A Short Course on

Performance-Based Reservoir Characterization

Material Balance:

This module has several major objectives — the introduction of the black oil and dry gas material balance equations, the development and demonstration of simple material balance analysis. In addition, this module introduces the concept of "material balance time," a critical component of the general methodology for the analysis of production data.

The following topics in material balance are covered:

- "Black oil" material balance equations
  - Constant compressibility formulation ($p>p_b$)
  - Solution gas drive formulation (full relation)
- "Dry gas" material balance equations
  - "No-influx" formulation
  - General formulation
- Development of the:
  - "Black oil" material balance equations
  - "Dry gas" material balance equations
  - "Material balance time" functions (oil and gas cases)
Material Balance: "Black Oil" Case

- Constant Compressibility Case: \((p>p_b)\)

\[
(p_i - \bar{p}) = \frac{1}{Nc_t} \frac{B_o}{B_{oi}} N_p
\]

- Application:
  1. Plot:
     \(\bar{p}\) versus \(N_p\)
  2. Extrapolate the straight-line trend to \(\bar{p} = 0\), this yields the MOVABLE oil-in-place, \(N_{p,mov}\).
  3. The slope of the straight-line trend can be used to estimate the original oil-in-place, \(N\) — however, this requires an accurate estimate of \(c_t\).

- Comment: Solving the constant compressibility oil MBE relation for \(N_p/N\) at \(\bar{p} = 0\), we obtain

\[
\frac{N_p}{N} = p_i c_t \frac{B_{oi}}{B_o}
\]

Example:

- \(p_i = 5000\) psia
- \(c_t = 10 \times 10^{-6}\) psia\(^{-1}\)
- \(B_{oi}/B_o = 0.95\)

Therefore:

\[
\frac{N_p}{N} = (5000\text{ psia})(10 \times 10^{-6}\text{ psia}^{-1})(0.95)
\]

or

\[
\frac{N_p}{N} = 0.0475 \text{ (or 4.8 percent)}
\]
Material Balance: "Black Oil" Case

- Solution Gas Drive Case: (Full Relation)

\[ N_p \left[B_o + (R_p - R_s)B_g\right] + W_p B_w = \]

(Water Influx (RB))

\[ N \left[(B_o - B_{oi}) + (R_{si} - R_s)B_g\right] \]

(Oil Expansion (RB))

\[ + mNB_{oi} \left[ \frac{B_g}{B_{gi}} - 1 \right] \]

(Gas Cap Expansion (RB))

\[ + (1 + m)NB_{oi} \frac{(c_w S_{wi} + c_f)}{1 - S_{wi}} (p_i - \bar{p}) \]

(Water Exp./Pore Comp.(RB))

\[ + W_e B_w \]

(Water Inlet (RB))

- Application:

1. Specialized plotting functions (Havlena-Odeh).
2. Or, use automated regression (should safeguard).

- Comment: Large number of unknowns (e.g., \(N, m, c_w, c_f, \) and \(W_e\)) can make application difficult. Make every effort to CHECK all data, and to verify material balance calculations using an independent technique such as numerical simulation.
Material Balance: Dry Gas Case

• Dry Gas Reservoir — No Influx

\[
\frac{\bar{p}}{\bar{z}} = \frac{p_i}{z_i} \left[ 1 - \frac{G_p}{G} \right]
\]

■ Application:
1. Plot:
\[
\frac{\bar{p}}{\bar{z}} \text{ versus } G_p
\]

2. Extrapolate the straight-line trend to \(\bar{p}/\bar{z} = 0\), this yields the gas-in-place, \(G\).

• Dry Gas Reservoir — General Formulation (includes compressibility and influx terms)

\[
\frac{\bar{p}}{\bar{z}} = \frac{p_i}{z_i} \left[ \frac{1}{1 - \left( c_w S_{wi} + c_f \right) \left( p_i - \bar{p} \right) - \left( W_e - W_p \right) B_w}{1 - S_{wi}} - \frac{(W_e - W_p)B_w}{GB_{gi}} \right] \left[ 1 - \frac{G_p}{G} \right]
\]

■ Application:
1. Plot:
\[
\frac{\bar{p}}{\bar{z}} = \left[ 1 - \left( c_w S_{wi} + c_f \right) \left( p_i - \bar{p} \right) - \left( W_e - W_p \right) B_w \right] \left( 1 - \frac{G_p}{GB_{gi}} \right) \text{ versus } G_p
\]

2. Extrapolate straight-line trend to y-axis=0, this yields the gas-in-place, \(G\).

■ Comment: This approach is rigorous, but the data required (e.g., \(c_w, c_f\), and \(W_e\)) can make application difficult.
Material Balance: "Black Oil" Case — Applications

- Application of the "Constant Compressibility" MBE:
  - The "pseudosteady-state flow equation" is given (without derivation) as:
    \[ \bar{p} = p_{wf} + qb_{pss} \]
    where:
    \[ b_{pss} = 141.2 \frac{\mu B_o}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] \] (circular reservoir)

- Analysis Relations
  - Average Pressure Formulation:
    \[ \bar{p} = p_i - 5.615 \frac{N_p B_o}{\phi h A c_t} \] (\( \bar{p} \) vs. \( N_p \))
  - Wellbore Pressure Formulation: (Variable-Rate)
    \[ \frac{p_i - p_{wf}}{q} = b_{pss} + \frac{1}{N c_t B_{oi}} \frac{N_p}{q} \quad \left[ \frac{p_i - p_{wf}}{q} \text{ vs. } \frac{N_p}{q} \right] \]
  - Wellbore Pressure Formulation: (Constant Rate)
    \[ p_{wf} = p_i - qb_{pss} - 0.23395 \frac{q B_o}{\phi h A c_t} t \] (\( p_{wf} \) vs. \( t \), \( t \) in hr)
Material Balance: "Dry Gas" Case — Applications

