Pity is not natural to man. Children are always cruel, savages also.

— Samuel Johnson (1763)

**Topic:** Linear Flow Solutions: Infinite and Finite-Acting Reservoir Cases

**Objectives:** (things you should know and/or be able to do)
- Be familiar with diffusivity equation for linear flow in porous media.
- Be able to derive the following real and Laplace domain solutions for a linear flow system:

  ■ **Laplace Domain Solutions**:
  a. "Infinite-acting" reservoir behavior
     \[ \bar{p}_{D,\text{inf}}(x_D,u) = \frac{1}{u\bar{u}} e^{-\bar{u}x_D} \]
  b. Bounded linear reservoir
     - "Prescribed Flux" at the outer boundary: \((pb) = \text{prescribed flux boundary}\)
     \[ \bar{p}_{D,pb}(x_D,u) = \frac{1}{u\bar{u}} \frac{\cosh[\bar{u}(1-x_D)]}{\sinh[\bar{u}]} - \bar{q}_{Dext}(x_D,u) \frac{1}{\bar{u}} \frac{\cosh[\bar{u}x_D]}{\sinh[\bar{u}]} \]
     - "No-flow" at the outer boundary: \((i.e., q_{Dext}=0)\)
     \[ \bar{p}_{D,nfb}(x_D,u) = \frac{1}{u\bar{u}} \frac{\cosh[\bar{u}(1-x_D)]}{\sinh[\bar{u}]} \quad (nfb = \text{"no-flow" boundary}) \]
  c. Bounded linear reservoir — "constant pressure" at the outer boundary
     \[ \bar{p}_{D,cpb}(x_D,u) = \frac{1}{u\bar{u}} \frac{\sinh[\bar{u}(1-x_D)]}{\cosh[\bar{u}]} \quad (cpb = \text{constant pressure boundary}) \]

  ■ **Real Domain Solutions:** (Special thanks to Taufan Maraendrajana for his help)
  a. "Infinite-acting" reservoir behavior
     \[ p_{D,\text{inf}}(x_D,t_D) = \frac{2}{\sqrt{\pi}} \frac{t_D}{\sqrt{2}} \exp\left[-\frac{x_D^2}{4t_D}\right] - x_D \text{erfc}\left[\frac{x_D}{2\sqrt{t_D}}\right] \]
  b. Bounded linear reservoir
     - "No-flow" at the outer boundary: \((i.e., q_{Dext}=0, nfb = \text{"no-flow" boundary})\)
     \[ p_{D,nfb}(x_D=0,t_D) = \sum_{n=-1}^{\infty} \left[ \frac{2\sqrt{t_D}}{\sqrt{\pi}} \exp\left[-\frac{a_n^2}{4t_D}\right] - a_n \left[ 1 - \text{erf}\left[\frac{a_n}{2\sqrt{t_D}}\right] \right] \right] \]
     \[ + \sum_{n=-\infty}^{-2} \left[ \frac{2\sqrt{t_D}}{\sqrt{\pi}} \exp\left[-\frac{a_n^2}{4t_D}\right] + a_n \left[ 1 + \text{erf}\left[\frac{a_n}{2\sqrt{t_D}}\right] \right] \right] \quad (a_n=2+2n) \]
  c. Bounded linear reservoir — "constant pressure" at the outer boundary
     \[ p_{D,cpb}(x_D=0,t_D) = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{2\sqrt{t_D}}{\sqrt{\pi}} \exp\left[-\frac{a_n^2}{4t_D}\right] - a_n \left[ 1 - \text{erf}\left[\frac{a_n}{2\sqrt{t_D}}\right] \right] \right] \]
     \[ + \sum_{n=-\infty}^{-1} (-1)^n \left[ \frac{2\sqrt{t_D}}{\sqrt{\pi}} \exp\left[-\frac{a_n^2}{4t_D}\right] + a_n \left[ 1 + \text{erf}\left[\frac{a_n}{2\sqrt{t_D}}\right] \right] \right] \quad (a_n=2n) \]
Lecture Outline:

- Development of the linear flow solutions for infinite and bounded linear systems
  - "Infinite-acting" reservoir case
  - "No-flow" at the outer boundary case
  - "Constant pressure" at the outer boundary case
- Laplace domain solutions are relatively straightforward—these developments are significantly less complicated than for the radial flow case.
- Development of the real domain solution for an infinite-acting linear flow system is also straightforward, use Laplace transform tables for inversion.
- Development of the real domain solution for the no-flow and constant pressure outer boundary cases is quite tedious, and requires an alternative form of integration due to negative values of the $a_n$-coefficients in the dimensionless pressure derivative results. Note that we use Laplace transform identities to integrate the dimensionless pressure derivative result to yield the dimensionless pressure result.

Reading Assignment:

- Review attached notes.
  - Pressure behavior in linear flow systems.

Exercises: For your own practice/skills building—do NOT turn in!

From the attached notes you are to rederive the following solutions (show all details).

- Derive the Laplace domain solutions as designated for a linear flow system, where the "well" is produced at a constant flow rate (inner boundary condition) in a homogeneous reservoir with the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior.
  - Bounded linear reservoir with a "no-flow" outer boundary.
  - Bounded linear reservoir with a constant pressure outer boundary.
Log-log Plot: Constant Well Rate Solutions for a Finite Linear Reservoir:
"No Flow" Outer Boundary Case (x_D=0)

Log-log Plot: Constant Well Rate Solutions for a Finite Linear Reservoir:
Constant Pressure Outer Boundary Case (x_D=0)
Pressure Behavior in Linear Flow Systems

(from Petroleum Engineering 620 Course Notes — 1996)

Petroleum Engineering 620
Fluid Flow in Reservoirs
Pressure Behavior in Linear Flow Systems

The Physical System

Outer Boundary Conditions

\[ a. \ p(x \to \infty, t) = p'_i \]
\[ b. \ q_{ext} = \frac{kA}{B_m} \left[ \frac{dp}{dx} \right]_{x=x_e} = \text{constant} \]
\[ c. \ p(x=x_e, t) = p_i \]

Development of the Dimensionless Diffusivity Equation

The diffusivity equation for linear flow is given as:

\[ \frac{x^2 p}{k} = \frac{\partial u}{\partial t} \frac{\partial p}{\partial x} \]

The objective is to develop a "dimensionless" form of Eq. 1. This dimensionless diffusivity equation is based on the following:

1. Dimensionless distance, \( x_D \), which is intuitively based on the length of the reservoir, \( x_e \).
2. Dimensionless pressure, \( p_D \), that satisfies the following mathematical conveniences:
   a. \( p_D(x_D=0, t_D=0) = 0 \) (the initial condition)
   b. \[ \left[ \frac{\partial p_D}{\partial x_D} \right]_{x_D=0} = -1 \] (the constant rate inner boundary condition)
   c. Dimensionless time, \( t_D \), which is made up of the "leftover" terms.

