

Recalling the inner boundary condition, Eq. 58, we have
\[ A \sqrt{\lambda} I_1(\mu) = -1 \]
(58)

We will solve Eqs. 61 and 58 simultaneously to determine \( A \) and \( B \). The algebra becomes a bit tedious, but we will show all steps. Solving for the \( A \) parameter we divide through Eq. 61 by \( \sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}}) \), then we divide through Eq. 58 by \( \sqrt{\lambda} k_1(\mu) \). These operations give
\[ \frac{A \sqrt{\lambda} r_{\text{rep}} I_1(\mu r_{\text{rep}})}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})} - \frac{B}{\sqrt{\lambda} k_1(\mu)} = \frac{\bar{\rho}_{\text{ext}}}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})} \]
\[ \frac{A}{\sqrt{\lambda} k_1(\mu)} - B = \frac{-1}{m} \frac{1}{\sqrt{\lambda} k_1(\mu)} \]
(62)

Subtracting Eq. 63 from Eq. 62 we have
\[ A \left[ \frac{\sqrt{\lambda} r_{\text{rep}} I_1(\mu r_{\text{rep}})}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})} - \frac{\sqrt{\lambda} I_1(\mu)}{\sqrt{\lambda} k_1(\mu)} \right] = \bar{\rho}_{\text{ext}} \frac{1}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})} + \frac{1}{m} \frac{1}{\sqrt{\lambda} k_1(\mu)} \]
Expanding to yield a uniform denominator on both sides
\[ A \left[ \frac{\sqrt{\lambda} r_{\text{rep}} I_1(\mu r_{\text{rep}})}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})} - \frac{\sqrt{\lambda} I_1(\mu)}{\sqrt{\lambda} k_1(\mu)} \right] = \frac{\bar{\rho}_{\text{ext}} \sqrt{\lambda} k_1(\mu) + \sqrt{\lambda} \sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}})}{\sqrt{\lambda} r_{\text{rep}} k_1(\mu r_{\text{rep}}) \sqrt{\lambda} k_1(\mu)} \]
Solving for \( A \) we have:

\[
A = \frac{\frac{1}{m} \sqrt{\bar{v}} \bar{W} \bar{D} k_1(\bar{v}_R \bar{D}) + \bar{F}_{\text{ext}} \sqrt{\bar{v}} k_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) \sqrt{\bar{v}_R} I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) \sqrt{\bar{v}_R} k_1(\bar{v}_R)}
\]

Factoring out the \( \sqrt{\bar{v}_R} \) terms and bringing out the \( \sqrt{\bar{v}} \) factor:

\[
A = \frac{1}{m} \frac{k_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) k_1(\bar{v}_R)}
\]

Comparing Eq. 64 to the result for the no-flow boundary case (Eq. 51) we have:

\[
A_{nf} = \frac{1}{m} \frac{k_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) k_1(\bar{v}_R)}
\]

where we find that Eq. 64 is identical to Eq. 51 for the \( q_{\text{ext}} = 0 \) case.

Solving for the \( B \) parameter we divide through Eq. 64 by \( \sqrt{\bar{v}_R} I_1(\bar{v}_R) \) and we divide through Eq. 38 by \( \sqrt{\bar{v}} I_1(\bar{v}_R) \), which gives:

\[
A = -B \sqrt{\bar{v}_R} k_1(\bar{v}_R) = \frac{\bar{F}_{\text{ext}}}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} \frac{1}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)}
\]

\[
A = -B \sqrt{\bar{v}} k_1(\bar{v}_R) = -\frac{1}{m} \frac{1}{\sqrt{\bar{v}} I_1(\bar{v}_R)}
\]

Subtracting Eq. 66 from Eq. 65 we have:

\[
B \left[ \frac{-\sqrt{\bar{v}_R} k_1(\bar{v}_R) + \sqrt{\bar{v}} k_1(\bar{v}_R)}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} \right] = \frac{\bar{F}_{\text{ext}}}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} - \frac{1}{m} \frac{1}{\sqrt{\bar{v}} I_1(\bar{v}_R)}
\]

Expanding to yield a uniform denominator on both sides gives:

\[
B \left[ \frac{-\sqrt{\bar{v}_R} k_1(\bar{v}_R) + \sqrt{\bar{v}} k_1(\bar{v}_R)}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} \right] = \left[ \frac{\bar{F}_{\text{ext}} \sqrt{\bar{v}} I_1(\bar{v}_R)}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} + \frac{m \sqrt{\bar{v}_R} I_1(\bar{v}_R)}{\sqrt{\bar{v}_R} I_1(\bar{v}_R)} \right]
\]

Solving for \( B \) we have:

\[
B = \frac{\frac{1}{m} \sqrt{\bar{v}_R} k_1(\bar{v}_R) \sqrt{\bar{v}_R} I_1(\bar{v}_R) - \sqrt{\bar{v}_R} I_1(\bar{v}_R) \sqrt{\bar{v}_R} k_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) \sqrt{\bar{v}_R} I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) \sqrt{\bar{v}_R} k_1(\bar{v}_R)}
\]

As before, factoring out the \( \sqrt{\bar{v}_R} \) terms and bringing out the \( \sqrt{\bar{v}} \) factor:

\[
B = \frac{1}{m} \frac{I_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) k_1(\bar{v}_R)}
\]

Comparing Eq. 67 with the result for the no-flow boundary case we recall Eq. 52:

\[
B_{nf} = \frac{1}{m} \frac{I_1(\bar{v}_R)}{\sqrt{\bar{v}} k_1(\bar{v}_R) I_1(\bar{v}_R) - \sqrt{\bar{v}} I_1(\bar{v}_R) k_1(\bar{v}_R)}
\]
where we find that Eq. 67 is identical to Eq. 52 for $q_{ext} = 0$. Having shown this for both A and B we have verified these results.

In order to determine the particular solution for this case, we substitute Eqs. 64 and 67 into the general solution (Eq. 29). This gives

$$\tilde{f}_B(r_0, m) = \frac{1}{m} \left( \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} I_1(\sqrt{r_0}) + I_0(\sqrt{r_0}) k_1(\sqrt{r_0}) \right)$$

$$+ \frac{1}{m} \tilde{f}_{D, ext} \left( \frac{m}{\sqrt{r_0}} \right) \left( \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} I_1(\sqrt{r_0}) + I_0(\sqrt{r_0}) k_1(\sqrt{r_0}) \right)$$

(68)

where the first part of Eq. 68 is exactly Eq. 53, the solution for the no-flow boundary case (i.e., $\tilde{f}_{D, ext} = 0$). Note that $\tilde{f}_{D, ext} = \tilde{f}(q_{ext} / r_0)$.

As with the previous cases we can consider the behavior of Eq. 68 as $m \to 0$. As we saw before

$$\sqrt{r_0} I_1(\sqrt{r_0}) = 1 \text{ as } m \to 0$$

$$\sqrt{r_0} k_1(\sqrt{r_0}) = 0 \text{ as } m \to 0$$

Combining these relations with Eq. 68 gives

$$\tilde{f}_B(r_0, m) = \frac{1}{m} k_0(\sqrt{r_0}) + \frac{1}{m} k_1(\sqrt{r_0}) I_0(\sqrt{r_0})$$

$$+ \frac{1}{m} \tilde{f}_{D, ext} \left( \frac{m}{\sqrt{r_0}} \right) \left( \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} I_1(\sqrt{r_0}) + I_0(\sqrt{r_0}) k_1(\sqrt{r_0}) \right)$$

(69)

The second term in Eq. 68 (or 69) should not be arbitrarily reduced without a comprehensive study of the interplay of individual terms. For example, reduction using the behavior of $I_1(x)$ and $k_1(x)$ as $x \to 0$ yields $I_0(\sqrt{r_0}) I_1(\sqrt{r_0})$, which tends to $1/\sqrt{\pi}$ as $m \to 0$. Considerable care must be exercised when making such reductions.
References — Radial Flow Solutions:

THE APPLICATION OF THE LAPLACE TRANSFORMATION TO FLOW PROBLEMS IN RESERVOIRS

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ABSTRACT

For several years the authors have felt the need for a source from which reservoir engineers could obtain fundamental theory and data on the flow of fluids through permeable media in the unsteady state. The data on the unsteady state flow are composed of solutions of the equation

$$\frac{\partial^2 P}{\partial t^2} + \frac{1}{r} \frac{\partial P}{\partial r} = \frac{\partial P}{\partial t}$$

Two sets of solutions of this equation are developed, namely, for "the constant terminal pressure case" and "the constant terminal rate case." In the constant terminal pressure case the pressure at the terminal boundary is lowered by unity at zero time, kept constant thereafter, and the cumulative amount of fluid flowing across the boundary is computed, as a function of the time. In the constant terminal rate case a unit rate of production is made to flow across the terminal boundary (from time zero onward) and the ensuing pressure drop is computed as a function of the time. Considerable effort has been made to compile complete tables from which curves can be constructed for the constant terminal pressure and constant terminal rate cases, both for finite and infinite reservoirs. These curves can be employed to reproduce the effect of any pressure or rate history encountered in practice.

Most of the information is obtained by the help of the Laplace transformations, which proved to be extremely helpful for analyzing the problems encountered in fluid flow. The application of this method simplifies the more tedious mathematical analyses employed in the past. With the help of Laplace transformations some original developments were obtained (and presented) which could not have been easily foreseen by the earlier methods.

INTRODUCTION

This paper represents a compilation of the work done over the past few years on the flow of fluid in porous media. It concerns itself primarily with the transient conditions prevailing in oil reservoirs during the time they are produced. The study is limited to conditions where the flow of fluid obeys the diffusivity equation. Multiple-phase fluid flow has not been considered.

A previous publication by Hurst1 shows that when the pressure history of a reservoir is known, this information can be used to calculate the water influx, an essential term in the material balance equation. An example is offered in the literature by Old2 in the study of the Jones Sand, Schuler Field, Arkansas. The present paper contains extensive tabulated data (from which work curves can be constructed), which data are derived by a more rigorous treatment of the subject matter than available in an earlier publication.3 The application of this information will enable those concerned with the analysis of the behavior of a reservoir to obtain quantitatively correct expressions for the amount of water that has flowed into the reservoirs, thereby satisfying all the terms that appear in the material balance equation. This work is likewise applicable to the flow of fluid to a well whenever the flow conditions are such that the diffusivity equation is obeyed.

DIFFUSIVITY EQUATION

The most commonly encountered flow system is radial flow toward the well bore or field. The volume of fluid which flows per unit of time through each unit area of sand is expressed by Darcy's equation as

$$v = \frac{K}{\mu} \frac{\partial P}{\partial r}$$

where $K$ is the permeability, $\mu$ the viscosity and $\partial P/\partial r$ the pressure gradient at the radial distance $r$. A material balance on a concentric element $AB$, expresses the net fluid traversing the surfaces $A$ and $B$, which must equal the fluid lost from within the element. Thus, if the density of the fluid is expressed by $\rho$, then the weight of fluid per unit time and per unit sand thickness, flowing past Surface $A$, the surface nearest the well bore, is given as

$$2\pi r \rho \frac{K}{\mu} \frac{\partial P}{\partial r} = 2\pi \rho \left( \frac{\partial P}{\partial r} \right)$$

The weight of fluid flowing past Surface $B$, an infinitesimal distance $dr$, removed from Surface $A$, is expressed as

$$2\pi \rho \left[ \frac{\partial P}{\partial r} + \frac{\partial \left( \frac{\partial P}{\partial r} \right)}{\partial r} dr \right]$$

References are given at end of paper.

Manuscript received at office of Petroleum Branch January 12, 1949.

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The difference between these two terms, namely,
\[ -2rK \frac{\partial}{\partial r} (\rho \frac{\partial P}{\partial r}) \]
is equal to the weight of fluid lost by the element AB, or
\[ -2r \rho \frac{\partial P}{\partial T} \]
where \( f \) is the porosity of the formation.

This relation gives the equation of continuity for the radial system, namely,
\[ K \frac{\partial}{\partial r} (\rho \frac{\partial P}{\partial r}) = f_r \frac{\partial \rho}{\partial T} \]  

From the physical characteristics of fluids, it is known that density is a function of pressure and that the density of a fluid decreases with decreasing pressure due to the fact that the fluid expands. This trend expressed in exponential form is
\[ \rho = A e^{-c(P - P_i)} \]  

where \( P \) is less than \( P_i \), and \( A \) is the compressibility of the fluid. If we substitute Eq. II-2 in Eq. II-1, the diffusivity equation can be expressed using density as a function of radius and time, or
\[ \left( \frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} \right) \frac{K}{\rho} = \frac{\partial^2 P}{\partial T} \]  

For liquids which are only slightly compressible, Eq. II-2 simplifies to \( \rho = \rho_0 [1 - \epsilon (P - P_i)] \) which further modifies Eq. II-3 to give
\[ \left( \frac{\partial^2 P}{\partial t^2} + \frac{1}{r} \frac{\partial P}{\partial r} \right) \frac{K}{\rho_0} = \frac{\partial^2 P}{\partial T} \]. Furthermore, if the radius of the well or field, \( R_w \), is referred to as a unit radius, then the relation simplifies to
\[ \frac{\partial^2 P}{\partial t^2} + \frac{1}{r} \frac{\partial P}{\partial r} = \frac{\partial^2 P}{\partial r^2} \]  

where \( t = KT/\rho_0 R_w ^2 \) and \( r \) now expresses the distance as a multiple of \( R_w \), the unit radius. The units appearing in this paper are always used in connection with Darcy's equation, so that the permeability \( K \) must be expressed in darcys, the time \( T \) in seconds, the porosity \( f \) as a fraction, the viscosity \( \mu \) in centipoises, the compressibility \( c \) as volume per volume per atmosphere, and the radius \( R_w \) in centimeters.

**LAPLACE TRANSFORMATION**

In all publications, the treatment of the diffusivity equation has been essentially the orthodox application of the Fourier-Bessel series. This paper presents a new approach to the solution of problems encountered in the study of flowing fluids, namely, the Laplace transformation, since it was recognized that Laplace transformations offer a useful tool for solving difficult problems in less time than by the use of Fourier-Bessel series. Also, original developments have been obtained which are not easily foreseen by the orthodox methods.

If \( P_{(t)} \) is a pressure at a point in the sand and a function of time, then its Laplace transformation is expressed by the infinite integral
\[ \bar{P}_{(s)} = \int_0^\infty e^{-st} P_{(t)} \, dt \]  

where the constant \( p \) in this relationship is referred to as the operator. If we treat the diffusivity equation by the process implied by Eq. III-1, the partial differential can be transformed to a total differential equation. This is performed by multiplying each term in Eq. II-4 by \( e^{-st} \) and integrating with respect to time between zero and infinity, as follows:
\[ \int_0^\infty e^{-st} \left( \frac{\partial^2 P}{\partial t^2} + \frac{1}{r} \frac{\partial P}{\partial r} \right) \, dt = \int_0^\infty e^{-st} \frac{\partial P}{\partial t} \, dt \]  

Since \( P \) is a function of radius and time, the integration with respect to time will automatically remove the time function and leave \( P \) a function of radius only. This reduces the left side to a total differential with respect to \( r \), namely,
\[ \int_0^\infty e^{-st} \frac{\partial P}{\partial r} \, dt = \int_0^\infty e^{-st} \frac{\partial^2 P}{\partial r^2} \, dt = \int_0^\infty \frac{dP}{dr} \, dt \]

and Eq. III-2 becomes
\[ \frac{d\bar{P}_{(s)}}{dr} + \frac{1}{r} \frac{d\bar{P}_{(s)}}{dr} = \int_0^\infty e^{-st} \frac{dP}{dt} \, dt \]

**FIG. 1A — SEQUENCE CONSTANT TERMINAL Pressures.**
**FIG. 1B — SEQUENCE CONSTANT TERMINAL Rates.**
Furthermore, if we consider that \( P_{\text{w}} \) is a cumulative pressure drop, and that initially the pressure in the reservoir is everywhere constant so that the cumulative pressure drop \( P_{\text{w}} = 0 \), the integration of the right hand side of the equation becomes

\[
\frac{dP}{dt} + \frac{1}{r} \frac{dP}{dr} = 0
\]

As this term is also a Laplace transform, Eq. III-2 can be written as a total differential equation, or

\[
\frac{dP_{\text{w}}}{dz} + \frac{1}{r} \frac{dP_{\text{w}}}{dr} = 0
\]

FIG. 2 - CONTOUR INTEGRATION IN ESTABLISHING THE CONSTANT TERMINAL RATE CASE FOR INFINITE EXTENT.

The next step in the development is to reproduce the boundary condition at the well bore or field radius, \( r = 1 \), as a Laplace transformation and introduce this in the general solution for Eq. III-3 to give an explicit relation

\[
\bar{P}_{\text{w}} = f(t)
\]

By inverting the term on the right by the Mellin's inversion formula, or other methods, we obtain the solution for the cumulative pressure drop as an explicit function of radius and time.

**ENGINEERING CONCEPTS**

Before applying the Laplace transformation to develop the necessary work-curves, there are some fundamental engineering concepts to be considered that will allow the interpretation of these curves. Two cases are of paramount importance in making reservoir studies, namely, the constant terminal pressure case and the constant terminal rate case. If we know the explicit solution for the first case, we can reproduce any variable pressure history at the terminal boundary to determine the cumulative influx of fluid. Likewise, if the rate of fluid influx varies, the constant terminal rate case can be used to calculate the total pressure drop. The constant terminal pressure and the constant terminal rate case are not independent of each other, as knowing the operational form of one, the other can be determined, as will be shown later.

**Constant Terminal Pressure Case**

The constant terminal pressure case is defined as follows: At time zero the pressure at all points in the formation is constant and equal to unity, and when the well or reservoir is opened, the pressure at the well or reservoir boundary, \( r = 1 \), immediately drops to zero and remains zero for the duration of the production history.

If we treat the constant terminal pressure case symbolically, the solution of the problem at any radius and time is given by \( P = P(r, t) \). The rate of fluid influx per unit sand thickness under these conditions is given by Darcy's equation

\[
q_{\text{r}} = \frac{2\pi K}{
\frac{dP}{dr} \bigg| r = 1
\}
\]

If we wish to determine the cumulative influx of fluid in absolute time \( T \), and having expressed time in the diffusivity equation as \( t = KT/\mu \), then

\[
Q_{\text{r}} = \int q_{\text{r}} \, dT = \int \frac{2\pi K \mu}{\rho \phi R_s^2} \left( \frac{\partial P}{\partial r} \bigg| r = 1 \right) \, dt
\]

FIG. 3 - CONTOUR INTEGRATION IN ESTABLISHING THE CONSTANT TERMINAL RATE CASE FOR LIMITED RESERVOIR.

where

\[
Q_{\text{r}} = \int \left( \frac{\partial P}{\partial r} \right) \bigg| r = 1 \, dt
\]

In brief, knowing the general solution implied by Eq. IV-3, which expresses the integration in dimensionless time, \( t \), of the pressure gradient at radius unity for a pressure drop of one atmosphere, the cumulative influx into the well bore or into the oil-bearing portion of the field can be determined by Eq. IV-2. Furthermore, for any pressure drop, \( \Delta P \), Eq. IV-2 expresses the cumulative influx as

\[
Q_{\text{r}} = 2\pi \mu \phi R_s^2 \Delta P Q_{\text{r}}
\]

per unit sand thickness.*

---

*The set of symbols now introduced and the symbols reported in Nummelin's earlier paper on water-drive are related as follows:

\[
G(a^2 \theta /R^2) = Q_{\text{r}} \quad \text{and} \quad G(a^2 \theta /R^2) = \int Q_{\text{r}} \, dt
\]

where

\[
d = a^2 \theta /R^2 = 1
\]
When an oil reservoir and the adjoining water-bearing formations are contained between two parallel and sealing faulting planes, the flow of fluid is essentially parallel to these planes and is "linear." The constant terminal pressure case can also be applied to this case. The basic equation for linear flow is given by

\[
\frac{\partial P}{\partial x} = \frac{Q_{t\alpha}}{\mu} \quad \text{...(IV-5)}
\]

where now \(t = K/T/\omega c \) and \(x \) is the absolute distance measured from the plane of influx extending out into the water-bearing sand. If we assume the same boundary conditions as in radial flow, with \(P = P(x,t) \) as the solution, then by Darcy's law, the rate of fluid influx across the original water-oil contact per unit of cross-sectional area is expressed by

\[
Q_{t\alpha} = \frac{K}{\mu} \left( \frac{\partial P}{\partial x} \right)_{x=0} \quad \text{...(IV-6)}
\]

The total fluid influx is given by

\[
Q_{t\alpha} = \int_0^T \int_0^\infty \left( \frac{\partial P}{\partial x} \right)_{x=0} \, dt \quad \text{...(IV-7)}
\]

where \(Q_{t\alpha} \) is the general solution for linear flow and is equal to

\[
Q_{t\alpha} = \int_0^T \left( \frac{\partial P}{\partial x} \right)_{x=0} \, dt \quad \text{...(IV-8)}
\]

Therefore, for any over-all pressure drop \(\Delta P\), Eq. IV.7 gives

\[
Q_{t\alpha} = \frac{\Delta P}{\mu} Q_{(t)} \quad \text{...(IV-9)}
\]

per unit of cross-sectional area.

### Constant Terminal Rate Case

In the constant terminal rate case it is likewise assumed that initially the pressure everywhere in the formation is constant but that from the time zero onward the fluid is withdrawn from the well bore or reservoir boundary at a unit rate. The pressure drop is given by \(P = P_{t\alpha 0}\), and at the boundary of the field, where \(r = 1\), \(\partial P/\partial r\)\(r=1\) = -1. The minus sign is introduced because the gradient for the pressure drop relative to the radius of the well or reservoir is negative. If the cumulative pressure drop is expressed as \(\Delta P\), then

\[
\Delta P = Q_{(t)} P_{t\alpha 0} \quad \text{...(IV-10)}
\]

where \(Q_{(t)}\) is a constant relating the cumulative pressure drop with the pressure change for a unit rate of production. By applying Darcy's equation for the rate of fluid flowing into the well or reservoir per unit sand thickness

\[
Q_{(t)} = \frac{-2\pi K}{\mu} \left( \frac{\partial P}{\partial r} \right)_{r=1} = \frac{-2\pi K}{\mu} \frac{Q_{(t)}}{r=1} = \frac{-2\pi K}{\mu} \frac{Q_{t\alpha 0}}{r=1} \quad \text{...(IV-11)}
\]

which simplifies to \(Q_{(t)} = \frac{Q_{t\alpha 0}}{r=1}\). Therefore, for any constant rate of production the cumulative pressure drop at the field radius is given by

\[
\Delta P = \frac{Q_{t\alpha 0}}{2\pi K} P_{t\alpha 0} \quad \text{...(IV-12)}
\]

Similarly, for the constant rate of production in linear flow, the cumulative pressure drop is expressed by

\[
\Delta P = \frac{Q_{t\alpha 0}}{K} P_{t\alpha 0} \quad \text{...(IV-13)}
\]

where \(Q_{t\alpha 0}\) is the rate of water encroachment per unit area of cross-section, and \(P_{t\alpha 0}\) is the cumulative pressure drop at the sand face per unit rate of production.

### Superposition Theorem

With these fundamental relationships available, it remains to be shown how the constant pressure case can be interpreted for variable terminal pressures, or in the constant rate case, for variable rates. The linearity of the diffusivity equation allows the application of the superposition theorem as a sequence of constant terminal pressures or constant rates in such a fashion that it reproduces the pressure or production history at the boundary, \(r = 1\). This is essentially Duhamel's principle, for which reference can be made to transient electric circuit theory in texts by Karman and Blot, and Bush. It has been applied to flow of fluids by Muskat, Schilthuis, and Hurst, in employing the variable rate case in calculating the pressure drop in the East Texas Field.

The physical significance can best be realized by an application. Fig. 1-A shows the pressure decline in the well bore or a field that has been flowing and for which we wish to obtain the amount of fluid produced. As shown, the pressure history is reproduced as a series of pressure plateaus which represent a sequence of constant terminal pressures. Therefore, by the application of Eq. IV.4, the cumulative fluid produced in time \(t\) by the pressure drop \(\Delta P_{t\alpha}\) operative since zero time, is expressed by \(Q_{t\alpha} = 2\pi K r_0^2 \Delta P_{t\alpha} Q_{(t)}\). If we next consider

![Graph showing pressure decline and cumulative production vs. time](image)
the pressure drop $\Delta P$, which occurs in time $t$, and treat this as a separate entity, but take cognizance of its time of inception $t$, then the cumulative fluid produced by this increment of pressure drop is $Q_{ct} = 2\pi R^2 \Delta P \cdot Q_{ct-1}$. By superimposing all these effects of pressure changes, the total influx in time $t$ is expressed as

$$Q_{ct} = 2\pi R^2 \int_0^t \frac{d\Delta P}{dt} Q_{ct-t} \, dt' \quad \ldots \quad (IV.13)$$

when $t > t$. To reproduce the smooth curve relationship of Fig. 1-A, these pressure plateaus can be taken as infinitesimally small, which give the summation of Eq. IV.13 by the integral

$$Q_{ct} = 2\pi R^2 \int_0^t \frac{d\Delta P}{dt} Q_{ct-t} \, dt' \quad \ldots \quad (IV.14)$$

By considering variable rates of fluid production, such as shown in Fig. 1-B, and reproducing these rates as a series of constant rate plateaus, then by Eq. IV.11 the pressure drop in the well bore in time $t$, for the initial rate $q_0$ is $\Delta P = q_0 \cdot P_{ct}$. At time $t_n$, the comparable increment for constant rate is expressed as $q_n = q_0$, and the effect of this increment rate on the corresponding increment of pressure drop is $\Delta P = (q_n - q_0) \cdot P_{ct-n}$. Again by superimposing all of these effects, the determination for the cumulative pressure drop is expressed by

$$\Delta P = q_{ct} \cdot P_{ct} + [q_{ct} - q_0] P_{ct-t} + [q_{ct-t} - q_0] P_{ct-t'} + \ldots$$

$$P_{ct-t} + [q_{ct-t} - q_0] P_{ct-t'} + \ldots \quad (IV.15)$$

If the increments are infinitesimal, or the smooth curve relationship applies, Eq. IV.15 becomes

$$\Delta P = q_{ct} \cdot P_{ct} + \int_0^t \frac{d}{dt} q_{ct} P_{ct-t} \, dt' \quad \ldots \quad (IV.16)$$

If $q_{ct} = 0$, Eq. IV.16 can also be expressed as

$$\Delta P = \int_0^t q_{ct} P_{ct-t} \, dt' \quad \ldots \quad (IV.17)$$

where $P_{ct}$ is the derivative of $P_{ct}$ with respect to $t$.

Since Eqs. IV.13 and IV.15 are of such simple algebraic forms, they are most practical to use with production history in making reservoir studies. In applying the pressure or rate plateaus as shown in Fig. 1, it must be realized that the time interval for each plateau should be taken as small as possible, so as to reproduce within engineering accuracy the trend of the curves. Naturally, if an exact interpretation is desired, Eqs. IV.14 and IV.16 apply.

**FUNDAMENTAL CONSIDERATIONS**

In applying the Laplace transformation, there are certain fundamental operations that must be clarified. It has been stated that if $P_{ct}$ is a pressure drop, the transformation for $P_{ct}$ is given by Eq. III-3, as

$$\overline{P}_{ct} = \int_0^\infty e^{-st} P_{ct} \, dt$$

To visualize more concretely the meaning of this equation, if the unit pressure drop at the boundary in the constant terminal pressure case is employed in Eq. III-1, its transform is given by

$$\overline{P}_{et} = \int_0^\infty e^{-st} 1 \, dt = \frac{e^{-st}}{p} \bigg|_0^\infty = \frac{1}{p} \quad (V.1)$$

The Laplace transformations of many transcendental functions have been developed and are available in tables, the most complete of which is the tract by Campbell and Foster. It is therefore often possible after solving a total differential such as Eq. III-3 to refer to a set of tables and transforms and determine the inverse of $\overline{P}_{et}$ or $P_{ct}$. It is frequently necessary to simplify $\overline{P}_{et}$ before an inversion can be made. However, Mellin's inversion formula is always applicable, which requires analytical treatment whenever used.

There are two possible simplifications for $\overline{P}_{et}$ when time is small or time is large. This is evident from Eq. III-3, where $p$ can be interpreted by the operational calculus as the operator $dt/dt$. Therefore, if we consider this symbolic relation, then if $t$ is large, $p$ must be small, or inversely, if $t$ is small, $p$ will be large. To understand this, if $\overline{P}_{et}$ is expressed by an involved Bessel relationship, the substitution for $p$ as a small or large value will simplify $\overline{P}_{et}$ to give $P_{ct}$ for the corresponding times.

Mellin's inversion formula is given on page 71 of Carslaw and Jaeger.
THE APPLICATION OF THE LAPLACE TRANSFORMATION TO FLOW PROBLEMS IN RESERVOIRS

\[ P(t) = \frac{1}{2\pi i} \int \frac{e^{\lambda t} P(\lambda)}{\gamma - \lambda i} \, d\lambda \]

where \( \overline{P}(\lambda) \) is the transform of \( P(\lambda) \). Where this report is concerned with pressure drops, the above can be written as

\[ \gamma + i\infty \]

\[ P(t) = \frac{1}{2\pi i} \int \frac{(e^{\lambda t} - e^{\lambda_0 t}) \overline{P}(\lambda)}{\gamma - \lambda i} \, d\lambda \, . \quad (V-2) \]

The integration is in the complex plane \( \lambda = x + iy \), along a line parallel to the \( y \)-axis, extending from minus to positive infinity, and a distance \( \gamma \) removed from the origin, so that all poles are to the left of this line, Fig. 2. The reader who has a comprehensive understanding of contour integrals will recognize that this integral is equal to the integration around a semi-circle of infinite radius extending to the left of the line \( x = \gamma \), and includes integration along the “cuts,” which joins the poles to the semi-circle. Since the integration along the semi-circle in the second and third quadrant is zero for radius infinity and \( t > 0 \), this leaves the integration along the “cuts” and the poles, where the latter, as expressed in Eq. V-2, are the residuals.