- Application of the "Dry Gas" MBE:
  - The dry gas form of the "pseudosteady-state flow equation" (in pseudopressure form) is given (without derivation) as:
    \[
    \bar{p}_p = p_{w} + q_g b_{pss}
    \]
  - where the constant \( b_{pss} \) is the same as the liquid case. The "pseudopressure" function is given as:
    \[
    p_p = \left[ \frac{\mu_g z_i}{p_i} \right] \int_{p_{base}}^{p} \frac{p}{\mu_g z} dp
    \]

- Analysis Relations
  - Wellbore Pressure Formulation: (Variable-Rate)
    \[
    \frac{p_{pi} - p_{w}}{q_g} = \frac{1}{Gc_{ti}} \tilde{t}_a + b_{pss} \quad \left[ \frac{p_{pi} - p_{w}}{q_g} \right. \text{ vs. } \tilde{t}_a
    \]
  - The "material balance pseudotime" function is defined by:
    \[
    \tilde{t}_a = \frac{\mu_g c_{ti}}{q_g} \int_{0}^{t} \frac{q_g}{\mu_g(\bar{p}) c_{ti}(\bar{p})} dt
    \]
  - The pseudotime function, \( \tilde{t}_a \), is explicitly a function of the average reservoir pressure — hence \( \tilde{t}_a \) is implicitly a function of the gas-in-place. Simply put, we must know \( G \) in order to compute \( \tilde{t}_a \).

Recalling the "dry gas" MBE, we have:

\[
\frac{\bar{p}}{z} = \frac{p_i}{z_i} \left[ 1 - \frac{G_p}{G} \right]
\]
Material Balance Notes
(from Department of Petroleum Engineering Course Notes — 1984)

- Black Oil Cases
  - Undersaturated Oil
  - Solution Gas Drive
- Dry Gas Case
VOLUMETRIC OIL RESERVOIRS

I. Undersaturated Reservoir

Objective: To derive a material balance equation for an undersaturated reservoir

A. Assumptions
   1. $P > P_B$
   2. No original or final gas cap
   3. No water influx or production

B. Derivation of Material Balance Equation
BY VOLUMETRIC BALANCE

ORIGINAL VOLUME = FINAL VOLUME

ORIGINAL VOLUME = \text{NB}_0\text{I}

FINAL VOLUME = (N-N_P)B_0 + VOLUME OCCUPIED BY WATER AND ROCK EXPANSION AS PRESSURE DECLINES

- ROCK AND WATER EXPANSION IMPORTANT IN UNDERSATURATED RESERVOIRS

FROM DEFINITION OF COMPRESSIBILITY

\[ c_w = -\frac{1}{V_w} \left( \frac{dV_w}{dP} \right) = -\frac{1}{V_{wi}} \frac{\Delta V_w}{\Delta P} \]

THUS, CHANGE IN RESERVOIR WATER VOLUME DUE TO PRESSURE CHANGE:

\[ \Delta V_w = -c_w V_{wi} \Delta P \]

AS PRESSURE DECREASES, MATRIX SUPPORTING STRUCTURE COLLAPSES INTO PORE SPACE

\[ c_f = -\frac{1}{V_p} \left( \frac{dV_p}{dP} \right) = -\frac{1}{V_{pi}} \frac{\Delta V_p}{\Delta P} \]
THUS, CHANGE IN PORE VOLUME DUE TO PRESSURE CHANGE:

\[ \Delta V_P = -c_F V_{PI} \Delta P \]

TOTAL CHANGE IN WATER VOLUME AND PORE VOLUME:

\[ \Delta V_W + \Delta V_P = -\left[ c_W V_{WI} + c_F V_{PI} \right] \Delta P \]

= \Delta V_{TOTAL}

NOTE THAT

\[ V_W = S_W V_P \]

\[ V_{WI} = S_{WI} V_{PI} \]

THUS

\[ \Delta V_{TOTAL} = -\left[ c_W S_{WI} + c_F \right] V_{PI} \Delta P \]

ALSO

\[ V_{PI} = \frac{NB_{OI}}{1 - S_{WI}} \]

THUS

\[ \Delta V_{TOTAL} = \frac{NB_{OI}}{1 - S_{WI}} \left[ c_W S_{WI} + c_F \right] \Delta P \]
THE VOLUMETRIC BALANCE BECOMES:

\[ NB_{OI} = (N-N_P)B_O - \frac{NB_{OI}}{1-S_{WI}} \left[ C_W S_{WI} + C_F \right] \Delta P \]

SOLVING FOR N:

\[ N = \frac{N_P B_O}{B_O + B_{OI} \left( \frac{C_W S_{WI} + C_F}{1-S_{WI}} \right) (P_I-P) - B_{OI}} \]  \hspace{1cm} (V-1)

TO SIMPLIFY, NOTE:

\[ C_O = - \frac{1}{V} \left( \frac{dV}{dP} \right) = - \frac{1}{V} \left( \frac{\Delta V}{\Delta P} \right) \]

IF \( V_{SC} \) IS VOLUME OF OIL IN STOCK TANK (STANDARD CONDITIONS)

\[ \frac{1}{V} \left( \frac{\Delta V}{\Delta P} \right) = \frac{1}{V_{I}/V_{SC}} \left( \frac{V/V_{SC} - V_I/V_{SC}}{(P-P_I)} \right) \]

\[ = \frac{1}{B_{OI}} \frac{(B_O-B_{OI})}{(P-P_I)} \]

THEN

\[ B_O-B_{OI} = c_O B_{OI} (P_I-P) \]
SUBSTITUTING INTO EQUATION V-1

\[ N = \frac{N_PB_O}{B_{OI}\left[c_O + \frac{C_WS_{WI} + C_F}{1-S_{WI}}\right](P_I-P)} \]

DEFINE

\[ C_E = C_O + \frac{C_WS_{WI} + C_F}{1-S_{WI}} = \frac{C_O S_{OI} + C_WS_{WI} + C_F}{(1-S_{WI})} \]

THUS

\[ N = \frac{N_PB_O}{B_{OI}C_E(P_I-P)} \] (V-2)