Beginning with the intuitive definition of the dimensionless distance, \( x_D \), we have

\[ x_D = \frac{x}{x_e} \] (2) or \[ x = x_e x_D \] (3)
Substituting Eq. 3 into Eq. 1, we obtain

\[ \frac{1}{x_e^2} \frac{d}{dx_e} \left[ \frac{dP}{dx_e} \right] = \frac{\phi \mu \xi u t}{k} \frac{dP}{dt} \]

Factoring out the \( x_e \) terms we have

\[ \frac{1}{x_e^2} \frac{d}{dx_e} \left[ \frac{dP}{dx_e} \right] \]

Moving the \( x_e^2 \) term to the right-hand-side to isolate the spatial derivative term gives us

\[ \frac{d}{dx_e} \left[ \frac{dP}{dx_e} \right] = \frac{\phi \mu \xi u t x_e^2}{k} \frac{dP}{dt} \tag{4} \]

Defining the dimensionless pressure, \( P_B \), we have

\[ P_B = \frac{1}{\text{Re}_h} (P_e - P) \quad \text{(Note that } P(t=0), P_e(t=0) = 0) \tag{5} \]

or

\[ P = P_e - \text{Re}_h P_B \tag{6} \]

Substituting Eq. 6 into Eq. 4 gives

\[ \frac{d}{dx_e} \left[ \frac{d}{dx_e} \left( P_e - \text{Re}_h P_B \right) \right] = \frac{\phi \mu \xi u t x_e^2}{k} \frac{d}{dt} \left( P_e - \text{Re}_h P_B \right) \]

Reducing, we have

\[ (-\text{Re}_h) \frac{d}{dx_e} \left[ \frac{dP}{dx_e} \right] = (-\text{Re}_h) \frac{\phi \mu \xi u t x_e^2}{k} \frac{dP}{dt} \]

or finally, we have

\[ \frac{\xi^2}{x_e^2} \frac{dP}{dx_e} = \frac{\phi \mu \xi u t x_e^2}{k} \frac{dP}{dt} \tag{7} \]

The issue remains—how do we determine the "characteristic" pressure, \( \text{Re}_h \)? The answer lies in the inner boundary condition.
Recalling the inner boundary condition,

\[ Q = \frac{kA}{R \mu} \left[ \frac{dP}{dx} \right]_{x=0} \]  

(inner boundary condition) \hspace{1cm} (8)

Solving for the \( \frac{dP}{dx} \) term we have

\[ \left[ \frac{dP}{dx} \right]_{x=0} = \frac{98 \mu}{kA} \]

Substituting \( x = x_e x_D \) (Eq. 3) and \( p = p_l - 12h \right) \) (Eq. 6) we obtain

\[ \left[ \frac{x (p_l - 12h x_D)}{2(x_e x_D)} \right]_{x_D=0} = \frac{98 \mu}{kA} \]

factoring

\[ \frac{-12h}{x_e} \left[ \frac{dP}{dx_D} \right]_{x_D=0} = \frac{98 \mu}{kA} \]

Solving for \( \frac{dP}{dx_D} \)

\[ \left[ \frac{dP}{dx_D} \right]_{x_D=0} = -\frac{1}{12h} \frac{98 \mu}{kA} x_e \] \hspace{1cm} (9)

As a mathematical convenience, we impose the following condition

\[ \left[ \frac{dP}{dx_D} \right]_{x_D=0} = -1 \]  

(dimensionless inner boundary condition) \hspace{1cm} (10)

Equating Eqs. 9 and 10, and solving for \( 12h \), we have

\[ 12h = \frac{98 \mu}{kA} x_e \] \hspace{1cm} (11)

Substituting Eq. 11 into Eq. 5 gives us

\[ B = \frac{kA}{98 \mu x_e} (p_l - p) \] \hspace{1cm} (12)
Recalling Eq. 7, we have
\[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial t} = \frac{\phi u c_x^2}{k \mu} \frac{\partial \phi}{\partial t} \]  (7)

Defining our dimensionless time, \( t_0 \), based on what is "leftover" in Eq. 7, we obtain
\[ t_0 = \frac{k \mu c_x^2}{\phi u c_x^2} \]  (18)

Substituting Eq. 13 into Eq. 7 gives
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t_0} \]  (14)

where
\[ x_0 = x / x_e \]  (2)

\[ \phi_0 = \frac{kA}{qB\mu x_e} (\rho_i - \rho) \] (where \( k, u, B \) are constant)  (12)

and
\[ t_0 = \frac{k \mu c_x^2}{\phi u c_x^2} t \] (where \( k, \phi, u, c_x \) are constant)  (13)

The initial and boundary conditions are given by:

**Initial Condition (Uniform Pressure Distribution)**
\[ p(x, t=0) = \rho_i \]  (15)

**Inner Boundary Condition (Constant Wall Rate)**
\[ q = \frac{kA}{B\mu} \left[ \frac{\partial p}{\partial x} \right]_{x=0} \]  (16)

**Outer Boundary Conditions**

a. **Infinite-Acting Reservoir**
\[ p(x=\infty, t) = \rho_i \]  (17)

b. **Specified-Flux Outer Boundary Condition**
\[ q_{\text{ext}} = \frac{kA}{B\mu} \left[ \frac{\partial p}{\partial x} \right]_{x=x_e} = \text{constant} \]  (18)

c. **Constant Pressure Outer Boundary Condition**
\[ p(x=x_e, t) = \rho_e \]  (19)
The dimensionless forms of the initial and boundary conditions are:

**Initial Condition**: (specified)

\[ p_B(x_D, t_D = 0) = \frac{kA}{qBw} (p_i - p_e) = 0 \]

or

\[ p_B(x_D, t_D = 0) = 0 \] \hspace{1cm} (20)

**Inner Boundary Condition**: (specified)

Recalling Eq. 10, we have

\[
\begin{bmatrix}
\frac{dx_B}{dx} \\
\frac{dx_D}{dx}
\end{bmatrix}
\bigg|_{x = 0} = -1
\]

\hspace{1cm} (10)

**Outer Boundary Conditions**:  

a. **Infinite-Acting Reservoir**

\[ p_B(x_D \to \infty, t_D) = \frac{kA}{qBw} (p_i - p_e) = 0 \]

or

\[ p_B(x_D = \infty, t_D) = 0 \] \hspace{1cm} (21)

b. **Specified-Flux Outer Boundary Condition**

Solving Eq. 18 for \((xp/xx)x=x_e\), we have

\[
\begin{bmatrix}
\frac{dx_P}{dx} \\
\frac{dx_D}{dx}
\end{bmatrix}
\bigg|_{x = x_e} = \frac{q_{ext} Bw}{kA}
\]

Substituting Eqs. 3 and 6 into the above relation gives

\[
\begin{bmatrix}
\frac{dx_B}{dx} \\
\frac{dx_D}{dx}
\end{bmatrix}
\bigg|_{x_D = 1} = -1 \cdot \frac{q_{ext} Bw x_e}{kA} = - q_{ext}
\]

or

\[
\begin{bmatrix}
\frac{dx_B}{dx} \\
\frac{dx_D}{dx}
\end{bmatrix}
\bigg|_{x_D = 1}
\]