Certain fundamental relationships in the Laplace transformations are found useful:

**Theorem A** — If \( \overline{P}(\lambda) \) is the transform of \( P(t) \), then

\[ \int_0^\infty e^{-\lambda t} P(t) \, dt = e^{-\lambda t} P(t) \bigg|_0^{\infty} + \frac{1}{p} \int_0^\infty e^{-\lambda t} P(t) \, dt \]

\[ = e^{-\lambda t} P(t) \bigg|_0^{\infty} \]

or the transform of \( \frac{dP(t)}{dt} = \overline{P}(\lambda) - \overline{P}(\lambda e^{\lambda}) \), provided \( e^{-\lambda t} P(t) \)

approaches zero as time approaches infinity.

**Theorem B** — The transform of \( \int_0^\infty e^{-\lambda t} P(t) \, dt \) is expressed by

\[ \int_0^\infty e^{-\lambda t} \int_0^t P(t') \, dt' \, dt = \frac{-e^{-\lambda t}}{p} \left[ \int_0^\infty e^{-\lambda t} P(t) \, dt \right] + \frac{1}{p} \int_0^\infty e^{-\lambda t} P(t) \, dt \]

or the transform of the integration \( P(t') \) with respect to \( t' \)

from zero to \( t \) is \( \overline{P}(\lambda)/p \), if \( e^{-\lambda t} \int_0^t P(t') \, dt' \) is zero for time infinity.

**Theorem C** — The transform for \( e^{\lambda t} P(t) \) is equal to

\[ \int_0^\infty e^{-\lambda t} e^{\lambda t} P(t) \, dt = \int_0^\infty e^{-\lambda t - \lambda t} P(t) \, dt = \overline{P}(\lambda e^\lambda) \]

if \( p - c \) is positive.

**Theorem D** — If \( \overline{P}(p) \) is the transform of \( P(1) \), and \( \overline{P}(p) \) is the transform of \( P(t) \), then the product of these two transforms is the transform of the integral

\[ \int_0^t P(t) \, P(t') \, dt' \]

This integral is comparable to the integrals developed by the superposition theorem, and of appreciable use in this paper.

**CONSTANT TERMINAL PRESSURE AND CONSTANT TERMINAL RATE CASES, INFINITE MEDIUM**

The analytics for the constant terminal pressure and rate cases have been developed for limited reservoirs° when the exterior boundary is considered closed or the production rate through this boundary is fixed. In determining the volume of water encroached into the oil-bearing portion of reservoirs, few cases have been encountered which indicated that the sands in which the oil occurs are of limited extent. For the most part, the data show that the influx behaves as if the water-bearing parts of the formations are of infinite extent, because within the productive life of oil reservoirs, the rate of water encroachment does not reflect the influence of an exterior boundary. In other words, whether or not the water sand is of limited extent, the rate of water encroachment is such as if supplied by an infinite medium.
Computing the water influx for an infinite reservoir with the help of Fourier-Bessel expansions, an exterior boundary can be assumed so far removed from the field radius that the production for a considerable time will reflect the infinite case. Unfortunately, the poor convergence of these expansions invalidates this approach. An alternative method consists of using increasing values for exterior radius, evaluating the water influx for each radius separately, and then drawing the envelope of these curves, which gives the infinite case, Fig. 5. In such a procedure, each of the branch curves reflects a water reservoir of limited extent. Inasmuch as the drawing of an envelope does not give a high degree of accuracy, the solutions for the constant terminal pressure and constant terminal rate cases for an infinite medium are presented here, with values for \( Q_{10} \) and \( P_{10} \), calculated directly.

The constant terminal pressure case was first developed by Nicholson\(^8\) by the application of Green's function to an instantaneous circular source in an infinite medium. Goldstein\(^9\) presented this solution by the operational method, and Smith\(^9\) employed Carslaw's contour method in its development. Carslaw and Jaeger\(^10\) later gave the explicit treatment of the constant terminal pressure case by the application of the Laplace transformation. The derivation of the constant terminal rate case is not given in the literature, and its development is presented here.

**The Constant Rate Case**

As already discussed, the boundary conditions for the constant rate case in an infinite medium are that (1) the pressure drop \( P_{r=0} \) is zero initially at every point in the formation, and (2) at the radius of the field \( r = 1 \) we have

\[
\left( \frac{\partial P}{\partial r} \right)_{r=1} = -1 \text{ at all times.}
\]

A reference to a text on Bessel functions, such as Karman and Biot,\(^4\) pp. 61-63, shows that the general solution for Eq. III-3 is given by

\[
\overline{P}_{r,p} = A \, I_0 \left( r \sqrt{p} \right) + B \, K_0 \left( r \sqrt{p} \right) . \quad (VI-1)
\]

where \( I_0 \left( r \sqrt{p} \right) \) and \( K_0 \left( r \sqrt{p} \right) \) are modified Bessel functions of the first and second kind, respectively, and of zero order. A and B are two constants which satisfy a second order differential equation. Since \( \overline{P}(r,p) \) is the transform of the pressure drop at a point in the formation, and because at a point not yet affected by production the absolute pressure equals the initial pressure, it is required that \( \overline{P}(r,p) \) should approach zero as \( r \) becomes large. As shown in Karman and Biot,\(^4\) \( I_0 \left( r \sqrt{p} \right) \) becomes increasingly large and \( K_0 \left( r \sqrt{p} \right) \) approaches zero as the argument \( (r \sqrt{p}) \) increases. Therefore, to obey the initial condition, the constant A must equal zero and (VI-1) becomes

\[
\overline{P}_{r,p} = B \, K_0 \left( r \sqrt{p} \right) . \quad . . . . . \quad (VI-2)
\]

To fulfill the second boundary condition for unit rate of production, namely \( \left( \frac{\partial P}{\partial r} \right)_{r=1} = -1 \), the transform for unitly gives

\[
\left( \frac{\partial \overline{P}}{\partial r} \right)_{r=1} = -\frac{1}{p} \quad . . . . . \quad (VI-3)
\]

by Eq. V-1. The differentiation of the modified Bessel function of the second kind, Watson's Bessel Functions,\(^9\) W.B.F., p. 79, gives \( K_0'(x) = -K_1(x) \). Therefore, differentiation Eq.

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**FIG. 7 — RADIAL FLOW, CONSTANT TERMINAL RATE CASE, CUMULATIVE PRESSURE DROP VS. TIME P(T) VS. t**

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VI-2, with respect to \( r \) at \( r = 1 \), gives

\[
\left( \frac{\partial \overline{P}}{\partial r} \right)_{r=1} = -B \sqrt{p} K_v(\sqrt{p})
\]

and since

\[
\left( \frac{\partial \overline{P}}{\partial r} \right)_{r=1} = -\frac{1}{p}
\]

the constant \( B = 1/p^n K_r(\sqrt{p}) \). Therefore, the transform for the pressure drop for the constant rate case in an infinite medium is given by

\[
\overline{P}_{r(\overline{r})} = \frac{K_v(r \sqrt{p})}{p^n K_r(\sqrt{p})} \quad \ldots \ldots \quad (VI-4)
\]

To determine the inverse of Eq. VI-4 in order to establish the pressure drop at radius unity, we can resort to the simplification that for small times the operator \( p \) is large. Since

\[
K_v(z) = \sqrt{\frac{z}{2\pi}} e^{-z} \quad \ldots \ldots \quad (VI-5)
\]

for \( z \) large, W.B.F., p. 202, then

\[
\overline{P}_{r(\overline{r})} = \frac{1}{p^{n/2}} \quad \ldots \ldots \quad (VI-6)
\]

The inversion for this transform is given in Campbell and Foster, Eq. 516, as

\[
P_{r(t)} = \frac{2}{\sqrt{\pi}} t^{n/2} \quad \ldots \ldots \quad (VI-7)
\]

In brief, Eq. VI-7 states that when \( t = KT/\mu c R^2 \) is small, which can be caused by the boundary radius for the field, \( R_0 \) being large, the pressure drop for the unit rate of production approximates the condition for linear flow.

To justify this conclusion, the treatment of the linear flow equation, Eq. IV-5, by the Laplace transformation gives

\[
d^2 \overline{P}_{r(\overline{r})} \overline{d}x^2 = p \overline{P}_{r(\overline{r})} \quad \ldots \ldots \quad (VI-8)
\]

for which the general solution is the expression

\[
\overline{P}(0 \sqrt{p}) = 1/p^{n/2} \quad \ldots \ldots \quad (VI-9)
\]

By repeating the reasoning already employed in this development, the transform for the pressure drop at \( x = 0 \) gives

\[
\overline{P}(0 \sqrt{p}) = 1/p^{n/2}
\]

which is identical with (VI-6) with \( p \) the operator of \( t = KT/\mu c \).

The second simplification for the transform (VI-4) is to consider \( p \) small, which is equivalent to considering time, \( t \), large. The expansions for \( K_v(z) \) and \( K_r(z) \) are given in Carslaw and Jaeger, p. 248.

\[
K_v(z) = -L_v(z) \{ \log \frac{z}{2} + \gamma \} + \frac{1}{2} z^2 + \quad (VI-10)
\]

\[
+ \left( \frac{1}{2} + \frac{1}{3} \right) \left( \frac{z}{2} \right)^4 + \left( \frac{1}{2} + \frac{1}{3} \right) \left( \frac{z}{2} \right)^4 + \quad (VI-10)
\]

The logarithmic term consists of natural logarithms. When \( z \) is small

\[
K_v(z) \approx - \left[ \log \frac{z}{2} + \gamma \right] \quad \ldots \ldots \quad (VI-12)
\]

\[
K_r(z) \approx 1/2 \quad \ldots \ldots \quad (VI-13)
\]

Therefore, Eq. VI-4 becomes

\[
\overline{P}_{r(\overline{r})} = -\frac{1}{2} \left[ \log \frac{p}{2} + \frac{2}{p} \left( \log 2 - \gamma \right) \right] \quad (VI-14)
\]

The inversion for the first term on the right is given by Campbell and Foster, Eq. 892, and the inverse of the second term by

\[
P_{r(t)} = \frac{1}{2} \left[ \log 4t - \gamma \right] \quad \ldots \ldots \quad (VI-15)
\]

The solution given by Eq. VI-15 is the solution of the continuous point source problem for large time \( t \). The relationship has been applied to the flow of fluids by Bruce, Elkins, and others, and is particularly applicable for study of interference between flowing wells.

The point source solution originally developed by Lord Kelvin and discussed in Carslaw can be expressed as

\[
P_{r(t)} = \frac{1}{2} \int e^{-n} \frac{e^n}{n} dt = \frac{1}{2} \left\{ -Ei \left( -\frac{1}{4t} \right) \right\} \quad (VI-16)
\]

often referred to as the logarithmic integral or the Ei-function. Its values are given in Tables of Sine, Cosine, and Exponential Integrals, Volumes I and II, Federal Works Agency, W.P.A., City of New York. For large values of the time, \( t \), Eq. VI-16 reduces to \( P_{r(t)} = \frac{1}{2} \left[ \log 4t - \gamma \right] \) which is Eq. VI-15, and this relation is accurate for values of \( t > 100 \).
By this development it is evident that the point source solution does not apply at a boundary for the determination of the pressure drop when \( t \) is small. However, when the radius, \( R_o \), is small, such as a well radius, even small values of the absolute time, \( T \), will give large values of the dimensionless time \( t \), and the point source solution is applicable. On the other hand, in considering the pressure drop at the periphery of a field (in which case \( K_o \) can have a large numerical value) the value of \( t \) can be easily less than 100 even for large values of absolute time, \( T \). Therefore, for intermediate times, the rigorous solution of the constant rate case must be used, which we will now proceed to obtain.

To develop the explicit solution for the constant terminal rate case, it is necessary to invert the Laplace transform, Eq. VI-4, by the Mellin's inversion formula. The path of integration for this transform is described by the "cut" along the negative real axis, Fig. 2, which gives a single valued function on each side of the "cut." That is to say that Path AB required by Eq. V-2 is equal to the Path AD and CB, both of which are described by a semi-circle of radius infinity. Since its integration is zero in the second and third quadrants, this leaves the integration along Paths DO and OC equal to AB. The integration on the upper portion of the "cut" can be obtained by making \( \lambda = u^2 e^{ix} \) which yields

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-\lambda t} \left( \frac{\lambda}{\pi} \right)^{\frac{1}{2}} K_0(\sqrt{\lambda \sigma}) d\lambda = \frac{1}{\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \left( \frac{e^{-u^2 t} - e^{-u^2 t}}{u^2 e^{ix}} \right) K_0(\sqrt{\lambda \sigma}) d\lambda
\]

\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \left( \frac{e^{-u^2 t} - e^{-u^2 t}}{u^2 e^{ix}} \right) K_0(\sqrt{\lambda \sigma}) d\lambda
\]

\[
= -1 \frac{1}{\pi} \int_{-\infty}^{\infty} \left( e^{-u^2 t} - e^{-u^2 t} \right) K_0(\sqrt{\lambda \sigma}) d\lambda
\]

\[
= -1 \int_{-\infty}^{\infty} \left( e^{-u^2 t} - e^{-u^2 t} \right) K_0(\sqrt{\lambda \sigma}) du
\]

Using Eq. VI-18 yields the relationship

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \left( e^{-u^2 t} - e^{-u^2 t} \right) \left[ I_0(u) J_0(u \sigma) - J_0(u) Y_0(u \sigma) \right] du = \frac{u^2}{\pi} \left[ I_0(u) + Y_0(u) \right] \quad \quad (VI-20)
\]

The integration along Paths DO and OC is the sum of the relations VI-19 and VI-20, or

\[
P_{(r, u)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( e^{-u^2 t} - e^{-u^2 t} \right) \left[ I_0(u) J_0(u \sigma) - J_0(u) Y_0(u \sigma) \right] du = \frac{u^2}{\pi} \left[ I_0(u) + Y_0(u) \right] \quad \quad (VI-21)
\]

Initially, that is at time zero, the cumulative pressure drop at any point in the formation is zero, \( P_{(r, u)} = 0 \). Hence, the pressure drop since zero time equals:

\[
P_{(r, u)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - e^{-u^2 t} \right) \left[ I_0(u) Y_0(u \sigma) - J_0(u) J_0(u \sigma) \right] du
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - e^{-u^2 t} \right) \left[ I_0(u) + Y_0(u) \right] \quad \quad (VI-22)
\]

By the recurrence formula given in W.B.F., p. 77

\[
J_0(u) = J_0(u) - I_0(u) \quad \quad (VI-23)
\]

Equation VI-22 simplifies to

\[
P_{(r, u)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - e^{-u^2 t} \right) \left[ I_0(u) + Y_0(u) \right] \quad \quad (VI-24)
\]

Constant Terminal Pressure Case

As already shown, the transform of the pressure drop in an infinite medium is \( P_{(r, u)} = B K_0(\sqrt{\sigma} r) \). In the constant terminal pressure case it is assumed that at all times the pressure drop at \( r = 1 \) will be unity, which is expressed as a transform by Eq. V-1

\[
\tilde{P}_{(r, u)} = 1/P
\]

By solving for the constant \( B \) at \( r = 1 \), we find \( B = 1/P K_0(\sqrt{\sigma} r) \), so that the transform for the pressure at any point in the reservoir is expressed by

\[
P_{(r, u)} = \frac{K_0(\sqrt{\sigma} r)}{p K_0(\sqrt{\sigma} r)} \quad \quad (VI-25)
\]

The comparable solution of VI-25 for a cumulative pressure drop can be developed as before by considering the paths of Fig. 2, with a pole at the origin, to give the solution.
P(r, u) - P(r, u₀) =
\frac{2}{r} \int u' \left[ J_\nu(u) Y_\nu(u r) - Y_\nu(u) J_\nu(u r) \right] du
\left( \frac{\partial P}{\partial r} \right) r = 1 \quad . \quad (VI-26)

If we are interested in the cumulative fluid influx at the field radius, r = 1, then the relationship, Eq. IV-3 applies, or

\bar{Q}(t) = \int \bar{P}(t) \frac{\partial P}{\partial r} r = 1 \quad . \quad (IV-3)

The determination of the transform of the gradient of the pressure drop at the field's edge follows from Eq. VI-25;

\left( \frac{\partial \bar{P}(r, u)}{\partial r} \right)_r = \frac{-K_0(V \bar{P})}{p' \nu K_n(V \nu \bar{P})}

since \( K_n^*(x) = -K_n(x) \). Since the pressure drop \( P(r, u) \) corresponds to the difference between the initial and actual pressure, the transform of the gradient of the actual pressure at \( r = 1 \) is given by

\left( \frac{\partial \bar{P}}{\partial r} \right)_r = \left( \frac{-\bar{P}(r, u)}{p' \nu K_n(V \nu \bar{P})} \right)

which corresponds to the integrand of Eq. IV-3. Further, from the definition given by Theorem B, namely, if \( \bar{F}(p) \) is the transform of \( P(r, u) \), then the transform of \( \int P(r, u) \) dt is given by \( \bar{F}(p)/p \) and the Laplace transform for \( Q(t) \) is expressed by

\bar{Q}(t) = \frac{K_0(V \bar{P})}{p' \nu K_n(V \nu \bar{P})} \quad . \quad (VI-27)

The application of the Mellin's inversion formula to Eq. VI-27 follows the paths shown in Fig. 2, giving

\bar{Q}(t) = \frac{4}{\nu} \int (1 - e^{-\nu t}) dt \quad . \quad (VI-28)

With respect to the transform \( \bar{Q}(t) \), there is the simplification that for time small, \( p \) is large, or Eq. VI-27 reduces to

\bar{Q}(t) \approx \frac{1}{n \nu} \quad . \quad (VI-29)

and the inversion is as before

\bar{Q}(t) = \frac{2}{\sqrt{\nu}} \quad . \quad (VI-30)

which is identical to the linear flow case. For all other values of the time, Eq. VI-28 must be solved numerically.

**Relation Between \( \bar{Q}(p) \) and \( \bar{P}(p) \)**

It is evident from the work that has already gone before, that the Laplace transformation and the superimposition theorem offer a basis for interchanging the constant terminal pressure to the constant terminal rate case, and vice versa. In any reservoir study the essential interest is the analyses of the flow either at the well bore or the field boundary. The purpose of this work is to determine the relationship between \( Q(t) \), the constant terminal pressure case, and \( P(t) \), the constant terminal rate case, which explicitly refer to the boundary \( r = 1 \). Therefore, if we conceive of the influx of fluid into a

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**Table I — Radial Flow, Constant Terminal Pressure and Constant Terminal Rate Cases for Infinite Reservoirs**

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<th>( Q(t) )</th>
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well or field as a constant rate problem, then the actual cumulative fluid produced as a function of the cumulative pressure drop is expressed by the superposition relationship in Eq. IV-14 as

$$ Q_{ct} = 2 \pi ho R_s c \int_0^t \frac{dP}{dt} Q_{ct} dt' $$  \hspace{1cm} (IV-14)$$

when $dP$ is the cumulative pressure drop at the wellbore affected by producing the well at constant rate which is established by

$$ \Delta P = \frac{Q_{ct}}{2K} $$  \hspace{1cm} (IV-11)$$

The substitution of Eq. IV-11 in IV-14 gives

$$ Q_{ct} = \frac{\pi \rho RT c R_s}{K} \int_0^t \frac{dP}{dt} Q_{ct} dt' $$

Since the rate is constant, $Q_{ct} = \pi c R_s \times T$, and as $t=K/T$, this relation becomes

$$ t = \int_0^t \frac{dP}{dt} Q_{ct} dt' $$  \hspace{1cm} (VI-31)$$

To express Eq. VI-31 in transformation form, the transform for $1/t^p$, Campbell and Foster, Eq. 408.1. The transform for $P_{ct}$ at $r = 1$ in $P_{ct}$, and it follows from Theorem A that the transform of $-\frac{dP_{ct}}{dt}$ is $\pi c R_s$, as the cumulative pressure drop $P_{ct}$, for constant rate is zero at time zero. Finally from Theorem D, the transform for the integration of the form Eq. VI-31 is equal to the product of the transforms for each of the two terms in the integrand, or

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PETROLEUM TRANSACTIONS, AIME 335

A. F. VAN EVERDINGEN AND W. HURST

T.P. 2732
\[
\frac{1}{p^2} = \bar{P}(t, Q) = \ldots \ldots (VI-32)
\]

Evidence of this identity can be confirmed by substituting Eqs. VI-4 and VI-27 in Eq. VI-32. In brief, Eq. VI-32 is the relationship between constant terminal pressure and constant terminal rate cases. If the Laplace transformation for one is known, the transform for the other is established. This interchange can only take place in the transformations and the final solution must be by inversion.

**Computation of \( P(t) \) and \( Q(t) \)**

To plot \( P(t) \) and \( Q(t) \) as work-curves, it is necessary to determine numerically the value for the integrals shown in Eqs. VI-24 and VI-28. In treating the infinite integrals for \( P(t) \) and \( Q(t) \), the only difficult parts are in establishing the integrals for small values of \( u \). For larger values of \( u \) the integrands converge fairly rapidly, and Simpson's rule for numerical integration has proved sufficiently accurate.

To determine the integration for \( Q(t) \) in the region of the origin, Eqs. VI-28 can be expressed as

\[
Q(t) = \frac{4}{s^2} \int_0^t \frac{1-e^{-ut}}{u[1+(Y'_u)(u) + Y'_u(u)]} \, du \ldots \ldots (VI-33)
\]

where the value for \( \delta \) is taken such that \( 1 - e^{-ut} \approx ut \), which is true for \( ut \) equal to or less than 0.02, or \( s = \sqrt{0.02/7} \) and the simplification for Eq. VI-33 becomes

\[
Q(t) = \frac{4t}{s^2} \int_0^t \frac{1}{u[1+(Y'_u)(u) + Y'_u(u)]} \, du
\]

For \( u \) less than 0.02, \( Y'_u = 1 \), and

\[
Y'_u(u) \approx \frac{2}{u} \{ \log u + \gamma \} \approx \frac{2}{u} \{ \log u - 0.11593 \}
\]

As the logarithmic term is most predominant in the denominator for small values of \( u \), this equation simplifies to

\[
Q(t) = \frac{t}{s} \int_0^s \frac{du}{u[\log u - 0.11593]} = \frac{t}{s} \int_0^s \frac{du}{[0.11593 - \log \delta]}
\]

The integration for \( P(t) \) close to the origin is expressed by

\[
P(t) = \frac{4}{s^2} \int_0^t (1-e^{-ut}) \, du \ldots \ldots (VI-35)
\]

For \( u \) equal to or less than 0.02, \( J_0(u) = 0 \), and \( Y_0(u) = 2/t u \) so that Eq. VI-35 reduces to

\[
P(t) = \frac{1}{2} \int_0^t \frac{1-e^{-nt}}{n} \, dn \ldots \ldots (VI-37)
\]

Further,

\[
\int_0^t (1-e^{-nt}) \, dn = \frac{1}{n} \int_0^t (1-e^{-nt}) \, dn - \int_0^t (1-e^{-nt}) \, dn
\]

Since Euler's constant \( \gamma \) is equal to

\[
\gamma = \int_1^\infty (1-e^{-nt}) \, dt - \int_0^\infty e^{-nt} \, dt
\]

Substitution of this relation in Eq. VI-38 gives

\[
\int_0^t (1-e^{-nt}) \, dt = \gamma + \int_0^t e^{-nt} \, dt - \int_0^t (1-e^{-nt}) \, dt
\]

and since the second term on the right is the Ei-function already discussed in the earlier part of this work, Eq. VI-37 reduces to

\[
P(t) = \frac{1}{2} \left[ \gamma - \text{Ei}(-t) + \log \frac{t}{n} \right] \ldots \ldots (VI-39)
\]

### TABLE III — Constant Terminal Rate Case Radial Flow — Limited Reservoirs

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( R_1 = 1.5 )</th>
<th>( R_2 = 2.0 )</th>
<th>( R_3 = 2.5 )</th>
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<th>( R_5 = 4.0 )</th>
<th>( R_6 = 4.5 )</th>
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<td>( \beta_1 = 4.32325 )</td>
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<td>( \beta_4 = 4.3184 )</td>
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<td>( \beta_6 = 4.3184 )</td>
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<td>( P_0 )</td>
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<td>( 2.0 \times 10^{-6} )</td>
<td>( 2.5909 ( \times 10^{-6} )</td>
<td>( 2.0 \times 10^{-6} )</td>
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<td>( 2.5909 ( \times 10^{-6} )</td>
</tr>
</tbody>
</table>

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The values for the integrands for Eqs. VI-24 and VI-28 have been calculated from Bessel Tables for or greater than 0.02 as given in W.B.F., pp. 666-697. The calculations have been somewhat simplified by using the square of the modulus of \( |\mathbf{H}_{\nu}(u)| = |J_{\nu}(u) + iY_{\nu}(u)| \) and \( |\mathbf{H}_{\nu}(u)| = |J_{\nu}(u) + iY_{\nu}(u)| \) which are the Bessel functions of the third kind or the Hankel functions.

Table I shows the calculated values for \( P_{\alpha}(t) \) and \( P_{\beta}(t) \) to three significant figures, starting at \( t = 0.01 \), the point where linear flow and radial flow start deviating. \( P_{\beta}(t) \) is calculated only to \( t = 1.000 \) since beyond this range the point source solution of Eq. VI-15 applies. The values for \( Q_{\alpha}(t) \) are given up to \( t = 10^{5} \).

The reader may reproduce these data as he sees fit; Fig. 4 is an illustrative plot for \( Q_{\alpha}(t) \), and Fig. 7 is a semi-logarithmic relationship for \( P_{\alpha}(t) \).

**LIMITED RESERVOIRS**

As already mentioned, the solutions for limited reservoirs of radial symmetry have been developed by the Fourier-Bessel type of expansion.\(^4\)\(^n\) Their introduction here is not only to show how the solutions may be arrived at by the Laplace transformation, but also to furnish data for \( P_{\alpha}(t) \), and \( Q_{\alpha}(t) \) curves when such cases are encountered in practice.

No Fluid Flow Across Exterior Boundary

The first example considered is the constant terminal pressure case for radial flow of limited extent. The boundary conditions are such that at the well bore or field's edge, \( r = 1 \), the cumulative pressure drop is at, and at some distance removed from this boundary at a point in the reservoir \( r = R \), there exists a restriction such that no fluid can flow past this barrier so that at that point \( \left( \frac{\partial P}{\partial r} \right)_{r=R} = 0 \).