C. CONSIDERATIONS

1. Eqs. V-1 or V-2 should be used for estimating 00IP above bubble point where rock and water expansion not negligible

2. Difficulty in measuring \( C_F \) and \( C_W \) may limit accuracy
D. Example — Use of Material Balance to Determine Original Oil in Place in Under-saturated Reservoir

Problem

Determine the original oil in place for the undersaturated reservoir for which data are summarized below.

\[
\begin{align*}
N_p &= 1.4 \times 10^6 \text{ STB} \\
B_o &= 1.46 \text{ RB/STB} \\
B_{oi} &= 1.39 \text{ RB/STB} \\
c_w &= 3.71 \times 10^{-6} \text{ PSI}^{-1} \\
c_f &= 3.52 \times 10^{-6} \text{ PSI}^{-1} \\
S_{wi} &= 32\% 
\end{align*}
\]

The reservoir was discovered at an initial pressure of 4300 PSI. Pressure has declined to 2450 PSI.

Solution

From equation V-1

\[
N = \frac{N_p B_o}{B_o + B_{oi} \left( \frac{c_w S_{wi} + c_f}{1 - S_{wi}} \right) (P_i - P) - B_{oi}}
\]
N = \frac{1.4 \times 10^6(1.46)}{1.46 + 1.39 \left( \frac{3.71 \times 10^{-6}(0.32) + 3.52 \times 10^{-6}}{1-0.32} \right) (4300-2450) - 1.39}

= 2.33 \times 10^7 \text{ STB}

NOTE:

IF $C_F$ IS ASSUMED TO BE 0:

N = \frac{1.4 \times 10^6(1.46)}{1.46 + 1.39 \left( \frac{3.71 \times 10^{-6}(0.32) + 0}{1-0.32} \right) (4300-2450) - 1.39}

= 2.74 \times 10^7 \text{ STB}

CALCULATED VALUE OF N IS SIGNIFICANTLY INCREASED IF $C_F$ NEGLECTED
II. Saturated Reservoir - Solution Gas Drive

Objective: To derive a material balance equation for a solution gas drive reservoir and to apply it to estimate original oil in place (OOIP)

A. Assumptions

1. $P \leq P_B$
2. No original gas cap
3. No water influx or production
4. Negligible rock and water expansion

B. Derivation of Material Balance Equation

\[
\begin{align*}
\text{OIL VOLUME}\quad N_{BOI} \\
\text{GAS VOLUME}\quad (N-N_P)B_O \\
\text{ORIGINAL CONDITIONS} \quad \text{LATER CONDITIONS}
\end{align*}
\]

By volumetric balance

Original volume = Final volume
ORIGINAL OIL VOLUME = $NB_{OI}$
ORIGINAL FREE GAS VOLUME = 0
FINAL OIL VOLUME = $(N-N_p)B_0$

- DETERMINE FINAL FREE GAS VOLUME BY PERFORMING A GAS BALANCE

ORIGINAL DISSOLVED GAS = $NR_{SI}$
FINAL DISSOLVED GAS = $(N-N_p)R_S$
GAS PRODUCED = $G_p$

THEREFORE,

FINAL FREE GAS = $NR_{SI} - (N-N_p)R_S - G_p$

- CONVERT TO RESERVOIR CONDITIONS

FINAL FREE GAS = $(NR_{SI} - (N-N_p)R_S - G_p)B_G/5.61$

THE VOLUMETRIC BALANCE BECOMES:

$NB_{OI} = (N-N_p)B_0 + (NR_{SI}-(N-N_p)R_S-G_p)B_G/5.61$

SOLVING FOR $N$

$$N = \frac{N_pB_0 - (N_pR_S-G_p)B_G/5.61}{B_0 + (R_{SI}-R_S)B_G/5.61 - B_{OI}} \quad (V-3)$$
TO SIMPLIFY, NOTE THAT

\[ B_T = B_0 + (R_{SI} - R_S)B_G/5.61 \]

\[ R_P = G_P/N_P \]

ALSO,

\[ B_{OI} = B_{TI} \quad \text{(NO GAS EVOLVED AT } P_B) \]

SUBSTITUTING INTO EQUATION V-3

\[ N = \frac{N_P\{B_T + (R_P - R_{SI})B_G/5.61\}}{B_T - B_{TI}} \quad \text{(V-4)} \]

SUBSTITUTING FOR \( B_0 \) IN THE NUMERATOR ONLY, AN ALTERNATIVE FORM IS

\[ N = \frac{N_P\{B_0 + (R_P - R_S)B_G/5.61\}}{B_T - B_{TI}} \quad \text{(V-5)} \]
III. Derivation of Material Balance Equation

- From Volumetric Balance

**Initial Volume = Final Volume**

\[ \frac{G(B_{G1})}{5.61} = (G - G_p)(B_G/5.61) + (W_E - W_p)(B_W) \]

Where

- \( B_G \) = Gas Formation Volume Factor, RCF/SCF

\[ \frac{G(B_{G1})}{5.61} = (G - G_p)(B_G/5.61) - W_p(B_W) + W_E(B_W) \]

This can be rearranged to be

\[ \frac{G(B_G - B_{G1})}{5.61} = \frac{G_p(B_G)}{5.61} + W_p(B_W) - W_E(B_W) \]
THE GENERALIZED FORM OF THE MATERIAL BALANCE EQUATION

\[
\frac{G_P(B_G)/5.61 + W_P(B_W)}{(B_G-B_{G1})/5.61} = G + \frac{W_E(B_W)}{(B_G-B_{G1})/5.61}
\]

(X-6)

A. PRESSURE DEPLETION CASE - NO WATER INFLUX

\[W_E = W_P = 0\]

therefore:

\[G(B_G-B_{G1})/5.61 = G_P(B_G)/5.61\]

or

\[G_P = G(1-B_{G1}/B_G)\]  