\hspace{1cm} (in general, \( q_{ext}(t_D) \) is a function of \( t_D \)) \hspace{1cm} (22)

c. **Constant Pressure Outer Boundary Condition**

\[ p_B(x_D = 1, t_D) = \frac{kA}{qBw} (p_i - p_e) = 0 \]

or

\[ p_B(x_D = 1, t_D) = 0 \] \hspace{1cm} (23)
Development of Solutions to the Dimensionless Diffusivity Equation for Linear Flow

The dimensionless diffusivity equation for linear flow is given by

\[
\frac{d^2 \Phi}{dx_D^2} = \frac{\Phi}{t_D} \tag{14}
\]

where

\[
x_D = \frac{x}{x_e} \tag{2}
\]

\[
P_0 = \frac{P_e k \Lambda}{g \mu \epsilon x_e} \tag{12}
\]

\[
t_D = t_{Dc} \frac{k}{\phi \mu \epsilon x_e^2} \tag{13}
\]

where the \( t_{Dc} \) and \( P_{Dc} \) coefficients are given in the table below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{Dc} )</td>
<td>1</td>
<td>2.637 x 10^{-3}</td>
<td>3.527 x 10^{-6} (t in hours)</td>
</tr>
</tbody>
</table>
| \( P_{Dc} \) | 1           | 1.127 x 10^{-3} | 8.527 x 10^{-5} |}

The initial and boundary conditions are given by:

Initial Condition: (Uniform Pressure Distribution)

\[
P_b(x_D, t=0) = 0 \tag{20}
\]

Inner Boundary Condition: (Constant Well Rate)

\[
\left[ \frac{\sqrt{\Phi}}{x_D} \right]_{x_D=0} = -1 \tag{10}
\]

Outer Boundary Conditions:

a. Infinite-Acting Reservoir

\[
P_b(x_D, \infty, t_D) = 0 \tag{21}
\]

b. Specified-Flux Outer Boundary Condition

\[
\left[ \frac{\sqrt{\Phi}}{x_D} \right]_{x_D=1} = -q_{ext} (t_D) \tag{22}
\]

c. Constant Pressure Outer Boundary Condition

\[
P_b(x_D=1, \ t_D) = 0 \tag{23}
\]
Laplace Transform Approach: General Solution

Taking the laplace transform of the dimensionless diffusivity equation we have

$$\frac{d^2 \overline{P_0}}{d \overline{x}^2} = u \overline{P_0}$$

(from the initial condition)

which reduces to

$$\frac{d^2 \overline{P_0}}{d \overline{x}^2} = u \overline{P_0}$$ \hspace{1cm} (24)

Taking the laplace transform of the boundary conditions we have

\textit{Inner Boundary Condition: (Constant Well Rate)}

$$\left[ \frac{d \overline{P_0}}{d \overline{x}_D} \right]_{\overline{x}_D = 0} = \frac{-1}{u}$$ \hspace{1cm} (25)

\textit{Outer Boundary Conditions:}

a. \textit{Infinite-Acting Reservoir}

$$\overline{P_0} (x_D \rightarrow \infty, u) = 0$$ \hspace{1cm} (26)

b. \textit{Specified-Flux Outer Boundary Condition}

$$\left[ \frac{d \overline{P_0}}{d \overline{x}_D} \right]_{x_D = 1} = -\overline{q}_{\text{dext}}(u)$$ \hspace{1cm} (27)

c. \textit{Constant Pressure Outer Boundary Condition}

$$\overline{P_0} (x_D = 1, u) = 0$$ \hspace{1cm} (28)

General Solution:

Recalling Eq. 24 we have

$$\frac{d^2 \overline{P_0}}{d \overline{x}^2} = u \overline{P_0}$$ \hspace{1cm} (24)

Using the trial solution $$\overline{P_0} = e^{m \overline{x}_D}$$, Eq. 24 becomes

$$m^2 e^{m \overline{x}_D} = u e^{m \overline{x}_D}$$
Eliminating the $e^{mx_0}$ terms, we obtain

$$m^2 = m$$

or

$$m^2 - m = 0$$  \hspace{1cm} (29)

Resolving Eq. 29 gives

$$m = \pm \sqrt{m}$$  \hspace{1cm} (30)

Where Eq. 30 gives the following general solution

$$\tilde{f}(m,x_D) = c_1 e^{\sqrt{m} x_D} + c_2 e^{-\sqrt{m} x_D}$$  \hspace{1cm} (31)

We also require $d(\tilde{f}(m,x_D))/dx_D$, which is given as

$$\frac{d}{dx_D} \tilde{f}(m,x_D) = c_1 \sqrt{m} e^{\sqrt{m} x_D} - c_2 \sqrt{m} e^{-\sqrt{m} x_D}$$  \hspace{1cm} (32)

**Laplace Transform Approach: Particular Solution - Infinite Acting Reservoir Case**

Combining the inner boundary condition (Eq. 25) and the derivative of the general solution with respect to $x_D$ (Eq. 32), we have

$$\left[ c_1 \sqrt{m} e^{\sqrt{m} x_D} - c_2 \sqrt{m} e^{-\sqrt{m} x_D} \right] = \frac{-1}{m} \quad \text{at} \ x_D = 0$$  \hspace{1cm} (33)

Combining the outer boundary condition (Eq. 26) and the general solution (Eq. 31) gives us

$$\lim_{x_D \to \infty} \left[ c_1 e^{\sqrt{m} x_D} + c_2 e^{-\sqrt{m} x_D} \right] = 0$$  \hspace{1cm} (34)

For "boundedness" (i.e., non-infinite values), Eq. 34 requires that $c_1 \neq 0$ (note that $c_2 e^{-\infty} = 0$); obviously Eq. 34 is indeterminate in terms of $c_2$. Therefore, we use Eq. 33 to determine $c_2$. 
Solving Eq. 33 for \( c_2 \) gives us

\[
c_2 = \left[ \frac{1}{m^{3/2}} e^{\sqrt{m} x_0} \right] = \frac{1}{m^{3/2}} \quad (35)
\]

And recalling,

\[
c_2 = 0 \quad (36)
\]

Substituting Eqs. 35 and 36 into the general solution, Eq. 31, we obtain

\[
\bar{\phi}(\mu, x_0) = \frac{1}{m^{3/2}} e^{-\sqrt{m} x_0} \quad (37)
\]

Laplace Transform Approach: Particular Solution

Specified-Flux Outer Boundary

From Eq. 33, we write the general inner boundary condition as

\[
c_1 \sqrt{\mu} - c_2 \sqrt{\mu} = \frac{-1}{m} \quad (38)
\]