The general solution of Eq. VI-1 still applies, but to fulfill the boundary conditions it is necessary to re-determine values for constants \( A \) and \( B \). The transformation of the boundary condition at \( r = 1 \) is expressed as

\[
\frac{1}{p} AL_1(\sqrt{p} R) + BK_1(\sqrt{p} R) = 0 \quad \ldots \quad (VII-1)
\]

and at \( r = R \) the condition is

\[
0 = AL_1(\sqrt{p} R) - BK_1(\sqrt{p} R) \quad \ldots \quad (VII-2)
\]

since it is shown in W.B.F. p. 79, that \( K_1(z) = -K_1(z) \), and \( L_1(z) = L_1(z) \). The solutions for \( A \) and \( B \) from these two simultaneous algebraic expressions are

\[
A = K_1(\sqrt{p} R) / p\left[ K_0(\sqrt{p} R) L_1(\sqrt{p} R) + K_1(\sqrt{p} R) L_0(\sqrt{p} R) \right]
\]

and

\[
B = I_1(\sqrt{p} R) / p\left[ K_1(\sqrt{p} R) L_1(\sqrt{p} R) + K_1(\sqrt{p} R) L_0(\sqrt{p} R) \right]
\]

By substituting these constants in Eq. VII-1, the general solution for the transform of the pressure drop is expressed by

\[
\tilde{P}_{\alpha}(p) = \frac{K_1(\sqrt{p} R) L_1(\sqrt{p} R) + I_1(\sqrt{p} R) K_0(\sqrt{p} R)}{p\left[ K_0(\sqrt{p} R) L_0(\sqrt{p} R) + I_0(\sqrt{p} R) K_1(\sqrt{p} R) \right]}
\]

(VII-3)

To find \( Q(t) \) the cumulative fluid produced for unit pressure drop, then the transform for the pressure gradient at \( r = 1 \) is obtained as follows:

\[
-\left( \frac{\partial \tilde{P}}{\partial r} \right)_{r=1} = \frac{I_1(\sqrt{p} R) K_1(\sqrt{p} R) - K_1(\sqrt{p} R) I_1(\sqrt{p} R)}{p^2\left[ K_0(\sqrt{p} R) L_0(\sqrt{p} R) + I_0(\sqrt{p} R) K_1(\sqrt{p} R) \right]}
\]

where the negative sign is introduced in order to make \( Q(t) \)

---

### TABLE III — Continued

<table>
<thead>
<tr>
<th>( R = 6 )</th>
<th>( R = 6.6 )</th>
<th>( R = 7.0 )</th>
<th>( R = 7.0 )</th>
<th>( R = 8.0 )</th>
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<td>( \beta_1 = 0.6472 )</td>
<td>( \beta_1 = 0.6864 )</td>
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<td>( \beta_1 = 0.7563 )</td>
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<td>( \beta_1 = 0.7963 )</td>
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<td>( \beta_1 = 1.3580 )</td>
<td>( \beta_1 = 1.3705 )</td>
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<td>( t )</td>
<td>( P_{\alpha}(t) )</td>
<td>( P_{\beta}(t) )</td>
<td>( t )</td>
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</tr>
</tbody>
</table>

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positive. Theorem B shows that the integration with respect to time introduces an additional operator \( p \) in the denominator to give

\[
\tilde{Q}(\omega) = \left[ I_c(\sqrt{\nu} R) K_v(\sqrt{\nu} R) - K_v(\sqrt{\nu} R) I_c(\sqrt{\nu} R) \right] p^{\nu} \left[ K_v(\sqrt{\nu} R) I_c(\sqrt{\nu} R) + I_c(\sqrt{\nu} R) K_v(\sqrt{\nu} R) \right]
\]

(VII-4)

In order to apply Mellin’s inversion formula, the first consideration is the roots of the denominator of this equation which indicate the poles. Since the modified Bessel functions for positive real arguments are either increasing or decreasing, the bracketed term in the denominator does not indicate any poles for positive real values for \( p \). At the origin of the plane of Fig. 2 a pole exists and this pole we shall have to investigate. Thus, the substitution of small and real values for \( \nu \) (Eqs. VI-12 and VI-13) in Eq. VII-4, gives

\[
\tilde{Q}(\omega) = \frac{(R^r-1)}{2p}
\]

\( p \to 0 \)

**TABLE IV -- Constant Terminal Rate Case Radial Flow**

<table>
<thead>
<tr>
<th>( \lambda = 1.6269 )</th>
<th>( \lambda = 1.7246 )</th>
<th>( \lambda = 1.8226 )</th>
<th>( \lambda = 1.9204 )</th>
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<tbody>
<tr>
<td>( R = 1.5 )</td>
<td>( R = 2.0 )</td>
<td>( R = 2.5 )</td>
<td>( R = 3.0 )</td>
</tr>
<tr>
<td>( t )</td>
<td>( F_{(1)} )</td>
<td>( t )</td>
<td>( F_{(1)} )</td>
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<tr>
<td>0.0</td>
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<td>1.0</td>
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<td>0.889</td>
<td>0.846</td>
</tr>
<tr>
<td>2.0</td>
<td>0.881</td>
<td>0.819</td>
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</tr>
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<td>3.0</td>
<td>0.845</td>
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<td>0.556</td>
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**TABLE IV -- Continued**

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<th>( \lambda = 2.4448 )</th>
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<tr>
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<td>0.377</td>
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and by the application of Mellin's inversion formula applied at the origin, then

\[
\frac{1}{2\pi i} \oint e^{\lambda t} \left( \frac{R^*-1}{\lambda} \right) d\lambda = \frac{1}{2} \oint e^{\lambda t} Q(\lambda) d\lambda = \frac{1}{2\pi i} \oint e^{\lambda t} [L_i(uR) Y_i(a_R) - Y_i(uR) J_i(a_R)] du - \frac{1}{2\pi i} \oint u^i [L_i(uR) Y_i(a_R) - Y_i(uR) J_i(a_R)] \frac{d\lambda}{\lambda}.
\]

(VII-5)

An investigation of the integration along the negative real “cut” both for the upper and lower portions, Fig. 2, reveals that Eq. VII-4 is an even function for which the integration along the paths is zero. However, poles are indicated along the negative real axis and these residuals together with Eq. VII-5 make up the solution for the constant terminal pressure case for the limited radial system. The residuals are established by the Mellin’s inversion formula by letting \( \lambda = u e^{i\pi} \); then by Eqs. VI-18

\[
\frac{1}{2\pi i} \oint e^{\lambda t} Q(\lambda) d\lambda = \frac{1}{2\pi i} \oint e^{\lambda t} \left[ L_i(uR) Y_i(a_R) - Y_i(uR) J_i(a_R) \right] du.
\]

(VII-6)

where \( a, a_n, a_R, a_n \) are the roots of

\[
L_i(uR) Y_i(a_R) - Y_i(uR) J_i(a_R) = 0.
\]

(VII-7)

and the poles are represented on the negative real axis by \( \lambda = -a^*_R \), Fig. 3. The residuals of Eq. VII-6 are the series expansion

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**TABLE IV — Continued**

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<th>( R = 20 )</th>
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<th>( \lambda = 0.8808 )</th>
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**TABLE IV — Continued**

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Therefore, the solution for \( Q(t) \) is expressed by

\[
Q_{ct} = -\alpha t \cdot R' + \frac{1}{2} \sum_{a_n} \alpha_n \frac{e^{-\alpha_n t}}{a_n} \left[ J_n\left(\alpha_n R\right) \right]
\]

Equation (VII-10)

This is essentially the solution developed in an earlier work, but Eq. VII-10 is more rapidly convergent than the solution previously reported.

The values of \( Q_{ct} \) for the constant terminal pressure case for a limited reservoir have been calculated from Eq. VII-10 for \( R = 1.5 \) to 10 and are tabulated in Table 2. A reproduction of a portion of these data is given in Fig. 5. As Eq. VII-10 is rapidly convergent for \( t \) greater than a given value, only two

### TABLE IV — Continued

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terms of the expansion are necessary to give the accuracy needed in the calculations.

Likewise from the foregoing work it can be easily shown that the transform of the pressure drop at any point in the formation in a limited reservoir for the constant terminal rate case, is expressed by

\[ \bar{P}_{(r,s)} = \frac{[K_r(\sqrt{p} R) L_r(\sqrt{p} R) + I_r(\sqrt{p} R) K_r(\sqrt{p} R)]}{p^n R}\left[ I_r(\sqrt{p} R) K_r(\sqrt{p} R) - K_r(\sqrt{p} R) I_r(\sqrt{p} R) \right] \]

(VII-11)

An examination of the denominator of Eq. VII-11 indicates that there are no roots for positive values of \( p \). However, a double pole exists at \( p = 0 \). This can be determined by expanding \( K_r(z) \) and \( K_r(z) \) to second degree expansions for small values of \( z \) and third degree expansions for \( I_r(z) \) and \( I_r(z) \), and substituting in Eq. VII-11. It is found for small values of \( p \), Eq. VII-11 reduces to

\[ \bar{P}_{(r,s)} = \frac{1}{p^n} \frac{R^4}{(R^4 + 1)} \log \frac{R}{R - 1} \log R \left( \frac{R^4}{R^4 - 1} + \frac{2}{p^n (R^4 - 1)} \right) \]

(VII-12)

This equation now indicates both a single and double pole at the origin, and it can be shown from tables or by applying Cauchy's theorem to the Mellin's formula that the inversion of Eq. VII-12 is

\[ \bar{P}_{(r,s)} = \frac{2}{R^4} \left[ \frac{R^4}{R^4 - 1} \right] \frac{R^4}{4} \int R^4 \log R - 2R^4 - 1 \]

(VII-13)

which holds when the time, \( t \), is large

As in the preceding case, there are poles along the negative real axis, Fig. 3, and the residues are determined as before by letting \( \lambda = u^2 e^{ir} \), and Eqs. VI-18 give

### TABLE IV — Continued

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When developing the solution by means of the Laplace transformation, it is assumed that the exterior boundary \( r = R \), \( \tilde{F}(R,p) = 0 \), which fixes the pressure at the exterior boundary as constant. Since the above-quoted references contain complete details, the final solutions are only quoted here for completeness’ sake.

Cylindrical source:

\[
\frac{P_{111}}{R} = \log R - 2 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n R}}{\lambda_n J_n'(\lambda_n R)} \frac{1}{\lambda_n} J_n'(\lambda_n R) \quad (VII.20)
\]

where \( \lambda_n \) is the root established from

\[
J_n(\lambda_n R) - Y_n(\lambda_n R) J_n(\lambda_n) = 0 \quad (VII.21)
\]

Point source:

\[
P_{111} = \log R - 2 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n R}}{\alpha_n} \frac{1}{\alpha_n} J_n'\alpha_n R \quad (VII.22)
\]

where the root \( \alpha_n \) is determined from \( J_n(\alpha_n R) = 0 \), W.B.F., p. 748. Table 4 is the summary of the calculated \( P(t) \) employing Eq. VII.20 for \( R = 1.5 \) to 50, the cylinder source solution, which applies for small as well as large times. The data given for \( R = 60 \) to 3,000 are calculated from the point source solution Eq. VII.22. Plots of these data are given in Fig. 7.

SPECIAL PROBLEMS

The work that has gone before shows the facility of the Laplace transformation in deriving analytical solutions. Not yet shown is the versatility of the Laplace transformation in arriving at solutions which are not easily foreseen by the orthodox methods. One such solution derived here has shown to be of value in the analysis of flow tests.

When making flow tests on a well, it is often noticed that the production rates, as measured by the fluid accumulating in the stock tanks, are practically constant. Since it is desired to obtain the relation between flowing bottom hole pressure and the rate of production from the formation, it is necessary to correct the rate of production as measured in the flow tanks for the amount of oil obtained from the annulus between casing and tubing. To arrive at the solution for this problem, we use the basic equation for the constant terminal rate case given by Eq. IV.11, where \( q_{ct} \) is the constant rate of fluid produced at the stock tank corrected to reservoir conditions, but \( P_{ct} \) is a pseudo pressure drop which is adjusted mathematically for the unloading of the fluid from the annulus to give the pressure drop occurring in the formation.

It is assumed that the unloading of the annulus is directly reflected by the change in bottom hole pressure as exerted by a hydrostatic head of oil column in the casing. Therefore, the rate of unloading of the annulus \( q_{ct} \), expressed in cc. per second corrected to reservoir conditions, is equal to

\[ q_{ct} = \frac{C}{dA} \frac{dP}{dt} \quad (VIII.1) \]

where \( C \) is the volume of fluid unloaded from the annulus per atmosphere bottom hole pressure drop per unit sand thickness. The rate of fluid produced from the formation is then given by \( q_{ct} = q_{ct} - q_{ct} \). As the bottom hole pressure is continuously changing, the problem becomes one of a variable rate. The substitution of the form of Eq. IV.11 in the superposition theorem, Eq. IV.16, gives

As a variation on the condition that \( \frac{dP}{dr} = 0 \), we may assume that the pressure at \( r = R \) is constant. In effect, this assumption helps to explain approximately the pressure history of flowing a well at a constant rate when, upon opening, the bottom hole pressure drops very rapidly and then levels out to become constant with time. The case has been developed by Hursil using a cylinder source and by Muskat using a point source solution.
\[ \Delta P = \frac{\mu}{2xK} \int \left[ q_{w} - q \alpha \right] P'_{(t-t')} \, dt' \]

and from Eq. VIII-1

\[ \Delta P = \frac{\mu}{2xK} \int \left[ q_{w} - C \cdot \frac{\alpha \Delta P}{dP} \right] P'_{(t-t')} \, dt' \quad \text{(VIII-2)} \]

Since \( T = \frac{2\pi r_s^4 \nu}{K} \), and the unit rate of production at the surface corrected to reservoir conditions is \( q_{w} = \frac{\alpha r_s^4}{2xK} \), Eq. VIII-2 becomes

\[ \Delta P = \int \left[ q_{w} - C \cdot \frac{\alpha \Delta P}{dP} \right] P'_{(t-t')} \, dt' \quad \text{(VIII-3)} \]

where \( C = \frac{2\pi r_s^4 \nu}{K} \).

Eq. VIII-3 presents a unique situation and we are confronted with determination of \( \Delta P \), the actual pressure drop, appearing both in the integrand and in the left side of the equation. The Laplace transformation offers a means of solving for \( \Delta P \) which, by orthodox methods, would be difficult to accomplish.

It will be recognized that Theorem D, from Chapter V, is applicable. Therefore, if Eq. VIII-3 can be changed to a Laplace transformation, \( \Delta P \) can be solved explicitly. If we express the transform of the constant rate \( q_{w} \) as \( q/P \), the transform of \( P'_{(t)} \) as \( P'_{(p)} \), and the transform of \( \Delta P \) as \( \Delta P' \), so that the transform for \( d\Delta P/dt \) is \( p \Delta P' \), then it follows that

\[ \Delta P' = \left[ \frac{q}{p} - C \cdot p \Delta P' \right] p \cdot P'_{(p)} \quad \text{and on solution gives} \]

\[ \Delta P' = \frac{q \cdot P'_{(p)}}{1 + C \cdot p \cdot P'_{(p)}} \quad \text{and on solution gives} \] (VIII-5)

Since \( q = q_{w} \cdot p / 2xK \), then the term \( \frac{p \cdot P'_{(p)}}{2xK} \) in Eq. VIII-5 can be interpreted as the transform of the pseudo pressure drop for the unit rate of production at the stock tank.

No mention has been made as to what value can be substituted for \( P'_{(p)} \). If we wish to apply the cylinder source, Eq. VI-4 applies, namely,

\[ P'_{(p)} = \frac{K_{s} \sqrt{p}}{p^{\gamma} K_{s} \sqrt{p}} \quad \text{and on solution gives} \] (VIII-6)

However, from the previous discussion it has been shown that for wells, \( t \) is usually large since the well radius is small, and the point source solution of Lord Kelvin's applies, namely,

\[ P'_{(p)} = \frac{1}{2} \sqrt{\frac{\pi}{4t}} \int_{-\infty}^{\infty} e^{-u^2} \, du \quad \text{and on solution gives} \] (VI-16)

the Ei-function. Therefore, to apply this expression in Eq. VIII-5, it is necessary to obtain the Laplace transform of the point source solution of Eq. VI-16. By an interchange of variables, this equation becomes

\[ P'_{(p)} = \frac{1}{2} \int_{0}^{\infty} e^{-\frac{u^2}{2t}} \, dt \quad \text{and on solution gives} \]

and it will be recognized from Campbell and Foster, Eq. 920.1, that the integrand is the transform for \( K_{s} \sqrt{p} \). Further, the integration with respect to time follows from Theorem B, Chapter V, so that the transform of Eq. VIII-7 is the relation

\[ \overline{P}_{(p)} = \frac{K_{s} \sqrt{p}}{p} \quad \text{and on solution gives} \] (VIII-8)

The same result can be gleaned from Eq. VIII-6 since for \( t \) large, \( p \) is small and \( K_{s} \sqrt{p} \approx 1 / \sqrt{p} \). Substitution of this approximation in Eq. VIII-6 yields Eq. VIII-8. Therefore, introducing the expression for \( P'_{(p)} \) in Eq. VIII-5 gives

\[ \Delta P' = \frac{q \cdot K_{s} \sqrt{p}}{1 + C \cdot p \cdot K_{s} \sqrt{p}} \quad \text{and on solution gives} \]

which it is necessary only to find the inverse of

\[ \overline{P}_{(p)} = \frac{K_{s} \sqrt{p}}{p} \quad \text{and on solution gives} \]

to obtain values for \( P_{(p)} \), the cumulative pressure drop for unit rate of production in the stock tank which automatically takes cognizance of the unloading of the annulus.

The inverse of the form of VIII-10 by the Mellin's inversion formula can be determined by the path described in Fig. 2. The analytical determination is identical with the constant terminal rate case given in Section VI. Therefore, the cumulative pressure drop in the well bore, for a unit rate of production at the surface, corrected for the unloading of the fluid in the casing, is the relation

\[ P_{(p)} = \int_{0}^{\infty} \frac{(1 - e^{-u^2}) J_{\nu}(u)}{u \left[ 1 + u^{2} - 1 J_{\nu}(u) \right]^{2} + (u^{2} C_{r}^{2} - 1 J_{\nu}(u))^{2}} \, du \quad \text{and on solution gives} \]

Fig. 8 presents a plot of the computed values for \( P_{(p)} \), corresponding to \( C \) from 1,000 to 75,000. It can be observed that the greater the unloading from the casing, the smaller the actual pressure drop is in a formation due to the reduced rate of fluid produced from the sand. For large times, however, all curves become identical with the point source solution which is the envelope of these curves. After a sufficient length of time, the change in bottom hole pressure is so slow that the rate of production from the formation is essentially that produced by the well, and the point source solution applies.

ACKNOWLEDGMENTS

The authors wish to thank the Management of the Shell Oil Co., for permission to prepare and present this paper for publication. It is hoped that this information, once available to the industry, will further the analysis and understanding of the behavior of oil reservoirs.

The authors acknowledge the help of H. Rainbow of the Shell Oil Co., whose suggestions on analytic development were most helpful, and of Miss L. Patterson, who contributed the greatest amount of the calculations with untrivial effort.

REFERENCES


*Note:* This book came to our notice only after the text of this paper was prepared and for that reason references to its contents are incomplete. The careful reader will observe that, for instance, equation (VI-21) in this paper is similar to equation (16), p. 283 when k and s are given unit values; also that “Limited Reservoirs” contains equations quite similar to those appearing in Section 126, “The Hollow Cylinder,” of Carslaw and Jaeger’s book. ***
Decline Curve Analysis Using Type Curves: Water Influx/Waterflood Cases

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ABSTRACT
Description of Paper
In this paper we develop a decline type curve for the analysis and interpretation of production data from reservoir systems experiencing natural water influx or pressure support via induced waterflooding. The physical model for this case is a single well centered in a bounded circular reservoir with a constant pressure at the inner boundary and prescribed flux at the outer boundary.

The influx at the outer boundary is initially zero (the no-flow boundary condition) and is abruptly changed to either a constant flux at some prescribed time (the "step" rate case) or the boundary influx increases slowly from time zero to a fixed value at large times (the "ramp" rate case). The "step" rate concept illustrates the production "humps" which are often associated with waterflood operations, while the "ramp" rate case is used to model natural water influx as well as slow starting waterfloods.

Our new type curves can be used to estimate the efficiency of a waterflood as well as the potential recovery from waterflooding. Verification of the boundary influx model is provided using single and multiphase numerical simulation cases and example applications of the new type curves are provided using field data.

Results, Observations & Conclusions
Results
The major results of this work are

- Development of a type curve methodology for the analysis and interpretation of production data from water influx/waterflood systems.
- Comparison of production performance for "ideal" water influx/waterflood cases using the new type curves and numerical simulation models—for both single- and multiphase flow conditions.
- Demonstration of the new type curves for typical field data from reservoirs with low permeability carbonate and sandstone reservoirs undergoing waterflood.

Observations
We note that our solution provides a good representation of performance observed in the field (although a very long production period is typically required to observe clear trends of secondary production decline). We also observed reasonable agreement between the multiphase numerical simulation cases and the performance of our single-phase reservoir model.

Another observation that may be of significant practical value is the similarity of the performance of our model with the production decline performance for a single well in a closed, dual porosity reservoir. In particular, the "ramp" rate influx case is virtually identical to the several cases modeled for a dual porosity reservoir—which implies that water-influx pressure support could be misinterpreted as a dual porosity reservoir response, and vice-versa. This observation clearly warrants further study.

Conclusions
We conclude that our new approach, though derived and implemented as a single-phase solution, should be applicable for oil/water systems which maintain a mobility ratio near unity. We also conclude that the new model, while certainly idealized, should accurately represent a closed injection system such as a 5- or 9-spot pattern.

INTRODUCTION
The decline curve analysis of "secondary" performance data resulting from external pressure support such as natural water influx or pattern waterflood has remained largely an empirical effort. This paper provides the rigorous development and application of an influx supported reservoir model—but assumes single-phase displacement (a unit mobility ratio).

While the assumption of single-phase displacement might seem an overwhelming limitation, our validation using single- and multi-phase numerical simulation cases as well as field data suggest that our model and our decline type curve approach are valid.

Most of the previous efforts for decline type curve analysis focused on primary, rather than secondary depletion. As such, we will consider the analysis of primary depletion data well-known and proceed on to our objective of analyzing secondary depletion data—but not before briefly reviewing the pertinent literature.

Decline Curve Analysis Using Type Curves
Fetkovich introduced the concept of a "decline type curve" which unifies the analytical reservoir solutions for a well produced at a constant bottomhole pressure with the empirical exponential, hyperbolic, and harmonic depletion equations. This work focused primarily on developments for the liquid case—with considerable effort also placed on correlating the gas reservoir.
depletion performance (a unique correlation for gas reservoir performance remains unresolved).

Fetkovich, et al.\textsuperscript{8} presented a variety of field case histories using decline type curve analysis and further verified the concept of estimating reservoir properties (i.e., permeability, skin factor, and original oil-in-place) from production data. The case of a generally varying bottomhole pressure was not addressed, although a "reinitialization" process was demonstrated in ref. 1 for cases of a single change in the bottomhole flowing pressure.

Towards the matter of resolving decline type curve analysis for systems of variable-rate/variable-pressure histories, McCray,\textsuperscript{9} Palacio and Blasingame,\textsuperscript{4} and Doublet, et al.\textsuperscript{9} present methods to rigorously account for a general rate and pressure history. Ref. 6 focuses on the general problem of a variable-rate history for production data analysis, ref. 4 considers gas reservoir systems, and ref. 5 provides in-depth application of decline type curve analysis to field cases from a variety of oil reservoirs.

As we noted above, the decline type curve analysis of primary depletion performance data can yield estimates of the following reservoir properties (ref. 5):

- Reservoir properties
  - Formation permeability, \(k\)
  - Skin factor (for near well damage or stimulation), \(s\)
- In-place fluid volumes
  - Original oil-in-place, \(N\)
  - Reservoir drainage area, \(A\)

For the analysis of primary depletion data we typically use the "Fetkovich/McCray" type curve for rate functions versus time as illustrated in Fig. 1. We also use a new "Fetkovich/McCray" type curve which features the rate and rate integral functions versus cumulative production. This type curve is presented in Fig. 2 and will be shown to be an excellent diagnostic tool for the analysis of both primary, as well as secondary performance data.

Water Influx Models

The base model for "natural" water influx was provided by van Everdingen and Hurst\textsuperscript{10} and later, and in more detail, by Matthews and Russell.\textsuperscript{11} This model considers a bounded circular reservoir with a constant pressure (i.e., initial pressure) at the outer boundary.

This solution has been widely applied as an "edgewater" drive mechanism, where influx is assumed to encroach the well uniformly across the reservoir boundary, which is held at the initial pressure of the reservoir. This solution is given by

\[
\bar{p}(x, t) = \frac{K_0 (\bar{r} - d^2)}{\bar{d}_d (\bar{r} - d^2) - K_0 (\bar{r} - d^2)} - \frac{K_0 (\bar{r} - d^2)}{\bar{d}_d (\bar{r} - d^2) + \bar{d}_d (\bar{r} - d^2)}
\]

Coats\textsuperscript{8} proposed a "bottomwater" drive model and Olarewaju\textsuperscript{9} later proposed a combination model for both an edge and bottomwater drive. Both of these models again consider the case of a bounded circular reservoir, with the boundary pressure (bottom or edge, or both) held constant. Example applications are demonstrated, though these applications are for material balance calculations, not decline curve analysis.

Naturally Fractured (Dual Porosity) Reservoir Models

Although discussion of naturally fractured reservoirs may seem somewhat misplaced in a study of the influence of fluid influx on reservoir performance, we will demonstrate both the relevance and the significance of this topic. The topic is relevant because the fluid influx mechanisms involved in the transfer of fluids from the matrix to the fracture system in a naturally fractured reservoir is conceptually similar to influx into a homogeneous reservoir system from an aquifer or pattern waterflood.

The topic of production performance of naturally fractured reservoirs is significant because the conceptual similarity of the fluid influx mechanisms may yield a production response in a naturally fractured reservoir system that appears quite similar to the production response in a homogeneous reservoir system with fluid influx. The lack of evidence for this comparison in the literature should not be taken lightly when compared with the empirical observations of many operators who claim that their waterflood production is "fracture-controlled."

Our goal is to attempt to resolve— or at least partially explain the empirical observation that many waterflood projects exhibit "dual porosity" reservoir behavior. In order to resolve this issue we must compare the solution for a single well centered in a circular naturally fractured reservoir with no-flow at the reservoir outer boundary to our proposed influx solution for a homogeneous reservoir. The naturally fractured reservoir solution is given in ref. 10.

DaPrat, et al.\textsuperscript{11} developed a variety of decline type curves for naturally fractured (or dual porosity) reservoirs. Unfortunately, when the parameters for the dual porosity model are added (i.e., \(A\) and \(a\)), we are unable to develop a single correlating trend during boundary-dominated flow as was done in the original work by Fetkovich.\textsuperscript{1}

**SOLUTION FOR A WELL IN A BOUNDED CIRCULAR RESERVOIR WITH PRESCRIBED FLUX AT THE RESERVOIR BOUNDARY**

**Boundary Flow Models**

The Laplace domain solution for a bounded circular reservoir was originally proposed by van Everdingen and Hurst\textsuperscript{10} and the derivation and real domain inversion were later provided in detail by Matthews and Russell.\textsuperscript{11} for the "no-flow" and "constant pressure" outer boundary cases.

We propose an additional solution of prescribed flux at the outer boundary where flux can "slowly" start from time zero for the "ramp" rate case, or abruptly change from a zero rate to a final rate at an arbitrary time in the "step" rate case. The schematic for the "step" rate case is shown in Fig. 3, and the "ramp" rate case is shown in Fig. 4.

These models were chosen for their similarities with natural water influx and induced waterflood processes. In particular, we assume that the "step" rate case reflects a waterflood process that starts sometime during primary depletion and the "ramp" rate case models natural water influx, or a "slow to respond" waterflood.

The rate models are given in dimensionless form by

**No-Flow Condition (No Flux Across the Boundary)**

\[ q_{\text{Dex}}(t) = 0 \]  \hspace{1cm} (2)

**"Step" Rate Condition (Impulse Start of Boundary Flux)**

\[ q_{\text{Dex}}(t) = q_{\text{Dex}} \cdot U(t - \text{Duration}) \]  \hspace{1cm} (3)

where \(U(t - \tau)\) is the "unit step" function.

**"Ramp" Rate Condition (Smooth Start of Boundary Flux)**

\[ q_{\text{Dex}}(t) = q_{\text{Dex}} \cdot [1 - \exp(-\tau/\text{Duration})] \]  \hspace{1cm} (4)

The boundary flux models are carried generically through the derivation of the new reservoir flow solution as the dimensionless boundary flux rate, \(q_{\text{Dex}}(t_0)\), of its Laplace transform, \(\hat{q}_{\text{Dex}}(s)\). The solution is described and presented in the next section.

**New Solution for Well Performance in a Bounded Circular Reservoir with Prescribed Flux at the Reservoir Boundary**

As noted in the section above, our approach is to redefine the van Everdingen and Hurst\textsuperscript{10}/Matthews and Russell\textsuperscript{11} Laplace domain solution for a well in a bounded circular reservoir assuming a generic flux term at the reservoir boundary \(q_{\text{Dex}}(t_0)\). This reservoir configuration is shown in Fig. 5 and the details of the solution developments are given in Appendix A.

The new Laplace domain solution for a constant flowrate at the well is given as
\[ \bar{P}_D(r_D=1,u) = \frac{1}{2} \left[ \begin{array}{c} K_0(\mu u) \sinh(1) I_1(\mu u) \\
- K_1(\mu u) I_0(\mu u) \\
\end{array} \right] + \frac{1}{4} \bar{q}_{D_{est}}(u) \left[ \begin{array}{c} K_0(\mu u) \sinh(1) I_1(\mu u) \\
- K_1(\mu u) I_0(\mu u) \\
\end{array} \right]\] 

Where the first part of Eq. 5 is exactly the bounded (no-flow) circular reservoir solution presented by van Everdingen and Hurst. More specifically, we have

\[ \bar{P}_D(r_D=1,u) = "No\ Flow\ " \text{Boundary Solution} \]

\[ + \text{Boundary Flux Solution} \]

So if \( q_{D_{est}}(u)=0 \), then the "boundary flux" portion of the solution is eliminated and we are left with a primary depletion system, which certainly makes sense.

Relative to our intended applications, the major assumptions for this solution are

- Single-phase (unit mobility ratio) displacement.
- Well is centered in a closed symmetrical reservoir pattern (i.e., a circle, square, or hexagon).
- Boundary flux is uniformly distributed around the perimeter of the boundary (i.e., boundary flux does not enter at a point, but rather along the entire "edge" of the reservoir).

In contrast to the solution for a constant rate at the well, our applications (decline type curves) require a constant bottomhole pressure at the well. For the solution of any linear partial differential equation, the conversion of the constant rate dimensionless pressure solution to the constant pressure dimensionless rate solution in the Laplace domain (using convolution) is given by van Everdingen and Hurst as

\[ \bar{q}(u) = \frac{1}{u^2} \bar{P}(u) \]

Rather than deal with the algebra of substituting Eq. 5 into Eq. 7 (which is simple, but will require us to carry a much larger expression), we simply program this sequence into our numerical Laplace transform inversion sequence.

In this work we use the Gaver-Stehfest numerical inversion algorithm in which the modified Bessel functions are evaluated using the algorithms provided by Cody and Stoltz. Both of these algorithms were tested for sensitivity to argument sizes and to the algorithm control parameters. The results obtained in this work should be considered accurate for all cases and all ranges of data.

Constant Rate Well Performance

In this section we present plots of the results from the inversion of the constant rate Laplace domain solution (Eq. 5). Our goal is to demonstrate the various boundary flux features relative to the pseudosteady-state performance trends.

Fig. 6 presents the results of the "step" rate boundary flux case where a special dimensionless pressure function is plotted versus the Fekovich dimensionless decline time function. These dimensionless variables are given by

\[ f_D = \frac{2}{r_D} \left[ \frac{1}{\ln r_D - 1} \right] \]

\[ f_D = \frac{1}{\ln r_D - 1} \left[ \bar{P}_D \right] \]

These \( f_D \) and \( f_D \) functions correlate the pressure responses during pseudosteady-state flow and clearly illustrate the deviation caused by the introduction of flux at the boundary. The most obvious features on Fig. 6 are the "sharp turns" that the solution makes when boundary flux begins, especially the \( q_{D_{est}}=0 \) cases where the boundary flux matches the production—which of course is the definition of the steady-state flow condition.

We also note that for cases of \( q_{D_{est}}=0 \) that the \( f_D \) function eventually returns to a "depletion" trend, implying that injection does not match production and the system is again depleting. We will investigate the constant well pressure case in the next section, but from Fig. 6 (and Fig. 7) we observe that there will be a "primary" depletion stem, recharge, then finally a "secondary" depletion stem (for cases where \( q_{D_{est}}=0 \)).

Fig. 7 illustrates the performance of the "ramp" boundary flux case where the boundary flux increases "slowly" from zero at time zero to some \( q_{D_{est}}=0 \) at large times. Again, the schematic of the ramp rate case is shown in Fig. 4.

