A Short Course on

Performance-Based Reservoir Characterization

Pseudosteady-State Performance:
This module provides a concise introduction to the solutions for pseudosteady-state performance in a radial reservoir system. The goals of this module are to introduce the concept of pseudosteady-state flow behavior, then provide the solutions for this flow regime—as well as schematic plots which illustrate pseudosteady-state performance.

- Introduction of the pseudosteady-state flow concept
- Illustrations of Pseudosteady-State Performance in Radial Flow Systems
Pseudosteady-State Performance:

- \( p_r - p_{wf} \) flow relation:

\[
p_r = p_{wf} + 141.2 \frac{qB\mu}{kh} \left[ \frac{r_e^2}{r_e^2 - r_w^2} \ln \left( \frac{r}{r_w} \right) - \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} + s \right]
\]

- Comment: Fundamental relation, explains log(r) behavior near the well during pseudosteady-state flow conditions.

- \( \bar{p} - p_{wf} \) flow relations:

  **Circular Reservoir Geometry:**

\[
\bar{p} = p_{wf} + 141.2 \frac{qB\mu}{kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right]
\]

  **General Reservoir Geometry:**

\[
\bar{p} = p_{wf} + 141.2 \frac{qB\mu}{kh} \left[ \frac{1}{2} \ln \left( \frac{A}{\gamma r_e^2 C_A} \right) + s \right]
\]

- Comment: Base relations for production engineering—simple pressure-rate calculations, IPR, etc.

- \( p(r,t) \) solution for pseudosteady-state flow conditions:

\[
p_r = p_i - 141.2 \frac{qB\mu}{kh} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} - \frac{3}{4} \right] - 5.615 \frac{qB}{V_p C_t} t
\]

- Comment: \( p(r,t) \) relation provides a mechanism for the prediction of pressure behavior at the well and in the reservoir as a function of time.
Illustrations of Pseudosteady-State Performance in Radial Flow Systems

(from Petroleum Engineering 412 Course Notes — 1997)
Fig. 5.2 Radial flow under semi steady state conditions.

Fig. 5.4 Radial flow under steady state conditions.

Figure 6.1 The Calculated History of the Steady-State Pressure Distribution (after Hurst, 1934).

Figure 6.3 Pressure Distribution in a Closed Circular Reservoir Produced at a Constant Pressure (after Hurst, 1934).

Figure 4 - Reservoir Pressure Distribution during Log Linear Rate Transient Flow Drawdown

Figure 7 - Reservoir Pressure Distribution during Constant Wellbore Pressure Transient Flow Drawdown

Figure 52 - Reservoir Pressure Distribution During Constant Rate Post-Transient Flow Drawdown, Homogeneous Reservoirs

Figure 57 - Reservoir Pressure Distribution During Constant Wellbore Pressure Post-Transient Flow Drawdown, Homogeneous Reservoirs

Figure 3.3 -- Variation of radius corresponding to the average reservoir pressure with time. (Note: These are numerical simulation results, not analytical results—hence the postulate that \( r(p)/r_e = 0.54928 \) ... should be both correct and unique.)

Derivation of the Pseudosteady-State Flow Relations for a Radial System

● Physical Considerations
● Material Balance Considerations
● Pseudosteady-State Solutions of the Radial Flow Diffusivity Equation
  ■ \( p_r - p_{wf} \) Formulation
  ■ \( \bar{p} - p_{wf} \) Formulation
  ■ \( \bar{p} - \bar{r} \) Concept (i.e., \( r(\bar{p}) \))
  ■ \( p(r,t) \) Solution for Pseudosteady-State Flow Conditions

(from Petroleum Engineering 412 Course Notes — 1997)
Pseudosteady-State Flow in a Radial System

Physical Considerations

The physical concept of pseudosteady-state is defined as the condition where the pressure at all points in the reservoir changes at the same rate. Mathematically, this condition is given by:

$$\frac{d}{dt}[P(r,t)] = \text{constant}$$

(1)

Physically, this condition is illustrated by

where,

- $P_{w1}$ = wellbore pressure at time $t_1$
- $P_r$ = average reservoir pressure at time $t_1$
- $P_e$ = external boundary pressure at time $t_1$

Objectives: (for pseudosteady-state flow conditions)

1. Derive a pressure change relation (i.e., $\frac{dP}{dt}$) using the material balance relation.
2. Derive a relation between the average reservoir pressure, $P_r$, and the wellbore flowing pressure, $P_{w1}$.
3. Derive a pressure, radius, time (i.e., $P(r,t)$) solution of the radial flow diffusivity equation.
Material Balance Considerations

Recalling the material balance relation for a slightly compressible liquid, we have

\[ \bar{P} = P_i - \frac{B}{NB_i c_t} Np \]  

or, noting that \( NB_i = \nu p \), we obtain

\[ \bar{P} = P_i - \frac{B}{\nu p c_t} Np \]  

For a cylindrical reservoir, we have

\[ \nu p = \phi h \pi (r_e^2 - r_w^2) \]  

Substituting Eq. 4 into Eq. 3 gives us

\[ \bar{P} = P_i - \frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} Np \]  

Recalling the definition of the cumulative production, \( Np \), we have

\[ Np = \int_0^t q(t) \, dt \]  

Therefore,

\[ \frac{d}{dt} Np = q \]  

Taking the derivative of Eq. 5 with respect to time

\[ \frac{d}{dt} \bar{P} = -\frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} q \]  