Combining the outer boundary condition (Eq. 27) and the derivative of the general solution with respect to \( x_0 \) (Eq. 32) gives us

\[
\left[ c_1 \sqrt{\mu} e^{\sqrt{\mu} x_0} - c_2 \sqrt{\mu} e^{-\sqrt{\mu} x_0} \right] \bigg|_{x_0 = 1} = -\bar{\Phi}_{\text{Dext}}(\mu)
\]

or reducing, we have

\[
c_1 \sqrt{\mu} e^{\sqrt{\mu}} - c_2 \sqrt{\mu} e^{-\sqrt{\mu}} = -\bar{\Phi}_{\text{Dext}}(\mu) \quad (39)
\]

Factoring out the \( \sqrt{\mu} \) terms from the left-hand-side of Eq. 38

\[
c_1 \sqrt{\mu} e^{\sqrt{\mu}} = -\frac{1}{\sqrt{\mu}} \bar{\Phi}_{\text{Dext}}(\mu) \quad (40)
\]

Factoring out the \( \sqrt{\mu} e^{\sqrt{\mu}} \) terms from the left-hand-side of Eq. 39

\[
c_1 \sqrt{\mu} e^{-2\sqrt{\mu}} = -\frac{e^{-\sqrt{\mu}} \bar{\Phi}_{\text{Dext}}(\mu)}{\sqrt{\mu}} \quad (41)
\]
Solving Eq. 40 for $c_1$ gives us
\[ c_1 = c_2 - \frac{1}{m\sqrt{\lambda}} \]  
Substituting Eq. 42 into Eq. 41 gives
\[ c_2 - \frac{1}{m\sqrt{\lambda}} - c_2 e^{-2\sqrt{\lambda}u} = -\frac{e^{-\sqrt{\lambda}u}}{\sqrt{\lambda}} \tilde{f}_{\text{ext}}(u) \]
Rearranging
\[ c_2 (1 - e^{-2\sqrt{\lambda}u}) = \frac{1}{m\sqrt{\lambda}} - \frac{e^{-\sqrt{\lambda}u}}{\sqrt{\lambda}} \tilde{f}_{\text{ext}}(u) \]
Or finally, solving for $c_2$, we have
\[ c_2 = \frac{1}{m\sqrt{\lambda}} \frac{1}{1 - e^{-2\sqrt{\lambda}u}} - \frac{e^{-\sqrt{\lambda}u}}{\sqrt{\lambda} (1 - e^{-2\sqrt{\lambda}u})} \tilde{f}_{\text{ext}}(u) \]  
Solving Eq. 42 for $c_2$, we obtain
\[ c_2 = c_1 + \frac{1}{m\sqrt{\lambda}} \]  
Substituting Eq. 44 into Eq. 43 and solving for $c_1$, gives
\[ c_1 = \frac{1}{m\sqrt{\lambda}} \left( \frac{1}{1 - e^{-2\sqrt{\lambda}u}} - 1 \right) - \frac{e^{-\sqrt{\lambda}u}}{\sqrt{\lambda} (1 - e^{-2\sqrt{\lambda}u})} \tilde{f}_{\text{ext}}(u) \]
Factoring
\[ c_1 = \frac{1}{m\sqrt{\lambda}} \left[ \frac{1}{(1 - e^{-2\sqrt{\lambda}u})} - 1 \right] - \frac{e^{-\sqrt{\lambda}u}}{\sqrt{\lambda} (1 - e^{-2\sqrt{\lambda}u})} \tilde{f}_{\text{ext}}(u) \]  
Substituting Eqs. 44 and 45 into Eq. 31 gives us the particular solution for this case
\[ \tilde{f}_D(u,x) = \frac{1}{m\sqrt{\lambda}} \left[ \left( \frac{1}{1 - e^{-2\sqrt{\lambda}u}} - 1 \right) e^{\sqrt{\lambda}x_D} + \frac{1}{1 - e^{-2\sqrt{\lambda}u}} e^{-\sqrt{\lambda}x_D} \right] \]
\[ - \tilde{f}_{\text{ext}}(u) \frac{1}{\sqrt{\lambda}} \frac{e^{-\sqrt{\lambda}u}}{(1 - e^{-2\sqrt{\lambda}u})} \left[ e^{\sqrt{\lambda}x_D} + e^{-\sqrt{\lambda}x_D} \right] \]
The particular solution, Eq. 46, can be reduced algebraically when we recognize that the exponential terms can be expressed by the hyperbolic sine and cosine functions (i.e., \( \sinh(x) \) and \( \cosh(x) \)). We begin by noting the following simplifications:

\[
\frac{1}{(1-e^{-2\sqrt{u}})} = \frac{e^\sqrt{u}}{e^{2\sqrt{u}} - e^{-2\sqrt{u}}} \tag{47}
\]

and

\[
\frac{1}{(1-e^{-2\sqrt{u}})} - 1 = \frac{e^{-\sqrt{u}}}{e^{\sqrt{u}} - e^{-\sqrt{u}}} \tag{48}
\]

and finally

\[
\frac{e^{-\sqrt{u}}}{(1-e^{-2\sqrt{u}})} = \frac{1}{e^{\sqrt{u}} - e^{-\sqrt{u}}} \tag{49}
\]

We also required the following identities

\[
z \sinh(x) = e^x - e^{-x} \tag{50}
\]

\[
z \cosh(x) = e^x + e^{-x} \tag{51}
\]

Substituting Eqs. 47-49 into Eq. 46 gives us

\[
\bar{\rho}(u,x_0) = \frac{1}{m \sqrt{u}} \left[ \frac{e^{-\sqrt{u}}}{e^{\sqrt{u}} - e^{-\sqrt{u}}} e^{\sqrt{u}x_0} + \frac{e^{\sqrt{u}}}{e^{\sqrt{u}} - e^{-\sqrt{u}}} e^{-\sqrt{u}x_0} \right]
\]

\[-\bar{\varphi}_{Dext}(u) \frac{1}{\sqrt{u}} \frac{1}{e^{\sqrt{u}} - e^{-\sqrt{u}}} (e^{\sqrt{u}x_0} + e^{-\sqrt{u}x_0})
\]

Collecting

\[
\bar{\rho}(u,x_0) = \frac{1}{m \sqrt{u}} \frac{e^{\sqrt{u}(1-x_0)} + e^{-\sqrt{u}(1-x_0)}}{e^{\sqrt{u}} - e^{-\sqrt{u}}}
\]

\[-\bar{\varphi}_{Dext}(u) \frac{1}{\sqrt{u}} \frac{1}{e^{\sqrt{u}} - e^{-\sqrt{u}}} (e^{\sqrt{u}x_0} + e^{-\sqrt{u}x_0}) \tag{52}\]
Substituting the definitions of sinh(x) and cosh(x), Eqs. 50 and 51, where appropriate in Eq. 52 gives us

\[
\bar{f}_0(m, x_0) = \frac{1}{m \sqrt{\nu}} \left[ \frac{z \cosh [\sqrt{\nu} (1-x_0)]}{z \sinh [\nu/2]} \right]
- \bar{a}_0(m) \frac{1}{\sqrt{\nu}} \frac{z \cosh [\sqrt{\nu} x_0]}{z \sinh [\nu/2]}.
\]

