Objectives: (things you should know and/or be able to do)

Real Domain Solutions (via Inversion of the Laplace Domain Solutions):

- Be able to derive the following particular solutions in the real domain using the appropriate Laplace transform solutions for an unfractured well produced at a constant flowrate in a homogeneous reservoir for the following outer boundary conditions:

  - "Infinite-acting" reservoir behavior (line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] \]

  - "Infinite-acting" reservoir behavior (the so-called "log approximation," also a line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4}{e^{r_D} r_D^2} \right] \]

  - Bounded circular reservoir — "no-flow" at the outer boundary
    \[ p_D(t_D, r_D, r_{e_D}) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{e_D}^2}{4t_D} \right] + \frac{2t_D}{r_{e_D}^2} \exp \left[ -\frac{r_{e_D}^2}{4t_D} \right] + \left[ \frac{r_D^2}{2r_{e_D}^2} - \frac{1}{4} \right] \exp \left[ -\frac{r_{e_D}^2}{4t_D} \right] \]

and its "well testing" derivative function, \( p_D' = \frac{d}{dt_D} [p_D(r_D, t_D)] \) is given by

\[ p_D'(t_D, r_D, r_{e_D}) = \frac{1}{2} \exp \left[ -\frac{r_D^2}{4t_D} \right] + \frac{2t_D}{r_{e_D}^2} \exp \left[ -\frac{r_{e_D}^2}{4t_D} \right] + \frac{1}{2t_D} \left[ \frac{r_D^2}{4} - \frac{r_{e_D}^2}{8} \right] \exp \left[ -\frac{r_{e_D}^2}{4t_D} \right] \]
Solutions of the Radial Flow Diffusivity Equation in the Real Domain

from Department of Petroleum Engineering Course Notes (1994)
Solutions for a Bounded Circular Reservoir: Infinite-Acting, No-Flow, and Constant Pressure Boundary Cases

a. Infinite-Acting Reservoir Case:
\[
\frac{\Phi}{\nu} = \frac{1}{\mu} \frac{k_0}{v_0} \frac{1}{r_0^2} \text{ (cylindrical source solution)}
\]
(1)

b. Line Source Solution:
Recalling Eq. 2 we have
\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{k_0}{v_0} \frac{1}{r_0^2} \text{ (line source solution)}
\]
(2)

\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{1}{2} \ln \left( \frac{4r_0}{l^2} \right) \text{ (as } a \to \infty, \text{ log approximation)}
\]
(3)

c. No-Flow Boundary Case:
\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{k_0}{v_0} \left( \frac{1}{2} \ln \left( \frac{l}{r_0} \right) - \frac{1}{2} \ln \left( \frac{l}{r_0} \right)^2 \right)
\]
(4)

where for \( a \to 0 \) Eq. 8 reduces to
\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{k_0}{v_0} \frac{1}{r_0^2} \text{ (line source)}
\]
(5)

d. Constant Pressure Outer Boundary Case:
\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{k_0}{v_0} \left( \frac{1}{2} \ln \left( \frac{l}{r_0} \right) - \frac{1}{2} \ln \left( \frac{l}{r_0} \right)^2 \right)
\]
(6)

where for \( a \to 0 \) Eq. 6 reduces to
\[
\frac{\Phi}{\nu(a_{\infty})} = \frac{1}{\mu} \frac{k_0}{v_0} \frac{1}{r_0^2} \text{ (line source)}
\]
(7)