(X-7)
SINCE \[ B_G = \frac{V_{R,C.}}{V_{S,C.}} = \frac{P_{S,C.}}{P_{R,C.}} \frac{T_{R,C.}}{T_{S,C.}} \frac{Z_{R,C.}}{Z_{S,C.}} \]

\[ B_G/B_{G1} = \frac{P_1}{Z_1}(z/P) = \text{constant} \]

THEN \[ G_p = G(1 - (P/z)(z_1/P_1)) \]

THIS CAN BE REARRANGED TO BE

\[ (P_1/z_1)(1-G_p/G) = P/z \] \hspace{1cm} (X-8)

THIS SUGGESTS A PLOT OF \( (P/z) \) VS. \( G_p \)

WHEN \( (P/z) = 0 \), NOTE THAT

\[ P_1/z_1 (1-G_p/G) = 0 \]

OR

\[ G_p = G \]

ALSO, THE PLOT SHOULD BE LINEAR AND THUS READILY EXTRAPOLATED TO \( P/z = 0 \).
B. Example Problem - Determination of OGIP and Drive Mechanism Using a P/z Plt.

Problem

An isopach map of the "Zapata Sand" in the Woodford Field in Atascosa County, Texas indicated an original gas in place of 44 mmscf. Production from the field has resulted in the following:

<table>
<thead>
<tr>
<th>Reservoir Pressure (Psia)</th>
<th>Gp (MMSCF)</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>0.00</td>
<td>0.80</td>
</tr>
<tr>
<td>3500</td>
<td>2.46</td>
<td>0.73</td>
</tr>
<tr>
<td>3000</td>
<td>4.92</td>
<td>0.66</td>
</tr>
<tr>
<td>2500</td>
<td>7.88</td>
<td>0.60</td>
</tr>
<tr>
<td>2000</td>
<td>11.20</td>
<td>0.55</td>
</tr>
<tr>
<td>200</td>
<td>---</td>
<td>0.94</td>
</tr>
</tbody>
</table>

The z factors were derived from fluid analysis data. A volumetric type depletion is suspected.

Perform a P/z plot to confirm original gas in place estimates and the suspected drive mechanisms.
**Solution**

<table>
<thead>
<tr>
<th>( G_p ) (MMSCF)</th>
<th>( P/z ) (PSIA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>5000</td>
</tr>
<tr>
<td>2.46</td>
<td>4795</td>
</tr>
<tr>
<td>4.92</td>
<td>4545</td>
</tr>
<tr>
<td>7.88</td>
<td>4167</td>
</tr>
<tr>
<td>11.20</td>
<td>3636</td>
</tr>
</tbody>
</table>

(See Graph)

The characteristics of the best straight line fit of the data set are:

A) \( G = 43.6 \) MMSCF (by extrapolation)

B) The curve gives some indication of nonlinearity but still fits the expected gas-in-place estimate

C) The straight line and good agreement with original gas-in-place estimate confirm the volumetric depletion characteristics
G = 43.6 MMSCF
(Appendix A)

Derivation of "Material Balance" Time
(Liquid Flow Case)

From:
APPENDIX A

DEVELOPMENT OF MATERIAL BALANCE TIME CONCEPT

In this appendix, we present the pseudosteady-state flow equation for slightly compressible liquids derived from a material balance relation. This derivation rigorously illustrates the necessity for using the material balance time function, $t_{mb}$, for pseudosteady-state flow.

In words, a material balance based on the amount of oil in a reservoir is given by

\[
\text{Amount of oil originally in reservoir} = \text{Amount of oil in reservoir at a later time} + \text{Change in hydrocarbon pore volume due to rock and water expansion.}
\]

This can be expressed mathematically as

\[
NB_{oi} = (N - N_p)B_o + \Delta V_r + \Delta V_w \quad \ldots \quad \text{(A-1)}
\]

We can express the change in rock and water volume in terms of formation and water compressibility respectively. The general definition of isothermal compressibility is

\[
c = -\frac{1}{V} \left[ \frac{\partial V}{\partial p} \right]_T
\]

We can calculate the slope, $\frac{\partial V}{\partial p}$, by approximating the tangent to the curve (volume versus pressure) by a chord slope, i.e.

\[
c \approx -\frac{1}{V_i} \left[ \frac{V - V_i}{p - p_i} \right]
\]

and rearranging this equation to obtain an expression for change in volume gives

\[
\Delta V = -c V_i(p - p_i) \quad \ldots \quad \text{(A-2)}
\]

Since the initial volume for rock is the initial pore volume, $V_{pi}$, Eq. (A-2) can be used to express change in rock volume as

\[
\Delta V_r = -c_f V_{pi}(p - p_i) \quad \ldots \quad \text{(A-3)}
\]

and since the initial volume for water is given by $V_{pi}S_{wi}$, the change in water volume is

\[
\Delta V_w = -c_w V_{wi}(p - p_i) = -c_w V_{pi}S_{wi}(p - p_i) \quad \ldots \quad \text{(A-4)}
\]