Which reduces to

\[
\bar{f}_0(m, x_0) = \frac{1}{m \sqrt{\nu}} \frac{\cosh [\sqrt{\nu} (1-x_0)]}{\sinh [\nu/2]} - \bar{a}_0(m) \frac{1}{\sqrt{\nu}} \frac{\cosh [\sqrt{\nu} x_0]}{\sinh [\nu/2]}
\]

(Laplace Transform Approach: Particular Solution - Constant Pressure Outer Boundary)

We again begin with the general (constant rate) inner boundary condition given by Eq. 38 (or Eq. 33). Recalling Eq. 33, we have

\[
c_1 \sqrt{\nu} - c_2 \sqrt{\nu} = -\frac{1}{m}
\]  

The constant pressure outer boundary condition (in the Laplace domain) is given by Eq. 28. Recalling Eq. 28 and the general solution given by Eq. 31, we have

\[
\bar{f}_0(x_0 = 1, m) = 0 \quad \text{(Constant Pressure Outer Boundary Condition)}
\]

and

\[
\bar{f}_0(x_0, m) = c_1 e^{\sqrt{\nu} x_0} + c_2 e^{-\sqrt{\nu} x_0}
\]

Evaluating Eq. 31 at \( x_0 = 1 \) gives us

\[
\bar{f}_0(x_0 = 1, m) = c_1 e^{\sqrt{\nu}} + c_2 e^{-\sqrt{\nu}} = 0
\]
Dividing through Eq. 38 by $V_{12}$ gives us Eq. 40. Recalling
Eq. 40, we have
\[ c_1 - c_2 = \frac{-1}{m\sqrt{\mu}} \]  \hspace{1cm} (40)
Dividing through Eq. 54 by $e^{2\sqrt{\mu}}$ gives us
\[ c_1 + c_2 e^{-2\sqrt{\mu}} = 0 \]  \hspace{1cm} (55)
Subtracting Eq. 40 from Eq. 55, we obtain
\[ c_2 e^{-2\sqrt{\mu}} + c_2 = \frac{1}{m\sqrt{\mu}} \]
\[ c_2 (e^{-2\sqrt{\mu}} + 1) = \frac{1}{m\sqrt{\mu}} \]
Solving for $c_2$ gives us
\[ c_2 = \frac{1}{m\sqrt{\mu}} \cdot \frac{1}{1 + e^{-2\sqrt{\mu}}} \]  \hspace{1cm} (56)
Solving Eq. 55 for $c_1$, we have
\[ c_1 = -c_2 e^{-2\sqrt{\mu}} \]  \hspace{1cm} (57)
Substituting Eq. 56 into Eq. 57, we obtain
\[ c_1 = -\frac{1}{m\sqrt{\mu}} \cdot \frac{e^{-2\sqrt{\mu}}}{1 + e^{-2\sqrt{\mu}}} \]  \hspace{1cm} (58)
Substituting Eqs. 56 and 58 into the general solution (Eq. 38), we have
\[ \Phi_0 (n, x_0) = \frac{1}{m\sqrt{\mu}} \left[ -\frac{e^{-2\sqrt{\mu}}}{e^{\sqrt{\mu}x_0} + 1} + \frac{1}{1 + e^{-2\sqrt{\mu}}} \right] \]  \hspace{1cm} (59)
The first exponential term can be expressed as
\[ \frac{e^{-2\sqrt{\mu}}}{(1 + e^{-2\sqrt{\mu}})} = \frac{e^{-\sqrt{\mu}}}{e^{\sqrt{\mu} + e^{-\sqrt{\mu}}} \]  \hspace{1cm} (60)
And the other exponential term in Eq. 59 reduces as follows
\[ \frac{1}{(1 + e^{-2\sqrt{\mu}})} = \frac{e^{\sqrt{\mu}}}{e^{\sqrt{\mu} + e^{-\sqrt{\mu}}} \]  \hspace{1cm} (61)
Substituting Eqs. 60 and 61 into Eq. 59 gives us
\[
\tilde{\rho}_0(m, x_0) = \frac{1}{m \sqrt{\pi}} \left[ -\frac{e^{-\sqrt{\mu}}}{e^{\sqrt{\mu}} + e^{-\sqrt{\mu}}} \right. \\
\left. e^{\sqrt{\mu} x_0} + \frac{e^{\sqrt{\mu}}}{e^{\sqrt{\mu}} + e^{-\sqrt{\mu}}} e^{-\sqrt{\mu} x_0} \right]
\]

Collecting terms we have
\[
\tilde{\rho}_0(m, x_0) = \frac{1}{m \sqrt{\pi}} \frac{-e^{-\sqrt{\mu}(1-x_0)} + e^{\sqrt{\mu}(1-x_0)}}{e^{\sqrt{\mu}} + e^{-\sqrt{\mu}}}
\]

or, rearranging, we have
\[
\tilde{\rho}_0(m, x_0) = \frac{1}{m \sqrt{\pi}} \frac{e^{\sqrt{\mu}(1-x_0)} - e^{-\sqrt{\mu}(1-x_0)}}{e^{\sqrt{\mu}} + e^{-\sqrt{\mu}}}
\]

(62)

Substituting the definitions of \(\sinh(x)\) and \(\cosh(x)\) (Eqs. 50 and 51, respectively), where appropriate in Eq. 62, we have
\[
\tilde{\rho}_0(m, x_0) = \frac{1}{m \sqrt{\pi}} \frac{\sqrt{\mu}(1-x_0)}{e^{\sqrt{\mu}} - e^{-\sqrt{\mu}}}
\]

which gives us the following final form
\[
\tilde{\rho}_0(m, x_0) = \frac{1}{m \sqrt{\pi}} \frac{\sinh[\sqrt{\mu}(1-x_0)]}{\cosh[\sqrt{\mu}]}
\]

(63)
Laplace Transform Approach: Summary of Laplace Domain Solutions

<table>
<thead>
<tr>
<th>Case</th>
<th>( \bar{\theta}(\mu, x_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infinite-Acting Reservoir</td>
<td>( \frac{1}{\mu} \frac{1}{\sqrt{\mu}} ) ( e^{-\sqrt{\mu} x_0} )</td>
</tr>
<tr>
<td>Specified-Flux Outer Boundary Condition</td>
<td>( \frac{1}{\mu} \frac{1}{\sqrt{\mu}} ) ( \cosh[\sqrt{\mu}(1-x_0)] ) ( \frac{1}{\sqrt{\mu}} ) ( \frac{1}{\sinh[\sqrt{\mu}]} ) ( -\bar{a}_{\text{Dext}}(\mu) ) ( \frac{1}{\sqrt{\mu}} ) ( \cosh[\sqrt{\mu} x_0] ) ( \frac{1}{\sinh[\sqrt{\mu}]} )</td>
</tr>
<tr>
<td>Constant Pressure Outer Boundary Condition</td>
<td>( \frac{1}{\mu} \frac{1}{\sqrt{\mu}} ) ( \frac{1}{\cosh[\sqrt{\mu}]} ) ( \sinh[\sqrt{\mu}(1-x_0)] )</td>
</tr>
</tbody>
</table>

Real Domain Solution: Infinite-Acting Reservoir

Recalling the Laplace transform solution for this case (i.e., Eq. 57) we have

\[
\bar{\theta}(\mu, x_0) = \frac{1}{\mu} \frac{1}{\sqrt{\mu}} \ e^{-\sqrt{\mu} x_0}
\]

(57)

The inverse Laplace transform of Eq. 57 is obtained using a table lookup.

