Objectives: (things you should know and/or be able to do)

- Be familiar with and be able to derive the single-phase, pseudosteady-state flow relations for compressible liquids in a radial flow system. In particular, you should be able to derive the following:

  - $p_r-p_{wf}$ flow relation:
    
    $$p_r = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{kh} \left[ \ln \left( \frac{r}{r_w} \right) - \frac{1}{2} \frac{(r^2-r_w^2)}{(r_e^2-r_w^2)} + s \right]$$

  - $p-p_{wf}$ flow relation:
    
    $$\bar{p} = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] \quad \text{(Well Centered in a Circular Reservoir)}$$
    
    $$\bar{p} = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{kh} \left[ \frac{1}{2} \ln \left( \frac{4 A}{e \gamma} \frac{1}{r_w^2} C_A \right) + s \right] \quad \text{(General Formulation)}$$

where:

- $\gamma = 0.577216 \ldots$ (Euler's Constant)
- $C_A =$ Dietz "shape factor" (e.g., $C_A = 31.62$ for a well in a circular reservoir)
Objectives: (things you should know and/or be able to do)

- Be familiar with and be able to derive the single-phase, pseudosteady-state flow relations for compressible liquids in a radial flow system. In particular, you should be able to derive the following:

  - \( p(r,t) \) solution for pseudosteady-state flow conditions:

    \[
    p_r = p_i - \frac{qB\mu}{2\pi kh} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \frac{(r^2-r_w^2)}{(r_e^2-r_w^2)} - \frac{3}{4} \right] - \frac{qB}{V_pc_t} t \\
    \]

    (Darcy Units)

    \[
    p_r = p_i - 141.2 \frac{qB\mu}{kh} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \frac{(r^2-r_w^2)}{(r_e^2-r_w^2)} - \frac{3}{4} \right] - 5.615 \frac{qB}{V_pc_t} t \\
    \]

    (Field Units)

    where for field units we use \( t \) in days, and \( V_p \) in ft\(^3\).

  - \( p(r_w,t) \) solution for pseudosteady-state flow conditions:

    \[
    p_{wf} = p_i - 141.2 \frac{qB\mu}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] - 5.615 \frac{qB}{V_pc_t} t \\
    \]

    (Field Units)
Pseudosteady-State Flow Relations for a Radial System

from Department of Petroleum Engineering Course Notes (1997)
(Derivation of the Pseudosteady-State Flow Relations for a Radial System)

**Physical Considerations**

The physical concept of pseudosteady-state is defined as the condition where the pressure at all points in the reservoir changes at the same rate. Mathematically, this condition is given by:

\[
\frac{d}{dt} \left[ p(r,z) \right] = \text{constant}
\]  

(1)

Physically, this condition is illustrated by:

\[ q = \text{constant} \]

**Material Balance Considerations**

Recalling the material balance relation for a slightly compressible liquid, we have

\[
\bar{p} = p_i - \frac{B}{\phi h \pi c_t} N_p
\]

(2)

or, noting that \( \frac{\partial S_i}{\partial p} = \nu_p \) we obtain

\[
\bar{p} = p_i - \frac{B}{\nu_p c_t} N_p
\]

(3)

For a cylindrical reservoir, we have

\[
N_p = \phi h \pi (r_e^2 - r_w^2)
\]

(4)

Substituting Eq. 4 into Eq. 3 gives us

\[
\bar{p} = p_i - \frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} N_p
\]

(5)

Recalling the definition of the cumulative production, \( N_p \), we have

\[
N_p = \int_0^t q(t) dt
\]

(6)

Therefore,

\[
\frac{dN_p}{dt} = q
\]

(7)

Taking the derivative of Eq. 5 with respect to time

\[
\frac{d\bar{p}}{dt} = -\frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} q
\]

(8)

*Note: all derivations are in "cgs" units unless otherwise noted."
(Derivation of the Pseudosteady-State Flow Relations for a Radial System)

**Pseudosteady-State Flow Solutions for the Radial Flow**

**Diffusivity Equation**

The governing partial differential equation for flow in porous media is called the "diffusivity" equation. The diffusivity equation for a slightly compressible liquid is given (without derivation) as:

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \frac{du_t}{k} \frac{dp}{dt}
\]  \hspace{1cm} (9)

The significant assumptions made in Eq. 9 are:
- slightly compressible liquid (constant compressibility)
- constant fluid viscosity
- single-phase liquid flow
- gravity and capillary pressure are neglected
- constant permeability
- horizontal radial flow (no vertical flow)

If we assume that the flowrate, \( q \), is constant then \( dp/dt \) is also constant—hence \( d/dt \) is constant as well. Assuming \( q \) is constant, then

\[
\frac{dp}{dt} = \frac{dp}{dr} = -\frac{8}{\phi \pi (r_e^2 - r_i^2) c_t}
\]  \hspace{1cm} (10)

Substituting Eq. 10 into Eq. 9 (we note that partial derivatives are now expressed as ordinary derivatives), this gives

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \left[ \frac{du_t}{k} \right] \left[ -\frac{8}{\phi \pi (r_e^2 - r_i^2) c_t} \right] q
\]

or, reducing,

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -\frac{8 \varphi \mu}{\pi k \eta (r_e^2 - r_i^2)}
\]  \hspace{1cm} (11)

**Defining**

\[
c = \frac{q \varphi \mu}{\pi \eta (r_e^2 - r_i^2)}
\]  \hspace{1cm} (12)

Substituting Eq. 12 into Eq. 11 we have

\[
\frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -c
\]  \hspace{1cm} (13)

Separating

\[
d \left[ r \frac{dp}{dr} \right] = -cr dr
\]

Integrating (indefinite integration)

\[
\int d \left[ r \frac{dp}{dr} \right] = -c \int r dr
\]

Completing

\[
r \frac{dp}{dr} = -\frac{c}{2} r^2 + c_1
\]  \hspace{1cm} (14)

Multiplying through Eq. 14 by \( 1/r \) gives us

\[
\frac{dp}{dr} = -\frac{c}{2} \frac{r}{r} + c_1 \frac{1}{r}
\]  \hspace{1cm} (15)

For pseudosteady-state we assume a closed reservoir, that is

\[
\left[ \frac{dp}{dr} \right]_{r_e} = 0
\]

or

\[
\left[ \frac{dp}{dr} \right] = 0 = -\frac{c}{2} r_e + c_1 \frac{1}{r_e}
\]

Solving for \( c_1 \) gives

\[
c_1 = \frac{c}{2} \frac{r_e^2}{z}
\]  \hspace{1cm} (16)
Derivation of the Pseudosteady-State Flow Relations for a Radial System

Substituting Eq. 16 into Eq. 15 gives

\[
\frac{dp}{dr} = \frac{c}{Z} \left[ \frac{1}{r} \left( e^z - r \right) \right] \tag{17}
\]

Multiplying through Eq. 17 by \( dr \) gives us

\[
dp = \frac{c}{Z} \left[ \frac{1}{r} \left( e^z - r \right) \right] dr
\]

Integrating across the reservoir, we have

\[
\int_{R_{out}}^{R_{in}} dp = \frac{c}{Z} \int_{R_{out}}^{R_{in}} \left[ \frac{1}{r} \left( e^z - r \right) \right] dr
\]

\[
\int_{R_{out}}^{R_{in}} dp = \frac{c}{Z} \int_{R_{out}}^{R_{in}} \left[ \frac{1}{r} \left( e^z - r \right) \right] dr
\]

Completing the integration

\[
P_r - P_{out} = \frac{c}{Z} \left[ \frac{e^z \ln(r)}{r_{in}} - \frac{e^z \ln(r)}{r_{out}} \right]
\]

or

\[
P_r - P_{out} = \frac{c}{Z} \left[ e^z \ln\left( \frac{r_{in}}{r_{out}} \right) - \frac{1}{2} (r_{in}^2 - r_{out}^2) \right]
\]

Recalling Eq. 12

\[
c = \frac{q_{ss}}{\pi k h (r_{in}^2 - r_{out}^2)}
\]