Combining Eqs. (A-3) and (A-4) gives

\[
\Delta V_r + \Delta V_w = -c_f V_{pi}(p - p_i) - c_w V_{pi}S_{wi}(p - p_i)
\]
Grouping terms condenses this equation to
\[
\Delta V_r + \Delta V_w = - (c_w S_{wi} + c_f) V_{pi}(p - p_i) \quad \text{...........................................(A-5)}
\]
We can now substitute Eq. (A-5) into Eq. (A-1), giving
\[
NB_{oi} = (N - N_p) B_o - (c_w S_{wi} + c_f) V_{pi}(p - p_i) \quad \text{...........................................(A-6)}
\]
The original oil in place in reservoir volumes, \( NB_{oi} \), can be expressed in terms of pore volume and initial water saturation
\[
NB_{oi} = V_{pi}(1 - S_{wi})
\]
or
\[
V_{pi} = \frac{NB_{oi}}{(1 - S_{wi})} \quad \text{...........................................(A-7)}
\]
We can substitute Eq. (A-7) for the pore volume term in Eq. (A-6), giving
\[
NB_{oi} = (N - N_p) B_o - (c_w S_{wi} + c_f) \frac{NB_{oi}}{(1 - S_{wi})} (p - p_i) \quad \text{...........................................(A-8)}
\]
We assume the pressure, \( p \), represents the average pressure, \( \bar{p} \), in our volumetric system. Rearranging Eq. (A-8) gives
\[
NB_{oi} = (N - N_p) B_o + (c_w S_{wi} + c_f) \frac{NB_{oi}}{(1 - S_{wi})} (p_i - \bar{p})
\]
or, grouping terms containing \( N \) on the left-hand side of the equation, we have
\[
N (B_{oi} - B_o) = - N_p B_o + (c_w S_{wi} + c_f) \frac{NB_{oi}}{(1 - S_{wi})} (p_i - \bar{p}) \quad \text{...........................................(A-9)}
\]
We can express the change in oil volume, \( B_{oi} - B_o \), in terms of the isothermal compressibility of oil, \( c_o \). Recall that the definition of \( c_o \) is
\[
c_o = - \frac{1}{B_o} \left[ \frac{\partial B_o}{\partial p} \right]_T
\]
As before, we can calculate the slope, \( \frac{\partial B_o}{\partial p} \), by approximating the tangent to the curve (formation volume factor versus pressure) by a chord slope, \( i.e. \)
\[
c_o = - \frac{1}{B_{oi}} \left[ \frac{B_o - B_{oi}}{p - p_i} \right]
\]
Rearranging this equation gives the following expression for the change in oil volume, \( B_{oi} - B_o \)
\[
B_o - B_{oi} = - c_o B_{oi} (p - p_i) = c_o B_{oi} (p_i - \bar{p}) \quad \text{...........................................(A-10)}
\]
We can now substitute Eq. (A-10) into Eq. (A-9) for the change in oil volume, \( B_{oi} - B_o \)

\[-Nc_o B_{oi} (p_i - \bar{p}) = -N_p B_o + (c_w S_{wi} + c_f) \frac{N B_{oi}}{(1 - S_{wi})} (p_i - \bar{p})\]

which can be rearranged to give

\[N_p B_o = Nc_o B_{oi} (p_i - \bar{p}) + (c_w S_{wi} + c_f) \frac{N B_{oi}}{(1 - S_{wi})} (p_i - \bar{p})\]

\[N_p B_o = N B_{oi} (p_i - \bar{p}) \left[ c_o + \frac{(c_w S_{wi} + c_f)}{(1 - S_{wi})} \right]\]

or finally,

\[N_p B_o = \frac{N B_{oi}}{(1 - S_{wi})} (p_i - \bar{p}) \left[ c_o (1 - S_{wi}) + c_w S_{wi} + c_f \right] \]

\[\text{(A-11)}\]

If we use the expression derived by Perrine\(^1\) and Martin\(^2\) for total system compressibility, \( c_t \),

\[c_t = c_o (1 - S_{wi}) + c_w S_{wi} + c_f = c_o S_{oi} + c_w S_{wi} + c_f \]

\[\text{(A-12)}\]

we can simplify Eq. (A-11) and write

\[N_p B_o = \frac{N B_{oi}}{(1 - S_{wi})} c_t (p_i - \bar{p}) \]

\[\text{(A-13)}\]

Recall that

\[V_{pi} = \frac{N B_{oi}}{(1 - S_{wi})} \]

\[\text{(A-7)}\]

therefore,

\[N_p = \frac{V_{pi}}{B_o} c_t (p_i - \bar{p}), \text{ where } V_{pi} = \phi h A, \text{ so we can write} \]

\[N_p = \frac{\phi h A}{B_o} c_t (p_i - \bar{p})\]

Rearranging gives

\[(p_i - \bar{p}) \frac{h}{B_o} = \frac{N_p}{\phi c_t A}\]

Multiply through by \( \frac{2\pi k}{q \mu} \) to obtain

\[(p_i - \bar{p}) \frac{2\pi k h}{q B_o \mu} = \frac{2\pi k}{\phi \mu c_t A} \frac{N_p}{q} \]

\[\text{(A-14)}\]

The definitions of dimensionless pressure, \( p_D \), and dimensionless time based on drainage area, \( t_{AD} \) are

\[p_D = \frac{2\pi k h}{q B_o \mu} (p_i - p_{wf}) \]

\[\text{(A-15)}\]
and

\[ t_{AD} = \frac{kt}{\phi \mu c_i A} \]  \hspace{2cm} (A-16)

Therefore, if we define a material balance time as \( t_{mb} = \frac{N_p}{q} \), Eq. (A-16) becomes

\[ 2\pi t_{AD,mb} = 2\pi \frac{k}{\phi \mu c_i A} \frac{N_p}{q} \]  \hspace{2cm} (A-17)

We can now substitute Eq (A-17) into Eq. (A-14) to obtain

\[ (p_i - \bar{p}) \frac{2\pi kh}{qB_o \mu} = 2\pi t_{AD,mb} \]  \hspace{2cm} (A-18)

Eq. A-18 is valid for all flow regimes (transient or pseudosteady-state). We can couple this relationship for \((p_i - \bar{p})\) with the general relationship for \((\bar{p} - p_{wf})\) derived by Camacho\(^3\), which is valid for pseudosteady-state flow and which is given below

\[ (\bar{p} - p_{wf}) \frac{2\pi kh}{qB_o \mu} = \frac{1}{2} \ln \left[ \frac{4}{e} \frac{A}{C_A r_w^2} \right] + s \]  \hspace{2cm} (A-19)

Adding Eqs. A-18 and A-19 eliminates \( \bar{p} \), giving

\[ (p_i - p_{wf}) \frac{2\pi kh}{qB_o \mu} = 2\pi t_{AD,mb} + \frac{1}{2} \ln \left[ \frac{4}{e} \frac{A}{C_A r_w^2} \right] + s \]  \hspace{2cm} (A-20)

Substituting the definition of dimensionless pressure, Eq. (A-15), into Eq. (A-20) gives the dimensionless form of the pseudosteady-state flow solution to the diffusivity equation,

\[ p_D = 2\pi t_{AD,mb} + \frac{1}{2} \ln \left[ \frac{4}{e} \frac{A}{C_A r_w^2} \right] + s \]  \hspace{2cm} (A-21)

This validates the use of the material balance time function, \( t_{mb} \), for pseudosteady-state flow.