<table>
<thead>
<tr>
<th>( \bar{f}(s) )</th>
<th>( f(t) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 ) ( \frac{1}{\sqrt{s}} ) ( e^{-a\sqrt{s}} )</td>
<td>( \frac{1}{\sqrt{\pi t}} ) ( e^{-a^2/4t} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 3.2.16, p. 246.</td>
</tr>
<tr>
<td>( \frac{1}{5} ) ( \frac{1}{\sqrt{s}} ) ( e^{-a\sqrt{s}} )</td>
<td>( \frac{2}{\sqrt{\pi t}} ) ( e^{-a^2/4t} ) ( -a \text{erfc}\left[\frac{a}{2\sqrt{t}}\right] )</td>
<td>Roberts and Kaufman: Eq. 3.2.22, p. 247.</td>
</tr>
</tbody>
</table>

* Inversion of this relation yields \( f(t) \) for this case.
Inverting Eq. 37 gives
\[ p_0(t_0, x_D) = \frac{2}{\sqrt{\pi}} e^{-\frac{X_D^2}{2t_0}} x_D \operatorname{erfc} \left[ \frac{x_D}{\sqrt{2t_0}} \right] \]  
(64)

**Real Domain Solution: Specified-Flux Outer Boundary**

Recalling the Laplace transform solution for this case (i.e., Eq. 33). This relation is given as
\[ \tilde{p}_0(u, x_D) = \frac{1}{\mu} \frac{1}{\sqrt{u}} \cosh [\sqrt{u} (1-x_D)] - \frac{\tilde{q}_0(u)}{\sqrt{u}} \frac{\cosh [\sqrt{u} x_D]}{\sinh [\sqrt{u} \mu]} \]  
(53)

Obviously the \( \tilde{q}_0(u) \) function must be known for the purpose of simplicity we will assume a "no flow" boundary condition (i.e., \( \tilde{q}_0(t_0) = 0 \)), this reduces Eq. 53 to
\[ \tilde{p}_0(u, x_D) = \frac{1}{\mu} \frac{1}{\sqrt{u}} \cosh [\sqrt{u} (1-x_D)] \]  
(65)

The inverse Laplace transform of Eq. 65 is obtained using the table lookup shown below.

<table>
<thead>
<tr>
<th>( f(s) )</th>
<th>( f(t) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\sqrt{s}} \frac{\cosh [\sqrt{s} x_D]}{\sinh [\sqrt{s} \mu]} )</td>
<td>( ? )</td>
<td>Not Found</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{s}} \frac{\cosh [\sqrt{s} x_D]}{\sinh [\sqrt{s} \mu]} )</td>
<td>( a^{-1} \theta_4 \left[ \frac{u}{2a}, \frac{\pm}{a^2} \right] )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, section 2, Eq. 7.2.25, p. 283.</td>
</tr>
</tbody>
</table>

where the 4th "Theta Function" is given as
\[ \theta_4 \left[ z, x \right] = \frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{x} \left( \frac{z+\frac{1}{2}+n}{2} \right)^2} \]  
(Roberts and Kaufman, p. xxvii)
Recall from the fundamentals of the Laplace transform that
\[ f(t) = \mathcal{L}^{-1} \left[ \frac{1}{s} \tilde{g}(s) \right] \quad \text{where} \quad \tilde{g}(s) = \mathcal{L}[g(t)] \]
and
\[ f(t) = \int_0^t g(t) \, dt \quad \text{i.e.,} \quad g(t) = f'(t) \]
Therefore if we can invert $\tilde{g}(s)$ to yield $g(t)$ (i.e., $f'(t)$), then we may be able to integrate $g(t)$ to yield $f(t)$.

Using the approach discussed above, and noting that $a=1$, and $v=1-x_D$, we have
\[ g(t_D, x_D) = \frac{1}{\sqrt{\pi t_D}} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2} \left[ \frac{(1-x_D) + 1/n}{2} \right]^2} \quad (66) \]
where
\[ g(t_D, x_D) = \frac{d}{dt_D} \quad (67) \]
Writing Eq. 66 in shorthand notation we have
\[ g(t_D, x_D) = \frac{1}{\sqrt{\pi t_D}} \sum_{n=-\infty}^{\infty} e^{-\frac{a_D^2}{4 t_D}} \quad (68) \]
where
\[ a_D = 2 \left[ \frac{(1-x_D) + 1/n}{2} \right] \quad (69) \]
Isolating a single term in Eq. 68 and generalizing gives us
\[ f(t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4 t}} \quad (70) \]
The integral of $f(t)$ in Eq. 70
\[ I(t) = \frac{z \sqrt{\pi t}}{\sqrt{\pi}} e^{-\frac{a^2}{4 t}} - a \text{erfc} \left[ \frac{a}{z \sqrt{4 t}} \right] \quad (71) \]
Integration of Eq. 70 requires considerable attention as the \( a \)-coefficient can either be positive or negative (i.e. + or -). The following sequence considers the \( a > 0 \) and \( a < 0 \) cases.

Recalling Eq. 70

\[
\bar{f}(t) = \frac{-a^2}{\sqrt{\pi t}} \quad e^{\frac{t}{4t}}
\]  

(70)

Defining a variable of substitution, \( z \), we have

\[
z = \frac{1}{\sqrt{t}} \quad \text{and} \quad dz = \frac{-1}{z} \quad t^{-3/2} \quad dt = -\frac{1}{z}
\]

and

\[
dz = -\frac{1}{z} \quad t^{-3/2} \quad dt = -\frac{1}{z} \quad \frac{1}{\sqrt{t^3}} \quad dt = -\frac{1}{z \sqrt{\pi t} \sqrt{t}} \quad dt
\]

or

\[
dz = -\frac{1}{z} \quad z^2 \quad dt
\]

Summarizing

\[
z = \frac{1}{\sqrt{t}} \quad (71) \quad z^2 = \frac{1}{t} \quad (72) \quad -\frac{z}{z^3} \quad dz = dt \quad (73)
\]

Integrating Eq. 70, we have

\[
I(t) = \int_{0}^{t} \frac{1}{\sqrt{\pi t^3}} \quad e^{-\frac{a^2}{4t}} \quad dt
\]  

(74)