Inversion of Eqs. 2 and 10 is accomplished by the use of Laplace transform tables, where the results of inversion are given below

\[
\begin{array}{c|c|c}
\mu \Phi & \frac{1}{k_0} & \frac{1}{E_1} \left( \frac{a^2}{4t} \right) \\
\hline
\text{Carroll and Jaeger:} & \frac{1}{k_0} & \frac{1}{E_1} \left( \frac{a^2}{4t} \right) \\
\text{Conduction of Heat in Solids, Table V, Eq. 26, p. 445.} & & \\
\text{Abramowitz and Stegun:} & \frac{1}{k_0} & \frac{1}{E_1} \left( \frac{a^2}{4t} \right) \\
\text{Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.120, p. 1028.} & & \\
\text{and} & \frac{1}{k_0} & \frac{1}{E_1} \left( \frac{a^2}{4t} \right) \\
\text{Roberts and Kaufman:} & \frac{1}{k_0} & \frac{1}{E_1} \left( \frac{a^2}{4t} \right) \\
\text{Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304.} & &
\end{array}
\]
Defining the so-called "well testing derivative" we have
\[ \phi'(r_0,t_0) = t_0 \frac{d}{dt_0} \left[ \phi(r_0,t_0) \right] \]  
\[ (12) \]
Substituting Eq. 16 into Eq. 17 we have
\[ \phi'(r_0,t_0) = \frac{1}{2} \exp\left(-r_0^2/4t_0\right) \]  
\[ (18) \]

**c. Log Approximation Solution:**
Recalling Eq. 3 we have
\[ \phi(r_0,\mu) = \frac{1}{\mu} \ln\left(\frac{4}{e^{2\phi} r_0^2 \mu} \right) \]  
\[ (13) \]
Expanding Eq. 3 into a more usable form, we have
\[ \phi(r_0,\mu) = \frac{1}{2} \ln\left(\frac{4}{\mu} \right) - \frac{1}{2} \ln\left(\frac{4}{e^{2\phi} r_0^2 \mu} \right) \]  
\[ (14) \]
Rather than attempt a derivative using \( \phi(r_0,\mu) \), we will simply differentiate the inversion result of Eq. 18. The inverse Laplace transform of the \( \ln(\mu) \) and constant terms in Eq. 19 we have
\[ \frac{1}{\mu} = 1 \quad \text{or} \quad \ln\left(\frac{4}{e^{2\phi} r_0^2 \mu} \right) \]

**References**
- Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.10, p. 822.
- Robarts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 4.1, p. 58.

Writing the \( \phi(r_0,t_0) \) inversion result for Eq. 19 is
\[ \phi(r_0,t_0) = \frac{1}{2} \ln\left(e^{2\phi} t_0^2 + \frac{4}{e^{2\phi} r_0^2} \right) \]  
\[ (20) \]
Collecting
\[ \frac{1}{2} \ln\left(e^{2\phi} t_0^2 + \frac{4}{e^{2\phi} r_0^2} \right) \]

Isolating the \( t_0^2 \) term in Eq. 20 we have
\[ \phi(r_0,t_0) = \frac{1}{2} \ln\left(\frac{4}{e^{2\phi} r_0^2} \right) + \frac{1}{2} \ln\left(\frac{1}{e^{2\phi} t_0^2} \right) \]  
\[ (21) \]
Substituting Eq. 21 into Eq. 17 to determine the well testing derivative we have
\[ \phi'(r_0,t_0) = \frac{1}{2} \ln\left(\frac{4}{e^{2\phi} r_0^2} \right) + t_0 \left[ \frac{1}{2} \ln\left(\frac{1}{e^{2\phi} t_0^2} \right) \right] \]
which reduces to
\[ \phi'(r_0,t_0) = \frac{1}{2} \ln\left(\frac{4}{e^{2\phi} r_0^2} \right) + \frac{1}{2} \ln\left(\frac{1}{e^{2\phi} t_0^2} \right) \]  
\[ (22) \]