REFERENCES used in APPENDIX A


(Appendix A)

Development and Use of the Material Balance Time for Boundary-Dominated Liquid Flow

From:

APPENDIX A

DEVELOPMENT AND USE OF THE MATERIAL BALANCE TIME FOR BOUNDARY-DOMINATED LIQUID FLOW

DEVELOPMENT

The original material balance approach for boundary-dominated flow (i.e., pseudosteady-state) was developed by Blasingame and Lee,1 and the following derivation builds upon their developments.

From the definition of liquid compressibility, the following material balance can be derived

\[ q_o = \frac{A \phi_c h_t}{5.615 B_o} \frac{d \bar{p}}{dt} \]  \hspace{1cm} (A-1)

Since we are considering the single-phase liquid case, the total compressibility, \( c_t \), is assumed to be constant. Therefore, the integration of Eq. A-1 yields

\[ \int_0^t q_o \, dt = \frac{A \phi_c h_t}{5.615 B_o} \int_{p_i}^{p} \frac{d \bar{p}}{p_i} \]

Completing the integration, we have

\[ N_p = \frac{A \phi_c h_t}{5.615 B_o} (p_i - \bar{p}) \]

Solving this relation for the pressure difference, it follows that

\[ (p_i - \bar{p}) = \frac{5.615 N_p B_o}{A \phi_c h_t} = \frac{1}{N c_t B_o} N_p \]  \hspace{1cm} (A-2)

where Eq. A-2 can also be rearranged to yield

\[ (p_i - \bar{p}) \frac{h}{B_o} = \frac{5.615 N_p}{A \phi_c h_t} \]

multiplying through by \( \frac{k_o}{141.2 q_o \mu_o} \) gives

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{5.615 k_o}{141.2 \phi \mu_o c_r A} \frac{N_p}{q_o} \]

Defining the material balance time function, \( \bar{t} = \frac{N_p}{q_o} \), it follows that

\[ (p_i - \bar{p}) \frac{k_o h}{141.2 q_o B_o \mu_o} = \frac{2 \pi (0.00633) k_o}{\phi \mu_o c_r A} \bar{t} \]

Redefining the dimensionless time based on drainage area, we have

\[ \bar{t}_{DA} = \frac{0.00633 k_o}{\phi \mu_o c_r A} \bar{t} \]  \hspace{1cm} (A-3)
Substituting Eq. A-3 into the pressure relation gives us

\[ \frac{k_o h}{141.2 q_o B_o \mu_o} = 2\pi \tilde{t}_{DA} \]  

(A-4)

The most important characteristic of Eq. A-4 is that this relation is always valid—regardless of time, flow regime, or production scenario—whether the well experiences constant or variable flowing bottomhole pressures, or constant or variable flowrate. This is due to the fact that Eq. A-4 is derived directly from material balance relations and is exact.

It has been shown\(^2\) that for a constant production rate of single-phase liquid, the flow equation for the pressure response under boundary dominated flow can be written as

\[ \frac{(p_i - p_{wf})}{141.2 q_o B_o \mu_o} = \frac{k_o h}{141.2 q_o B_o \mu_o} = 2\pi \tilde{t}_{DA} + \frac{1}{2} \ln \left[ \frac{4}{e^{\frac{A}{c_A^2}}} \right] \]  

(A-5)

Although Eq. A-5 was derived for the constant rate case (variable \(p_{wf}\)), this relation has been shown\(^1\) to yield a very good approximation for the case where the flowing bottomhole pressure is constant.

The substitution of Eqs. A-4 and A-5 yields

\[ \frac{(p_i - p_{wf})}{141.2 q_o B_o \mu_o} = 2\pi \tilde{t}_{DA} + \frac{1}{2} \ln \left[ \frac{4}{e^{\frac{A}{c_A^2}}} \right] \]  

(A-6)

The previous considerations imply that Eq. A-6 is valid for the pseudosteady-state flow regime—for any rate or pressure profile. Substituting Eq. A-3 into A-6 gives the following

\[ \frac{(p_i - p_{wf})}{q_o} = m t + b_{pss} \]  

(A-7)

where

\[ m = \frac{2\pi (0.00633) k_o B_o \mu_o}{\frac{141.2}{c_A^2}} = \frac{5.615 B_o}{\frac{1}{\mu_o} B_{oi}} = \frac{1}{\phi c_A^2} \]  

(A-8)

and

\[ b_{pss} = 141.2 B_o \mu_o \frac{1}{k_o h} \left[ \frac{1}{2} \ln \left( \frac{4}{e^{\frac{A}{c_A^2}}} \right) \right] \]  

(A-9)

Rearranging Eq. A-7, we obtain our final form for behavior of a well producing a variable-rate profile during pseudosteady-state (or boundary dominated) flow conditions. This result is given by

\[ \frac{q_o}{(p_i - p_{wf})} b_{pss} = \frac{1}{1 + \frac{m}{b_{pss}}} \tilde{t} \]  

(A-10)

**USE**

The group on the left-hand-side (LHS) of Eq. A-10 is exactly the dimensionless decline rate variable, \(q_{DD}\), as presented by Fetkovich.\(^3,4\) The second term in the denominator of the right-hand-side (RHS) group in the same equation is defined as, \(t_{DD}\), therefore

\[ \tilde{t}_{DD} = \left( \frac{m}{b_{pss}} \right) \tilde{t} = \frac{2\pi \tilde{t}_{DA} + W_e B_w}{\frac{1}{2} \ln \left( \frac{4}{e^{\frac{A}{c_A^2}}} \right)} \]  

(A-11)
The only difference between \( t_{Dd} \) and the Fetkovich\(^3\) dimensionless decline time function, \( t_{Dd} \), is that material balance time, \( \bar{t} \), is substituted for production time, \( t \). Fetkovich assumed a circular reservoir, the denominator on the RHS of Eq. A-11 is given by 
\[
\ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4}
\]
which results from using the appropriate \( C_A \) value for a circular reservoir. However, Fetkovich actually chose to use 1/2 rather than 3/4 within the argument of the logarithm, as he obtain a better correlation of \( q_{Dd}^{-t_{Dd}} \) trends.