Substituting Eq. 20 into Eq. 19, we obtain

\[
P_r - P_{out} = \frac{q_{ss}}{2 \pi k h} \left[ \frac{1}{r_{in}} \left( e^z \ln\left( \frac{r_{in}}{r_{out}} \right) - \frac{1}{2} (r_{in}^2 - r_{out}^2) \right) \right]
\]

Expanding through with the \( 1/(r_{in}^2 - r_{out}^2) \) term gives

\[
P_r - P_{out} = \frac{q_{ss} \pi k h}{2 \pi k h} \ln\left( \frac{r_{in}}{r_{out}} \right) - \frac{1}{2} (r_{in}^2 - r_{out}^2)
\]

\[
= \frac{q_{ss} \pi k h}{2 \pi k h} \ln\left( \frac{r_{in}}{r_{out}} \right) - \frac{1}{2} (r_{in}^2 - r_{out}^2)
\]

Which reduces to

\[
P_r = \frac{q_{ss}}{2 \pi k h} \frac{1}{r_{out}^2 - r_{in}^2} \int_{R_{out}}^{R_{in}} \left[ \frac{e^z \ln(r)}{r_{out}^2} - \frac{1}{2} (r_{in}^2 - r_{out}^2) \right] dr
\]

Solving Eq. 21 for \( P_r \) gives us

\[
P_r = \frac{q_{ss}}{2 \pi k h} \frac{1}{r_{out}^2 - r_{in}^2} \int_{R_{out}}^{R_{in}} \left[ \frac{e^z \ln(r)}{r_{out}^2} - \frac{1}{2} (r_{in}^2 - r_{out}^2) \right] dr
\]
Substituting Eq. 26 into Eq. 25 gives

\[
\frac{\Delta P}{r^2 - r_w^2} \int_{r_w}^{r} \left[ \frac{\rho_1 u + \frac{g \rho_1 u}{2 \pi \eta h} \left( \frac{r^2}{r_w^2} - 1 \right) - \frac{1}{2} \frac{r^2}{(r^2 - r_w^2)} \right] r \, dr
\]

(27)

Separating,

\[
\frac{\Delta P}{r^2 - r_w^2} = \frac{z}{r_w^2} \int_{r_w}^{r} r \, dr
\]

\[
+ \frac{z}{r^2 - r_w^2} \frac{g \rho_1 u}{2 \pi \eta h} \left( \frac{r^2}{r_w^2} - 1 \right) \int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr
\]

\[
- \frac{z}{r^2 - r_w^2} \frac{g \rho_1 u}{2 \pi \eta h} \frac{1}{2(r^2 - r_w^2)} \int_{r_w}^{r} r^3 \, dr
\]

\[
+ \frac{z}{r^2 - r_w^2} \frac{g \rho_1 u}{2 \pi \eta h} \frac{r_w^2}{2(r^2 - r_w^2)} \int_{r_w}^{r} r \, dr
\]

Isolating terms and evaluating each integral, we have

\[
\int_{r_w}^{r} r \, dr = \frac{1}{2} (r^2 - r_w^2)
\]

(29)

\[
\int_{r_w}^{r} r^3 \, dr = \frac{1}{4} (r^4 - r_w^4)
\]

(30)

\[
\int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = \frac{1}{2}
\]

Starting with the fundamental form of the logarithm integral, we have

\[
\int x \ln(x/c) \, dx
\]

... integration by parts \( \int u \, dv = uv - \int v \, du \)

\[
mu = \ln(x/c) \quad dv = x \, dx \]

\[
du = \frac{1}{x} \quad v = \frac{x^2}{2}
\]

Then

\[
\int x \ln(x/c) \, dx = \frac{x^2 \ln(x/c)}{2} - \frac{1}{2} \int x \, dx
\]

Reducing

\[
\int x \ln(x/c) \, dx = \frac{x^2 \ln(x/c)}{2} - \frac{x^2}{4}
\]