**Solution for a No-Flow Outer Boundary:**
It is not possible to invert the complete solution (Eq. 4) for this case so we will attempt an approximate solution of the Line source form (Eq. 5). Recalling Eq. 5 we have
\[ \phi(r_0,\mu) = \frac{1}{\mu} \ln(\mu r_0) + \frac{1}{\mu} \ln(\mu r_0) \quad I_0(\mu r_0) \]  
\[ (5) \]
We immediately recognize that the first term in Eq. 5 is the solution for an infinite-acting reservoir, and given the linearity of the inverse Laplace transform, we can invert Eq. 5 to yield
\[ \phi(r_0,t_0) = \phi_{in}(t_0) + \int \frac{1}{\mu} \ln(\mu r_0) \quad I_0(\mu r_0) \]  
\[ (23) \]
where
\[ \phi_{in}(t_0) = \frac{1}{2} \ln\left(\frac{1}{e^{2\phi} r_0^2} \right) \]  
\[ (24) \]
So what is our strategy to invert the second term in Eq. 23? First we will use recursion relations to express the \( I_0(\mu r_0) \) Bessel functions then consider a two-term expansion of the resulting \( I_0(\mu r_0) \) ratio. Recall that as \( \varepsilon \to 0 \) that \( I_0(\varepsilon) \to 1 \) and \( I_0(\varepsilon) \to 0 \), which permits polynomial expansions.

From Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.1.15, p. 875) we have
\[ I_0(\varepsilon) \sim I_0(\varepsilon) + I_0(\varepsilon) I_0(\varepsilon) = \frac{1}{\varepsilon} \]
Using \( n = 0 \)
\[ \int_0^\infty I_0(\varepsilon) k_0(\varepsilon) = \frac{1}{\varepsilon} \]
\[ (25) \]
Using \( n = 1 \) we have
\[
I_n(z) \frac{d}{dz} I_n(z) + I_n(z) \frac{d}{dz} k_n(z) = 1 \frac{d}{dz}
\]
(26)

Equating Eqs. 25 and 26 we have
\[
I_0(z) \frac{d}{dz} I_0(z) + I_0(z) k_0(z) + I_1(z) \frac{d}{dz} k_1(z) - I_1(z) k_1(z) - I_0(z) k_0(z) = 1 \frac{d}{dz}
\]
or solving for \( k_0(z) \)
\[
k_0(z) = \frac{I_0(z)}{I_0(z) - I_1(z)} \frac{d}{dz} k(z) - k_0(z)
\]
(28)

Recalling the first recursion relation in Eq. 9.6.26, p. 336, Abramowitz and Stegun, Handbook of Mathematical Functions, we have
\[
I_n(z) = I_{n-1}(z) - I_{n+1}(z) = \frac{z}{n} I_{n-1}(z)
\]
Rearranging
\[
I_n(z) = \frac{z}{n} I_{n-1}(z)
\]
(28)

Substituting Eq. 28 into Eq. 26 we have
\[
k_0(z) = \frac{z}{I(0)} [k_1(z) - k_0(z)]
\]
(29)

Expanding
\[
\frac{I_0(b)}{I_0(a)} = \frac{z}{a} \left[ 1 + \frac{z^2}{4} - \frac{z^4}{8} \right]
\]
Neglecting the \( z^4 \) term we have
\[
\frac{I_0(b)}{I_0(a)} = \frac{z}{a} \left[ 1 + \frac{z^2}{4} \right]
\]
(30)

Recalling that \( b \) = \( \sqrt{r_0} \) and \( a = \sqrt{r_0} \) and substituting Eq. 37 into Eq. 30
\[
p_b(\psi, \phi) = p_0(\psi, \phi) + \frac{z}{a} \left[ 1 + \frac{z^2}{4} \right]
\]
(31)

Continuing the expansion
\[
p_b(\psi, \phi) = p_0(\psi, \phi) + \frac{z^2}{4} \left[ \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} \right] + \frac{z^4}{8} \left[ \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} \right] + \frac{z^6}{16} \left[ \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} \right] + \cdots
\]
(32)

for reference, we note the Laplace transform of Eq. 38
\[
p_0(\psi, \phi) = \frac{z}{a} \left[ \frac{k_0(\sqrt{r_0})}{\sqrt{r_0}} \right]
\]
(33)
Multiply through Eq. 29 by the laplace transform parameter \( \mu \), giving

\[
\mu \bar{Q}_0(s, \mu) = k_0 \left( \frac{\mu}{R_0} \right) + k_2 \left( \frac{\mu}{R_0} \right) - k_0 \left( \frac{\mu}{R_0} \right) + \left( \frac{r^2 - r_0^2}{8} \right) \left( \frac{\mu}{R_0} \right) - \left( \frac{r^2 - r_0^2}{8} \right) \left( \frac{\mu}{R_0} \right) \tag{40}
\]