From Fig. 7 we immediately note that the deviation from the initial depletion trend (i.e., \( q_{D_{est}}=0 \)) is not as "abrupt" as in the step rate case—which is expected, but we also note that the ramp rate pressure trends converge to the same trends as the step rate case at large times. While not obvious, but rather subtle, we recognize that the dimensionless pressure stems for the ramp rate boundary flux case appear to be similar to the performance for a naturally fractured reservoir.

Responses for a well producing at a constant flowrate in a bounded naturally fractured (dual porosity) reservoir system are shown in ref. 10. This observation will be further investigated in the next section when we study the reservoir performance for production at a constant bottomhole pressure.

Constant Pressure Well Performance

In this section, we provide a variety of decline type curve solutions developed using our new reservoir model which includes the "step" and "ramp" boundary flux cases. These type curves are summarized below.

### Summary of Decline Type Curves

<table>
<thead>
<tr>
<th>Case Type</th>
<th>Plotting Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Rate-Time&quot;</td>
<td>( q_D ) versus ( t_D )</td>
</tr>
<tr>
<td>&quot;Rate Integral-Time&quot;</td>
<td>( q_D ) versus ( t_D )</td>
</tr>
<tr>
<td>&quot;Rate Integral Derivative-Time&quot;</td>
<td>( q_D ) versus ( t_D )</td>
</tr>
<tr>
<td>&quot;Rate-Cumulative&quot;</td>
<td>( q_D ) versus ( N_{PD} )</td>
</tr>
<tr>
<td>&quot;Rate Integral-Cumulative&quot;</td>
<td>( q_D ) versus ( N_{PD} )</td>
</tr>
</tbody>
</table>

The plotting functions are defined in the Nomenclature section as well as in Appendix B.

### "Step" Rate Boundary Flux Case

We first consider the "step" rate boundary flux case where the flux at the external boundary is "switched" on to the dimensionless terminal rate, \( q_{D_{est}}=0 \), at a prescribed time, \( t_{init} \). This action provides the reservoir with an instantaneous surge of fluid influx across the reservoir boundary which manifests itself in production spikes as illustrated in Fig. 8. These "spikes" are fairly common features on production plots for waterflood projects and we consider this as confirmation that our "step" rate model may give some insight into waterflood performance and recovery.

Interpreting the \( q_{D_{est}}=0 \) and \( t_{init} \) parameters quantitatively is somewhat difficult as we are using a single-phase displacement model to characterize the production behavior from a multiphase system. However, we believe that the \( q_{D_{est}}=0 \) term can be used to represent the "strength" of the influx, with the limiting value of \( q_{D_{est}}=0 \) occurring for the steady-state flow case, where there is a 1:1 replacement.

The \( t_{init} \) parameter is the time that the external flux was initiated and reflects the recovery rate, but not the overall recovery (assuming production to total depletion). This effect will be more evident on the rate-cumulative and rate integral-cumulative type curves as we will show later.

### Rate-Time Type Curve

As mentioned previously, the "rate-time" type curve is presented in Fig. 8, where the production spikes due to the start of boundary flux are clearly shown. This type curve shows a
variety of cases of influx strength and several different start times in order to illustrate the various responses one could expect from waterflood a reservoir currently under primary producing conditions. Given the nature of the rate spikes shown on this plot, in addition to the erratic (i.e., noisy) behavior of production data in general, it may be difficult to obtain a good match of production data on Fig. 8.

**Rate Integral-Type Curve**

Fig. 9 shows the rate integral responses for the "step" rate boundary flux case. In this figure we note that the trends are much smoother than the rate case and boundary recharge is not reflected as a spike, but rather as an abrupt deviation from the depletion trend. For a particular influx strength \( q_{\text{influx}} \) we note that all of the solutions converge to the same unit-slope trends during secondary depletion. This will be a useful feature for correlating data functions.

**Rate Integral Derivative-Type Curve**

In Fig. 9 we present the rate integral derivative responses for the "step" rate boundary flux case. Although this figure provides an interesting array of trends showing transition from primary to secondary depletion, it is highly unlikely that this figure will find broad use in practice due to the noisy behavior of production rate data. We do not "close the door" on this figure as either an analysis or diagnostic plot, but we do not expect this plot to find significant application in practice.

**Rate-Cumulative Type Curve**

Our goal is to diagnose, and if possible, analyze the production rate responses from a particular well in terms of our new boundary influx model. Fig. 11 shows dimensionless rate versus dimensionless cumulative production performance for the "step" boundary flux case—where the effect of various start times is shown to only affect the rate of recovery, not the recovery itself. We note this because the total dimensionless cumulative production is constant for a particular value of influx strength \( q_{\text{influx}} \).

We also note that given sufficient primary and secondary depletion data, it should be possible to estimate the primary recovery (where the \( q_{\text{influx}} \) trends converge to \( N_{\text{PD}}=1 \)) as well as the secondary recovery (where the \( q_{\text{influx}} \) trends converge to the maximum \( N_{\text{PD}} \) value for a particular influx strength \( q_{\text{influx}} \)).

As noted earlier for the rate-time plot, the rate data may be too erratic for consistent matching on this plot.

**Rate Integral-Cumulative Type Curve**

We previously mentioned that production rate data are typically erratic and as such, we developed the dimensionless rate integral versus dimensionless cumulative production type curve. This type curve is shown in Fig. 12. Compared to Fig. 11, we note that Fig. 12 does not reflect the production spike due to the start of boundary flux. Fig. 12 also shows the convergence of the trends to a single trend of cumulative recovery, relative to a particular value of influx strength \( q_{\text{influx}} \).

Again comparing Figs. 11 and 12, we note that the decline trends on Fig 12 appear sharper, which while contrary to intuition about an integral function being smoother than the input function, should help us to interpret specific decline regimes. The "sharpness" of the \( q_{\text{influx}} \) versus \( N_{\text{PD}} \) trends, relative to the \( q_{\text{influx}} \) versus \( N_{\text{PD}} \) trends is more likely due to nature of the \( N_{\text{PD}} \) function than the \( q_{\text{influx}} \) trend. Regardless, Fig. 12 is an excellent diagnostic plot and typically gives better resolution of data trends than any other type curve in this work.

"Ramp" Rate Boundary Flux Case

In this section we consider the case of a "ramped" boundary flux where the flux at the external boundary begins slowly from time zero and builds to the dimensionless terminal rate, \( q_{\text{influx}} \), at approximately \( t_{\text{start}} \). Recalling Eq. 4, we have

\[
q_{\text{influx}}(t) = q_{\text{influx}}[1 - \exp(-t/t_{\text{start}})]
\]

where we note that \( t_{\text{start}} \) is more of a correlation parameter than a physical time.

The net effect of the boundary influx given by Eq. 4, as opposed to a step change in the boundary flux rate, is that the features on the type curves for the "ramp" rate case are much smoother than those of the step rate case, as shown in Fig. 13. There are no spikes responses in production, but rather a flattening of the production, then a secondary depletion trend, where the duration of the "flattened" portion of the trend is a unique function of the transition rate, \( q_{\text{influx}} \). This is analogous to the model for interporosity flow in a naturally fractured (dual porosity) reservoir and will be discussed in a later section of this paper.

**Ramp-Time Curve**

The "rate-time" type curve for the "ramp" rate boundary influx case is presented in Fig. 13. In Fig. 13 we note that the \( q_{\text{influx}} \) trends are very smooth, especially as they deviate from the primary depletion trend. In addition, the trends are uniformly distributed, implying a consistent (i.e., predictable) behavior of the reservoir for this case. Recall that we consider this case to represent natural water influx and a "slow to respond" waterflood pattern, so it is expected that the trends should be uniform, without specific effects which indicate how the influx was initiated, as was the case for the production spikes of the "step" rate boundary flux model.

**Rate Integral-Time Curve**

In Fig. 14 we present the decline type curve of the dimensionless rate integral responses versus the dimensionless time function for the "ramp" rate boundary flux case. As with the "step" rate case, we again note the smooth and uniform behavior of the \( q_{\text{influx}} \) versus \( t_{\text{influx}} \) trends. We also note that the \( q_{\text{influx}} \) functions converge to single trend for a given value of influx strength \( q_{\text{influx}} \), although in this case there appears to be a considerably longer "transition zone" for the point of deviation from the primary depletion trend to the fully developed secondary depletion trend. This "dispersed" behavior is an artifact of the "ramp" rate profile given by Eq. 4. Recall that \( t_{\text{start}} \) is not a physical parameter, but rather, just a variable in the model and its influence may or may not be consistent with natural water influx behavior as we propose.

**Rate Integral Derivative-Time Curve**

We provide the rate integral derivative-time type curve for completeness rather than practical applications, as shown in Fig. 15. It is difficult to say whether field data would be of sufficient quality to apply the rate integral derivative function, but Doublet, et al. showed that the integral derivative function can be applied regularly to field data, particularly for cases with large quantities of "early time" data.

As with the "step" rate case (Fig. 10), Fig. 15 clearly illustrates the transition from primary to secondary performance, as well as the unified behavior we expect for fully developed secondary performance.

**Rate-Cumulative Time Curve**

The rate-cumulative type curve for the "ramp" rate boundary influx case is shown in Fig. 16 and we note the uniform behavior of each set of \( q_{\text{influx}} \) trends. As with the step rate case, this type curve can be used to estimate the primary and secondary recovery for a particular influx strength \( q_{\text{influx}} \). Again, we note that the rate data may be too erratic for consistent matching on this plot, and we recommend the development of a rate integral-cumulative production type curve.

**Rate Integral-Cumulative Time Curve**

The rate integral-cumulative type curve for the "ramp" rate boundary flux case is shown in Fig. 17. As compared to the "step" rate case, we again note that this type curve \( (q_{\text{influx}})N_{\text{PD}} \) has much "sharper" features during the transition to the secondary depletion limits than does the \( q_{\text{influx}} \) versus \( N_{\text{PD}} \) trends shown in Fig. 16. We again find that the influx strength
(q_{Dea,∞}) controls the secondary recovery.

When we consider the q_{Dea} versus N_{p DS} trends for the "step" and "ramp" boundary influx cases (Figs. 12 and 17), we note distinct differences only during the transition from primary depletion to fully developed secondary depletion. The step rate cases show a sharper (and flatter) transition region, while the ramp rate case shows smooth continuous trends. This behavior suggests that we should be able to distinguish the "step" and "ramp" rate models using field data.

Naturally Fractured Reservoir Performance

We noted earlier in our comments regarding the "ramp" rate cases (Figs. 7 and 13) that the constant rate and constant pressure performance data appear to behave similar to the responses for a naturally fractured (or dual porosity) reservoir system (ref. 11). In this section we prepare decline type curve plots of the constant pressure performance for a well centered in a circular naturally fractured reservoir with a "no-flow" outer boundary. These are specialized plots (p_{f}=50) where the model parameters (λ and φ) control the rate response in much the same fashion as the q_{Dea,∞} and t_{Df,∞} parameters.

These specialized plots will be used to compare the performance of the "ramp" rate boundary flux cases with the performance of the naturally fractured reservoir system in order to establish any similarities. Once this presentation and comparison is made we will summarize the significant findings at the end of this section.

Rate-Time Type Curve

The "rate-time" type curve (Fig. 18) shows the influence of the interporosity flow coefficient, λ, and the fracture storativity, φ, on the bounded circular reservoir solution. Clearly, the λ parameter is analogous to t_{Df,∞} and the φ parameter is analogous to q_{Dea,∞}.

Comparing Figs. 13 and 18 we note very similar trends in the primary depletion stems, the " recharge" or transition stems, and the secondary depletion stems. This suggests that the case of a well with boundary flux may "act" like a well in a closed dual porosity reservoir system. While we will confirm this by comparison with additional type curve functions, the obvious question is how do we distinguish performance from such different mechanisms? Especially since many operators believe that their waterflood is "fracture controlled," even when there is no other evidence of natural fractures (well logs, core, well tests, etc.). The answer lies in the geological description of the reservoir as well as the careful monitoring of the waterflood performance.

Rate Integral-Time Type Curve

Fig. 19 illustrates the rate integral performance of the naturally fractured reservoir solution and we again see the similarities between this and the ramp rate case (Fig. 14), including the broad transition from primary to secondary depletion.

Rate Integral Derivative-Time Type Curve

Although a bit overwhelming, Fig. 20 shows the rate integral derivative type curve for the naturally fractured reservoir case and we clearly see the transition from primary to secondary depletion. Comparing to the ramp rate case, Fig. 15, we note many similar features, in particular the amplitude of the rate integral derivative functions appears to be in excellent agreement. This observation further confirms our contention that a ramp rate boundary influx case can be used to model boundary-dominated flow conditions in a dual porosity reservoir, and vice-versa.

Rate-Cumulative Type Curves

As we continue our comparison of the ramp rate boundary influx case and the naturally fractured reservoir case we come to Fig. 21, the rate-cumulative type curve. As before, we again note the excellent agreement between the dual porosity reservoir case and the ramp rate boundary influx case (Fig. 16).

Rate Integral-Cumulative Type Curve

For completeness we also show the rate integral-cumulative type curve in Fig. 22 and compare this plot to the plot for the ramp rate boundary influx case given in Fig. 17. These two plots are quite similar and also serve our argument that one reservoir system could be confused with the other.

Summary Comments: Naturally Fractured Reservoir Cases

The goal of this section was to substantiate our contention that the ramp rate boundary flux model yields a rate performance profile very similar to that of a well in a closed dual porosity system. We believe that these two models are quite similar and can in fact be confused. Whether or not one solution could be "tuned" to model the other, or whether or not we can establish an exact correlation between these two cases is a matter for further study.

What is relevant, and perhaps even urgent, is the realization that a dual porosity reservoir system and a water influx system can have very similar rate-time responses. How an operator distinguishes one case from the other may have a profound effect on reservoir development and production operations. This issue also remains a matter for further study.

Validation--Numerical Simulation Cases

In order to verify our type curve analysis methods we used a commercially available simulation package to model both single- and two-phase flow through a 3-D cartesian grid system in a 40-are reservoir area with homogeneous and isotropic properties. For the single- and two-phase water injection cases, a five-spot geometry was used with constant rate injectors at the pattern corners and a well producing at constant bottomhole pressure in the center of the grid. Although this violates our assumption of uniform boundary influx to some extent, the amount of model anisotropy is minimal, and the injected fluid front stays fairly uniform. As we will see, even when we introduce large amounts of anisotropy (i.e., field data), the type curves generated using analytical solutions that assume uniform boundary influx still work extremely well.

To be consistent, the cartesian grid system was also used for the aquifer influx case. An 1x1x1 tapered grid system was formulated so that the cell block size decreased in the area of the producing well, with the smallest cell block containing the producer. As a result, we obtained excellent matches not only for the boundary-dominated production data, but for the transient production data as well.

The majority of the PVT and rock and fluid data are from the Ninth SPE Comparative Solution Project. The grid geometry, fluid contacts, and relative permeabilities have been changed for this study. For our model validation we used a water-wet, black oil reservoir case. The oil-water relative permeabilities for the two-phase cases were generated using relations developed by Brooks and Corey. For an irreducible water saturation of 22 percent and a pore size distribution index of 2, we obtained the oil-water relative permeabilities shown below:

<table>
<thead>
<tr>
<th>S_w</th>
<th>k_{ow}</th>
<th>k_{ow}</th>
</tr>
</thead>
<tbody>
<tr>
<td>.22</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>.30</td>
<td>0.000</td>
<td>0.797</td>
</tr>
<tr>
<td>.40</td>
<td>0.003</td>
<td>0.560</td>
</tr>
<tr>
<td>.50</td>
<td>0.017</td>
<td>0.358</td>
</tr>
<tr>
<td>.60</td>
<td>0.056</td>
<td>0.201</td>
</tr>
<tr>
<td>.70</td>
<td>0.156</td>
<td>0.092</td>
</tr>
<tr>
<td>.80</td>
<td>0.306</td>
<td>0.029</td>
</tr>
<tr>
<td>.90</td>
<td>0.578</td>
<td>0.004</td>
</tr>
<tr>
<td>1.00</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The pertinent reservoir, rock, and fluid properties for all simulation cases are summarized in the table below. The reader can find the pertinent PVT data in ref. 15.

Geometry

Square cartesian grid system

Gridblock size distribution in X- and Y-direction, ft

453.6, 142.4, 44.6, 14.0, 4.4, 2.0, 4.4, 14.0, 44.6, 142.4, and 453.6
cumulative type curve (Fig. 28) show the result of changing the value of the boundary influx term. The terminal endpoint boundary influx for the data that lies along the $t_{d, start}=10$ trend was 30 STBW/D, while the data along the $t_{d, start}=30$ trend resulted from a value of $q_{est, oo}=12$ STBW/D. This indicates that in the absence of a high degree of reservoir heterogeneity, the strength of the boundary influx and the response time at the producing well are directly related. This is an obvious, but important result for multiphase flow analysis as it proves that the dimensionless start time on the type curve is not only a function of the boundary influx start time and the associated rock and fluid properties as shown in Eq. B-2, but it is also directly related to the strength of the boundary influx. This result indicates that the type curves can be useful in diagnosing possible waterflow operational problems.

We also note that for the two-phase case, the dimensionless boundary influx starting time can not be predicted with any accuracy due to changes in fluid compressibility and other associated fluid property variations, however, this in no way affects the usefulness of the methodology as a qualitative diagnostic tool.

Case 3 - Natural Water Influx (Water Displacing Oil)
Natural water influx was modelled using the Carter-Tracy method\textsuperscript{17} which was the aquifer inflow option available in the commercial simulation package that was utilized for this work. This method yields fair approximations to the analytical solution for aquifer inflow provided by van Everdingen and Hurst.\textsuperscript{4}

A fairly small water aquifer ($r_{eq}=5$ -- aquifer radius is five times reservoir radius) was used for this simulated case and the results are matched on the type curves formulated for a ramp change in boundary influx, as we would expect the response to be smoother than for the step change in boundary influx caused by water injection. The rate-time type curve (Fig. 29), rate integral-time type curve (Fig. 30), and the rate integral-cumulative type curve (Fig. 31) indicate that the reservoir receives only minimal support from an aquifer this small, as we would expect. From Fig. 30, the rate integral-time type curve, we see that the data trend follows the exponential stem until a dimensionless boundary influx start time of approximately $t_{d, start}=3$, and falls just below the $q_{est, oo}=0.5$ trend for the remainder of production history. In this case, as the reservoir is depleting rather quickly without ever reaching steady-state flow conditions, we can perform a quick analysis and estimate ultimate recovery as we will do for the field data cases to follow.

Type Curve Analysis Results

Summary of Type Curve Analysis

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\textit{Variable} & \textit{Rate-Time} & \textit{Rate Integral-} & \textit{Rate Integral- Cum.} \\
 & (Fig. 29) & Time (Fig. 30) & (Fig. 31) \\
\hline
$[\text{ta}]_{\text{max}}$ & 690 Days & 690 Days & -- \\
$[\text{ta}_{\text{max}}]_{\text{max}}$ & 295 STBW/D & -- & -- \\
$[\text{ta}_{\text{max}}]_{\text{max}}$ & 295 STBW/D & 295 STBW/D & -- \\
$[N_{\text{p}}]_{\text{max}}$ & 2640 & 2640 & 2640 \\
$[N_{\text{p}}]_{\text{max}}$ & 2640 & 2640 & 2640 \\
$[\text{ta}]_{\text{max}}$ & -- & 3 & 3 \\
$[\text{ta}]_{\text{max}}$ & -- & 3 & 3 \\
$[\text{ta}]_{\text{max}}$ & -- & 0.5 & 0.5 \\
$[\text{ta}]_{\text{max}}$ & -- & 0.5 & 0.5 \\
\hline
\end{tabular}
\end{center}

Summary

Although the results have shown that there are some limitations to the use of analytical models for the analysis and interpretation of water influx and/or waterflooding pressure-supported reservoir systems, it has been shown that the new type curves provide us with an excellent tool for estimating the flow mechanisms at work within the reservoir. We have obtained acceptable, and what is even more important, expected results from the type curve matching of simulated production data. The true utility of our methods will become more obvious as we attempt to match actual field data.
VALIDATION—FIELD DATA CASES

Another goal of this work is to verify the application of these type curves to field data. One issue that we encountered is that the necessary rate-time data functions must be fairly accurate, and they must be of very long duration—in most cases 20 years or more. This unfortunately makes our work a post-mortem tool, one that is applied after the fact to evaluate performance and interpret the presence and influence of external pressure support.

On the other hand, this work may, as the last field data example shows, give unique insight into the reservoir drive mechanism(s) in force over time. Which will in turn provide the operator with know ledge of how to develop future projects, as well as focus remedial pressure-support efforts.

Recall also that this work was derived from a single-phase analytical solution, and as such, assumes unit mobility ratio displacement. However, as we saw the successful matches of multiphase numerical simulation data in the previous section, we should assume that these methods will work on field data cases as well.

North Robertson (Clearfork) Unit, Gaines Co., TX

The North Robertson (Clearfork) Unit, located in Gaines County on the northeast edge of the Central Basin Platform in the Permian Basin of west Texas, produces from the Glorieta and Upper, Middle, and Lower Clearfork reservoirs. The Lower Clearfork (LCF) reservoir is the dominant section, producing >70 percent of the oil in most areas of the Unit. The Clearfork is a Leonardian shallow-shelf carbonate consisting of a massive dolomite section with varying degrees of anhydrite cement. The depositional sequence was cyclic, and coupled with strong diagenesis, resulted in a thick, heterogeneous reservoir interval.

The North Robertson (Clearfork) Field was developed on a nominal 40-acre well spacing beginning in 1956 and the dominant reservoir producing mechanism was solution-gas drive. Between 1987 and 1991, 116 new wells were drilled and existing producers were converted to water injection as a full-field waterflood program was initiated on 20-acre nominal spacing.

NRU (Clearfork) Unit, Fluid Property, and Production Data

Reservoir Properties:
- Average reservoir depth = 6600 ft
- Wellbore radius, \( r_w \) = 0.31 ft
- Estimated gross pay interval = 1300 ft
- Estimated net pay thickness, \( h \) = 230 ft
- Average porosity, \( \phi \) (fraction) = 0.08
- Average water saturation, \( S_{wtr} \) = 0.25
- Average formation permeability, \( k \) = 1.5 md
- Original nominal well spacing = 40 acres
- Current nominal well spacing = 20 acres

Fluid Properties:
- Average oil formation volume factor, \( B \) = 1.30 RB/STB
- Average oil viscosity, \( \mu \) = 1.3 cp
- Average total compressibility, \( c_t \) = 20.0 x 10^{-6} psi^{-1}

Production Parameters:
- Initial reservoir pressure (LCF), \( p_i \) = 2800 psia
- Flowing bottomhole pressure, \( p_{fof} \) = unknown

NRU Well No. 4202

NRU Well 4202 was drilled in 1962, and completed in the Lower, Middle, and Upper Clearfork sections. The well initially tested at 141 STB/D and had produced approximately 210 MSTB as of May 1995. The response to the unit waterflood occurs at approximately 9,000 days and is identified by a sharp increase in the oil rate.

Type Curve Analysis Results

Summary of Type Curve Analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Rate-Time</th>
<th>Rate Integral-Cum.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{p} )</td>
<td>3250 Days</td>
<td>54 STB/D</td>
</tr>
<tr>
<td>( q_{max} )</td>
<td>54 STB/D</td>
<td>54 STB/D</td>
</tr>
<tr>
<td>( N_{p} )</td>
<td>16,000 STB</td>
<td>( r_w ) = 1.0 \times 10^{-4}</td>
</tr>
<tr>
<td>( t_{d} )</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( q_{d} )</td>
<td>&gt; 0.5</td>
<td>&gt; 0.5</td>
</tr>
</tbody>
</table>

Discussion

Rate-Time Type Curve Analysis: Fig. 32
The most obvious feature on this plot is the production spike which resulted from the initiation of the waterflood—this behavior clearly suggests the use of the "step" rate boundary flux model for the analysis of this well. The primary depletion data match the exponential primary depletion trend quite well and the secondary depletion data appear to match the \( t_{d} = 3 \) trend, but have not established a depletion trend on a particular value of \( q_{d} \).

Rate Integral-Time Type Curve Analysis: Fig. 33
These data match both the primary depletion stem and appear to match the \( t_{d} = 3 \) trend as the rate data did, again with no trend for \( q_{d} \) being having been established. The data do not appear to match a particular transient stem, although \( r_w = 1.0 \times 10^{-4} \) does seem reasonable.

Rate Integral-Cumulative Type Curve Analysis: Fig. 34
The rate integral data match the primary depletion stem quite well and the \( N_{p} \) value of 16,000 STB is actually the estimate of "movable oil" from primary depletion. This value compares well with the value of 175,000 STB, which was the total cumulative oil production at the time the first significant rate increase due to water injection was noted. As with the rate integral-time type curve, the results of this type curve match indicate that \( t_{d} = 3 \), and there is no clear trend established for \( q_{d} \), but we can assume that \( q_{d} = 0.5 \).

Summary
The "waterflood" type curve analysis for NRU Well 4202 gives very consistent matches for each of the type curves. All of the analyses were force matched for consistency (appropriate parameters are "forced" or held constant during the matching process). Qualitatively, the waterflood support appears strong, but the flood should have been initiated earlier.

Emmons (San Andres) Unit, Ector Co., TX

The Emmons Unit, located in Ector County on the Central Basin Platform of the Permian Basin of west Texas is located within the South Cowden Field. Production is from the San Andres platform carbonate, which is a massive dolomite section with a fairly high degree of vertical and lateral heterogeneity, however, not as severe as that associated with the Clearfork at North Robertson.

First production in the area was in 1948, and waterflood operations were initiated in the late 1960's. The Emmons Unit was formed in 1970 and contains approximately 30 oil producing wells and 36 water injection wells. The first response to water injection for the Unit was noted during 1971. The reservoir in this area is thought to be aquifer-supported.

Emmons (San Andres) Unit, Rock, Fluid, and Production Data

Reservoir Properties:
- Average reservoir depth = 4700 ft
- Wellbore radius, \( r_w \) = 0.3 ft (est.)
- Estimated gross pay interval = 240 ft
- Estimated net pay thickness, \( h \) = 170 ft
Average porosity, $\phi$ (fraction) = 0.10
Average irr. water saturation, $S_{wir}$ = 0.25 (est.)
Average formation permeability, $k$ = 60 md
Original nominal well spacing = 40 acres
Current nominal well spacing = 10 and 20 acres

**Production Parameters:**
Initial reservoir pressure, $p_i$ = 1700 psia
Flowing bottomhole pressure, $p_{wf}$ unknown

**Emmons Well 101**
There are no fluid or completion data available for this particular well. Emmons 101 had produced approximately 645 MSTB of oil as of May 1995. The total cumulative oil production at the time the first response to water injection was noted was 310,000 STB. This example was selected because of its seemingly erratic rate performance that was significantly clarified using the rate integral functions.

**Type Curve Analysis Results**

**Summary of Type Curve Analysis**

<table>
<thead>
<tr>
<th>Variable</th>
<th>&quot;Rate-Time&quot;</th>
<th>&quot;Rate Integral-Cumulative Time&quot;</th>
<th>Cum. (Fig. 35)</th>
<th>Cum. (Fig. 36)</th>
<th>Cum. (Fig. 37)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{Dd,1}$</td>
<td>2200 Days</td>
<td>2200 Days</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_{Dd,1}$</td>
<td>105 STB/D</td>
<td>105 STB/D</td>
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</tr>
<tr>
<td>$N_{Dd,1}$</td>
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<td>300,000 STB</td>
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<td></td>
</tr>
<tr>
<td>$t_{Dd,start}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$q_{Dd,cum}$</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
</tr>
</tbody>
</table>

**Discussion**

**Rate-Time Type Curve Analysis:** Fig 35
The rate-time data are quite erratic during secondary performance—perhaps due to the character of the reservoir, but probably due to problems associated with production operations (i.e., wellbore fill, plugging, scale-up, injection wells). During secondary production the rate data oscillates by a factor of 5. A reasonable match was made on the primary exponential depletions after the secondary depletions appear to match the $t_{Dd,start}$=3 trend, but due to the large oscillations in rate we cannot estimate a value for $q_{Dd,cum}$.

**Rate Integral-Cumulative Type Curve Analysis:** Fig 36
These data match the transient stem well for $r_{Dd}=12$, as well as the primary depletions exponential decline. The data seem to establish a secondary trend near the $t_{Dd,start}$=3 stems. The integral rate function removed most of the influence of the rate fluctuations, and our best estimate for $q_{Dd,cum}$ would be >0.875, possibly even steady-state recharge. As aquifer support is probable in this area, the additional of waterflood pressure support could yield a response on the order of 1:1 replacement.

**Rate Integral-Cumulative Type Curve Analysis:** Fig 37
Although slightly erratic at early times, the rate integral data match the transient stem well ($r_{Dd}=12$) and the estimate of "movable oil" from primary depletion is 300,000 STB. This compares very favorably with the 310,000 STB of total cumulative oil production at the time the first waterflood response was noted. Similar to the rate-time and the rate-integral-time analyses, our match on this type curve match gives $t_{Dd,start}$=3, and we again assume $q_{Dd,cum}$>0.875.

**Summary**
Although no fluid or completion data are available for this well we believe that the waterflood type curve analysis for Emmons Well 101 is reasonably accurate. We especially take note of how apparent to be a fairly substantial aquifer support effect. Coupling our uncertainty about the reservoir with the erratic nature of the rate data, we were still able to achieve a consistent analysis on each of the curves. As with the previous case, the analyses were forced matched for consistency.

**La Cira Field (Area 07) Colombia, South America**
La Cira Field is located in the Magdalena Middle Basin, Colombia, South America. The field is an elongated anticline with the major axis in the north-south direction. La Cira produces from three distinct reservoir intervals, with approximately 80% of the cumulative oil produced from Zone C, which is a fluvial sandstone of low to moderate permeability located within the Mugrosa Formation deposited during the Eocene and Oligocene Ages.

The first production from the field occurred in the 1920's, and La Cira originally produced under a solution-gas drive mechanism until water injection was initiated in 1936. The field currently produces approximately 8,000 STB/D at a total water injection rate of 50,000 BW/D.