Making the substitution of \( z = \sqrt{\pi t} \) into Eq. 74 we have

at \( t=0 \) \quad \( z = \infty \)

at \( t=t \) \quad \( z = z \)

\[
I(z) = \int_{0}^{z} \left[ e^{\frac{z}{\sqrt{\pi t^3}}} \right] e^{-\frac{a^2}{4t}} \quad \left[ -\frac{z}{z^3} \quad dz \right]
\]
Reducing

\[ I(z) = \frac{-z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{z^2} e^{-\frac{a^2}{4} z^2} \, dz \]

or

\[ I(z) = \frac{z}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\left[\frac{a}{z}\right]^2} \, dz \]  \hspace{1cm} (75)

Integrating by parts, Eq. 75 becomes

\[ \int uv = \mu v - \int v du \]

\[ \mu = e^{-\left[\frac{a}{z}\right]^2} = e^{-\frac{a^2}{4} z^2} \]

\[ du = \frac{a^2}{4} \int_{z}^{\infty} e^{-\left[\frac{a}{z}\right]^2} \, dz \]

\[ dv = \frac{z^2}{(-z+1)} \]

\[ u = \frac{1}{(-z+1)} \frac{z^{-1}}{(-z+1)} = \frac{1}{z} \]

\[ I(z) = \frac{z}{\sqrt{\pi}} \left[ \left[ e^{-\frac{a^2}{4} z^2} \right]_{z}^{\infty} - \int_{z}^{\infty} \left[ e^{-\frac{a^2}{4} z^2} \right] \, dz \right] \]

or

\[ I(z) = \frac{z}{\sqrt{\pi}} \frac{a^2}{z^2} \int_{z}^{\infty} e^{-\left[\frac{a}{z}\right]^2} \, dz \]  \hspace{1cm} (76)

which can be written as

\[ I(z) = I_1(z) - I_2(z) \]  \hspace{1cm} (77)

where

\[ I_1(z) = \frac{z}{\sqrt{\pi}} e^{-\left[\frac{a}{z}\right]^2} \]  \hspace{1cm} (78)

and

\[ I_2(z) = \frac{a^2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\left[\frac{a}{z}\right]^2} \, dz \]  \hspace{1cm} (79)
Defining another variable of substitution, \(w\), for Eq. 79, we have

\[ \omega = \frac{a}{z} \quad (80) \]
\[ dw = \frac{a}{z} \quad (82) \]

or

\[ z = \frac{a}{w} \quad (81) \]
\[ dz = \frac{a}{w} \quad (83) \]

Before proceeding, we immediately note that 2 cases are possible; \(a \geq 0\) and \(a < 0\), this gives

\[ a \geq 0: \]
\[ z = z; \quad w = w \]
\[ z = \infty; \quad w = \infty \]

\[ a < 0: \]
\[ z = z; \quad w = w \]
\[ z = \infty; \quad w = -\infty \]

Making these substitutions, we obtain

\[ I_2(x) = \frac{a^2}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{a}{z} \frac{x}{z}\right)^2} \, dz = \left\{ \begin{array}{ll}
\frac{a^2}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \, dw & (a \geq 0) \\
\frac{a^2}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \, dw & (a < 0)
\end{array} \right. \]

or

\[ I_2(x) = \left\{ \begin{array}{ll}
\frac{a}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \, dw & (a \geq 0) \\
\frac{a}{\sqrt{\pi}} \int_0^\infty e^{-w^2} \, dw & (a < 0)
\end{array} \right. \]
From Abramowitz and Stegun, Handbook of Mathematical Functions, we have

\[ \text{erf}(w) = \frac{2}{\sqrt{\pi}} \int_{0}^{w} e^{-t^2} dt \]  \hspace{1cm} [\text{Eq. 7.1.1, p. 297}]

\[ \text{erfc}(w) = \frac{2}{\sqrt{\pi}} \int_{w}^{\infty} e^{-t^2} dt = 1 - \text{erf}(w) \]  \hspace{1cm} [\text{Eq. 7.1.2, p. 297}]

Splitting the integral in Eq. 85, we have

\[ \int_{w}^{-\infty} e^{-w^2} dw = \int_{w}^{0} e^{-w^2} dw + \int_{0}^{-\infty} e^{-w^2} dw \]

\[ = -\int_{0}^{\infty} e^{-w^2} dw + \int_{0}^{\infty} e^{-w^2} dw \]

From Abramowitz and Stegun,

\[ \int_{0}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \]  \hspace{1cm} [\text{Eq. 7.4.1, p. 302}]

or

\[ \int_{0}^{-\infty} e^{-t^2} dt = -\frac{\sqrt{\pi}}{2} \]

Also

\[ \int_{0}^{\infty} e^{-w^2} dw = \frac{\sqrt{\pi}}{2} \text{erf}(w) \]

Reconstructing

\[ \frac{2}{\sqrt{\pi}} \int_{w}^{-\infty} e^{-w^2} dw = -\frac{2}{\sqrt{\pi}} \int_{w}^{0} e^{-w^2} dw + \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-w^2} dw \]

\[ = -\text{erf}(w) - 1 \]  \hspace{1cm} (86)
And
\[ \int_0^\infty e^{-w^2} \, dw = 1 - \text{erf}(w) \] (87)

Putting these results back into the expressions for \( I_2(z) \) gives (recall that \( w = \frac{a}{z} \))

\[ I_2(z) = \begin{cases} 
  a \left[ 1 - \text{erf} \left( \frac{a}{z} \right) \right] & (a > 0) \\
  a \left[ -\text{erf} \left( \frac{a}{z} \right) - 1 \right] & (a < 0)
\end{cases} \] (88)

or

\[ I_2(z) = \begin{cases} 
  a \left[ 1 - \text{erf} \left( \frac{a}{z^2} \right) \right] & (a > 0) \quad (87) \\
  a \left[ -\text{erf} \left( \frac{a}{z^2} \right) - 1 \right] & (a < 0) \quad (88)
\end{cases} \]

Combining Eqs. 87 and 88 into Eq. 77, then into Eq. 76

\[ I(z) = \begin{cases} 
  \sqrt{\frac{2}{\pi z}} \, e^{-\left( \frac{a}{z^2} \right)^2} - a \left[ 1 - \text{erf} \left( \frac{a}{z^2} \right) \right] & (a > 0) \quad (89) \\
  \sqrt{\frac{2}{\pi z}} \, e^{-\left( \frac{a}{z^2} \right)^2} - a \left[ -\text{erf} \left( \frac{a}{z^2} \right) - 1 \right] & (a < 0) \quad (90)
\end{cases} \]