We will take the inverse laplace transform of Eqs. 39 and 40 using the following tables

<table>
<thead>
<tr>
<th>( \frac{1}{\mu} k_0 (\mu/a^2) )</th>
<th>( \frac{1}{\mu} )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{E_1 (\mu/a^2)} )</td>
<td>( \frac{1}{\mu} )</td>
<td>Carslaw and Jaeger: Conduction of Heat in Solids, Table V, Eq. 26, p. 493.</td>
</tr>
<tr>
<td>( k_2 (\mu/a^2) )</td>
<td>( \frac{1}{\mu} )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.1, Eq. 31.1,120, p. 1028.</td>
</tr>
<tr>
<td>( k_0 (\mu/a^2) )</td>
<td>( \frac{\mu}{a^2} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 504.</td>
</tr>
<tr>
<td>( k_2 (\mu/a^2) )</td>
<td>( \frac{\mu}{a^2} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.13, p. 506.</td>
</tr>
<tr>
<td>( k_0 (\mu/a^2) )</td>
<td>( \frac{\mu}{a^2} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.14, p. 506.</td>
</tr>
<tr>
<td>( k_1 (\mu/a^2) )</td>
<td>( \frac{\mu}{a^2} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.15, p. 506.</td>
</tr>
</tbody>
</table>

Inverting Eq. 39 term-by-term using the previous tables gives

\[
Q_0(r_0, t_0) = \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{2} E_2 \left( \frac{r_0^2}{4t_0} \right) + \frac{z t_0}{r_0^2} \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

\[
+ \left( \frac{r^2 - r_0^2}{8} \right) \left( \frac{1}{z t_0} + \frac{z}{r_0^2} \right) \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

\[
- \left( \frac{r^2 - r_0^2}{8} \right) \frac{1}{z t_0} \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

Collecting

\[
Q_0(r, t) = \frac{1}{2} E_1 \left( \frac{r^2}{4t} \right) - \frac{1}{2} E_2 \left( \frac{r^2}{4t} \right)
\]

\[
+ \left( \frac{z t_0}{r_0^2} - \frac{1}{2} \right) \left( \frac{1}{z t_0} + \frac{z}{r_0^2} \right) \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

where

\[
c = \left( \frac{r_0^2}{4} \right) - \frac{z t_0}{r_0^2}
\]

Cancelling the c/zt_0 terms

\[
\frac{Q_0(r, t_0)}{r_0^2} = \frac{1}{2} E_1 \left( \frac{r^2}{4t_0} \right) - \frac{1}{2} E_2 \left( \frac{r^2}{4t_0} \right) + \frac{z t_0}{r_0^2} \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

\[
+ \frac{z}{r_0^2} \left( \frac{r^2 - r_0^2}{8} \right) \exp \left( -\frac{r_0^2}{4t_0} \right)
\]

Which yields the following reduction

\[
\frac{Q_0(r, t_0)}{r_0^2} = \frac{1}{2} E_1 \left( \frac{r^2}{4t_0} \right) - \frac{1}{2} E_2 \left( \frac{r^2}{4t_0} \right) + \frac{z t_0}{r_0^2} \exp \left( -\frac{r_0^2}{4t_0} \right) + \left( \frac{r^2 - \frac{1}{4} r_0^2}{4t_0} \right) \exp \left( -\frac{r_0^2}{4t_0} \right) \tag{41}
\]