**La Cira Field (07 Area), Fluid Property and Production Data**

**Reservoir Properties:**
Average reservoir depth = 2700 ft
Estimated wellbore radius, $r_w$ = 0.25 ft (est)
Estimated net pay thickness, $h$ = 100 ft
Average porosity, $\phi$ (fraction) = 0.22
Estimated irr. water saturation, $S_{wir}$ = 0.27
Average formation permeability, $k$ = 20-300 md

**Fluid Properties:**
API gravity, $\delta$API = 24.0
Average oil formation volume factor, $B_o$ = 1.082 RB/STB
Average oil viscosity, $\mu$ = 1.0 DP
Average total compressibility, $c_t$ = 6.1x10^-6 psi^-1

**Production Parameters:**
Bubb ability point pressure, $p_b$ = 957 psia
Initial reservoir pressure, $p_i$ = 1180 psia (140)
Flowing surface tubing pressure, $p_{wf}$ = 50 psia (1/94)
Maximum injection BHP, $p_{inj}$ = 3200 psia (1/94)

**La Cira Well No. 1210**
La Cira Well 1210 is located in the 07 Area in the southeastern section of the field. This area is bounded by inverse faults to the east and west, and by normal faults to the north and south. The first oil production from this section of La Cira Field was in the 1940's, and water injection was initiated in 1971. Total cumulative oil production from the well was approximately 500,000 STB at that time.

**Type Curve Analysis Results**

**Summary of Type Curve Analysis**

<table>
<thead>
<tr>
<th>Variable</th>
<th>&quot;Rate-Time&quot;</th>
<th>&quot;Rate Integral-Cumulative Time&quot;</th>
<th>Cum. (Fig. 38)</th>
<th>Cum. (Fig. 39)</th>
<th>Cum. (Fig. 40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{Dd,1}$</td>
<td>820 Days</td>
<td>820 Days</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_{Dd,1}$</td>
<td>480 STB/D</td>
<td>480 STB/D</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_{Dd,1}$</td>
<td>410,000 STB</td>
<td>410,000 STB</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_{Dd,start}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$q_{Dd,cum}$</td>
<td>= 0.5</td>
<td>= 0.5</td>
<td>= 0.5</td>
<td>= 0.5</td>
<td>= 0.5</td>
</tr>
<tr>
<td>$t_{Dd,start}$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$q_{Dd,cum}$</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
<td>&gt; 0.875</td>
</tr>
</tbody>
</table>

**Discussion**

**Rate-Time Type Curve Analysis:** Fig 38
The rate-time data are quite good in this case and a fairly complete analysis could be done from the rate-time match alone. The data follows the exponential trend until a time which corresponds closely with the point at which the waterflooding was initiated in other areas of the field. It appears that this area was receiving pressure support from outside the pattern, and it is obvious that the faults surrounding this area are non-sealing as was indicated by Cubillos-Gutiérrez. A good match was made on the primary exponential depletions and the first trend of.
secondary depletion data match on $t_{D_{ext, oo}}=1$, with a value of $q_{D_{ext, oo}}=0.5$. This trend appears to correspond to the time during which the well was receiving support from outside its pattern.

The second secondary depletion trend matches the $t_{D_{ext}}=10$ trend, with $q_{D_{ext, oo}}=0.875$. This corresponds to the time during which the well was also receiving injection support from within its own pattern. An acceptable transient match cannot be made on the rate-time plot, however, the data appear to fall on the $r_{D_{p}}=12$ line.

**Rate Integral-Time Type Curve Analysis:** Fig 39

These data match the transient stem quite well for $r_{D_{p}}=12$, as well as the primary depletion exponential decline stem. As was noted above, the “first” secondary depletion trend clearly deviates from the exponential trend on the $t_{D_{ext}}=1$ stem along the $q_{D_{ext, oo}}=0.5$ trend, which is indicative of marginal support from outside the well’s pattern. We can clearly see that when water injection was initiated in the pattern containing Well 1210, the data trend switches to the $t_{D_{ext}}=10$ stem at a dimensionless boundary influx support value of at least 0.875. The data fall slightly above the $q_{D_{ext, oo}}=0.875$ line due to the out of pattern support that the well receives.

**Rate Integral-Cumulative Type Curve Analysis:** Fig 40

An excellent transient match is noted on the $r_{D_{p}}=12$ stem, and the depletion stem matches are even more obvious than on the previous type curve plots. The estimate of "movable oil" from primary depletion is 410,000 STB, which compares fairly well with the estimated total cumulative oil production of 500,000 STB prior to pattern waterflood. The discrepancy may be due to the degree of support the well received from outside its pattern, or perhaps just because this is allocated production data. Similar to the rate-time and the rate integral-time analyses, our match on rate integral-cumulative type curve gives $t_{D_{ext}}$ values of 1 and 10, and we see that once again $q_{D_{ext, oo}}$ appears to be 0.5 and $>0.875$, respectively.

**Summary**

It is obvious that La Cira Well 1210 is receiving good injection support from within its pattern, as well as marginal support from outside the area. The data could be reinitialized to the point at which the pattern waterflood begins to obtain an estimate of what the relative support contributions may be from both inside and outside the well’s immediate vicinity. The type curve matches give a clear indication of the reservoir mechanisms at work in the 07 Area of La Cira Field. This analysis gives an excellent indication of how useful these analyses techniques can be.

**SUMMARY AND CONCLUSIONS**

This work presents the development of an analytical solution for a well centered in a bounded circular reservoir with a general, time-dependent boundary flux. The well can be produced at either constant rate or constant pressure. For the constant well pressure case, a variety of type curve solutions are presented for modelling the primary and “secondary” rate decline performance during. Secondary performance is considered to be the portion of the data affected by boundary influx.

We have successfully compared the analytical solution with results from numerical simulation and field performance data. The analysis and interpretation of the secondary performance portion of field data is considered only qualitative in that our model only accounts for timing and strength of the boundary influx, but not multiphase displacement effects. However, using multiphase reservoir simulation, we have demonstrated the viability of analyzing multiphase flow data with our idealized, single-phase decline type curve solutions.

The principal conclusions of this work are:

1. Decline type curves for single-phase fluid displacement with prescribed flux at the outer boundary in a bounded circular reservoir were developed and successfully demonstrated for the analysis of simulated and field data. The correlating parameters for this model are the timing and strength of the outer boundary flux.

2. The “step” rate boundary flux model exhibits a production "hump" characteristic of waterflood response. Such “humps” observed from our limited field data were generally very erratic and gave poor matches on the rate-time plot. However, the integral rate-time plots show much better agreement, although the span of secondary performance data for our field cases was fairly short and little can be concluded about the efficiency and overall recovery.

3. The “ramp” rate boundary flux model yields a much smoother response than the step rate model, and we consider this to be the representative solution for natural water influx or a slow-starting waterflood.

4. The most striking observation is that the secondary performance portion of the "ramp" rate solution is virtually identical to the response for a well in a bounded dual porosity reservoir. Clearly, this is a significant observation, as pressure support supplied by natural water influx or waterflooding could be misinterpreted as dual porosity reservoir performance, and vice-versa.

5. The new rate-cumulative and rate integral-cumulative-type curves can be used to quickly estimate the primary and secondary reservoirs. These type curves illustrate all of the features of transient and depletion flow, although the cumulative axis is somewhat compressed due to the nature of this function.

The overall goal of this paper was to present a simplified model to characterize the natural water influx/waterflood process in order to develop type curves for the decline curve analysis of secondary performance data. Obviously, a simple approach will not always work. This topic remains relatively unexplored and warrants further investigation—especially in regard to multiphase fluid displacement, but also in regard to the development and application of different models for boundary flux.

**RECOMMENDATIONS**

This work summarizes our efforts in developing analytical models for the analysis and interpretation of water influx and/or waterflood pressure supported reservoir systems. Our goal was to provide decline type curves for the diagnosis and analysis of production response in such systems. To that end believe that we have succeeded.

However, the new reservoir model must be applied cautiously—first as a diagnostic tool, and second as an analysis tool. As such, several questions remain. In particular:

- What do the $t_{D_{int}}$ and $q_{D_{ext, oo}}$ variables represent in a practical sense? Other than the start and strength of influx, respectively—can we interpret reservoir properties from these variables?
- How do we interpret multiphase reservoir performance data using a single-phase reservoir model?
- Are the boundary flux models that we propose consistent with field behavior/practices in general?

We enthusiastically encourage application of these type curves, particularly where large bodies of production data are available. But, we warn that the application and interpretation of these methods must be considered qualitative—until such time that more descriptive models are developed, in which multiphase system properties are incorporated directly into the model, rather than in "humped" variables such as $t_{D_{int}}$ and $q_{D_{ext, oo}}$.

**FUTURE WORK**

In this work we solved the water influx/waterflood problem by using a prescribed boundary flux rate, rather than a prescribed boundary pressure—where either could be constant or a specified function of time. We also made the assumption of single-phase fluid displacement (or implicitly, a unit mobility ratio).
Obviously, the assumption of single-phase flow must be investigated more thoroughly, and if possible, new solutions should be developed.

Likewise, other models for boundary flux could be attempted, but a more productive path would likely be to consider models for the outer boundary pressure condition as a function of time. This would be fairly straightforward to develop using the methodology discussed in Appendix A, and this approach could also be verified by numerical simulation as we have shown for the prescribed boundary flux cases.

As a starting point, we propose the following outer boundary pressure models (in dimensionless form)

**Constant Pressure Condition (Constant at Initial Pressure)**

\[ P_{\text{ext}}(t) = 0 \]  \hspace{1cm} (10)

**“Step” Pressure Condition (Impulse Change of Pressure)**

\[ P_{\text{ext}}(t) = P_{\text{ext,0}} \delta(t - t_{\text{D}}) \]  \hspace{1cm} (11)

where \( \delta(t - t) \) is the “unit step” function.

**“Ramp” Pressure Condition (Smooth Start of Pressure)**

\[ P_{\text{ext}}(t) = P_{\text{ext,0}} [1 - \exp(-t/t_{\text{D}})] \]  \hspace{1cm} (12)

**NOMENCLATURE**

**Field Variables**

- **Formation and Fluid Parameters:**
  - \( A \): reservoir drainage area, \( \text{ft}^2 \)
  - \( B \): formation volume factor, \( \text{RB/STB} \)
  - \( c_t \): total system compressibility, \( \text{psi}^{-1} \)
  - \( \phi (VC)_f \): fracture compressibility, \( \text{psi}^{-1} \)
  - \( \phi (VC)_m \): matrix compressibility, \( \text{psi}^{-1} \)
  - \( [(\phi (VC)_f) + (\phi (VC)_m) = \text{total reservoir compressibility}, \text{psi}^{-1} \)
  - \( \phi \): porosity, fraction
  - \( h \): formation thickness, \( \text{ft} \)
  - \( S_{\text{wirr}} \): irreducible water saturation, fraction
  - \( k \): formation permeability, \( \text{md} \)
  - \( r_e \): reservoir drainage radius, \( \text{ft} \)
  - \( r_w \): wellbore radius, \( \text{ft} \)
  - \( r_{10} \): apparent wellbore radius (includes formation damage or stimulation effects), \( \text{ft} \)
  - \( \mu \): fluid viscosity, \( \text{cp} \)

- **Pressure/Rate/Time Parameters:**
  - \( b \): Fetkovich/Arps decline curve exponent
  - \( N \): original oil in place, STB
  - \( N_p \): cumulative oil production, STB
  - \( q \): oil flow rate, \( \text{STB/D} \)
  - \( q_t \): oil rate integral as defined by McCray
  - \( q_{\text{ext,0}} \): terminal (endpoint) boundary influx, \( \text{STB/D} \)
  - \( p \): pressure, psia
  - \( p_{\text{f}} \): initial reservoir pressure, psia
  - \( p_{\text{bf}} \): flowing bottomhole pressure, psia
  - \( \Delta p \): pressure drop, psia
  - \( r \): radial distance, \( \text{r-direction} \)
  - \( t \): time, \( \text{days} \)
  - \( t_{\text{D}} \): starting time for the boundary influx, \( \text{days} \)
  - \( \tau \): dummy variable of integration

**Dimensionless Variables:**

- **Reservoir Parameters:**
  - \( C_A \): reservoir shape factor
  - \( \gamma \): Euler's Constant = 0.577216

- **Time & Distance Parameters:**
  - \( f(x) = \frac{\lambda + \omega(1 - \omega)\mu}{\lambda + (1 - \omega)\mu} \): dimensionless pseudosteady-state "interporosity" function
  - \( N_P D_d \): dimensionless decline cumulative production function
  - \( \pi = \text{circumference to diameter ratio} = 3.1415926 ... \)

- **Pressure Parameters:**
  - \( P_{\text{D}} = \frac{k h}{3.1415926 A p} \): dimensionless pressure function for the constant flow rate case
  - \( P_{\text{Di}} \): dimensionless pressure integral function
  - \( P_{\text{Dd}} \): dimensionless pressure integral-derivative function
  - \( P_{\text{Dd}} \): dimensionless PSS pressure integral function
  - \( P_{\text{Ddd}} \): dimensionless PSS pressure integral-derivative function

- **Decline Parameters:**
  - \( q_o = \frac{141.2 B_p}{k h (p_{\text{w}} - p_{\text{f}})} \): dimensionless pressure function for the constant flow rate case
  - \( q_{\text{Di}} \): dimensionless rate integral function
  - \( q_{\text{Dd}} \): dimensionless rate integral-derivative function
  - \( q_{\text{Dd}} \): dimensionless PSS pressure function
  - \( q_{\text{Ddd}} \): dimensionless PSS pressure integral-derivative function

- **Other:**
  - \( r_p \): dimensionless terminal (endpoint) boundary influx
  - \( r_a \): dimensionless drainage radius of reservoir
  - \( s \): skin factor for near well damage or stimulation
  - \( t \): dimensionless time
  - \( t_{\text{D}} \): dimensionless time as defined by Fetkovich
  - \( t_{\text{D}} \): dimensionless time based on drainage area
  - \( t_{\text{D}} \): dimensionless time based on drainage area
  - \( t_{\text{D}} \): dimensionless time as defined by Fetkovich

**Special Functions:**

- \( I_0(x) \): modified Bessel function, 1st kind, zero order
- \( I_1(x) \): modified Bessel function, 1st kind, 1st order
- \( K_0(x) \): modified Bessel function, 2nd kind, zero order
- \( K_1(x) \): modified Bessel function, 2nd kind, 1st order

**Special Subscripts:**

- \( Dd \): dimensionless decline variable
- \( ext \): condition at the external boundary of the reservoir
- \( MP \): match point
- \( PSS \): pseudosteady-state
- \( i \): integral
- \( id \): integral derivative

**ACKNOWLEDGMENTS**

We acknowledge the permission to publish field data provided by Fina Oil and Chemical, Co. (Western Division, USA). We also acknowledge the technical assistance of Mr. P.K. Pande (Fina Oil and Chemical Co.) regarding the acquisition and interpretation
of these field data cases.

We gratefully acknowledge the technical and computing support services provided by the Department of Petroleum Engineering at Texas A&M University, as well as the financial support of the United States Department of Energy (DOE) for funding provided through the DOE Class II Oil Program.

REFERENCES

Decline Type Curve Analysis References


Water Influx Models


Naturally Fractured Reservoir Models


General


APPENDIX A - DERIVATION OF THE BOUNDARY FLUX MODEL--BOUNDED CIRCULAR RESERVOIR

Governing Equation, Initial & Boundary Conditions

The fundamental partial differential equation for the flow of a single-phase fluid in a homogeneous porous medium (with a radial flow geometry) is given in "dimensionless" form as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p_D}{\partial r} \right] + \frac{\partial^2 p_D}{\partial \theta^2} + \frac{\partial}{r} \frac{\partial}{\partial \theta} p_D = 0
\]

(A-1)

The "initial" and "boundary" conditions for our problem are given as follows

**Initial Condition** (Uniform Pressure Distribution)

\[
p_D(0, \theta) = 0
\]

(A-2)

**Inner Boundary Condition** (Constant Flowrate at the Well)

\[
\left[ r \frac{\partial p_D}{\partial r} \right]_{r=r_w} = 1
\]

(A-3)

We will solve the condition of production at a constant bottomhole pressure directly from the constant rate solution as provided by van Everdingen and Hurst.6 This Laplace domain relation is given by

\[
\tilde{q}_D(u) = \frac{1}{u} \frac{1}{\tilde{q}_D(0)}
\]

(A-4)

**Outer Boundary Condition** (Prescribed Flux at the Boundary)

\[
\left[ r \frac{\partial p_D}{\partial r} \right]_{r=r_o} = q_{Dext}(t_D)
\]

(A-5)

Where the following flux models will be used

**No-Flow Condition** (No Flux Across the Boundary)

\[
q_{Dext}(t) = 0
\]

(A-6)

**Step" Rate Condition** (Impulse Start of Boundary Flux)

\[
q_{Dext}(t_0) = q_{Dext} \delta(t-t_0)
\]

(A-7)

Where \( \delta(t-t_0) \) is the "unit step" function.

**"Ramp" Rate Condition** (Smooth Start of Boundary Flux)

\[
q_{Dext}(t) = q_{Dext} \{1 - exp\{-(t-t_0)\}/\tau_{Dmax}\}
\]

(A-8)

The negative sign (-) is required for Eqs. A-7 and A-8 because flow is across the boundary, towards the well. This orientation is seen by the reservoir as injection--just as we wish to model.
Development of the General Solution

In order to develop our particular solutions of Eq. A-1 we will use the Laplace transform approach popularized by van Everdingen and Hurst. For completeness, we present the definition of the Laplace transform

\[ \mathcal{L} \{ f(t) \} = \tilde{f}(u) = \int_0^\infty f(t)e^{-ut}dt \]

Where \( u \) is the Laplace transform variable. Applying the Laplace transform to partial derivatives yields

\[ \mathcal{L} \left\{ \frac{\partial^2 \tilde{f}}{\partial x^2} \right\} = \frac{d^2 \tilde{f}}{dx^2} \quad \text{(space derivative relation)} \]

\[ \mathcal{L} \left\{ \frac{\partial \tilde{f}}{\partial t} \right\} = u \tilde{f} - f(t=0) \quad \text{(time derivative relation)} \]

We note that the Laplace transform of the first partial derivative with respect to time will require knowledge of \( f(t=0) \) (and higher derivatives of \( f(t=0) \) for higher partial derivatives). This is how we incorporate the initial condition into the solution of Eq. A-1. This will be demonstrated in the next steps.

Taking the Laplace transform of our governing partial differential equation (Eq. A-1) and using the identities given above, we have

\[ \frac{1}{\sqrt{D}} \frac{d}{d\sqrt{D}} \left[ \sqrt{D} \frac{d \tilde{P}}{d\sqrt{D}} \right] = \tilde{u} \tilde{P} - \tilde{P} \frac{d}{d\sqrt{D}} (P_{D,D,t=0}) \quad \cdots \quad (A-9) \]

Substituting the Initial Condition (Eq. A-2) into Eq. A-9 gives

\[ \frac{1}{\sqrt{D}} \frac{d}{d\sqrt{D}} \left[ \sqrt{D} \frac{d \tilde{P}}{d\sqrt{D}} \right] = \tilde{u} \tilde{P} \quad \cdots \quad (A-10) \]

Defining a variable of substitution, \( z = \sqrt{D} \tilde{P} \)

\[ \text{or} \quad \sqrt{D} \frac{d}{d\sqrt{D}} = \tilde{z} \frac{d}{dz} \quad \cdots \quad (A-11) \]

or

\[ \sqrt{D} \frac{d}{d\sqrt{D}} \left[ \sqrt{D} \frac{d \tilde{P}}{d\sqrt{D}} \right] = \tilde{z} \frac{d}{dz} \quad \cdots \quad (A-12) \]

where

\[ \frac{d}{d\sqrt{D}} \left[ \sqrt{D} \frac{d \tilde{P}}{d\sqrt{D}} \right] = \frac{d}{dz} \quad \cdots \quad (A-13) \]

Applying the chain rule to Eq. A-10 (for the \( z \)-terms) and then multiplying through by \( \tilde{P} \) gives

\[ \frac{1}{\sqrt{D}} \frac{d}{d\sqrt{D}} \left[ \sqrt{D} \frac{d \tilde{P}}{d\sqrt{D}} \right] = \tilde{u} \tilde{P} \quad \cdots \quad (A-14) \]

Substituting Eqs. A-12 and A-13 into Eq. A-14 gives

\[ \tilde{P} = \frac{d^2 \tilde{P}}{dz^2} \quad \cdots \quad (A-15) \]

Expanding the left-hand-side (LHS) of Eq. A-15, we have

\[ \frac{d^2 \tilde{P}}{dz^2} + \frac{d \tilde{P}}{dz} = \tilde{P} \quad \cdots \quad (A-16) \]

Eq. A-16 is our final form of the original differential equation. To this point, we have reduced the original partial differential equation (Eq. A-1) to an ordinary differential equation (Eq. A-10), and then by a variable of substitution to a solvable form. We say solvable because, by inspection, we recognize that Eq. A-16 is actually a form of Bessel’s modified differential equation. Consulting an appropriate reference (in this case, Abramowitz and Stegun19-p. 374, Eq. 9.6.1) we find that the general solution of Eq. A-16 is

\[ \tilde{P}(z) = A I_0(\sqrt{D}z) + B K_0(\sqrt{D}z) \quad \cdots \quad (A-17) \]

Substituting \( z = \sqrt{D} \tilde{P} \) gives

\[ \tilde{P}(rD,t) = A I_0(\tilde{u}rD) + B K_0(\tilde{u}rD) \quad \cdots \quad (A-18) \]

Taking the derivative of Eq. A-18 with respect to \( rD \) yields

\[ \frac{d \tilde{P}}{d(rD)} = A \tilde{u} I_1(\tilde{u}rD) - B \tilde{u} K_1(\tilde{u}rD) \quad \cdots \quad (A-19) \]

Multiplying through Eq. A-19 by \( rD \) gives

\[ rD \frac{d \tilde{P}}{d(rD)} = A \tilde{u} rD I_1(\tilde{u}rD) - B \tilde{u} rD K_1(\tilde{u}rD) \quad \cdots \quad (A-20) \]

We will use Eq. A-20 to solve for the various particular solutions of interest to us, as dictated by the inner and outer boundary conditions.

Development of the Particular Solutions

In this section our efforts are focused on solving the particular solution (in the Laplace domain) for the case of a well produced at a constant rate with a prescribed flux term (\( q_{Des}(\tilde{u}) \)) at the outer boundary. After solving the constant rate problem in the Laplace domain, we will then use the van Everdingen and Hurst6 identity (Eq. A-4) to yield the solution for a well produced at a constant bottomhole pressure.

As with the development of any Laplace transform solution we now require the Laplace transforms of the associated boundary conditions. Taking the Laplace transform of each of the boundary conditions we obtain the following expressions

**Inner Boundary Condition** (Constant Flowrate at the Well)

\[ \left[ rD \frac{d \tilde{P}}{d(rD)} \right]_{rD=1} = \frac{-1}{u} \quad \cdots \quad (A-21) \]

**Outer Boundary Condition** (Prescribed Flux at the Boundary)

\[ rD \frac{d \tilde{P}}{d(rD)} = \mathcal{L} \{ q_{Des}(\tilde{u}) \} = \tilde{Q}_{Des}(u) \quad \cdots \quad (A-22) \]

Where the following flux models are used

**No-Flow Condition** (No Flux Across the Boundary)

\[ \tilde{Q}_{Des}(u) = 0 \quad \cdots \quad (A-23) \]

**Step** Rate Condition (Impulse Start of Boundary Flux)

\[ \tilde{Q}_{Des}(u) = \frac{1}{u} q_{Des,\infty} \exp(-\sqrt{2}u \sqrt{t}) \quad \cdots \quad (A-24) \]

**Ramp** Rate Condition (Smooth Start of Boundary Flux)

\[ \tilde{Q}_{Des}(u) = \frac{1}{u} q_{Des,\infty} \left[ 1 - \frac{1}{u} \right] \quad \cdots \quad (A-25) \]

Our approach for solving the particular solution (i.e., the \( A \) and \( B \) coefficients for a particular set of boundary conditions) is to use Eq. A-20 and the inner and outer boundary flux equations (Eqs. A-21 and A-22, respectively).

Substituting Eq. A-21 into Eq. A-20 gives

\[ A \tilde{u} I_1(\tilde{u}rD) - B \tilde{u} K_1(\tilde{u}rD) = \tilde{Q}_{Des}(u) \quad \cdots \quad (A-26) \]

Substituting Eq. A-22 into Eq. A-20 gives

\[ A \tilde{u} I_1(\tilde{u}rD) - B \tilde{u} K_1(\tilde{u}rD) = \tilde{Q}_{Des}(u) \quad \cdots \quad (A-27) \]

Solving Eqs. A-26 and A-27 simultaneously (after a tedious bit of algebra) we have

\[ \tilde{P}(rD,u) = \frac{1}{u} \left[ \frac{K_0(\tilde{u}rD) I_1(\tilde{u}rD) + K_1(\tilde{u}rD) I_0(\tilde{u}rD)}{u \tilde{u} K_1(\tilde{u}rD) - u \tilde{u} I_1(\tilde{u}rD)} \right] \quad \cdots \quad (A-28) \]

Solving Eq. A-28 at the well (\( rD=1 \)), we obtain

\[ \tilde{P}(rD=1,u) = \frac{1}{u} \left[ \frac{K_0(\tilde{u}rD) I_1(\tilde{u}rD) + K_1(\tilde{u}rD) I_0(\tilde{u}rD)}{u \tilde{u} K_1(\tilde{u}rD) - u \tilde{u} I_1(\tilde{u}rD)} \right] \quad \cdots \quad (A-29) \]

The utility of our approach of using a general boundary flux term, \( \tilde{Q}_{Des}(u) \), is that we can use arbitrary models for the boundary flux (in addition to Eqs. A-23 to A-25). We can also check our algebra by comparing Eqs. A-28 and A-29 with the no-flow boundary condition (\( \tilde{Q}_{Des}(u)=0 \)) developed by Matthews and
Recalling this solution we have

\[ \bar{p}(r_D, u) = \frac{1}{\nu} K_0(\nu r_D) I_1(\nu r_D) + K_1(\nu r_D) I_0(\nu r_D) \]  

(A-30)

Where Eq. A-30 is exactly the same as the first term of Eq. A-28. The reduction to Eq. A-29 from Eq. A-30 is trivial, obtained by substituting \( r_D = 1 \). These comparisons verify our algebra (at least for the no-flow case).

Our next step is to use the van Everdingen and Hurst result for "converting" constant rate solutions to constant well pressure solutions in the Laplace domain. From Eq. A-4 we have

\[ \bar{q}(u) = \frac{1}{\nu} \bar{P}(u) \]  

(A-4)

Rather than generate an even more complicated algebraic expression than Eq. A-29 (the wellbore solution), we prefer to simply substitute the results of Eq. A-29 into Eq. A-4 during the programmation of these solutions into a computer algorithm for the numerical inversion of the \( \bar{p}(r_D = 1, u) \) and \( \bar{q}(u) \) functions. As is standard in the industry, we have used the Gaver-Stehfest numerical inversion algorithm.

For the evaluations of the modified Bessel functions in the equations derived above, we have used the algorithms provided by Cody and Stoltz where these algorithms should provide extended precision accuracy (18-digits). By comparison, the correlations provided in Abramowitz and Stegun are often single-precision correct (7-digits) and often yield numerical instabilities for very large and very small arguments.

### APPENDIX B - PLOTTING FUNCTIONS FOR DECLINE CURVE ANALYSIS USING TYPE CURVES: WATER INFUX/WATERFLOOD CASES

This appendix presents the plots and plotting functions that are used to generate the decline type curves for the water influx/waterflood cases. The dimensionless "decline" variables (time, rate, and pressure functions) are presented in terms of the traditional dimensionless variables \( t_{D2}, q_{D2}, p_{D2} \) with corresponding field variable definitions.

#### Dimensionless Decline Time

In this work we use standard definition given by Fetkovich for the "dimensionless decline time" as our base time function. We recall that the Fetkovich definitions of the dimensionless decline variables \( t_{D2} \) and \( q_{D2} \) provide a unique correlation of boundary-dominated flow data (i.e., the exponential decline term) and as such, this approach clearly indicates the onset and deviation of boundary-dominated flow behavior from production data.

The dimensionless decline time, \( t_{D2} \), is given as

\[ t_{D2} = \frac{2}{r_{D2}} \left[ \frac{1}{\ln r_{D2} - \frac{1}{2}} \right] t_D \]  

(B-1)

where in terms of real variables we have

\[ t_{D2} = 0.00633 \frac{k}{\phi \mu c A} \left[ \frac{2 \pi}{\ln r_{D2} - \frac{1}{2}} \right] t \]  

(B-2)

Before proceeding to the rate and pressure functions we note a minor discrepancy in these definitions (as was noted by Doublet, McCollum, and Blasingame) where the 1/2 term should be 3/4. This was originally noted by Elhig-Economides and Ramey and although this discrepancy with theory is minor—it is also worth noting.

We continue to use 1/2 rather than 3/4 for type curve development in order to be compatible with existing literature, and we note that the effect of this discrepancy is of little (if any) practical consequence.