Therefore

\[
\int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = \left[ \frac{1}{2} \frac{r^2 \ln \left( \frac{r}{r_w} \right)}{r_w} - \frac{1}{4} \right]_{r_w}^{r}
\]

\[
= \frac{1}{2} \frac{r^2 \ln \left( \frac{r}{r_w} \right)}{r_w} - \frac{1}{4} \left( \frac{r^2}{r_w^2} - 1 \ln \left( \frac{r}{r_w} \right) \right)
\]

or finally, we have

\[
\int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = \frac{1}{2} \frac{r^2 \ln \left( \frac{r}{r_w} \right)}{r_w} - \frac{1}{4} \left( r^2 - r_w^2 \right)
\]

(31)

Obviously, the integral of the logarithm term will require a little work to resolve, we could simply look up the appropriate result in a suitable text—but deriving the required result will be enlightning.
(Derivation of the Pseudosteady-State Flow Relations for a Radial System)

Substituting Eqs. 29-31 into Eq. 28 gives

$$
\bar{p}_r = \frac{z}{(r^2-r_2^2)} \left[ \frac{r^2 \ln \left( \frac{r}{r_2} \right)}{2} - \frac{1}{4} (r^2-r_2^2) \right] + \frac{z}{(r^2-r_2^2)} \left[ \frac{q \mu r_2^2}{2 \pi k h} \left( \frac{r^2}{r_2^2} \right) \right] + \frac{z}{(r^2-r_2^2)} \left[ \frac{q \mu r_2^2}{2 \pi k h} \left( \frac{1}{2} \right) \right] - \frac{z}{(r^2-r_2^2)} \left[ \frac{q \mu r_2^2}{2 \pi k h} \left( \frac{1}{4} \right) \right]
$$

Reduction

$$
\bar{p}_r = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left( \frac{r^2}{r_2^2} \right) \left( \frac{1}{2} \right) - \frac{1}{4} (r^2-r_2^2)
$$

Collecting

$$
\bar{p}_r = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left( \frac{r^2}{r_2^2} \right) \left( \frac{1}{2} \right) - \frac{1}{4} (r^2-r_2^2)
$$

or “finally”

$$
\bar{p}_r = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left[ \frac{r^2}{r_2^2} \ln \left( \frac{r}{r_2} \right) - \frac{1}{2} \right] - \frac{1}{4} (r^2-r_2^2) + \frac{r_2^2}{z (r^2-r_2^2)}
$$

Eq. 32, which is given in “Darcy” units, is our fundamental linking relation between the wellbore and average reservoir pressures during pseudosteady-state flow. However, \( \bar{p}_r \) (the average reservoir pressure at a given radius, \( r \)) is of little use except as a rigorous linking relation for pressures in the reservoir.

In contrast, if we consider \( \bar{p}_e \) (i.e., \( \bar{p}_e \) at \( r = r_2 \)) we obtain the average reservoir pressure based on the entire reservoir volume. Such a result can be directly coupled with the material balance equation to develop a time-pressure relation for pseudosteady-state flow.

Evaluating Eq. 32 at \( r = r_2 \) we have

$$
\bar{p}_e = \bar{p}_e = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left[ \ln \left( \frac{r_2}{r_2} \right) - \frac{1}{2} \right] - \frac{1}{4} (r^2-r_2^2) + \frac{r_2^2}{z (r^2-r_2^2)}
$$

Assuming that \( r_2 \gg r_1 \), then

$$
\frac{r_2^2}{(r^2-r_2^2)} \approx 1 \quad \frac{(r_2^2+r_1^2)}{r_2^2-r_2^2} \approx 0 \quad \frac{r_2^2}{(r^2-r_1^2)} \approx 0
$$

Substituting these expressions into Eq. 33, we obtain

$$
\bar{p} = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left[ \ln \left( \frac{r_2}{r_2} \right) - \frac{1}{2} \right] - \frac{1}{4} (r^2-r_2^2) + \frac{r_2^2}{z (r^2-r_2^2)}
$$

or

$$
\bar{p} = \bar{p}_{w, f} + \frac{q \mu r_2^2}{2 \pi k h} \left[ \ln \left( \frac{r_2}{r_2} \right) - \frac{3}{4} \right]
$$
Summarizing our results so far (using generalized units systems):