Segmenting the solution into particular flow requires

\[
\frac{Q_0(r, t_0)}{r_0^2} = \left( \frac{1}{2} E_1 \left( \frac{r^2}{4t_0} \right) + \frac{z t_0}{r_0^2} \exp \left( -\frac{r_0^2}{4t_0} \right) \right) / \text{Infinite-Acting-Reservoir Term (Reservoir Size)}
\]

\[
\left( \frac{1}{2} E_2 \left( \frac{r^2}{4t_0} \right) + \left( \frac{r^2 - \frac{1}{4} r_0^2}{4t_0} \right) \exp \left( -\frac{r_0^2}{4t_0} \right) \right) / \text{Reservoir Shape Effects Terms} \tag{42}
\]
Due to conflicting results obtained by inverting Eq. 40 term-by-term, we will proceed by differentiating Eq. 42.

Note that
\[
\frac{d}{dt_0} \left[ \frac{dE(x)}{dx} \right] = \frac{\sqrt{x}}{\sqrt{t_0}} \frac{dE(x)}{dx} = \frac{\sqrt{x}}{\sqrt{t_0}} \left[ -\exp(-x) \right]
\]

and
\[
\frac{d}{dt_0} \exp(x) = \frac{\sqrt{x}}{\sqrt{t_0}} \exp(x)
\]

Differentiating Eq. 42 term-by-term:
\[
\frac{d}{dt_0} \left[ \frac{dE(x)}{dx} \right] = \frac{\sqrt{x}}{\sqrt{t_0}} \frac{dE(x)}{dx} \exp(-\frac{r^2}{4t_0})
\]

or
\[
\frac{d}{dt_0} \left[ \frac{dE(x)}{dx} \right] = \frac{\sqrt{x}}{\sqrt{t_0}} \exp(-\frac{r^2}{4t_0})
\]

Similarly, for \( \frac{d}{dt_0} \left[ \frac{dE(x)}{dx} \right] \) we have
\[
\frac{d}{dt_0} \left[ \frac{dE(x)}{dx} \right] = \frac{\sqrt{x}}{\sqrt{t_0}} \exp(-\frac{r^2}{4t_0})
\]

Next we have
\[
\frac{d}{dt_0} \left[ \frac{z}{\sqrt{t_0}} \exp(-\frac{r^2}{4t_0}) \right] = \frac{z}{\sqrt{t_0}} \left[ \exp(-\frac{r^2}{4t_0}) \right] + t_0 \frac{d}{dt_0} \left[ \exp(-\frac{r^2}{4t_0}) \right]
\]

\[
= \frac{z}{\sqrt{t_0}} \left[ 1 + \frac{r^2}{4t_0} \right] \exp(-\frac{r^2}{4t_0})
\]

Similarly
\[
\frac{d}{dt_0} \left[ \frac{z^2}{r^2t_0} \exp(-\frac{r^2}{4t_0}) \right] = \frac{z^2}{r^2t_0} \left[ \frac{r^2}{4t_0} \right] \exp(-\frac{r^2}{4t_0})
\]

Collecting the derivative terms we have
\[
\frac{d}{dt_0} \left[ \frac{E(x)}{\sqrt{t_0}} \right] = \frac{1}{2t_0} \exp(-\frac{r^2}{4t_0}) - \frac{1}{2t_0} \exp(-\frac{r^2}{4t_0})
\]

\[
+ \left[ \frac{r^2 + 1}{2t_0} \right] \exp(-\frac{r^2}{4t_0}) + \frac{1}{2t_0} \left[ \frac{r^2 - r_0}{4} \right] \exp(-\frac{r_0}{4t_0})
\]

Collecting further
\[
\frac{d}{dt_0} \left[ \frac{E(x)}{\sqrt{t_0}} \right] = \frac{1}{2t_0} \exp(-\frac{r^2}{4t_0}) + \left[ \frac{r^2 + 1}{2t_0} \right] \exp(-\frac{r^2}{4t_0}) + \frac{1}{2t_0} \left[ \frac{r^2 - r_0}{4} \right] \exp(-\frac{r_0}{4t_0})
\]