### Constant Pressure Dimensionless Flowrate Variables

The dimensionless decline flowrate function, \( q_{D2} \), is given by

- **Dimensionless Decline Flowrate**

\[ q_{D2} = \frac{\ln r_{D2} - \frac{1}{2}}{2} q_{D} \]  

(B-3)

or, in terms of real variables we have

\[ q_{D2} = 141.2 \frac{B}{k h_D p} \left[ \ln r_{D2} - \frac{1}{2} \right] q \]  

(B-4)

The analysis of rate data from field operations can be significantly influenced by both "random noise" from acquisition systems as well as by "systematic deviations" from operational practices (e.g., production allocation, well shut-ins, workovers, etc.). We often find ourselves with data that cannot be analyzed without smoothing. Rather than smooth the rate data for analysis, our approach has been to use the rate integral and rate integral derivative functions defined by McCray. This approach provides us with smooth data functions as well as unique trends on type curve solutions. The dimensionless decline flowrate integral function, \( q_{D2i} \), is given as

**Dimensionless Decline Flowrate Integral**

\[ q_{D2i} = N_{D2i} \frac{q_{D2}}{t_{D2}} \int_0^{t_{D2}} \frac{q_{D2}\, dt}{t_{D2}} \]  

(B-5)

where,

\[ N_{D2i} = \int_0^{t_{D2}} q_{D2}(t) \, dt = \frac{2}{r_{D2}} \int_0^{t_{D2}} q_D(t) \, dt \]  

(B-6)

In terms of the dimensionless flowrate integral, \( q_{D2i} \), we have

\[ q_{D2i} = \left[ \ln r_{D2} - \frac{1}{2} \right] q_{D2i} \]  

(B-7)

where,

\[ q_{D2i} = \frac{1}{t_{D2}} \int_0^{t_{D2}} q_D(t) \, dt \]  

(B-8)

Similarly, the dimensionless decline flowrate integral derivative function, \( q_{D2id} \), is given as

**Dimensionless Decline Flow Rate Integral Derivative**

\[ q_{D2id} = \frac{d q_{D2i}}{d \ln(t_{D2})} = -t_{D2} \frac{d q_{D2i}}{d t_{D2}} \]  

(B-9)

or

\[ q_{D2id} = q_{D2i} - q_{D2} \]  

(B-10)

or, in terms of the dimensionless rate integral derivative

\[ q_{D2id} = \left[ \ln r_{D2} - \frac{1}{2} \right] q_{D2id} \]  

(B-11)

where,

\[ q_{D2id} = -\frac{d q_{D2i}}{d \ln(t_{D2})} = -t_{D2} \frac{d q_{D2i}}{d t_{D2}} \]  

(B-12)

or

\[ q_{D2id} = q_{D2i} - q_{D2} \]  

(B-13)

### Constant Rate Dimensionless Pressure Variables

We have defined a "Fetkovich" type dimensionless pressure variable for the case of a well producing at a constant flowrate, where all data should overlay the same trend during pseudo-steady-state flow conditions. We have called these the "dimensionless PSS pressure" functions, where PSS means that the functions are unique during pseudo-steady-state.

The dimensionless PSS pressure concept forces all of the pressure data to converge to the same trend, similar in concept to the "exponential stem" on the original Fetkovich type curve

\[ \text{Constant Rate Dimensionless Pressure Variables} \]
where the well is assumed to produce at a constant bottomhole pressure. Except of course, in this case we consider production at a constant flow rate, rather than at a constant bottomhole pressure.

Our purpose in using these constant-rate "Fetkovich" variables is to validate the \( P_D \) functions (i.e., the numerical inversion of Eq. A-29) for a variety of \( r_D \) values. The constant-rate, dimensionless PSS variables are given below. We note that the so-called "pressure integral" functions are taken from ref. 21.

**Dimensionless PSS Pressure Function**

\[
P_D = \frac{1}{2} \left( \frac{H}{b} \right) \ln \left( \frac{r_D + 1/2}{r_D} \right) \tag{B-14}
\]

or, in terms of real variables we have

\[
P_D = \frac{1}{141.2} \left( \frac{r_D}{H} \right) \ln \left( \frac{r_D + 1/2}{r_D} \right) \tag{B-15}
\]

**Dimensionless PSS Pressure Derivative Function**

\[
p_D' = \frac{1}{r_D \sqrt{2 \pi}} P_D \tag{B-16}
\]

or

\[
p_D = \frac{1}{r_D \sqrt{2 \pi}} P_D \tag{B-17}
\]

where,

\[
p_D = \frac{1}{r_D \sqrt{2 \pi}} P_D \tag{B-18}
\]

**Dimensionless PSS Pressure Integral Function**

\[
p_D = \frac{1}{2} \int_0^{t_D} P_D(t) \, dt \tag{B-19}
\]

or

\[
p_D = \frac{1}{2} \int_0^{t_D} P_D(t) \, dt \tag{B-20}
\]

where,

\[
p_D = \frac{1}{2} \int_0^{t_D} P_D(t) \, dt \tag{B-21}
\]

**Dimensionless PSS Pressure Integral-Derivative Function**

\[
p_D = \frac{1}{2} \int_0^{t_D} \frac{dP_D}{dt} \, dt \tag{B-22}
\]

or

\[
p_D = \frac{1}{2} \int_0^{t_D} \frac{dP_D}{dt} \, dt \tag{B-23}
\]

where

\[
p_D = \frac{1}{2} \int_0^{t_D} \frac{dP_D}{dt} \, dt \tag{B-24}
\]

In this work we use the functions presented above to illustrate the constant-rate depletion performance (i.e., pseudosteady-state flow behavior) for the new solution (given in the Laplace domain by Eq. A-28). This approach provides a unique correlation during pseudosteady-state flow conditions and helps to distinguish the influence of boundary flux on well performance.

**APPENDIX C - DEVELOPMENT AND INTEGRATION OF EXTERNAL BOUNDARY FLUX MODELS**

In this section we discuss the models that we have chosen to prescribe the flux at the outer boundary of the reservoir, and in particular, the physical significance of the model parameters \( q_{D,\infty} \) and \( t_{D,\infty} \).

In this work we consider the reservoir as a sealed system with withdrawals taken from the well and recharge from sources located uniformly around the reservoir boundary. This scenario clearly describes the case of uniform water influx with the exception being that we allow "influx" to start at a prescribed time, \( t_{D,\text{start}} \), (rather than time zero) and we allow the "source" at the boundary to have a prescribed strength, \( q_{D,\infty} \), which can vary instantaneously or continuously from 0 to 1.

The previous concepts of water influx consider an aquifer supplying fluid at the initial reservoir pressure—which ultimately leads to steady-state flow, since for our case the well is produced at a constant pressure. Such "ideal" recharge is rarely found in practice, and in cases of waterflooding we suspect significant "leakage" to other portions of the reservoir, by "driest" zones (and fractures) as well as by poor balancing of injectants with pattern geometry.

While our model of a circular reservoir pattern with unit mobility (single-phase) displacement is highly idealized, the concept of constant rate injection, rather than constant pressure injection is not. In addition, our variable "start" times provide realistic control on the injection process. While validation of our models with field data is incomplete—we believe that the field data have conceptually verified our boundary flux models and our overall approach. Future work should consider both additional boundary flux conditions and multiphase fluid displacement.

**External Boundary Flux Models**

As with any analysis technique, our goal is to match field performance so that predictions of future performance can be made—relative to both the quality of the field data and the validity of the analysis model. As mentioned above, our goal is to accurately model reservoir performance for cases of both natural water influx and induced waterflow.

Our approach is to consider the flux across the reservoir boundary as constant or eventually constant, and to consider a maximum ratio of injection to production of 1:1 (steady-state conditions). This condition arises from the mathematics of combining the reservoir and boundary flux models—where a replacement ratio of higher than 1:1 causes instabilities, and ultimately failure of the \( P_D \) solution (Eq. A-29).

From our previous references in the text and Appendix A, our boundary flux models are given by

**No-Flow Condition (No Flux Across the Boundary)**

\[
q_{D,\infty}(t_D) = 0 \tag{A-6}
\]

**"Step" Rate Condition (Impulse Start of Boundary Flux)**

\[
q_{D,\infty}(t_D) = q_{D,\infty} \text{U}(t_D) \tag{A-7}
\]

where \( \text{U}(t_D) \) is the "unit step" function.

**"Ramp" Rate Condition (Smooth Start of Boundary Flux)**

\[
q_{D,\infty}(t_D) = q_{D,\infty} \text{[1-exp(-t_D/t_{D,\text{start}})]} \tag{A-8}
\]

As mentioned earlier, the negative sign (-) is required for Eqs. A-7 and A-8 because flow is across the boundary, towards the well.

The application of Eqs. A-7 and A-8 in type curve solutions require that we present "families" of solutions related to the \( t_{D,\text{start}} \) and \( q_{D,\infty} \) variables. However, from the standpoint of a correlation, we need to express the dimensionless start time, \( t_{D,\text{start}} \). In terms of the dimensionless decline time, \( t_D \), because we use the \( t_D \) format as the basis for our decline type curves. Recalling the definition of \( t_D \) (Eq. B-1) we have

\[
t_D = \frac{2}{r_D} \left[ \frac{1}{\ln \left( \frac{r_D}{r_D + 1/2} \right)} \right] \tag{B-1}
\]

Substituting the dimensionless start time, \( t_{D,\text{start}} \) into Eq. B-1
Decline Curve Analysis Using Type Curves: Water Influx/Waterflood Cases

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gives

\[ t_{D, start} = \frac{2}{r_{D}} \left[ \ln \left( \frac{r_{D}}{r_{p}} \right) - \frac{1}{2} \right] t_{D, start} \]  \hspace{1cm} (C-1)

Or, solving for the dimensionless start time, \( t_{D, start} \), we have

\[ t_{D, start} = \frac{2}{r_{D}} \left[ \ln \left( \frac{r_{D}}{r_{p}} \right) - \frac{1}{2} \right] t_{D, start} \]  \hspace{1cm} (C-2)

By convention, our start time family parameter is \( t_{D, start} \). The physical significance of the "start" time in its representation of the start of flux across the boundary—but the practical significance of the start time is its effect on recovery. Clearly, the earlier the start of boundary flux, the faster the recovery.

By contrast, the \( q_{D, start} \) may have little physical significance as the "strength" of the source—but as a practical matter the \( q_{D, start} \) term clearly indicates the efficiency of the flood/influx process.

APPENDIX D - PROCEDURES FOR THE ANALYSIS OF PRODUCTION DATA FROM WATER-INFLUX/WATERFLOOD RESERVOIR SYSTEMS

In this appendix, we discuss the analysis relations for analyzing the primary and secondary flow data using our new Fetkovich-style type curves for reservoir systems undergoing active water influx or waterflooding. To simplify our work, we will separate the analysis and interpretation sequence into two parts, the first part is for primary depletion and the second part is for evaluating secondary performance.

Evaluation of Primary Performance

The quantitative analysis of primary performance data using Fetkovich-style type curves is well documented in the literature (refs. 1-5) and the pertinent analysis relations will be summarized in this section. The qualitative evaluation of the primary performance data is also relevant—especially for estimating the reservoir drive mechanism(s) and identifying the need for remedial work and secondary recovery activities.

Our approach for the quantitative analysis of primary performance data focuses on the case of a well producing at a constant bottomhole pressure (as was the model assumption for our new type curve solutions). In practice, the case of a constant bottomhole flowing pressure is somewhat limited—but this approach is widely applied and is accepted as the standard for decline type curve analysis.

Type Curve Matching Procedure:

The procedures for "matching" data onto a type curve are well-known in the petroleum industry—especially for the analysis of transient well tests. For the case of using decline type curves, we simply prepare a plot of data grid-dotted to match the type curve (usually a log-log plot with 3-inch square log cycles) and overlay these data onto the appropriate (transient and depletion) trend(s) on the type curve. Once the data are "matched," any pairs of corresponding \( x \) and \( y \)-axis coordinates (real and dimensionless) are read as the "match points" and the "family" parameters (in our case, \( r_{D} \) and \( b \)) are recorded. This procedure is given below:

1. Assemble the well rates (STB/D) versus time (in days).
2. Compute the cumulative production and the "flowrate integral function" from the \( q \) versus \( t \) trend.

\[ N_p = \int_0^t q(\tau) d\tau \]  \hspace{1cm} (D-1)

And the flowrate integral function, \( q_{t} \), is defined as

\[ q_{t} = \int_0^t q(\tau) d\tau = \frac{N_p}{t} \]  \hspace{1cm} (D-2)

3. Prepare the following type curve plots on a scaled log-log grid (usually on 3-inch log cycles).
   a. Rate-time data (\( q \) vs. \( t \))
   b. Rate integral-time (\( q_{t} \) vs. \( t \))
   c. Rate-cumulative (\( q \) vs. \( N_p \))
   d. Rate integral-cumulative (\( q_{t} \) vs. \( N_p \))

4. Force match the depletion data trends onto the Arps \( b=0 \) (exponential) for each of the Fetkovich-type style curves being used: rate-time, rate-integral-time, rate-cumulative, and the rate integral-cumulative.

Once a match is obtained, record the "time" and "rate" axis match points as well as the \( r_{D} \) transient flow stem are recorded as follows

a. Rate Match Point: any \( (q)_{MP} - (q)_{DMP} \) pair
b. Time Match Point: any \( (q_{t})_{MP} - (q_{t})_{DMP} \) pair
c. Match of \( r_{D} \) transient flow stem

For consistency between the different analysis plots, the data on different plots should be "force matched," where the \( y \)-axis match point is estimated then held constant for each of the analysis plots.

Estimation of Reservoir Properties: \( (N, A, k, r_{sw}, s) \)

Once a match of the data and the type curve has been obtained, the time and rate axis "match points" can be used to estimate the oil-in-place, \( N \), and the reservoir drainage area, \( A \).

Original Oil-in-Place

\[ N = \frac{1}{c_{AP}} \frac{(q)_{MP} (q)_{DMP}}{(q_{t})_{MP} (q_{t})_{DMP}} \]  \hspace{1cm} (D-3)

where MP indicates that the "match point" values are used.

Reservoir Drainage Area

\[ A = 5.6148 \frac{N B}{g_{p} (1-S_{w, ini})} \]  \hspace{1cm} (D-4)

where the "effective" drainage radius, \( r_{e} \), is given by

\[ r_{e} = \sqrt{\frac{A}{\pi}} \]  \hspace{1cm} (D-5)

From the match of the data on a particular transient stem (a unique value of \( r_{D} \)), we can solve for the formation permeability, \( k \), the effective wellbore radius, \( r_{sw} \), and the skin factor, \( s \).

Formation Permeability

\[ k = 141.2 \frac{Bh}{h_{DMP}} \left[ \ln \left( \frac{r_{D}}{r_{p}} \right) - \frac{1}{2} \right] \left( \frac{(q)_{MP}}{(q_{t})_{DMP}} \right) \]  \hspace{1cm} (D-6)

or, for a generalized well geometry, we have

\[ k = 70.6 \frac{Bh}{h_{DMP}} \left[ \ln \left( \frac{4A}{\varepsilon C_{w, ini}^{2}} \right) \left( \frac{(q)_{MP}}{(q_{t})_{DMP}} \right) \right] \]  \hspace{1cm} (D-7)

where MP again refers to the "match point" values.

Effective Wellbore Radius

\[ r_{sw} = \frac{r_{e}}{r_{D}} \]  \hspace{1cm} (D-8)

Near-Well Skin Factor

\[ s = -\ln \frac{r_{sw}}{r_{sw}} \]  \hspace{1cm} (D-9)

Evaluation of Secondary Performance

Our idealized reservoir model is well suited for qualitative analysis of field performance as we can graphically compare field and model performance using very limited assumptions about the boundary flux. Unfortunately, the assumptions that we have made in our model prevent a detailed quantitative analysis from...
being made. Some of our assumptions are:

- Bounded circular reservoir with boundary flux acting uniformly around the reservoir,
- "Step" or "ramp" boundary flux conditions—where flow begins either abruptly at \( t_{\text{start}} \) or varies gradually from \( t=0 \),
- Constant producing pressure at the well, and
- Unit mobility, single-phase fluid displacement process.

Obviously the most significant assumption is that of single-phase flow. This would hardly be likely in practice, but the concept of uniform, "piston-like" displacement (assuming a unit mobility ratio) is not unrealistic and is regularly used in waterflood calculations. In fact, we even suggest that deviation of field performance from the idealized model should be attributed to non-uniform displacement, all other conditions being equal (i.e., non-compositional fluids (water and a black oil), a homogeneous reservoir, and a stable injection rate/production pressure profile).

Non-idealities such as multiphase flow, reservoir heterogeneities, and an unbalanced injection/production profile are common in practice, but we believe that our type curve approach for qualitatively evaluating water influx and waterflood performance is both consistent and representative of the physical process. Unfortunately, we cannot adjust our model to uniquely reflect non-idealities such as multiphase flow—and as such, we cannot quantitatively estimate properties that we do not model.

In particular, our model has only 2 parameters for "controlling" the water-influx/waterflood process—these are the starting time of the process, \( t_{\text{start}} \), and the terminal influx rate, \( q_{\text{est}, \infty} \). To assume that these two parameters could model all of the possible variations in fluid displacement as well as heterogeneous reservoir effects would be optimistic at best and foolish at worst. Nonetheless, we can use these parameters to estimate the effectiveness and the efficiency of the displacement process.

The influence of \( t_{\text{start}} \) (or \( t_{\text{start}} \), \( t_{\text{pump}} \), \( t_{\text{draw}} \)) is clearly evident on the type curves as this variable controls the rate of recovery, while \( q_{\text{est}, \infty} \) (or \( q_{\text{dest}, \infty} \)) represents the recovery efficiency by illustrating the "strength" of the influx source—with steady-state flow being the limiting case of influx strength (i.e., \( q_{\text{dest}, \infty} = 1 \)).

How one chooses to interpret well performance using our idealized reservoir model will depend on the reservoir data available (the geologic description, the pressure and rate history, and the rock and fluid property data), as well as the experience of the analyst. Our recommendation is to use this model as a "screening" tool prior to more sophisticated modeling using reservoir simulation, but we also encourage the user audience to use this model as demonstrated in this paper, as a qualitative tool for assessing the efficiency and performance of the water influx/waterflood process.
Figure 38 - Match of Rate versus Time Data for La Cima Well 1210 on the Fellowsich-Style "Waterflood" Type Curves for an Unfractured Well Centered in a Bounded Circular Reservoir (Constant Production Pressure and "Step" Rate Boundary Flux).

Figure 39 - Match of Rate Integral versus Time Data for La Cima Well 1210 on the Fellowsich-McCray-Style "Waterflood" Type Curves for an Unfractured Well Centered in a Bounded Circular Reservoir (Constant Production Pressure and "Step" Rate Boundary Flux).

Figure 40 - Match of Rate Integral versus Cumulative Production Data for La Cima Well 1210 on the Fellowsich-McCray-Style "Waterflood" Type Curves for an Unfractured Well Centered in a Bounded Circular Reservoir (Constant Production Pressure and "Step" Rate Boundary Flux).
Type Curve Package From:

SPE 30774

Decline Curve Analysis Using Type Curves:
Water Influx/Waterflood Cases

by L.E. Doublet and T.A. Blasingame,
Texas A&M University

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Model Legend: Fetkovich-Style Type Curve -- Unfractured Well Centered in a Bounded Circular Reservoir (Step Change in Influx (from Zero) at External Boundary)

Type Curves From:
SPE 30774
Decline Curve Analysis Using Type Curves: Water Influx/Waterflood Cases
by L.E. Douillet and T.A. Blasingame, Texas A&M University
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Legend:
- \( q_{Dd} \) vs. \( N_{pDd} \)
  - Fetkovich/Arps
  - \( q_{out} = 0.50 \)
  - \( q_{out} = 0.75 \)
  - \( q_{out} = 0.875 \)
  - \( q_{out} = 0.95 \)
  - \( q_{out} = 0.98 \)
  - \( q_{out} = 1.00 \)
  - (Steady-State)
Model Legend: Fatkovich-Style Type Curve
Unfractured Well Centered In a Bounded Circular Reservoir
("Ramp" Change in Influx (from Zero) at External Boundary)

Transient "Stems"
(Transient Radial Flow Region – Analytical Solutions)

Legend: $q_{Dd} vs. t_{Dd}$
- Fatkovich/Arps
- $q_{Out}=0.50$
- $q_{Out}=0.75$
- $q_{Out}=0.875$
- $q_{Out}=0.95$
- $q_{Out}=0.98$
- $q_{Out}=1.0$ (Steady-State)

Primary Depletion "Stem" (Exponential Decline)
Secondary Depletion "Stems"
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2c — Real Domain Solutions of the Radial Flow Diffusivity Equation for a Well Produced at a Constant Rate in a Bounded Circular Reservoir: Infinite and Finite-Acting Reservoir Cases

Resistance to tyrants is obedience to God.
— Thomas Jefferson (motto)

Topic: Real Domain Solutions of the Radial Flow Diffusivity Equation for a Well Produced at a Constant Rate in a Bounded Circular Reservoir: Infinite and Finite-Acting Reservoir Cases

Objectives: (things you should know and/or be able to do)
- Be able to derive the following particular solutions in the real domain using the appropriate Laplace transform solutions for an unfractured well produced at a constant flowrate in a homogeneous reservoir for the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior (line source solution)
    \[ p_D(t_D,r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] \]
  - "Infinite-acting" reservoir behavior (the so-called "log approximation," also a line source solution)
    \[ p_D(t_D,r_D) = \frac{1}{2} \ln \left[ \frac{4}{e^r} \frac{r_D}{r_D} \right] \]
  - Bounded circular reservoir — "no-flow" at the outer boundary
    \[ p_D(t_D,r_D,r_{eD}) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{eD}^2}{4t_D} \right] + \frac{2t_D}{r_{eD}^2} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] + \frac{r_D^2}{2r_{eD}^2} - \frac{1}{4} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] \]
    and its "well testing" derivative function, \( p_D' = \frac{d}{dt_D} [p_D(r_D,t_D)] \) is given by
    \[ p_D'(t_D,r_D,r_{eD}) = \frac{1}{2} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] + \frac{2t_D}{r_{eD}^2} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] + \frac{1}{2t_D} \left[ \frac{r_D^2}{4} - \frac{r_{eD}^2}{8} \right] \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] \]
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    \[ p_D(t_D,r_D,r_{eD}) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{eD}^2}{4t_D} \right] + \frac{1}{8t_D} (r_{eD}^2 - r_D^2) \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] \]
    and its "well testing" derivative function, \( p_D' = \frac{d}{dt_D} [p_D(r_D,t_D)] \) is given by
    \[ p_D'(t_D,r_D,r_{eD}) = \frac{1}{2} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] - \frac{1}{2} \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] + \frac{1}{8t_D} (r_{eD}^2 - r_D^2) \left[ \frac{r_{eD}^2}{4t_D} - 1 \right] \exp \left[ \frac{-r_{eD}^2}{4t_D} \right] \]

Lecture Outline:
- Development of solutions in the real domain:
  - "Infinite-acting" reservoir behavior (line source solution)
    - Cylindrical source solution not directly invertible (in closed form).
  - Bounded circular reservoir — "no-flow" at the outer boundary
    - Inversion of line source solution using recursion relations and polynomial expansions for Bessel functions (for behavior near zero).
Lecture Outline: (Continued)

- Development of solutions in the real domain: (Continued)
  - Bounded circular reservoir — "no-flow" at the outer boundary (Continued)
    - Derivatives taken explicitly from real domain solution rather than Laplace transform solutions. Can check directly, term-by-term.
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    - Inversion of the line source solution using recursion relations and polynomial expansions for Bessel functions (for behavior near zero).
    - Derivatives taken explicitly from real domain solution rather than Laplace transform solutions. Can check directly, term-by-term.

- Discussion of Applications
  - Modelling of well performance (transient and pseudosteady-state performance, variable-rate superposition).
  - Development of short- and long-term analysis relations.

Reading Assignment:

- Review attached notes.
  - Solution of the Dimensionless Radial Flow Diffusivity Equation:
    - Real domain solutions via inversion of the Laplace transform solutions.

Exercises: For your own practice/skills building—do **NOT** turn in!

From the attached notes you are to rederive the following, and show all details.

- Starting from the Laplace transform solutions, derive the real domain solution(s) for an unfractured well produced at a constant flowrate (inner boundary) in a homogeneous reservoir with the following outer boundary condition(s):
  - Bounded circular reservoir — "no-flow" at the outer boundary
  - Bounded circular reservoir — "prescribed" at the outer boundary
  - Bounded circular reservoir — "constant pressure" at the outer boundary
**Log-log Plot:** Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)

**Semilog Plot:** Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir—Various $r_D$: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Constant Pressure Outer Boundary: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Wellbore Pressure Solutions for a Bounded Circular Reservoir: Dimensionless Rate Functions—Radial Flow Case (SPE 25479)
Solution of the Dimensionless Radial Flow Diffusivity Equation:

- Real Domain Solutions via Inversion of the Laplace Transform Solutions

Solutions for a Bounded Circular Reservoir: Infinite-Acting, No-Flow, and Constant Pressure Boundary Cases

The Laplace transform solutions under consideration are

a. Infinite-Acting Reservoir Case:

\[ \tilde{p}_p(r_p, \mu) = \frac{1}{m} \frac{k_p(\sqrt{\mu} r_p)}{\sqrt{\mu} k_i(\sqrt{\mu})} \] (cylindrical source solution) \ (1)

\[ \tilde{p}_l(r_p, \mu) = \frac{1}{m} \frac{k_0(\sqrt{\mu} r_p)}{\sqrt{\mu}} \] (line source solution) \ (2)

\[ \tilde{p}_l(r_p, \mu) = \frac{1}{2m} \ln \left( \frac{4}{e^{\frac{\pi}{2}} \sqrt{\mu} r_p^2} \right) \] (as \( \mu \to 0 \), log approximation) \ (3)

b. No-Flow Boundary Case:

\[ \tilde{p}_p(r_p, \mu) = \frac{1}{m} \frac{k_0(\sqrt{\mu} r_p)}{\sqrt{\mu} k_i(\sqrt{\mu})} \tilde{I}_0(\mu \lambda_{rep}) + \frac{1}{m} \frac{k_i(\sqrt{\mu} \lambda_{rep}) \tilde{I}_0(\mu \lambda_{rep})}{\lambda_{rep}} \] \ (4)

where for \( \mu \to 0 \) Eq. 4 reduces to

\[ \tilde{p}_p(r_p, \mu) = \frac{1}{m} \frac{k_0(\sqrt{\mu} r_p)}{\sqrt{\mu} k_i(\sqrt{\mu})} \tilde{I}_0(\mu \lambda_{rep}) \] (line source) \ (5)

\[ \frac{1}{m} \frac{k_0(\sqrt{\mu} \lambda_{rep}) \tilde{I}_0(\mu \lambda_{rep})}{\lambda_{rep}} \]

\[ \] \ (6)

where for \( \mu \to 0 \) Eq. 6 reduces to

\[ \tilde{p}_p(r_p, \mu) = \frac{1}{m} \frac{k_0(\sqrt{\mu} r_p)}{\sqrt{\mu} k_i(\sqrt{\mu})} \tilde{I}_0(\mu \lambda_{rep}) \] (line source) \ (7)

\[ \frac{1}{m} \frac{k_0(\sqrt{\mu} \lambda_{rep}) \tilde{I}_0(\mu \lambda_{rep})}{\lambda_{rep}} \]

\[ \] \ (8)

Solutions for an Infinite-Acting Reservoir:

\[ \] \ (9)

Unfortunately, Eq. 1 cannot be inverted directly to yield a closed form, non-infinite series or integral solution. However, van Everdingen and Hurst give the following results

\[ \tilde{p}_p(r_p, \mu) = \frac{1}{\pi} \int_0^\infty \left[ 1 - e^{-\pi^2 m^2} \right] \tilde{J}_0(\mu y_{rep}) \tilde{J}_0(y_{rep}) \] \ (10)

and for the wellbore solution \( (r_p=1) \) Eq. 8 reduces to

\[ \tilde{p}_p(r_p=1, \mu) = 4 \pi \int_0^\infty \left[ 1 - e^{-\pi^2 m^2} \right] \frac{1}{m^3} \tilde{J}_0(y_{rep}) \] \ (11)
b. Line Source Solution:

Recalling Eq. 2, we have

\[ \phi_B(r, \theta, \phi) = \frac{1}{\mu} k_0(Hr \rho) \]

Multiplying through Eq. 2 by the Laplace transform parameter, \( u \), gives

\[ u \phi_B(r, \theta, \phi) = k_0(uHr \rho) \]

Recalling the time derivative theorem for Laplace transforms we have

\[ L \left\{ \frac{d[F(t)]}{dt} \right\} = uF(u) - f(t=0) \]

assuming that \( f(t=0) \), which is true by our initial condition, we can similarly write

\[ \frac{d[F(u)]}{du} = L^{-1} \left\{ uF(u) \right\} \]

or in terms of our problem we have

\[ \frac{d}{dt_0} \phi_B(r, \theta, \phi) = L^{-1} \left\{ k_0(uHr \rho) \right\} \]

Combining Eqs. 10 and 13

\[ \frac{d}{dt_0} \phi_B(r, \theta, \phi) = L^{-1} \left\{ k_0(uHr \rho) \right\} \]

Inversion of Eqs. 10 and 11 is accomplished by the use of Laplace transform tables, where the results of inversion are given below

<table>
<thead>
<tr>
<th>( \tilde{f}(u) )</th>
<th>( f(t) )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\mu} k_0(uHr \rho) )</td>
<td>( \frac{1}{\mu} E_i \left( \frac{a^2}{4t} \right) )</td>
<td>Carslaw and Jaeger: Conduction of Heat in Solids, Table V, Eq. 26, p. 495.</td>
</tr>
<tr>
<td>( k_0(uHr \rho) )</td>
<td>( \frac{1}{2t} \exp \left( \frac{-a^2}{4t} \right) )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Eq. 29.3.120, p. 1028, and Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304.</td>
</tr>
</tbody>
</table>

Making the appropriate substitutions

\[ \phi_B(r, \theta, \phi) = \frac{1}{\mu} E_i \left( \frac{r_0^2}{4t_0} \right) \]

and

\[ \frac{d}{dt_0} \phi_B(r, \theta, \phi) = \frac{1}{2t_0} \exp \left( \frac{-r_0^2}{4t_0} \right) \]
Defining the so-called "well testing derivative" we have
\[ p'_b(r_b, t_D) = t_D \frac{d}{d t_D} \left[ p_b(r_b, t_D) \right] \]  
\hfill (17)

Substituting Eq. 16 into Eq. 17 we have
\[ p'_b(r_b, t_D) = \frac{1}{2} \exp \left( -\frac{r_b^2}{4 t_D} \right) \]  
\hfill (18)

C. Log Approximation Solution:
Recalling Eq. 3 we have
\[ p_b(r_b, \mu) = \frac{1}{2 \mu} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \mu \right) \]  
\hfill (19)

Expanding Eq. 3 into a more usable form, we have
\[ p_b(r_b, \mu) = \frac{1}{2} \left[ -\frac{1}{\mu} \ln(\mu) + \frac{1}{\mu} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \right] \]  
\hfill (19)

Rather than attempt a derivative using \( p_b(r_b, \mu) \), we will simply differentiate the inversion result of Eq. 19. The inverse Laplace transform of the \( \ln(\mu) \) and constant terms in Eq. 19 we have

\[ \mathcal{F}[\ln(\mu)] = \frac{1}{\mu} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \]  
\[ \mathcal{F}[t] = \ln(t) + 8 \]  
\[ \mathcal{F}[e^{2 \delta t}] = \ln(e^{2 \delta t}) \]  
\[ \text{References} 
\]  
\[ \text{Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.98, p. 1027.} \]
\[ \text{and} \]
\[ \text{Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 4.1, p. 258.} \]

\[ \frac{1}{\mu} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \]  
\[ \text{or} \]
\[ \frac{1}{\mu} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \]  
\[ \text{or} \]
\[ \text{constant} \]

Writing the \( p_b(r_b, t_D) \) inversion result for Eq. 19 is
\[ p_b(r_b, t_D) = \frac{1}{2} \left[ \ln \left( e^{2 \delta t_D} \right) + \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \right] \]  
\hfill (19)

Collecting
\[ p_b(r_b, t_D) = \frac{1}{2} \ln \left( \frac{4}{e^{2 \delta} r_b^2} \right) \]  
\hfill (19)
Isolating the \( t_D \) term in Eq. 20 we have
\[
\rho_D(r_0, t_D) = \frac{1}{2} \ln(t_D) + \frac{1}{2} \ln\left(\frac{4}{\varepsilon r_0^2}\right)
\]  
(21)

Substituting Eq. 21 into Eq. 17 to determine the well testing derivative we have
\[
\rho'_D(r_0, t_D) = t_D \left[ \frac{d}{dt_D} \left( \frac{1}{2} \ln[t_D] \right) \right] + t_D \left[ \frac{d}{dt_D} \left( \frac{1}{2} \ln\left[ \frac{4}{\varepsilon r_0^2} \right] \right) \right]
\]
which reduces to
\[
\rho'_D(r_0, t_D) = t_D \left[ \frac{1}{2t_D} \right] = \frac{1}{2}
\]  
(22)

Solution for a No-Flow Outer Boundary:

It is not possible to invert the complete solution (Eq. 4) for this case, so we will attempt an approximate solution of the line source form (Eq. 5). Recalling Eq. 5 we have
\[
\rho_D(r_0, \omega) = \frac{1}{\mu} \kappa_0(\omega r_0) + \frac{1}{\mu} \kappa_1(\omega r_0) \frac{I_0(\omega r_0)}{I_1(\omega r_0)}
\]  
(5)

We immediately recognize that the first term in Eq. 5 is the solution for an infinite-acting reservoir, and given the linearity of the inverse Laplace transform, we can invert Eq. 5 to yield
\[
\rho_D(r_0, t_D) = \rho_{D, \text{inf}}(r_0, t_D) + \frac{1}{\mu} \frac{1}{I_1(\omega r_0)} \frac{I_0(\omega r_0)}{I_1(\omega r_0)}
\]  
(23)

where
\[
\rho_{D, \text{inf}}(r_0, t_D) = \frac{1}{2} \left( \frac{r_0^2}{4t_D} \right)
\]  
(24)

So what is our strategy to invert the second term in Eq. 23?
First we will use recursion relations to express the \( k_n(z) \) Bessel functions then consider a two-term expansion of the resulting \( I_0(z)/I_1(z) \) ratio. Recall that as \( z \to 0 \) that \( I_0(z) \to 1 \) and \( I_1(z) \to 0 \), which permits polynomial expansions.

From Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.15, p. 875) we have
\[
I_n(z)I_{n+1}(z) + I_{n+1}(z)k_n(z) = \frac{1}{z}
\]

Using \( n = 0 \)
\[
I_0(z)k_1(z) + I_1(z)k_0(z) = \frac{1}{z}
\]  
(25)
Using \( n = 1 \) we have
\[
I_1(z) k_2(z) + I_2(z) k_0(z) = \frac{1}{z}
\] (26)

Equating Eqs. 25 and 26 we have
\[
I_0(z) k_1(z) + I_1(z) k_0(z) = I_1(z) k_0(z) + I_2(z) k_1(z)
\]
\[
k_1(z)[I_0(z) - I_2(z)] = I_1(z)[k_0(z) - k_1(z)]
\]
or solving for \( k_1(z) \)
\[
k_1(z) = \frac{I_1(z)}{[I_0(z) - I_2(z)]} [k_0(z) - k_1(z)]
\] (27)

Recalling the first recursion relation in Eq. 9.6.26, p. 376, Abramowitz and Stegun, Handbook of Mathematical Functions, in terms of \( I_n(z) \)
\[
I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z)
\]
for \( n = 1 \) we have
\[
I_0(z) - I_2(z) = \frac{2}{z} I_1(z)
\]
rearranging
\[
\frac{I_1(z)}{I_0(z) - I_2(z)} = \frac{z}{2}
\] (28)

Substituting Eq. 28 into Eq. 27 we have
\[
k_1(z) = \frac{z}{2} [k_0(z) - k_1(z)]
\] (29)

Substituting Eq. 29 into Eq. 23 gives
\[
\lambda' P_{\lambda}(\rho, \phi, \phi_P) = P_{\lambda'}(\rho, \phi, \phi_P) + L^{-1} \left[ \frac{1}{n} \frac{\sqrt{n^2 + 1}}{z} \left[ k_0(z) - k_1(z) \right] \right] \frac{I_0(\sqrt{n^2 + 1})}{I_1(\sqrt{n^2 + 1})}
\] (30)

Considering the \( I_0(\phi) \) \( I_1(\phi) \) term, where \( b = \sqrt{n^2 + 1} \) and \( a = \sqrt{n^2 + 1} \), we will use the polynomial expansions for \( I_0(z) \) and \( I_1(z) \) taken from the general \( I_n(z) \) series given in Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.6.10, p. 375 we have
\[
I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{2304} + \ldots
\] (31)

and
\[
I_1(z) = \frac{z}{2} \left[ 1 + \frac{z^2}{8} + \frac{z^4}{192} + \frac{z^6}{9216} + \ldots \right]
\] (32)

using two term expansions for \( I_0(\phi) \) and \( I_1(\phi) \) we have
\[
I_0(\phi) = 1 + \frac{b^2}{4}
\] (33)

and
\[
I_1(\phi) = \frac{a}{z} \left( 1 + \frac{\phi^2}{8} \right)
\] (34)
Establishing the $I_0(b)/I_1(a)$ ratio using Eqs. 33 and 34 we have

$$\frac{I_0(b)}{I_1(a)} = \frac{\frac{2}{a} \left(1 + \frac{b^2}{4a}\right)}{\frac{1}{a} \left(1 + \frac{a^2}{8}\right)}$$

(35)

assuming $a^2/8 < 1$ we can express $(1 + a^2/8)^{-1}$ as a binomial series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \quad (|x| < 1)$$

From Abramowitz and Stegun, Handbook of

Mathematical Functions, Eq. 3.6.10, p. 15, we have

$$\frac{1}{1-x} = 1 - x + x^2 - x^3 + \ldots \quad (|x| < 1)$$

using a two term expansion of $(1 + a^2/8)^{-1}$ we have

$$\left(1 + \frac{a^2}{8}\right)^{-1} = 1 - \frac{a^2}{8}$$

(36)

Substituting Eq. 36 into Eq. 35 we have

$$\frac{I_0(b)}{I_1(a)} = \frac{2}{a} \left(1 + \frac{b^2}{4a} - \frac{a^2}{8}\right)$$

Expanding

$$\frac{I_0(b)}{I_1(a)} = \frac{2}{a} \left(1 + \frac{b^2}{4} - \frac{a^2}{8}\right)$$

neglecting the $a^2b^2/8a$ term we have

$$\frac{I_0(b)}{I_1(a)} = \frac{2}{a} \left(1 + \frac{b^2}{4} - \frac{a^2}{8}\right)$$

(37)

Recalling that $b = \sqrt{a}r_0$ and $a = \sqrt{a}r_0$ and substituting Eq. 37 into Eq. 30

$$p_b(r_0, t_0) = p_b^{\text{inf}}(r_0, t_0) + L^{-1}\left\{\frac{1}{a}[k_2(a)-k_0(a)] \frac{2}{M} \left(1 - \frac{a^2}{8} + \frac{b^2}{4}\right)\right\}$$

Cancelling the $1/a$ terms and using $a = \sqrt{a}r_0$ and $b = \sqrt{a}r_0$

$$p_b(r_0, t_0) = p_b^{\text{inf}}(r_0, t_0) + L^{-1}\left\{1 \frac{1}{k_2(\sqrt{a}r_0)-k_0(\sqrt{a}r_0)} \left(1 - \frac{a^2}{8} + \frac{b^2}{4}\right)\right\}$$

Continuing the expansion

$$p_b(r_0, t_0) = p_b^{\text{inf}}(r_0, t_0) + L^{-1}\left\{\frac{1}{M} k_2(\sqrt{a}r_0)\right\} - L^{-1}\left\{\frac{1}{M} k_0(\sqrt{a}r_0)\right\}$$

$$+ \left(\frac{a^2}{8} - \frac{b^2}{4}\right) L^{-1}\left\{k_2(\sqrt{a}r_0)\right\} - \left(\frac{a^2}{8} - \frac{b^2}{4}\right) L^{-1}\left\{k_0(\sqrt{a}r_0)\right\}$$

(38)

For reference, we note the Laplace transform of Eq. 38

$$p_b(r_0, s) = \frac{1}{M} k_2(\sqrt{a}r_0) + \frac{1}{M} k_2(\sqrt{a}r_0) - \frac{1}{M} k_0(\sqrt{a}r_0)$$

$$+ \left(\frac{a^2}{8} - \frac{b^2}{4}\right) k_2(\sqrt{a}r_0) - \left(\frac{a^2}{8} - \frac{b^2}{4}\right) k_0(\sqrt{a}r_0)$$

(39)
Multiplying through Eq. 39 by the laplace transform parameter $\mu$, November 94 gives

$$m\tilde{p}_0^2(k_0, \mu) = k_0(k_0 \tilde{r}_0) + k_2(k_2 \tilde{r}_0) - k_0(k_0 \tilde{r}_0) + (\frac{k_0^2 - \tilde{r}_0^2}{4})m k_2(k_2 \tilde{r}_0) - (\frac{k_0^2 - \tilde{r}_0^2}{8})m k_0(k_0 \tilde{r}_0) \quad (40)$$

We will take the inverse Laplace transform of Eqs. 39 and 40 using the following tables:

<table>
<thead>
<tr>
<th>$\tilde{F}(k)$</th>
<th>$F(t)$</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{k_0(k_0 a)}$</td>
<td>$1 \frac{E_1(a^2)}{z} (\frac{a^2}{4z})$</td>
<td>Carslaw and Jaeger: Conduction of Heat in Solids, Table V, Eq. 26, p. 495.</td>
</tr>
<tr>
<td>$k_0(k_0 a)$</td>
<td>$\frac{1}{z} \exp(-a^2) (\frac{a^2}{4z})$</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.120, p. 1028.</td>
</tr>
<tr>
<td>$\frac{1}{k_2(k_2 a)}$</td>
<td>$\frac{2z \exp(-a^2)}{a^2} (\frac{a^2}{4z})$</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.11, p. 304.</td>
</tr>
<tr>
<td>$k_2(k_2 a)$</td>
<td>$\frac{2z \Gamma(z, a^2)}{a^2} (\frac{a^2}{4z})$</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.13, p. 306.</td>
</tr>
</tbody>
</table>

or $\frac{z \Gamma(z, a^2)}{a^2} (\frac{a^2}{4z}) = (\frac{1}{2z} + \frac{z}{a^2}) \exp(-a^2) (\frac{a^2}{4z})$
Inverting Eq. 39 term-by-term using the previous table gives
\[ P(r_0, t_D) = \frac{1}{Z} e^i \left( \frac{r_0^2}{4t_D} \right) - \frac{1}{Z} e^i \left( \frac{r_D^2}{4t_D} \right) + \frac{Z t_D}{r_D^2} \exp \left( -\frac{r_D^2}{4t_D} \right) \]
\[ + \left( \frac{r_0^2 - r_D^2}{4B} \right) \left( \frac{1}{Z t_D} + \frac{Z}{r_D^2} \right) \exp \left( -\frac{r_D^2}{4t_D} \right) \]
\[ - \left( \frac{r_0^2 - r_D^2}{4B} \right) \frac{1}{Z t_D} \exp \left( -\frac{r_D^2}{4t_D} \right) \]

Collecting
\[ P(r_0, t_D) = \frac{1}{Z} e^i \left( \frac{r_0^2}{4t_D} \right) - \frac{1}{Z} e^i \left( \frac{r_D^2}{4t_D} \right) \]
\[ + \left[ \frac{Z t_D}{r_D^2} + c \left( \frac{1}{Z t_D} + \frac{Z}{r_D^2} \right) - \frac{1}{2B} \right] \exp \left( -\frac{r_D^2}{4t_D} \right) \text{ where } c = \frac{1}{2} \left( \frac{r_0^2}{4B} - \frac{r_D^2}{4B} \right) \]

Cancelling the \( \frac{c}{Z t_D} \) terms
\[ P(r_0, t_D) = \frac{1}{Z} e^i \left( \frac{r_0^2}{4t_D} \right) - \frac{1}{Z} e^i \left( \frac{r_D^2}{4t_D} \right) + \frac{Z t_D}{r_D^2} \exp \left( -\frac{r_D^2}{4t_D} \right) \]
\[ + \frac{Z}{r_D^2} \left( \frac{r_0^2 - r_D^2}{4B} \right) \exp \left( -\frac{r_D^2}{4t_D} \right) \]

Which yields the following reduction
\[ P(r_0, t_D) = \frac{1}{Z} e^i \left( \frac{r_0^2}{4t_D} \right) - \frac{1}{Z} e^i \left( \frac{r_D^2}{4t_D} \right) + \frac{Z t_D}{r_D^2} \exp \left( -\frac{r_D^2}{4t_D} \right) + \left( \frac{r_0^2 - 1}{2r_D^2} \right) \exp \left( -\frac{r_D^2}{4t_D} \right) \] (41)

Segmenting the solution into particular flow regimes
\[ P(r_0, t_D) = \frac{1}{Z} e^i \left( \frac{r_0^2}{4t_D} \right) + \frac{Z t_D}{r_D^2} \exp \left( -\frac{r_D^2}{4t_D} \right) \]
\[ \underbrace{\text{Infinite-Acting-Reservoir Term}}_{\text{(Reservoir Size)}} \]
\[ \underbrace{\text{Material Balance Term}}_{\text{(Reservoir Size)}} \]
\[ - \frac{1}{Z} e^i \left( \frac{r_D^2}{4t_D} \right) + \left( \frac{r_0^2 - 1}{z r_D^2} \right) \exp \left( -\frac{r_D^2}{4t_D} \right) \] (42)

Reservoir Shape Effects Terms
Due to conflicting results obtained by inverting Eq. 40 term-by-term, we will proceed by differentiating Eq. 42.

Note that
\[
\frac{d}{dt_d} \left[ \frac{1}{2} E_1 \left( \frac{r_p^2}{4 t_d} \right) \right] = \frac{1}{2} \frac{x}{x t_d} \left[ \frac{-\exp(-x)}{x} \right]
\]
(43)
and
\[
\frac{d}{dt_d} \exp(x) = \frac{1}{2} \frac{x}{x t_d} \left[ -\exp(-x) \right]
\]
(44)

Differentiating Eq. 42 term-by-term
\[
\frac{d}{dt_d} \left[ \frac{1}{2} E_1 \left( \frac{r_p^2}{4 t_d} \right) \right] = \frac{1}{2} \frac{x}{x t_d} \left[ \frac{-\exp(-x)}{x} \right] \exp \left( \frac{-r_p^2}{4 t_d} \right)
\]
\[
= \frac{1}{2} \left[ \frac{-r_p^2}{4 t_d^2} \exp \left( \frac{-r_p^2}{4 t_d} \right) \right]
\]
(45)

or
\[
\frac{d}{dt_d} \left[ \frac{1}{2} E_1 \left( \frac{r_p^2}{4 t_d} \right) \right] = \frac{1}{2} \frac{x}{x t_d} \exp \left( \frac{-r_p^2}{4 t_d} \right)
\]
(46)

Similarly for \( d/dt_d [E_1 (r_p^2/4 t_d)] \) we have
\[
\frac{d}{dt_d} \left[ \frac{1}{2} E_1 \left( \frac{r_0^2}{4 t_d} \right) \right] = \frac{1}{2} \frac{x}{x t_d} \exp \left( \frac{-r_0^2}{4 t_d} \right)
\]
(46)

Next we have
\[
\frac{d}{dt_d} \left[ \frac{x}{r_0^2} \frac{t_0}{x} \exp \left( \frac{-r_0^2}{4 t_0} \right) \right] = \frac{x}{r_0^2} \left[ \exp \left( \frac{-r_0^2}{4 t_0} \right) \frac{d}{dt_0} \frac{t_0}{x} \exp \left( \frac{-r_0^2}{4 t_0} \right) \right]
\]
\[
= \frac{-r_0^2}{x} \left[ 1 + \frac{t_0}{x} \frac{d}{dt_0} \exp \left( \frac{-r_0^2}{4 t_0} \right) \right] \exp \left( \frac{-r_0^2}{4 t_0} \right)
\]
\[
= \frac{z}{r_0^2} \left[ 1 + \frac{t_0}{x} \frac{d}{dt_0} \exp \left( \frac{-r_0^2}{4 t_0} \right) \right] \exp \left( \frac{-r_0^2}{4 t_0} \right)
\]
(47)

Similarly
\[
\frac{d}{dt_d} \left[ \left( \frac{r_p^2}{z r_0^2} - \frac{1}{4} \right) \exp \left( \frac{-r_0^2}{4 t_0} \right) \right] = \left( \frac{r_p^2}{z r_0^2} - \frac{1}{4} \right) \frac{r_0^2}{4 t_d} \exp \left( \frac{-r_0^2}{4 t_0} \right)
\]
\[
= \frac{1}{2} \frac{r_0^2}{x} \left[ \frac{r_p^2}{4} - \frac{r_0^2}{8} \right] \exp \left( \frac{-r_0^2}{4 t_0} \right)
\]
(48)
Collecting the derivative terms we have
\[
\frac{d}{dt} \left[ \Phi_0(r_0, t_0) \right] = \frac{1}{z} \exp \left( \frac{-r_0^2}{4t_0} \right) - \frac{1}{z} \exp \left( \frac{-r_0^2}{4t_0} \right)
\]
\[
+ \left[ \frac{z}{2t_0} + \frac{1}{2t_0^2} \right] \exp \left( \frac{-r_0^2}{4t_0} \right) + \frac{1}{2t_0^2} \left[ \frac{r_0^2 - r_0^2}{4t_0} \right] \exp \left( \frac{-r_0^2}{4t_0} \right)
\]

Collecting further
\[
\frac{d}{dt} \left[ \Phi_0(r_0, t_0) \right] = \frac{1}{z} \exp \left( \frac{-r_0^2}{4t_0} \right) + \left[ \frac{z}{2t_0^2} + \frac{1}{2t_0^2} \right] \exp \left( \frac{-r_0^2}{4t_0} \right)
\]

Multiplying through by \( t_0 \) we have
\[
\Phi_0'(r_0, t_0) = t_0 \frac{d}{dt} \left[ \Phi_0(r_0, t_0) \right]
\]
\[
\Phi_0'(r_0, t_0) = \frac{1}{z} \exp \left( \frac{-r_0^2}{4t_0} \right) + \frac{2}{z} \exp \left( \frac{-r_0^2}{4t_0} \right) + \frac{1}{z} \left[ \frac{r_0^2 - r_0^2}{4t_0} \right] \exp \left( \frac{-r_0^2}{4t_0} \right)
\]  

\[\text{Infinite-Acting Reservoir Term}\]
\[\text{Material Balance Term (Reservoir Size)}\]
\[\text{Reservoir Shape Effects Term}\]

Solution for Constant Pressure Outer Boundary:

Similar to the no-flow outer boundary case, we cannot directly invert Eq. 6, so we will attempt an approximate solution of the line source form (Eq. 7). Recalling Eq. 7 we have
\[
\Phi_0(r_0, u) = \frac{1}{M} k_0(\Phi_0 r_0) - \frac{1}{M} k_0(\Phi_0 r_0) I_0(\Phi_0 r_0) \quad \text{(line source)}
\]  

Recalling the polynomial approximation for \( I_0(x) \) (Eq. 31) we have
\[
I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \ldots
\]  

\[\text{Using a two-term approximation for the } \frac{I_0(\Phi_0 r_0)}{I_0(\Phi_0 r_0)} \]
\[
\frac{I_0(\Phi_0 r_0)}{I_0(\Phi_0 r_0)} = 1 + \frac{r_0^2}{4}
\]

\[\text{Using a two-term binomial series for } (1 + \frac{r_0^2}{4})^{-1} \text{ we have} \]
\[
\frac{I_0(\Phi_0 r_0)}{I_0(\Phi_0 r_0)} = (1 + \frac{r_0^2}{4})^{-1} - \frac{r_0^2}{16}
\]  

\[\text{or} \]
\[
\frac{I_0(\Phi_0 r_0)}{I_0(\Phi_0 r_0)} = \frac{1 + \frac{r_0^2}{4} - \frac{r_0^2}{4} - \frac{r_0^2}{16}}{4}
\]  

\[\frac{I_0(\Phi_0 r_0)}{I_0(\Phi_0 r_0)} = \frac{1 + \frac{r_0^2}{4} - \frac{r_0^2}{4} - \frac{r_0^2}{16}}{4} + \frac{r_0^2}{16}
\]
Substituting Eq. 50 into Eq. 47 gives us
\[
\frac{\bar{P}_0(r, t)}{M} = \frac{1}{M} \left( k_0(\sqrt{a}r) - k_0(\sqrt{a}r_0) \right) \left( 1 + \frac{\mu\nu_0^2 - \mu\nu_0^2 - \mu^2\nu_0^2}{16} \right)
\]
or
\[
\bar{P}_0(r, t) = \frac{1}{M} \left( k_0(\sqrt{a}r) - k_0(\sqrt{a}r_0) \right) \left( 1 + \frac{(\nu_0^2 - \nu_0^2) k_0(\sqrt{a}r_0) - \nu_0^2}{16} \right)
\]
for simplicity, we will ignore the \( \frac{(\nu_0^2 - \nu_0^2) k_0(\sqrt{a}r_0)}{16} \) term in Eq. 51, this gives
\[
\bar{P}_0(r, t) = \frac{1}{M} \left( k_0(\sqrt{a}r) - k_0(\sqrt{a}r_0) \right) \left( 1 + \frac{(\nu_0^2 - \nu_0^2) k_0(\sqrt{a}r_0)}{16} \right)
\]
From our previous efforts we recall that
\[
\frac{f(r)}{M} = \frac{1}{k_0(\sqrt{a}r)} - \frac{1}{k_0(\sqrt{a}r_0)}
\]
\[
\frac{1}{Z} \left( \frac{E_1(a^2)}{4t} \right)
\]
Inverting Eq. 52 term-by-term we have
\[
\bar{P}_0(r, t) = \frac{1}{Z} \left( \frac{E_1(\nu_0^2)}{4t_0} \right) - \frac{1}{Z} \left( \frac{E_1(\nu_0^2)}{4t_0} \right) + \left( \frac{(\nu_0^2 - \nu_0^2)}{8} \right) \frac{1}{E_1(\nu_0^2)} \frac{\exp(-\nu_0^2)}{4t_0}
\]
Differentiating Eq. 53 term-by-term. We simply recall Eqs. 45 and 46
\[
\frac{d}{dt_0} \left[ \frac{1}{2} E_1 \left( \frac{\nu_0^2}{4t_0} \right) \right] = \frac{1}{2t_0} \exp\left( -\frac{\nu_0^2}{4t_0} \right)
\]
\[
\frac{d}{dt_0} \left[ \frac{1}{2} E_1 \left( \frac{\nu_0^2}{4t_0} \right) \right] = \frac{1}{2t_0} \exp\left( -\frac{\nu_0^2}{4t_0} \right)
\]
Differentiating the last term in Eq. 53 we have
\[
\frac{d}{dt_0} \left[ \frac{(\nu_0^2 - \nu_0^2)}{8} \frac{1}{t_0} \exp\left( -\frac{\nu_0^2}{4t_0} \right) \right] = \frac{(\nu_0^2 - \nu_0^2)}{8} \frac{d}{dt_0} \left[ \frac{1}{t_0} \exp\left( -\frac{\nu_0^2}{4t_0} \right) \right]
\]
\[
= \frac{(\nu_0^2 - \nu_0^2)}{8} \left[ \exp\left( -\frac{\nu_0^2}{4t_0} \right) \frac{d}{dt_0} \left( \frac{1}{t_0} \right) + \frac{1}{t_0} \frac{d}{dt_0} \left[ \exp\left( -\frac{\nu_0^2}{4t_0} \right) \right] \right]
\]
\[
= \frac{(\nu_0^2 - \nu_0^2)}{8} \left[ \frac{\nu_0^2}{4t_0} - 1 \right] \exp\left( -\frac{\nu_0^2}{4t_0} \right)
\]
Collecting the derivatives we have
\[ \frac{d}{dt_o} \left[ \Phi_0 (r_o, t_o) \right] = \frac{1}{z t_o} \exp \left( -\frac{r_o^2}{4 t_o} \right) - \frac{1}{z t_o} \exp \left( -\frac{r_o^2}{4 t_o} \right) \]
\[ + \frac{(r_o^2 - r_i^2)}{B t_o^2} \left[ \frac{r_o^2}{4 t_o} - 1 \right] \exp \left( -\frac{r_o^2}{4 t_o} \right) \]

Multiplying through by \( t_o \) yields the well testing derivative
\[ \Phi'_0 (r_o, t_o) = \frac{1}{z} \exp \left( -\frac{r_o^2}{4 t_o} \right) - \frac{1}{z} \exp \left( -\frac{r_o^2}{4 t_o} \right) + \frac{(r_o^2 - r_i^2)}{B t_o^2} \left[ \frac{r_o^2}{4 t_o} - 1 \right] \exp \left( -\frac{r_o^2}{4 t_o} \right) \]

---

### Summary of Results:

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. infinite-acting reservoir</td>
<td></td>
</tr>
<tr>
<td>Laplace Domain / Cylindrical Source Solution</td>
<td>( \Phi_0 (r_o, u) = \frac{1}{M} \frac{k_o (\sqrt{u} r_o)}{\sqrt{u} k_1 (\sqrt{u} r_o)} )</td>
</tr>
<tr>
<td>Laplace Domain / Line Source Solution</td>
<td>( \Phi_0 (r_o, u) = \frac{1}{M} k_o (\sqrt{u} r_o) )</td>
</tr>
<tr>
<td>Laplace Domain / &quot;log&quot; approximation</td>
<td>( \Phi_0 (r_o, u) = \frac{1}{2M} \frac{\ln \left( \frac{\sqrt{u} r_o}{r_i} \right)}{1 + \frac{1}{u} \frac{1}{M} \frac{1}{k_1 (\sqrt{u} r_o)}} )</td>
</tr>
<tr>
<td>Real Domain / Cylindrical Source Solution</td>
<td>( \Phi_0 (r_o, t_o) = \frac{Z^{-1}}{M} \left{ \frac{1}{\sqrt{u} k_1 (\sqrt{u} r_o)} \right} ), not invertible</td>
</tr>
<tr>
<td>Real Domain / Line Source Solution</td>
<td>( \Phi_0 (r_o, t_o) = \frac{1}{Z} \frac{F_1 (\sqrt{u} r_o)}{4 t_o} )</td>
</tr>
<tr>
<td>Real Domain / Derivative of the Line Source Solution</td>
<td>( \Phi_0' (r_o, t_o) = \frac{1}{Z} \frac{\exp \left( -\frac{r_o^2}{4 t_o} \right)}{4 t_o} )</td>
</tr>
<tr>
<td>Real Domain / &quot;log&quot; approximation</td>
<td>( \Phi_0 (r_o, t_o) = \frac{1}{Z} \frac{\ln \left( \frac{\sqrt{u} r_o}{r_i} \right)}{e^{\frac{r_o^2}{t_o}} r_o^2} )</td>
</tr>
<tr>
<td>Real Domain / Derivative of the &quot;log&quot; approximation</td>
<td>( \Phi_0' (r_o, t_o) = \frac{1}{Z} )</td>
</tr>
</tbody>
</table>

b. bounded circular reservoir / no-flow outer boundary |
| Laplace Domain / Cylindrical Source | \( \Phi_0 (r_o, u) = \frac{1}{M} \frac{k_o (\sqrt{u} r_o) I_0 (\sqrt{u} r_o) + k_1 (\sqrt{u} r_o) I_0 (\sqrt{u} r_o)}{\sqrt{u} k_1 (\sqrt{u} r_o) - \sqrt{u} I_1 (\sqrt{u} r_o) k_1 (\sqrt{u} r_o)} \) |
| Laplace Domain / Line Source | \( \Phi_0 (r_o, u) = \frac{1}{M} \frac{k_o (\sqrt{u} r_o) + k_1 (\sqrt{u} r_o) I_0 (\sqrt{u} r_o)}{\sqrt{u} I_1 (\sqrt{u} r_o)} \) |
Case b, bounded circular reservoir / no-flow boundary - continued

Real Domain / Line Source Soln.
\[ p(r, t) = \frac{1}{2} E_i\left(\frac{r^2}{4t}\right) - \frac{1}{2} E_i\left(\frac{r_0^2}{4t_0}\right) + \frac{z}{2t_0} \exp\left(-\frac{r_0^2}{4t_0}\right) \]
\[ + \left(\frac{r^2}{r_0^2} - 1\right) \exp\left(-\frac{r^2}{4t_0}\right) \]

Real Domain / Derivative of Line Source Soln.
\[ p_t(r, t) = \frac{1}{2z_0} \exp\left(-\frac{r^2}{4t_0}\right) + \frac{z}{2t_0} \exp\left(-\frac{r_0^2}{4t_0}\right) \]
\[ + \frac{1}{2t_0} \left(\frac{r^2 - r_0^2}{4t_0}\right) \exp\left(-\frac{r_0^2}{4t_0}\right) \]

Case c, bounded circular reservoir / constant pressure boundary

Laplace Domain / Cyl. Source
\[ F_0(r, \theta) = \frac{1}{m} \frac{k_0(r_0 \theta_0) I_0(r \theta_0) - k_0(\theta_0 \theta_0) I_0(r \theta_0)}{I_0(\theta_0 \theta_0) - I_0(r \theta_0)} \]
\[ - \frac{k_0(r_0 \theta_0) I_0(r \theta_0) - k_0(\theta_0 \theta_0) I_0(r \theta_0)}{I_0(\theta_0 \theta_0) - I_0(r \theta_0)} \]

Laplace Domain / Line Source
\[ F_0(r, \theta) = \frac{1}{m} \frac{k_0(r \theta_0) I_0(r \theta_0) - k_0(\theta_0 \theta_0) I_0(r \theta_0)}{I_0(\theta_0 \theta_0) - I_0(r \theta_0)} \]

Real Domain / Line Source Soln.
\[ p_0(r, t) = \frac{1}{2} E_i\left(\frac{r^2}{4t_0}\right) - \frac{1}{2} E_i\left(\frac{r_0^2}{4t_0}\right) \]
\[ + \frac{1}{8t_0} \left(\frac{r_0^2 - r^2}{4t_0}\right) \exp\left(-\frac{r_0^2}{4t_0}\right) \]

Real Domain / Derivative of Line Source Soln.
\[ p_0'(r, t) = \frac{1}{2z_0} \exp\left(-\frac{r^2}{4t_0}\right) - \frac{1}{2z_0} \exp\left(-\frac{r_0^2}{4t_0}\right) \]
\[ + \frac{1}{8t_0} \left[\frac{r_0^2 - r^2}{4t_0}\right] \exp\left(-\frac{r_0^2}{4t_0}\right) \]
References — Radial Flow Solutions (Real Domain Solutions):

SPE 25479

Semi-Analytical Solutions for a Bounded Circular Reservoir—No Flow and Constant Pressure Outer Boundary Conditions: Unfractured Well Case

by T.A. Blasingame, Texas A&M U.