Regardless, Eq. A-10 can be reduced to
\[
q_{Dd} = \frac{1}{1 + \bar{t}_{Dd}} \quad \text{(A-12)}
\]

Eq. A-12 is a harmonic decline type of equation where the decline exponent is unity. This is evident by comparing Eq. A-12 to Arps\(^5\) original definition of a hyperbolic decline which in dimensionless form is
\[
q_{Dd} = \frac{1}{(1 + bt_{Dd})^{1/b}} \quad \text{(A-13)}
\]

When the decline exponent, \( b \), is taken as unity in the Arps\(^5\) equations we call this a harmonic decline. Given \( b=1 \), Eqs. A-12 and A-13 have exactly the same form.

Therefore if \( \bar{t} \) is correctly calculated, a scaled log-log plot of \( \frac{q_o}{(p_i - p_{wf})} \) versus \( \bar{t} \) will overlay the \( q_{Dd} \) versus \( t_{Dd} \) trend for a harmonic decline on Fetkovich's\(^3\) type curve. Once the match has been obtained, the "match point" relations for \( m \), \( b_{pss} \), and \( N \) are:
\[
b_{pss} = \frac{[q_{Dd}]_{MP}}{\left( \frac{q_o}{(p_i - p_{wf})} \right)_{MP}} \quad \text{(A-14)}
\]
\[
m = \frac{1}{N c_i B_{oi}} = b_{pss} \frac{[t_{Dd}]_{MP}}{[\bar{t}]_{MP}} \quad \text{(A-15)}
\]
\[
N = \frac{1}{c_{ii} B_{oi}} \left( \frac{\frac{q_o}{(p_i - p_{wf})}}{[q_{Dd}]_{MP}} \right) \quad \text{(A-16)}
\]

where MP denotes a match point value.

REFERENCES used in APPENDIX B


(Appendix B)

Development, Proof and Use of the Material Balance Pseudotime for Boundary-Dominated Gas Flow

From:
APPENDIX B

DEVELOPMENT, PROOF AND USE OF THE MATERIAL BALANCE PSEUDOTIME FOR BOUNDARY-DOMINATED GAS FLOW

DEVELOPMENT
Blasingame and Lee\(^1\) proposed a gas flow equation based on a modified pseudotime function to model the general variable-rate/variable pressure drop gas flow case. This was done on an empirical basis using numerical simulation as validation.

Analytical proofs are given below to provide the theoretical background for the modified gas flow equation proposed by Blasingame and Lee. Beginning with the material balance equation for a liquid, we have:

\[
\frac{(p_i - p_{wf})}{q_o} = \frac{1}{Nc_t} \frac{B_o}{B_{oi}} \frac{N_p}{q_o} = \frac{1}{Nc_t} \frac{B_o}{B_{oi}} \tilde{t} \tag{B-1}
\]

Blasingame and Lee proposed a relation similar in form to Eq. B-1 for the gas flow case, this result is given by:

\[
\frac{(p_{pi} - p_p)}{q_g} = \frac{1}{Gc_t} \tilde{t}_a \tag{B-2}
\]

Eq. B-2 is assumed to be valid when the pseudotime function is defined by

\[
\tilde{t}_a = \frac{\mu_g c_{ti}}{q_g} \int_0^t \frac{q_g}{\mu_g(\bar{p})c_i(\bar{p})} dt \tag{B-3}
\]

The pseudopressure functions in Eq. B-2 are normalized variables,\(^2\) which are defined by

\[
p_{pi} = \left[ \frac{\mu_{gi} z_i}{p_i} \right] \int_{P_{base}}^P \frac{p}{\mu_g z} dp
\]

and

\[
p_p = \left[ \frac{\mu_{gi} z_i}{p_i} \right] \int_{P_{base}}^\bar{p} \frac{p}{\mu_g z} dp
\]

The pseudosteady-state gas flow equation (given by Al Hussainy, \textit{et al.}\(^3\,4\)) is given by:

\[
\frac{(\bar{p}_p - p_{psf})}{q_g} = 141.2 \frac{\mu_{gi} B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left[ \frac{4 A}{C_{Ar} r_w^2} \right] \right] \tag{B-4}
\]

The addition of Eqs. B-3 and B-4 yields the following generalized relation for the variable-rate flow of gas in a reservoir during boundary-dominated flow conditions. This result is:

\[
\frac{\Delta p_p}{q_g} = m_a \tilde{t}_a + b_{a,psf} \tag{B-5}
\]
where
\[ \Delta p_p = (p_{pi} - p_{pwf}) \]  \hspace{1cm} (B-6)
\[ m_a = \frac{1}{Gc_t_i} \]  \hspace{1cm} (B-7)
\[ b_{a,poss} = 141.2 \frac{\mu_g B_{gi}}{k_g h} \left[ \frac{1}{2} \ln \left( \frac{4}{\epsilon_i} \right) \frac{A}{C_{A_{r_w}}^2} \right] \]  \hspace{1cm} (B-8)

PROOF

In this section we will provide the analytical proof of Eq. B-2. This process begins by using the definition of gas compressibility, which is given by:
\[ c_g = \frac{1}{\rho_g} \frac{d}{dp} (\rho_g) \]

The density of a real gas is given as:
\[ \rho_g = \frac{M}{RT} \frac{p}{z} \]