Multiplying through by \( t_0 \) we have
\[
\frac{d}{dt_0} \left[ \frac{E(x)}{t_0} \right] = \frac{1}{2} \exp(-\frac{r^2}{4t_0}) + \frac{z^2 t_0 \exp(-\frac{r^2}{4t_0})}{4} + \frac{1}{2} \left[ \frac{r^2 - r_0}{4} \right] \exp(-\frac{r_0}{4t_0})
\]

\[
\text{Solution for Constant Pressure Outer Boundary:}
\]

Similar to the no-flow outer boundary case, we cannot directly invert Eq. 46, so we will attempt an approximate solution of the line source term (Eq. 7). Recalling Eq. 7, we have
\[
I_0(r_0, \mu) = \frac{1}{2} k_0 \left( \frac{r_0}{2} \right) - \frac{1}{2} k_0 \left( \frac{r_0}{2} \right) I_0(\mu r_0) \quad (3)
\]

Recalling the polynomial approximation for \( I_0(z) \) (Eq. 51), we have
\[
I_0(z) = 1 + z^2 + \frac{z^4}{4} + \frac{z^6}{4} + ... + \frac{z^4}{32}
\]

Using a two-term approximation for the \( I_0(\mu r_0) / I_0(r_0) \)
\[
\frac{I_0(\mu r_0)}{I_0(r_0)} = 1 + \mu r_0^2 + \frac{1}{4} \mu^2 r_0^4
\]

Using a two-term binomial series for \( (1 + \mu r_0^2)^{-1} \), we have
\[
\frac{I_0(L_0)}{I_0(L_0)} = \left( 1 + \mu r_0^2 \right)^{-1} \left( 1 - \mu r_0^2 \right)
\]

or
\[
\frac{I_0(\mu r_0)}{I_0(r_0)} = \frac{1}{4} \frac{r_0^2 - \mu r_0^2}{4} - \frac{1}{16} \frac{r_0^4 - \mu r_0^4}{4}
\]
Collecting the derivatives we have
\[
\frac{d}{dt} \frac{p^*(t, r)}{r^2} = \frac{1}{2a^2} \exp\left(-\frac{r^2}{4t}\right) - \frac{1}{2a^2} \exp\left(-\frac{r_0^2}{4t}\right) + \frac{r_0^2 - r^2}{8} \frac{1}{2a^2} \exp\left(-\frac{r_0^2}{4t}\right)
\]

Multiplying through by \( t^2 \) yields the well testing derivative
\[
\frac{d}{dt} \frac{p^*(t, r)}{r^2} = \frac{1}{2} \exp\left(-\frac{r^2}{4t}\right) - \frac{1}{2} \exp\left(-\frac{r_0^2}{4t}\right) + \frac{(r_0^2 - r^2)}{8} \exp\left(-\frac{r_0^2}{4t}\right) \exp\left(-\frac{r_0^2}{4t}\right)
\]

Summary of Results:

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. infinite-acting reservoir</td>
<td>( \bar{p}_0(r, t) = \frac{r_0^2}{4a^2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Lapse Domain/Cylindrical Source Solution</td>
<td>( \bar{p}_0(r, t) = \frac{r_0^2}{4a^2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Lapse Domain/Line Source Solution</td>
<td>( \bar{p}_0(r, t) = \frac{r_0^2}{4a^2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Lapse Domain/&quot;Log&quot; approximation</td>
<td>( \bar{p}_0(r, t) \approx \frac{1}{2} \ln\left(\frac{L^2}{2\pi r^2} \frac{1}{a^2}\right) )</td>
</tr>
<tr>
<td>Real Domain/Cylindrical Source Solution</td>
<td>( \bar{p}_0(r, t) = \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Real Domain/Line Source Solution</td>
<td>( \bar{p}_0(r, t) = \frac{r_0^2}{4a^2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Real Domain/&quot;Log&quot; approximation</td>
<td>( \bar{p}_0(r, t) \approx \frac{1}{2} \ln\left(\frac{L^2}{2\pi r^2} \frac{1}{a^2}\right) )</td>
</tr>
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<td>Real Domain/&quot;Log&quot; approximation</td>
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</tr>
<tr>
<td>b. bounded circular reservoir (no-flow outer boundary)</td>
<td>( \bar{p}_0(r, t) = \frac{1}{2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Lapse Domain/Cylindrical Source Solution</td>
<td>( \bar{p}_0(r, t) = \frac{1}{2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
<tr>
<td>Lapse Domain/Line Source Solution</td>
<td>( \bar{p}_0(r, t) = \frac{1}{2} \exp\left(-\frac{r^2}{4t}\right) )</td>
</tr>
</tbody>
</table>
Case b, bounded circular reservoir (no-flow boundary — continued)
Real Domain / Line Source Soln.
\[ p(t_r, t_0) = \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] t_0^2 \exp \left( -\frac{r_0^2}{t_0^2} \right) \]
\[ + \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] \frac{1}{t_0} \exp \left( -\frac{r_0^2}{4t_0^2} \right) \]
\[ + \frac{1}{8t_0^2} \left[ \exp \left( -\frac{r_0^2}{4t_0^2} \right) - \exp \left( -\frac{r_0^2}{16t_0^2} \right) \right] \]

Real Domain / Derivative of Line Source Soln.
\[ \frac{dp}{dt}(t_r, t_0) = \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] t_0^2 \exp \left( -\frac{r_0^2}{t_0^2} \right) \]
\[ + \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] \frac{1}{t_0} \exp \left( -\frac{r_0^2}{4t_0^2} \right) \]
\[ + \frac{1}{8t_0^2} \left[ \exp \left( -\frac{r_0^2}{4t_0^2} \right) - \exp \left( -\frac{r_0^2}{16t_0^2} \right) \right] \]

Case c, bounded circular reservoir (constant pressure boundary)
Laplace Domain / Line Source
\[ \tilde{p}(r_0, \mu) = \frac{1}{\mu} \left[ k_0 \tilde{I}_0(\mu r_0) - k_0 \tilde{K}_0(\mu r_0) \right] \]
\[ - \frac{1}{\mu} \left[ \tilde{I}_0(\mu r_0) \tilde{K}_0(\mu r_0) \right] \]

Laplace Domain / Line Source Soln.
\[ p(t_r, t_0) = \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] t_0^2 \exp \left( -\frac{r_0^2}{t_0^2} \right) \]
\[ + \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] \frac{1}{t_0} \exp \left( -\frac{r_0^2}{4t_0^2} \right) \]
\[ + \frac{1}{8t_0^2} \left[ \exp \left( -\frac{r_0^2}{4t_0^2} \right) - \exp \left( -\frac{r_0^2}{16t_0^2} \right) \right] \]

Real Domain / Derivative of Line Source Soln.
\[ \frac{dp}{dt}(t_r, t_0) = \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] t_0^2 \exp \left( -\frac{r_0^2}{t_0^2} \right) \]
\[ + \frac{1}{2} \left[ \frac{E_1(k \sqrt{t_r})}{t_0^2} - \frac{E_1(k \sqrt{t_0})}{t_0^2} \right] \frac{1}{t_0} \exp \left( -\frac{r_0^2}{4t_0^2} \right) \]
\[ + \frac{1}{8t_0^2} \left[ \exp \left( -\frac{r_0^2}{4t_0^2} \right) - \exp \left( -\frac{r_0^2}{16t_0^2} \right) \right] \]
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir—Various $r_D$
Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Legend: No Flow Outer Boundary Case ($r_{eD} = 10^3$)
- Numerical Inversion Solutions
- Approximate $p_D$ Solutions (This Work)
- Approximate $p_D'$ Solutions (This Work)