Pressure at any radius:

\[ P_r = P_wf + \frac{q_s u}{\phi h k} \left[ \frac{r_e^2}{(r_e^2 - r_i^2)} \ln \left( \frac{r_i}{r_w} \right) - \frac{(r_e^2 - r_i^2)}{2(r_e^2 - r_i^2)} \right] \]  

Average Reservoir Pressure at any Radius:

\[ P_r = P_wf + \frac{q_s u}{\phi h k} \left[ \frac{r_e^2}{(r_e^2 - r_i^2)} \ln \left( \frac{r_i}{r_w} \right) - \frac{1}{2} \right] - \frac{(r_e^2 + r_i^2)}{2(r_e^2 - r_i^2)} + \frac{r_i^2}{2(r_e^2 - r_i^2)} \]  

Average Reservoir Pressure at \( r_e \) (Volumetric Average Pressure):

\[ P_r = P_wf + \frac{q_s u}{\phi h k} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{5}{4} \right] \]  

For a general reservoir geometry, Eq. 38 becomes:

\[ P_r = P_wf + \frac{q_s u}{\phi h k} \left[ \frac{1}{2} \ln \left( \frac{4}{c_i} \frac{A}{r_i^2 c_i} \right) \right] \]  

where:

\[ q = 0.577216 \ldots \text{ Euler's Constant} \]

\[ c_i \text{ Darcy's shape factor} \] (e.g., \( c_i = 31.62 \) for circular reservoir)

Table of Units Conversion Factors

<table>
<thead>
<tr>
<th>Factor</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_r )</td>
<td>( 2\pi )</td>
<td>( 2\pi \times 1.127 \times 10^{-3} ) or ( 7.081 \times 10^{-3} )</td>
<td>( 2\pi \times 6.327 \times 10^{-5} ) or ( 3.538 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

\( \phi \) concept

An interesting (and possibly useful) result is the concept of \( \bar{P} \), which could be the location of the average reservoir pressure. This development is only rigorously valid for a vertical well centered in a bounded circular reservoir. A graphical illustration of this concept is shown below.

Mathematically, \( \bar{P} \) is defined by equating the \( P \) relation (Eq. 35) with the average reservoir pressure identity, \( \bar{P} \) (Eq. 37). Equating Eqs. 35 and 37 gives:

\[ \frac{r_e^2}{(r_e^2 - r_i^2)} \ln \left( \frac{r_i}{r_w} \right) - \frac{(r_e^2 + r_i^2)}{2(r_e^2 - r_i^2)} = \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \]

We now assume \( \frac{r_e^2}{(r_e^2 - r_i^2)} \approx 1 \) and \( \frac{r_i^2}{(r_e^2 - r_i^2)} \approx 0 \) gives:

\[ c = \ln \left( \frac{r_i}{r_w} \right) - \ln \left( \frac{r_e}{r_w} \right) = \frac{r_i^2}{2(r_e^2 - r_i^2)} + \frac{3}{4} \]
(Derivation of the Pseudosteady-State Flow Relations for a Radial System)

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Assuming that \( R_e \gg R_w \) (i.e., \( (R_e^2 - R_w^2) \approx R_e^2 \) and rearranging, we have

\[
\ln \left[ \frac{R_e}{R_w} \right] - \frac{R_e^2}{2 \, R_w^2} + \frac{3}{4} = 0 \tag{39}
\]

Defining a dimensionless radius, \( R_D \), we obtain

\[
R_D = \frac{R_e}{R_w} \tag{40}
\]

Substituting Eq. 40 into Eq. 39 gives us

\[
\ln \left( \frac{R_D}{2} \right) - \frac{1}{2} \frac{R_D^2}{R_w^2} + \frac{3}{4} = 0 \tag{41}
\]

Solving Eq. 41 for \( R_D \), we obtain

\[
R_D = 0.54928 \ldots \quad \text{(or } R = 0.54928 \, \text{in} \, R_e) \tag{42}
\]