Substituting the gas density identity into the gas compressibility definition, we obtain:
\[ c_g = \frac{1}{\rho_g} \frac{d}{dp} (\rho_g) = \frac{1}{M \frac{p}{RT} \frac{1}{z}} \frac{d}{dp} \left[ \frac{M \frac{p}{RT} \frac{1}{z}}{\frac{p}{z}} \right] = \frac{1}{\rho_g} \frac{d}{dp} \left[ \frac{M \frac{p}{RT} \frac{1}{z}}{\frac{p}{z}} \right] \]  \hspace{1cm} (B-9)

Evaluating Eq. B-9 at the average reservoir pressure, then solving for the pressure derivative expression, we have:
\[ \frac{d}{dp} \left[ \frac{\bar{p}}{z} \right] = \frac{\bar{p}}{z} c_g \]  \hspace{1cm} (B-10)

Recalling the gas material balance equation, we have:
\[ \frac{\bar{p}}{z} = \frac{p_i}{z_i} \left[ 1 - \frac{\mu_g}{G} \right] \]

Taking the time derivative of the gas material balance equation, then solving for the gas flowrate gives
\[ q_g = \frac{dG_p}{d\bar{p}} = -\frac{Gz_i}{p_i} \frac{d}{dt} \left[ \frac{\bar{p}}{z} \right] \]  \hspace{1cm} (B-11)

Applying the chain rule to Eq. B-11, we obtain
\[ q_g = \frac{dG_p}{dp} = -\frac{Gz_i}{p_i} \frac{d}{dp} \left[ \frac{\bar{p}}{z} \frac{d\bar{p}}{dt} \right] \]  \hspace{1cm} (B-12)

Substituting Eq. B-10 into Eq. B-12 yields
\[ q_g = -\frac{Gz_i}{p_i} \frac{\bar{p}}{z} \frac{d\bar{p}}{dt} \]  \hspace{1cm} (B-13)

Substituting Eq. B-13 into Eq. B-3, we obtain
\[ \bar{t}_a = \frac{\mu_g c_{ti}}{q_g} \left[ -\frac{Gz_i}{p_i} \right] \int_0^t \frac{\bar{p}}{z} \frac{1}{\rho_g \bar{c}_g} \frac{d\bar{p}}{dt} dt \]  \hspace{1cm} (B-14)

We can assume that the formation compressibility is negligible compared to the fluid compressibility (i.e., \( \bar{c}_i \approx \bar{c}_g \)). Making this assumption, changing the variable of integration from time to pressure, and rearranging terms, Eq. B-14 becomes
\[ \tau_a = \frac{G_{c_{ii}}}{q_g} \left( \frac{\mu g z_i}{p_i} \right) \int_{p_i}^{\bar{p}} \frac{\bar{p}}{p \bar{g} z} \, dp \] .............................. (B-15)

Recalling the definition of normalized pseudopressure, Eq. B-15 may be re-written as

\[ \tau_a = G_{c_{i}} \frac{(p_{pi} - \bar{p}_p)}{q_g} \]

where this result can be rearranged to yield

\[ \frac{(p_{pi} - \bar{p}_p)}{q_g} = \frac{1}{G_{c_{i}}} \tau_a \] .............................. (B-16)

Eq. B-16 is identical to Eq. B-2—where this result provides the proof that Eq. B-3 is the exact definition of the pseudotime function for boundary-dominated gas flow.

**USE**

The pseudopressure-pseudotime relations developed above for the decline curve analysis of gas reservoir systems can also be written as "Fetkovich"-style variables. More specifically, we can rearrange Eq. B-5 to yield:

\[ \frac{q_g}{\Delta p_p} b_{a,pss} = \frac{1}{1 + \left[ \frac{m_a}{b_{a,pss}} \right] \tau_a} \] .............................. (B-17)

Eq. B-17 and the liquid flow relation (Eq. A-10) have identical forms and as such, Eq. B-17 can be written as

\[ q_{Dd} = \frac{1}{1 + \tau_{a,Dd}} \] .............................. (B-18)

where the dimensionless decline rate function, \( q_{Dd} \), is given by:

\[ q_{Dd} = \frac{q_g}{\Delta p_p} b_{a,pss} \] .............................. (B-19)

and the dimensionless decline time function, \( \tau_{a,Dd} \), is given by:

\[ \tau_{a,Dd} = \left[ \frac{m_a}{b_{a,pss}} \right] \tau_a \] .............................. (B-20)

Eq. B-18 (the gas flow identity) and the liquid flow (Eq. A-12) are identical in form, as is expected—therefore, all of the comments regarding the comparison of Eq. A-12 to the Arps\(^5\) relations (in particular, the harmonic decline case) apply for the gas case as well.

These results prove that when pseudopressure and the material balance pseudotime are used to model gas flow, the data trend must decline along the harmonic stem on the Fetkovich\(^6\),\(^7\) (liquid flow) type curve. If \( \tau_a \) is correctly calculated, then a scaled log-log plot of \( q_g/(p_{pi}-p_{pwf}) \) versus \( \tau_a \) will overlay the \( q_{Dd} \) versus \( t_{Dd} \) trend on the Fetkovich\(^6\),\(^7\) type curve—exactly on the \( b=1 \) stem.
The "match point" relations are used to estimate the gas-in-place and formation properties are given by:

\[
b_{a,pss} = \frac{[q_{DD}]_{MP}}{\frac{q_g}{(p_{pi}-p_{pwf})}_{MP}} \quad \text{(B-21)}
\]

\[
m_a = \frac{1}{Gc_{ti}} = b_{a,pss} \left[ \frac{[t_{DD}]_{MP}}{[\bar{t}_a]_{MP}} \right] \quad \text{(B-22)}
\]

\[
G = \frac{1}{c_{ti}} \left[ \frac{[\bar{t}_a]_{MP}}{[t_{DD}]_{MP}} \right] \frac{[q_{DD}]_{MP}}{q_g} \left( p_{pi}-p_{pwf} \right)_{MP} \quad \text{(B-23)}
\]

where MP denotes a match point value.

REFERENCES used in APPENDIX B