This is a preprint -- subject to correction.

ABSTRACT

This paper gives the development of semi-analytical (approximate) solutions for an unfractured well producing at a constant flow rate or a constant wellbore pressure from the center of a bounded circular reservoir that has a no flow or constant pressure outer boundary.

The utility of these solutions is that they provide explicit formulas for computing reservoir performance, as opposed to the Laplace transform solutions which have complicated Bessel functions and usually require numerical inversion. Or, for the cases that can be directly inverted, we are left with infinite series solutions, which often require not only evaluation of Bessel functions, but also the evaluation of multiple roots of Bessel functions!

This paper serves as mechanism to provide those interested with accurate and computationally simple methods for computing reservoir performance. These solutions are simple enough to compute on a hand calculator. Due to straightforward nature of these relations (transient, transition, and boundary dominated flow parts of the solution are given by individual terms in each relation). From these solutions we can also develop relations for the analysis and interpretation of reservoir performance data (well test and production data).

While the more rigorous (and hence, complex), Laplace transform solutions are relatively easy to compute and manipulate given modern computing environments, we believe that results such as these are useful for illustrating concepts as well as for computing reservoir performance (pressure and rate solutions) and for the analysis and interpretation of reservoir performance data.

INTRODUCTION

This paper confirms the validity of a semi-analytical solution for a well producing at a constant flow rate from the center of a bounded reservoir with a no flow outer boundary. This solution is given as

\[ p_r(t) = \frac{1}{2} E_1 \left( \frac{z^2}{4D} \right) - \frac{1}{2} E_1 \left( \frac{\rho^2}{4D} \right) + \frac{2\rho^2}{r_D} \exp \left( \frac{-z^2}{4D} \right) + C (1) \]

References and illustrations at end of paper

This solution was originally proposed by Homer, who claimed that this relation has no rigorous theoretical basis. Ramey proceeded to develop Eq. 1 from the limiting assumption of a composite region of zero permeability surrounding the inner portion of the reservoir. As it turns out, Eq. 1 is a sort of "zero-order" approximation to the correct solution, which is given exactly in Laplace space and must be approximated in order to obtain a explicit inverse Laplace transform (real space solution).

The utility of Eq. 1 lies not so much in its accuracy, but more in terms of its individual components. For instance, we immediately recognize that

\[ \frac{1}{2} E_1 \left( \frac{z^2}{4D} \right) \] is the transient flow term

and

\[ \frac{2\rho^2}{r_D} \] is the material balance term

Using this information we can conclude that Eq. 1 will correctly model transient flow and fully developed boundary dominated flow, where the pressure response is dominated by the volume of fluid in the reservoir system. The only period for which we can not be sure that Eq. 1 is applicable is the transition flow regime between transient and fully developed boundary dominated flow.

The most important aspect of this work is that we approach this problem systematically and create approximate solutions in real space from the rigorous Laplace transform solutions. In doing so we will find that the Homer/Ramey equation (Eq. 1) is actually a zero-order approximation for a more complete set of solutions that can be developed from the rigorous Laplace transform solutions.

Clearly, we must prove conclusively that the Homer/Ramey equation is a solution of the radial flow diffusivity equation. To do this we will start with the Laplace transform solutions for an unfractured well in a bounded circular reservoir, producing at either no flow or constant pressure outer boundary conditions and a constant flow rate at the well.
From this exercise we can expect several issues to arise. Since we will rely on polynomial expansions of the \( J_0(z) \) and \( J_1(z) \) functions to obtain invertible forms of the solutions in Laplace space, we expect that several forms of a solution can be developed by extending or truncating a particular set of terms.

Our rationale will be that for any relation we develop we must provide verification in the following fashion:

- First, we must prove that the proposed relation (in Laplace space) is accurate compared to the exact Laplace transform solution.
- Second, we must prove the approximate real space solution is accurate compared to the numerical inversion of the Laplace transform solution.
- Finally, we must also prove that the derivative of the approximate real space solution is accurate compared to the derivative function obtained by numerical inversion of the Laplace transform solution.

**LAPLACE TRANSFORM SOLUTIONS**

**No Flow Outer Boundary Case**

The schematic of the configuration of this reservoir along with a diagram of the pressure distribution within the reservoir are shown in Fig. 1. The exact Laplace transform solution for this case is developed in refs. 3.

\[
\bar{p}_D(r_m, \omega) = \frac{K_0(\bar{w}_D)}{u} \left( \frac{1}{\bar{w}_D} \right) I_0(\bar{u}_D) + \frac{K_1(\bar{w}_D)}{\bar{w}_D} I_1(\bar{u}_D) \quad (2)
\]

If we consider the behavior of the Bessel functions, \( J_0(z) \) and \( K_1(z) \), we find that as the Laplace transform parameter, \( u \), approaches zero (for "large" values of \( \omega \)), we have

\[
u \to 0 \quad \text{and} \quad \bar{w}_D \to 1
\]

and

\[
u \to 0 \quad \text{and} \quad \bar{w}_D \to 1\]

Combining these relations with Eq. 2 gives

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) I_0(\bar{u}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) I_1(\bar{u}_D) \quad (3)
\]

This approach and the result (Eq. 3) are discussed by Ozkan and Raghavan.4 We recognize that the first term in Eq. 3 is the Laplace transform solution for a well in an infinite-acting reservoir (i.e., the transient flow solution). This term is invertible and yields the familiar exponential integral solution.

In order to further reduce Eq. 3, we must devise a strategy to reduce the quotient and product of the Bessel functions to a more usable form, or at least to expand these functions with approximations that are valid for small arguments of the Bessel functions such that accurate expansions are obtained. Looking at the polynomial expansions for \( J_0(z) \) and \( K_1(z) \) we have

\[
I_0(z) = 1 + \frac{z^2}{6} + \frac{z^4}{240} + \frac{z^6}{65520} + \ldots \quad (4)
\]

and

\[
I_1(z) = 1 + \frac{z^2}{8} + \frac{z^4}{192} + \frac{z^6}{6144} + \ldots \quad (5)
\]

Due to the behavior of the behavior of the \( K_1(z) \) Bessel function we are forced to consider a relation to reduce \( K_1(z) \) terms to more invertible forms, particularly, \( K_0(z) \) and \( K_2(z) \) terms. The appropriate recurrence relation is

\[
K_1(z) = \frac{z}{2} (K_2(z) - K_0(z)) \quad (6)
\]

Using the first two terms of each expansion in Eqs. 4 and 5, and combining Eqs. 4, 5, and 6 with Eq. 3, we obtain the following approximation for the no flow outer boundary case which can be written as

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) I_0(\bar{u}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) I_1(\bar{u}_D) + \frac{1}{\bar{w}_D} K_2(\bar{w}_D) I_2(\bar{u}_D) 
\]

Truncating the \( \frac{1}{\bar{w}_D} K_2(\bar{w}_D) \) term, we obtain the "second approximation" for the no flow outer boundary case which is given as

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) I_0(\bar{u}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) I_1(\bar{u}_D) \quad (7)
\]

We suggest that although Eq. 7 (the first approximation) has more terms from the original expansion, we will show that Eq. 8 (the second approximation) is not only more compact, but in fact, just as accurate. We will demonstrate this later in a graphical comparison of Eqs. 2, 3, 7, and 8.

**Constant Pressure Outer Boundary Case**

Although this case is often thought to be physically unrealistic, we believe the solution and analysis relations generated via the solution may aid in the analysis and interpretation of reservoir performance data. The reservoir configuration and a diagram of the pressure distribution within the reservoir are shown in Fig. 2.

As with the no flow outer boundary case, the exact Laplace transform solution for this case is also developed in ref. 3. This result is

\[
\bar{p}_D(r_m, \omega) = \frac{K_0(\bar{w}_D)}{u} I_0(\bar{u}_D) + \frac{K_1(\bar{w}_D)}{\bar{w}_D} I_1(\bar{u}_D) + \frac{K_2(\bar{w}_D)}{\bar{w}_D} I_2(\bar{u}_D) \quad (8)
\]

Again, we consider the behavior of the Bessel functions \( J_1(z) \) and \( K_1(z) \) and we recall that as the Laplace transform parameter, \( u \), approaches zero (for "large" values of \( \omega \)), we have

\[
u \to 0 \quad \text{and} \quad \bar{w}_D \to 1
\]

and

\[
u \to 0 \quad \text{and} \quad \bar{w}_D \to 1\]

Combining these relations with Eq. 9 gives

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) I_0(\bar{u}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) I_1(\bar{u}_D) + \frac{1}{\bar{w}_D} K_2(\bar{w}_D) I_2(\bar{u}_D) \quad (9)
\]

Eq. 10 represents a new solution for this case. We must verify that the conditions placed on its development are valid, but we can assume that if these conditions are valid for the no flow outer boundary case then these conditions should also be valid for this case. We again recognize that the first term in Eq. 10 is the Laplace transform solution for a well in an infinite-acting reservoir (i.e., the transient flow solution) and we again note that this term is explicitly invertible and yields the familiar exponential integral solution.

As in the previous case, we must reduce the quotient and product terms involving Bessel functions into a more usable (hence, invertible) form. In this case we only need use the two term expansion of \( J_0(z) \) given by Eq. 4, and the recurrence relation given by Eq. 6. Combining Eqs. 4 and 6 with Eq. 10, we obtain the following first approximation for the constant pressure outer boundary case

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) + \frac{1}{\bar{w}_D} K_2(\bar{w}_D) + \frac{1}{\bar{w}_D} K_3(\bar{w}_D) \quad (10)
\]

Using the first two terms of each expansion in Eqs. 4 and 5, and combining Eqs. 4, 5, and 6 with Eq. 3, we obtain the following approximation for the constant pressure outer boundary case

\[
\bar{p}_D(r_m, \omega) = \frac{1}{\bar{w}_D} K_0(\bar{w}_D) + \frac{1}{\bar{w}_D} K_1(\bar{w}_D) + \frac{1}{\bar{w}_D} K_2(\bar{w}_D) + \frac{1}{\bar{w}_D} K_3(\bar{w}_D) \quad (11)
\]
Truncating the $\frac{2^2}{2^2} u$ term, we obtain the "second approximation" for the constant pressure outer boundary case which is given as
\[
\bar{p}_D(r_D u) = \frac{1}{u} K_0(\tilde{r}_D) - \frac{1}{u} K_2(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_0(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_2(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_0(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_2(\tilde{r}_D)
\]
As with the previous case, we must verify the applicability of the proposed approximations. As a verification, we will demonstrate a comparison of values computed using Eqs. 9, 10, 11, and 15 on a log-log plot for various values of the dimensionless drainage radius, $r_D$.

**REAL SPACE APPROXIMATE SOLUTIONS**

**No Flow Outer Boundary Case**
Recalling the first approximation for this case, Eq. 7, and expanding we have
\[
\bar{p}_D(r_D u) = \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{u} K_2(\tilde{r}_D) - \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_2(\tilde{r}_D)
\]
Truncating the $\frac{2^2}{2^2} u$ term from Eq. 13, we obtain the "second approximation" as
\[
\bar{p}_D(r_D u) = \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{u} K_2(\tilde{r}_D) - \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_2(\tilde{r}_D)
\]
Using standard tables of Laplace transforms we can invert Eq. 14 to yield
\[
\bar{p}_D(r_D u) = \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
Taking the logarithmic derivative of Eq. 15 gives
\[
p_D'(r_D u) = \bar{D}_D \frac{d}{dD} \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
Taking the logarithmic derivative of Eq. 16 gives
\[
p_D'(r_D u) = \frac{1}{2} \bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
We have also inverted Eqs. 13 and we found no significant improvement in estimates of the $p_D$ and function and we found less accurate results for the logarithmic derivative function, $p_D'$, for this case.

Recalling the Horner/Ramey relation (Eq. 1) we have
\[
p_D'(r_D u) = \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right) + s
\]
Taking the logarithmic derivative of Eq. 1 gives
\[
p_D'(r_D u) = \frac{1}{2} \exp \left(-\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
If we would consider Eq. 15 as a "first-order" approximation to the solution of the diffusivity equation then the Horner/Ramey equation (Eq. 1) would be a "zero-order" type of solution. At any rate, these derivations suggest that the Horner/Ramey equation is at least an approximate solution, and that the new solution that we have proposed should give some (and maybe considerable) improvement over the Horner/Ramey equation, for very little cost, an additional exponential term.

Eqs. 15 and 16 will be compared to Eqs. 1 and 17 for several values of $r_D$ as demonstrated on figures given in the Verification section of this paper.

**Constant Pressure Outer Boundary Case**
Recalling the first approximation for this case, Eq. 11, we have
\[
\bar{p}_D(r_D u) = \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{u} K_2(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_0(\tilde{r}_D) + \frac{1}{4} \frac{2^2}{2^2} K_2(\tilde{r}_D)
\]
Experience (comparison of Laplace and real space forms of this solution) suggests that the $\frac{2^2}{2^2} u$ term can be ignored. This gives
\[
\bar{p}_D(r_D u) = \frac{1}{u} K_0(\tilde{r}_D) + \frac{1}{u} K_2(\tilde{r}_D)
\]
Using standard tables of Laplace transforms we can invert Eq. 12 to yield
\[
p_D(r_D u) = \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right) + s
\]
Taking the logarithmic derivative of $p_D'$, $\bar{D}_D$ of Eq. 18 gives
\[
p_D'(r_D u) = \frac{1}{2} \exp \left(-\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
Ramey also presented a similar solution for this case, again obtained from the concept of a composite reservoir system. This solution is
\[
p_D(r_D u) = \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} E_1 \left(\frac{\tilde{r}_D}{4u}\right) + 2\bar{D}_D \exp \left(-\frac{\tilde{r}_D}{4u}\right) + s
\]
Taking the logarithmic derivative of Eq. 19 we obtain
\[
p_D'(r_D u) = \frac{1}{2} \exp \left(-\frac{\tilde{r}_D}{4u}\right) + \frac{1}{2} \exp \left(-\frac{\tilde{r}_D}{4u}\right)
\]
Again, we can consider our solutions (Eqs. 18 and 19) to be "first-order" approximations to the solutions of the diffusivity equation. With this, the Ramey equations (Eqs. 20 and 21) would
be "zero-order" types of solutions. Again, our relations are a bit more rigorous and should give better performance than the Homer/Flamey relations.

It is worthy of comment to point out that the explicit inversion of Eq. 11 has been performed, along with the development of a logarithmic derivative. Unfortunately, more terms in the expansion lead to slightly better performance overall, but the derivative behavior becomes quite erratic, especially during the transition from transient to boundary dominated flow. Similar observations were made for higher order developments for the no flow outer boundary case.

The point being that additional expansions have been attempted for both cases, with no significant improvement and with much worse behavior in the derivative functions. We are not implying that further expansions or different expansions will not offer improvement, nor are we attempting to dissuade would be explorers from this topic. We are only suggesting that additional expansions using the polynomial relations for the $I_0(z)$ and $I_1(z)$ Bessel functions, or other similar strategy, should offer no further improvement in these solutions.

**AN APPROXIMATION FOR PRODUCTION AT A CONSTANT WELLBORE PRESSURE**

Our objective is to create a general algorithm to compute the dimensionless flowrate for a well produced at a constant bottomhole pressure from the dimensionless pressure solution for a well producing at constant flowrate. We would like for this relation to be approximate for transient and transition flow, and exact for boundary dominated flow.

This concept will employ the constant wellbore pressure/constant flowrate relation in Laplace space proposed by van Everdingen and Hurst.\(^7\) This result is

$$q_{ed}^p(u) = -\frac{1}{u^2} \frac{1}{P_D^0}$$

(22)

Assuming a linear behavior in the dimensionless pressure function, $P_D^0$, we have

$$P_D^0(u) = b + au_D$$

(23)

Where Eq. 23 gives the following Laplace transform relation

$$\bar{P}_D^0(u) = \frac{a}{u} + \frac{b}{u^2}$$

(24)

Combining Eqs. 22 and 24, and rearranging gives

$$q_{ed}^p(u) = \frac{1}{b} \frac{1}{u + a/b}$$

Which has the following inverse Laplace transform

$$q_{ed}^p(t) = \frac{1}{b} \exp\left(\frac{-a}{b} t\right)$$

(25)

From Eq. 23, when we solve for $a$ and $b$ we obtain

$$a = P_D^0t$$

(26)

and

$$b = P_D^0 - P_D^0$$

(27)

Combining Eqs. 23, 26, and 27 we obtain

$$q_{ed}^p(t) = \frac{1}{P_D^0 - P_D^0} \exp\left(\frac{P_D^0 - P_D^0}{P_D^0 - P_D^0}\right)$$

(28)

We readily admit that Eq. 28 is only an approximation for transient and transitional flow, and that we will have to verify its application by comparison to more rigorous solutions. However, Eq. 28 is exact for boundary dominated flow where the pressure decline is linearly proportional to time.

**VERIFICATION**

If the purpose of this work is to provide the user with compact and accurate relations for the prediction, analysis, and interpretation of reservoir performance, then we must provide systematic validation of each concept and each relation.

This verification consists of the following three parts:

- Verification of approximations for a particular Laplace transform solution.
- Verification of approximations for a particular real space solution and its derivative.
- Verification of the approximation for a well produced at a constant wellbore pressure.

We note that the 'analytical' solution for the inverse of the Laplace transform solutions will be computed using the Stehfest inversion algorithm.\(^8\) This algorithm is accepted for giving sufficiently accurate results for analytical comparisons, for most cases. A Stehfest parameter of 12 was used for all cases in this work.

**Verification of Approximate Laplace Transform Solutions**

In this section, we will compare the individual approximations with the exact Laplace transform solutions. For the no flow outer boundary cases, Eqs. 3, 7, and 8 will be compared to the exact solution for this case, Eq. 2. And for the constant pressure outer boundary cases, Eqs. 10, 11, and 12 will be compared to the exact solution for this case, Eq. 9.

Fig. 3 illustrates the comparison for both the no flow and constant pressure outer boundary cases, for $r_D$ of values of 10, 100, and 1000. We immediately note excellent agreement between each approximation and its appropriate analytical solution. We even note that the approximations for the constant pressure outer boundary case agree with the straight line trends of the steady-state solutions where the dimensionless pressure function is constant in real space and is proportional to $1/u$ in Laplace space, as shown.

While these results are encouraging, the level of detail shown in Fig. 3 is not sufficient to make observations related to behavior of these solutions at a particular point or in a particular region. Therefore, we can use Fig. 4, which is a plot of the absolute relative error in percent, for each of the cases shown on Fig. 3.

We quickly note that there are no problems at larger times (small $u$) which suggests almost exact behavior of the approximations during boundary dominated flow. However, during transition from transient flow we see errors as high as 1 percent, which while hardly perceptible to the $P_D^0$ function, may cause significant deviation for the $P_D^0$ function. So the sweeping conclusion is that we can expect excellent performance during boundary dominated flow, but some deviation (possibly even significant deviation) during transition flow. But this deviation should only be observed primarily in the $P_D^0$ function.

**Verification of Approximate Real Space Solutions**

In this section, we will compare the individual real space approximations with the numerical inversion results using the Laplace transform solutions. Fig. 5 (log-log) and Fig. 6 (semi-log) illustrate the behavior of the $P_D^0$ functions for a variety of $r_D$ values. We quickly note that virtually no difference exists between the approximate and computed solutions for either of these plots. The approximate $P_D^0$ solutions are given by Eq. 15 for the no flow outer boundary case and by Eq. 18 for the constant pressure outer boundary case.
Fig. 7 illustrates the comparison of the pp and pp' solutions for the no flow outer boundary case, for various r_p values. In this figure we compare the performance of our solutions (Eqs. 15 and 16) with those of Ramey (Eqs. 1 and 17) and we note excellent agreement in the pp' approximations and their appropriate analytical solutions.

We note very good agreement in the pp' approximations and their appropriate analytical solutions, except in the region where the derivative functions turn to proceed into full boundary dominated flow. We believe that this error is certainly tolerable and that our pp' function is slightly better than the Horne/Ramey relations. However, we admit that from this plot, it appears that an average of the two functions would probably give better results than using either function alone.

In Fig. 8 we show the comparison of the pp and pp' solutions for the constant pressure outer boundary case, again for various r_p values. We compare the performance of our solutions (Eqs. 18 and 19) with the Horne/Ramey relations (Eqs. 20 and 21) and we again note excellent agreement in the pp' approximations and their appropriate analytical solutions.

From observing the behavior of the pp' approximations we note good agreement with the appropriate analytical solutions, except in the region where the derivative functions die off as the reservoir approaches steady-state flow conditions. We attempted other solutions (more complex exponential), but no other solution gave better performance than the cases shown in Fig. 8.

From a modeling standpoint, the behavior as the reservoir approaches steady-state would be extremely difficult to model due to the tendency of the solution to move quickly towards zero. Further effort in this area is warranted.

Another measure of the validity of these solutions would be to compare the pressure and pressure derivative distributions within the reservoir. Fig. 9 illustrates our comparison for this case (r_p = 1x10^3) where the pp and pp' functions are compared at various points within the reservoir. We again compare the performance of our solutions for a no flow outer boundary (Eqs. 15 and 16) with the Horne/Ramey relations (Eqs. 1 and 17). We note excellent agreement for all evaluation points, indicating that we can expect good performance for computing pressure distributions with these relations.

Verification of Constant Wellbore Pressure Approximation

We would like to compare our new relation for computing the rate profile assuming a constant wellbore pressure using the constant rate solution. This relation is given by Eq. 28 and a comparison of cases for the no flow and constant pressure outer boundary conditions are shown in Fig. 10.

While the assumptions going into Eq. 28 are somewhat limiting, we note excellent agreement of the approximate and analytical solutions for this case. These results suggest that this general solution can be used for the prediction, analysis, and interpretation of reservoir performance data (well test and production data) and for decline curve analysis and for water influx calculations.

SUMMARY AND CONCLUSIONS

We have developed and verified approximate solutions for the Laplace transform and real space solutions for an unfractured well centered in a bounded circular reservoir with a no flow or constant pressure outer boundary, with a constant flow rate at the well. We have also provided an approximation for computing the dimensionless solution for the constant wellbore pressure case.

Each of these solutions has been shown to be accurate compared to its "analytical" solutions, the numerical inversion of the Laplace transform solution. These results will find utility in the prediction, analysis, and interpretation of reservoir performance data (well test and production data) and could also be used for decline curve analysis and water influx calculations.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>formation volume factor, RB/STB</td>
</tr>
<tr>
<td>ε</td>
<td>total system compressibility, psi^-1</td>
</tr>
<tr>
<td>E(x)</td>
<td>first exponential integral</td>
</tr>
<tr>
<td>h</td>
<td>formation thickness, ft</td>
</tr>
<tr>
<td>I_0(x)</td>
<td>modified Bessel function of the first kind, zero order</td>
</tr>
<tr>
<td>I_1(x)</td>
<td>modified Bessel function of the first kind, first order</td>
</tr>
<tr>
<td>k</td>
<td>formation permeability, md</td>
</tr>
<tr>
<td>K_0(x)</td>
<td>modified Bessel function of the second kind, zero order</td>
</tr>
<tr>
<td>K_1(x)</td>
<td>modified Bessel function of the second kind, first order</td>
</tr>
<tr>
<td>K_2(x)</td>
<td>modified Bessel function of the second kind, second order</td>
</tr>
<tr>
<td>P_D</td>
<td>pressure function for the constant flow rate case</td>
</tr>
<tr>
<td>P_D''</td>
<td>logarithmic derivative of dimensionless pressure function for the constant flow rate case</td>
</tr>
<tr>
<td>P_D'</td>
<td>Laplace transform of dimensionless pressure function for the constant flow rate case</td>
</tr>
<tr>
<td>P_I</td>
<td>initial pressure, psia</td>
</tr>
<tr>
<td>P_wf</td>
<td>bottomhole flowing pressure, psia</td>
</tr>
<tr>
<td>P_r</td>
<td>pressure at some radius, psia</td>
</tr>
<tr>
<td>A_P</td>
<td>A_P = P_r - P_I, or A_P = P_r - P_wf, psi</td>
</tr>
<tr>
<td>q</td>
<td>flow rate, STB/day</td>
</tr>
<tr>
<td>q_{wp}</td>
<td>141.2 \frac{B_p}{kH P_r P_{wf}} \psi, dimensionless flow rate function</td>
</tr>
<tr>
<td>\psi</td>
<td>dimensionless pressure factor</td>
</tr>
<tr>
<td>r</td>
<td>radius, ft</td>
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<tr>
<td>R_D</td>
<td>dimensionless radius</td>
</tr>
<tr>
<td>r_s</td>
<td>drainage radius of the reservoir, ft</td>
</tr>
<tr>
<td>r_w</td>
<td>wellbore radius, ft</td>
</tr>
<tr>
<td>\phi</td>
<td>dimensionless skin factor</td>
</tr>
<tr>
<td>\tau</td>
<td>time, hrs</td>
</tr>
<tr>
<td>\tau_D</td>
<td>time, hrs</td>
</tr>
</tbody>
</table>

ACKNOWLEDGMENTS

We express gratitude and appreciation to all of the students who participated in the development and validation of the solutions in this work. In particular, the cunning and determination of Philip Likitsipua, Juan Carlos Palacio, and Min-Yu Shih are recognized and appreciated by the author.

REFERENCES


Figure 1 - Schematic of initial and boundary conditions as well as a typical reservoir pressure distribution for an unfractured well centered in a bounded circular reservoir with a no flow outer boundary, constant rate of production at the well.
Figure 2 - Schematic of initial and boundary conditions as well as a typical reservoir pressure distribution for an unfractured well centered in a bounded circular reservoir with a constant pressure outer boundary, constant rate of production at the well.
Figure 3 - Log-log plot of the Laplace space dimensionless pressure solutions and approximations for an unfractured well centered in a bounded circular reservoir producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 4 - Log-log plot of absolute relative error for the Laplace space dimensionless pressure solutions for an unfractured well centered in a bounded circular reservoir producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 5 - Log-log plot of analytical and approximate dimensionless pressure solutions for an unfractured well centered in a bounded circular reservoir producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 6 - Semilog plot of analytical and approximate dimensionless pressure solutions for an unfractured well centered in a bounded circular reservoir producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 7 - Log-Log plot of analytical and approximate dimensionless pressure and pressure derivative solutions for an unfractured well centered in a bounded circular reservoir (no flow outer boundary) producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 8 - Log-Log plot of analytical and approximate dimensionless pressure and pressure derivative solutions for an unfractured well centered in a bounded circular reservoir (constant pressure outer boundary) producing at a constant flow rate ($r_D=1$ [wellbore solution], $s=0$).
Figure 9 - Log-Log plot of analytical and approximate dimensionless pressure and pressure derivative solutions for an unfractured well centered in a bounded circular reservoir (no flow outer boundary) producing at a constant flow rate (various $r_D$, $s=0$).
Figure 10 - Log-log plot of analytical and approximate dimensionless flow rate solutions for an unfractured well centered in a bounded circular reservoir, producing at a constant wellbore pressure ($r_D=1$ wellbore solution), $s=0$.

Legend:
- Numerical Inversion Solutions
- Approximate Solutions (This Work)

Constant Pressure Outer Boundary Cases
No Flow Outer Boundary Cases

Legend:
- Numerical Inversion Solutions
- Approximate Solutions (This Work)