Recalling that \( z = 1/\sqrt{T} \) (Eq. 71), Eqs. 89 and 90 become

\[ I(t) = \begin{cases} 
  \frac{z \sqrt{T}}{\sqrt{\pi}} e^{-\left( \frac{a}{\sqrt{T^2}} \right)^2} - a \left[ 1 - \text{erf} \left( \frac{a}{\sqrt{T^2}} \right) \right] & (a > 0) \quad (91) \\
  \frac{z \sqrt{T}}{\sqrt{\pi}} e^{-\left( \frac{a}{\sqrt{T^2}} \right)^2} - a \left[ -\text{erf} \left( \frac{a}{\sqrt{T^2}} \right) - 1 \right] & (a < 0) \quad (92)
\end{cases} \]
Recalling our identity for the $a_n$'s (Eq. 69), we have

$$a_n = 2 \left[ \frac{(1-x_0)}{2} + \frac{1}{2} + n \right]$$  \hspace{1cm} (69)

And our final expression for $I(t)$ is given by

$$I(t) = \begin{cases} 
-\left[ \frac{a_t}{2\sqrt{\pi}} \right]^2 \\
\frac{z\sqrt{\pi}}{\sqrt{\pi}} e^{-\left[ \frac{a_t}{2\sqrt{\pi}} \right]^2} - a_n \left[ 1 - \text{erf} \left[ \frac{a_t}{2\sqrt{\pi}} \right] \right] & (a > 0) \hspace{1cm} (93) \\
\frac{z\sqrt{\pi}}{\sqrt{\pi}} e^{-\left[ \frac{a_t}{2\sqrt{\pi}} \right]^2} + a_n \left[ 1 + \text{erf} \left[ \frac{a_t}{2\sqrt{\pi}} \right] \right] & (a < 0) \hspace{1cm} (94)
\end{cases}$$

Recalling the derivative relations (Eqs. 66 and 67), we note that the $I_p$ solution is given by

$$I_p(t_0, t_0) = \int_0^{t_0} g(t_0, x_0) dt_0 \hspace{1cm} (95)$$

from Eq. 69, we have

**Case 1:**

$x_0 = 0$; $n = -2$; $a_{-2} > 0$ and at $n = -1$ $a_1 < 0$

**Case 2:**

$x_0 = 1$; $n = -1$; $a_{-1} < 0$ and at $n = 0$ $a_0 \geq 0$

Coupling Eqs. 93-95, and solving for $x_0 = 0$

$$I_p(t_0, 0) = \sum_{n=-1}^{\infty} \left[ \frac{z\sqrt{\pi}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{2\sqrt{\pi}} \right]^2} - a_n \left[ 1 - \text{erf} \left[ \frac{a_n}{2\sqrt{\pi}} \right] \right] \right]$$

$$+ \sum_{n=-\infty}^{-2} \left[ \frac{z\sqrt{\pi}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{2\sqrt{\pi}} \right]^2} + a_n \left[ 1 + \text{erf} \left[ \frac{a_n}{2\sqrt{\pi}} \right] \right] \right]$$

$$\hspace{1cm} (96)$$
Coupling Eqs. 93-95, and solving for $x_0=1$

$$p_o(t, x_0=1) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left( \frac{2n}{Z_{B}} e^{-\frac{(a_n)^2}{Z_{B}^2}} - a_n \left[ 1 - \text{erf} \left( \frac{a_n}{Z_{B}} \right) \right] \right)$$

$$+ \sum_{n=-\infty}^{-1} \left( \frac{2n+1}{Z_{B}} e^{-\frac{(a_n)^2}{Z_{B}^2}} + a_n \left[ 1 + \text{erf} \left( \frac{a_n}{Z_{B}} \right) \right] \right) \quad (97)$$

**Real Domain Solution: Constant Pressure Outer Boundary**

Recalling the Laplace transform solution for this case (i.e., Eq. 63). This relation is given as

$$p_o(u, x_0) = \frac{1}{u} \frac{\sinh [u(1-x_0)]}{\cosh [u]} \quad (63)$$

The inverse Laplace transform of Eq. 63 is obtained using the table lookup shown below.

<table>
<thead>
<tr>
<th>$f(s)$</th>
<th>$f(t)$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{5} \frac{\sinh [\sqrt{s}]}{\cosh [\sqrt{s}]}$</td>
<td>$-\frac{1}{\sqrt{\pi}} \Theta_1 \left( \frac{u}{2a^2} \right)$</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 9.2.22, p. 283</td>
</tr>
</tbody>
</table>

Where the 1st "Theta Function" is given by

$$\Theta_1(z, x) = \frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{1}{x} \left( \frac{z}{x} - \frac{x}{2} + n \right)^2} \quad (Roberts and Kaufman, p. xxvii)$$

For this case, $a=1$, and $u=(1-x_0)$, which gives

$$p_o(t, x_0) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{1}{t_0} \left( \frac{(1-x_0) - 1 + n}{2} \right)^2}$$

$$= \frac{1}{\sqrt{\pi t_0}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{1}{t_0} \left( \frac{(1-x_0) - 1 + n}{2} \right)^2} \quad (98)$$
Writing Eq. 98 in shorthand notation, we have

\[ g(t_d, x_d) = \frac{1}{\sqrt{\pi t_d}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{a_n^2}{4t_d}} \]  

(99)

where

\[ a_n = z \left[ \frac{(1-x_d)}{z} - \frac{1}{z} + n \right] \]  

(100)

The \((-1)^n\) term appears to be a problem—however, we can hold this term out of the integration.
In fact, Eq. 99 is identical in form to Eq. 68 (with the exception of the \((-1)^n\) term, as noted). As with the previous case, we will also consider \(x_d = 0\) and \(x_d = 1\) solutions.

**Case 1**: \(x_d = 0\); \(n = -1\), \(a_{-1} \leq 0\) and at \(n = 0\) \(a_0 \geq 0\)

**Case 2**: \(x_d = 1\); \(n = 0\), \(a_0 \leq 0\) and at \(n = 1\) \(a_1 \geq 0\)

The solutions are

**Case 1**:

\[ P_0(t_d, x_d = 0) = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{z \sqrt{t_d}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{z \sqrt{t_d}} \right]^2} - a_n \left[ 1 - \operatorname{erf} \left[ \frac{a_n}{z \sqrt{t_d}} \right] \right] \right] \]

(101)

\[ + \sum_{n=-\infty}^{-1} (-1)^n \left[ \frac{z \sqrt{t_d}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{z \sqrt{t_d}} \right]^2} + a_n \left[ 1 + \operatorname{erf} \left[ \frac{a_n}{z \sqrt{t_d}} \right] \right] \right] \]

**Case 2**:

\[ P_0(t_d, x_d = 1) = \sum_{n=1}^{\infty} (-1)^n \left[ \frac{z \sqrt{t_d}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{z \sqrt{t_d}} \right]^2} - a_n \left[ 1 - \operatorname{erf} \left[ \frac{a_n}{z \sqrt{t_d}} \right] \right] \right] \]

(102)

\[ + \sum_{n=-\infty}^{0} (-1)^n \left[ \frac{z \sqrt{t_d}}{\sqrt{\pi}} e^{-\left[ \frac{a_n}{z \sqrt{t_d}} \right]^2} + a_n \left[ 1 + \operatorname{erf} \left[ \frac{a_n}{z \sqrt{t_d}} \right] \right] \right] \]