Development of a \( p(r,t) \) Relation for Pseudosteady-State Flow

Our last objective is to develop a \( p(r,t) \) relation for pseudosteady-state flow in a banded circular reservoir. Recalling the material balance relation (Eq. 5) we have

\[
\frac{dp}{dt} = -\frac{q}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{5}
\]

For a constant flowrate, \( q \), we have

\[
\int_{0}^{t} \frac{dp}{dt} \, dt = q \, t \tag{43}
\]

Substituting Eq. 43 into Eq. 42,

\[
\frac{dp}{dt} = \frac{q}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{44}
\]

Recalling the average reservoir pressure identity for a well centered in a bounded circular reservoir, we have

\[
\bar{p} = \bar{p}_w + \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{45}
\]

Substituting Eq. 45 into Eq. 44 gives

\[
\bar{p}_w + \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{46}
\]

Rearranging,

\[
\bar{p}_w - \bar{p}_w = \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} + \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{47}
\]

Recalling the wellbore-reservoir pressure relation (Eq. 26), we have (upon slight rearranging)

\[
\frac{p_r - \bar{p}_w}{\bar{p}_w} = \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} + \frac{1}{2} \left( \frac{R_e^2 - R_w^2}{(R_e^2 - R_w^2)^2} \right) \quad \text{(Carnot units)} \tag{48}
\]

Subtracting Eq. 48 from Eq. 47 and solving for \( p_r \) gives us

\[
\frac{p_r - \bar{p}_w}{\bar{p}_w} = \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} + \frac{1}{2} \left( \frac{R_e^2 - R_w^2}{(R_e^2 - R_w^2)^2} \right) \quad \text{(Carnot units)} \tag{49}
\]

Assuming that \( R_e \gg R_w \) (i.e., \( (R_e^2 - R_w^2) \approx R_e^2 \)) gives us

\[
\frac{p_r - \bar{p}_w}{\bar{p}_w} = \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) + \frac{1}{2} \left( \frac{R_e^2 - R_w^2}{(R_e^2 - R_w^2)^2} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} - \frac{q \, h \, h \, \pi (R_e^2 - R_w^2) \, \ln \left( \frac{R_e}{R_w} \right) + \frac{1}{2} \left( \frac{R_e^2 - R_w^2}{(R_e^2 - R_w^2)^2} \right) - \frac{3}{4}}{\phi \, h \, \pi (R_e^2 - R_w^2) \, c_f} \quad \text{(Carnot units)} \tag{50}
\]
Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs

Fundamental Flow Lecture 4 — Pseudosteady-State Flow in a Circular Reservoir

(Derivation of the Pseudosteady-State Flow Relations for a Radial System)

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Summarying, we have the following relations in Darcy units

\[ \frac{p_r - p_w}{\frac{98}{2 \pi kh}} = \frac{\ln \frac{r}{r_w}}{4} - \frac{r_w^2}{2 \pi (r_e^2 - r_w^2)} \ln \frac{r}{r_w} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{r_e^2} \] - \frac{98}{V_pc t} \]

and

\[ \frac{p_r - p_w}{\frac{98}{2 \pi kh}} = \frac{\ln \frac{r}{r_w}}{4} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{r_e^2} \frac{3}{4} \] - \frac{98}{V_pc t} \]

In field units, we have

\[ \frac{p_r - p_w}{\frac{98}{kh}} = \frac{\ln \frac{r_e}{r_w}}{4} - \frac{r_w^2}{2 \pi (r_e^2 - r_w^2)} \ln \frac{r_e}{r_w} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{r_e^2} \]

- 5.615 \frac{98}{V_pc t} \quad (t \text{ in days, } V_pc \text{ in } ft^2) \]

and

\[ \frac{p_r - p_w}{\frac{98}{kh}} = \frac{\ln \frac{r_e}{r_w}}{4} + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{r_e^2} \frac{3}{4} \] - 5.615 \frac{98}{V_pc t} \]

For \( t \) in hours we use 5.615/24 = 0.23395.

Finally, for conditions at the well, we have

Darcy Units:

\[ \frac{p_r - p_w}{\frac{98}{2 \pi kh}} \left[ \frac{\ln \frac{r_e}{r_w}}{4} \right] - \frac{98}{V_pc t} \]

Field Units:

\( \left( t \text{ in days; } V_pc \text{ in } ft^2 \right) \)

\[ \frac{p_r - p_w}{\frac{98}{kh}} \left[ \frac{\ln \frac{r_e}{r_w}}{4} \right] - 5.615 \frac{98}{V_pc t} \]

\( \left( t \text{ in hours; } V_pc \text{ in } ft^2 \right) \)

Recall that the pore volume, \( V_pc \), is given by

\[ V_pc = \phi h A \left( r_e^2 - r_w^2 \right) = \phi h A \]

\[ 4 \]
Illustrations of Pseudosteady-State Performance in Radial Flow Systems

Figure 2 - Pressure Distribution during Constant Rate Transient Flow Drawdown

Figure 7 - Reservoir Pressure Distribution during Constant Wellbore Pressure Transient Flow Drawdown
\[ pr = p_i - 141.2 \frac{qB \mu}{kh} \left[ \ln \frac{r_e}{r} + \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} - \frac{3}{4} \right] - 5.615 \frac{qB}{V_p c_t} t \] (Field Units)

**Figure 52 - Reservoir Pressure Distribution During Constant Rate Post-Transient Flow Drawdown, Homogeneous Reservoirs**

*From: Blasingame, T.A.: Variable-Rate Analysis: Transient and Pseudosteady-State Methods of Interpretation and Application, M.S. Thesis, Texas A&M University (1986).*
Figure 57 - Reservoir Pressure Distribution During Constant Wellbore Pressure Post-Transient Flow Drawdown, Homogeneous Reservoirs.
Pressure Trends
from Department of Petroleum Engineering Course Notes (2012)
Pressure Distributions: Solutions

All relations given in FIELD units.

Steady-State Solution:

\[ p_r = p_w + 141.2 \frac{q_{sc} B \mu}{kh} \ln\left(\frac{r}{r_w}\right) \quad [p_r - p_{wf} \text{ form}] \]

\[ p_r = p_e - 141.2 \frac{q_{sc} B \mu}{kh} \ln\left(\frac{r_e}{r}\right) \quad [p_r - p_e \text{ form}] \]

Radius of Investigation:

\[ r_{inv} = 2.434 \times 10^{-2} \sqrt{\frac{k}{\phi \mu C_t}} t \]

Full Solution: \(q_{sc}=\text{constant}\)

\[ p_D = \frac{1}{141.2} \frac{kh}{qB\mu} (p_i - p_r) \]

\[ \approx \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{eD}^2}{4t_D} \right] + 2 \frac{t_D}{r_{eD}^2} \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] + \left[ \frac{r_D^2}{2r_{eD}^2} - \frac{1}{4} \right] \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] \]
Pressure Distributions for Transient Radial Flow

- Note the effect of the drawdown.
- Note that the buildup pressure trends retrace last drawdown trend.
- Recall that all measurements are at the wellbore, we cannot "see" in the reservoir — our analyses are inferred from wellbore measurements.

Legend:
- $pD_{DD}(r, t_1\text{Em}1 \text{ hr})$
- $pD_{DD}(r, t_1\text{E}0 \text{ hr})$
- $pD_{DD}(r, t_1\text{E}1 \text{ hr})$
- $pD_{DD}(r, t_1\text{E}2 \text{ hr})$
- $pD_{DD}(r, t_1\text{E}3 \text{ hr})$
- $pD_{BU}(r, t_p + \Delta t_1\text{E}1 \text{ Em}1 \text{ hr})$
- $pD_{BU}(r, t_p + \Delta t_1\text{E}0 \text{ hr})$
- $pD_{BU}(r, t_p + \Delta t_1\text{E}1 \text{ hr})$
- $pD_{BU}(r, t_p + \Delta t_1\text{E}2 \text{ hr})$
- $pD_{BU}(r, t_p + \Delta t_1\text{E}3 \text{ hr})$
Pressure Distributions: Pseudosteady-State

The physical concept of the PSEUDOSTEADY-STATE FLOW condition is defined as the condition where the pressure at all points in the reservoir changes at the same rate. Mathematically, this condition is given by:

$$\frac{d}{dt}[p(r,t)]_r = \text{constant}$$

![Diagram showing pressure distributions in a reservoir with a constant flow rate.](image)
**Pressure Distributions: Pseudosteady-State**

**Concept:** (pressure changes at the same rate at all points in the reservoir)

\[ \frac{dp}{dr} = \text{constant} \]

**Reservoir Pressure Schematic:**

[Graph showing pressure response for a single well in a closed circular reservoir, produced at a constant flowrate (liquid case).]

Pseudosteady-State Flow: Summary of Relations

$(p_r - p_{wf})$ Flow Relations: (Circular Reservoir)

$$p_r - p_{wf} = 141.2 \frac{qB \mu}{kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r}{r_w} \right) - \frac{1}{2} \left( \frac{r^2}{r_e^2} - \frac{r^2}{r_w^2} \right) + s \right]$$

$(\bar{p} - p_{wf})$ Flow Relations: ($\gamma = 0.577216$ Euler's constant)

$$\bar{p} = p_{wf} + 141.2 \frac{qB \mu}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right]$$ (Circular Reservoir)

$$\bar{p} = p_{wf} + 141.2 \frac{qB \mu}{kh} \left[ \frac{1}{2} \ln \left( \frac{4A}{e^\gamma r_w^2 CA} \right) + s \right]$$ (General Formulation)

Time-Dependent Pseudosteady-State Flow Relations:

$$p_r = p_i - 141.2 \frac{qB \mu}{kh} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \left( \frac{r^2}{r_e^2} - \frac{r^2}{r_w^2} \right) - \frac{3}{4} \right] - 5.615 \frac{qB}{V_p c_t} t$$

$$p_{wf} = p_i - 141.2 \frac{qB \mu}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] - 5.615 \frac{qB}{V_p c_t} t$$
Pseudosteady-State Flow: *Illustrative Behavior*

Figure 2: Reservoir Pressure Distribution — Constant Rate Transient Flow Drawdown.
Pseudosteady-State Flow: *Illustrative Behavior*

**Figure 7: Reservoir Pressure Distribution — Constant Wellbore Pressure Transient Flow Drawdown.**

Pseudosteady-State Flow: *Illustrative Behavior*

![Graph showing reservoir pressure distribution during constant rate post-transient flow drawdown, homogeneous reservoirs.](image)

\[ r_{inv} = 2.434 \times 10^{-2} \sqrt[4]{\frac{k}{\phi \mu c_t}} t \]

**Figure 52: Reservoir Pressure Distribution — Constant Rate Post-Transient Flow Drawdown, Homogeneous Reservoirs.**

Pseudosteady-State Flow: *Illustrative Behavior*

Figure 57: Reservoir Pressure Distribution — Constant Wellbore Pressure Post-Transient Flow Drawdown, Homogeneous Reservoirs.

\[ r_{inv} = 2.434 \times 10^{-2} \left( \frac{k}{\phi \mu c_t} \right)^{\frac{1}{2}} t \]

Reservoir Pressure Trends: Questions to Consider

Q1. Why study "reservoir pressure trends?"
A1. We can not measure pressure in the reservoir — only at the wellbore (or sandface). In order to estimate the behavior in the reservoir, we must use "model-based" pressure distributions.

Q2. Isn't the use of a simple model too limiting?
A2. Actually, no. Simple models are extremely consistent, and as such, even when "wrong," the "trend" behavior is typically quite representative.

Q3. What is the "radius of investigation?"
A3. For the infinite-acting radial flow case, the radius of investigation is the point in the reservoir where the logarithm of radius equation (straight line) intersects the initial reservoir pressure. It is a fictitious point, but it represents the "theoretical" location of the front of the pressure distribution front.

\[ r_{inv} = 2.434 \times 10^{-2} \sqrt{\frac{k}{\phi \mu c_t}} t \]