(Note: all derivations are in "cubic" units unless otherwise noted.)
Reudosteady-State Flow Solutions for the Radial Flow Diffusivity Equation

The governing partial differential equation for flow in porous media is called the "diffusivity" equation. The diffusivity equation for a "slightly compressible liquid" is given (without derivation) as

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \frac{\alpha \mu c_t}{k} \frac{dp}{dt} \tag{9}
\]

The significant assumptions made in Eq. 9 are:
- slightly compressible liquid (constant compressibility)
- constant fluid viscosity
- single-phase liquid flow
- gravity and capillary pressure are neglected
- constant permeability
- horizontal radial flow (no vertical flow)

If we assume that the flowrate, \( q \), is constant then \( dp/dt \) is also constant—hence \( dp/dt \) is constant as well. Assuming \( q \) is constant, then

\[
\frac{dp}{dt} = \frac{dp}{dt} = -\frac{B}{\phi \pi (r_e^2-r_w^2)c_t} \quad q = \text{constant} \tag{10}
\]

Substituting Eq. 10 into Eq. 9 (we note that partial derivatives are now expressed as ordinary derivatives), this gives

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \left[ \frac{\alpha \mu c_t}{k} \right] \left[ -\frac{B}{\phi \pi (r_e^2-r_w^2)c_t} q \right]
\]

or, reducing

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -\frac{q B m}{\pi k \pi (r_e^2-r_w^2)} \tag{11}
\]
Defining
\[ c = \frac{gBw}{\pi kh (r_e^2 - r_w^2)} \]  

(12)

Substituting Eq. 12 into Eq. 11 we have
\[ \frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -c \]  

(13)

Separating
\[ d \left[ r \frac{dp}{dr} \right] = -cr dr \]

Integrating (indefinite integration)
\[ \int d \left[ r \frac{dp}{dr} \right] = -c \int r \ dr \]

Completing
\[ r \frac{dp}{dr} = -c \frac{r^2}{2} + c_1 \]  

(14)

Multiplying through Eq. 14 by \( \frac{1}{r} \) gives us
\[ \frac{dp}{dr} = -c \frac{r}{2} + c_1 \frac{1}{r} \]  

(15)

For pseudosteady-state we assume a closed reservoir, that is
\[ \left[ \frac{dp}{dr} \right]_{r_e^2} = 0 \]

or
\[ \left[ \frac{dp}{dr} \right]_{r_e^2} = 0 = -c \frac{r_e^2}{2} + c_1 \frac{1}{r_e} \]

Solving for \( c_1 \), gives
\[ c_1 = -\frac{c}{2} r_e^2 \]  

(16)
Substituting Eq. 16 into Eq. 15 gives

$$\frac{dp}{dr} = \frac{c}{2} \left[ \frac{r_e^2}{r} - r \right]$$

Multiplying through Eq. 17 by dr gives us

$$dp = \frac{c}{2} \left[ \frac{r_e^2}{r} - r \right] dr$$

Integrating across the reservoir, we have

$$\int_{P_{wf}}^{P_r} dp = \frac{c}{2} \int_{rw}^{r} \left[ \frac{r_e^2}{r} - r \right] dr$$

Completing the integration

$$P_r - P_{wf} = \frac{c}{2} \left[ \frac{r_e^2 \ln(r)}{rw} \right]^r_{rw} - \frac{r_e^2}{2}$$

Or

$$P_r - P_{wf} = \frac{c}{2} \left[ \frac{r_e^2 \ln\left(\frac{r}{rw}\right)}{rw} - \frac{1}{2} \frac{(r^2 - r_w^2)}{rw} \right]$$

Recalling Eq. 12

$$C = \frac{q_8 \mu}{\pi kh (r_e^2 - r_w^2)}$$

Substituting Eq. 20 into Eq. 19, we obtain

$$P_r - P_{wf} = \frac{q_8 \mu}{2 \pi kh} \frac{1}{(r_e^2 - r_w^2)} \left[ \frac{r_e^2 \ln\left(\frac{r}{rw}\right)}{rw} - \frac{1}{2} \frac{(r^2 - r_w^2)}{rw} \right]$$

Expanding through with the $1/(r_e^2-r_w^2)$ term gives

$$P_r - P_{wf} = \frac{q_8 \mu}{2 \pi kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln\left(\frac{r}{rw}\right) - \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} \right]$$

Eq. 21 is our final result (in "Darcy" units).
Development of a $p-$Pwf Relation - Pseudostate-State Flow

In this section we develop the relationship between the average reservoir pressure, $\bar{p}$, and the wellbore flowing pressure, $p_{wf}$. The definition of the average reservoir pressure is given as

$$\bar{p} = \frac{\int r\,p_r\,dv}{\int r\,dv} \quad (22)$$

and for a cylindrical reservoir, we have

$$V = \phi h \pi (r^2 - r_w^2) \quad (23)$$
$$dv = \phi h \pi (2r)\,dr \quad (24)$$

Substituting Eq. (24) into Eq. (22) gives

$$\bar{p} = \frac{\phi h \pi}{\phi h \pi (r^2 - r_w^2)} \int r_p\, r\,dr$$

which reduces to

$$\bar{p} = \frac{Z}{(r^2 - r_w^2)} \int \bar{p}_r\, r\,dr \quad (25)$$

Solving Eq. (21) for $p_r$ gives us

$$p_r = p_{wf} + \frac{q_0 \mu}{2\pi k h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \frac{(r^2 - r_w^2)}{r_e^2 - r_w^2} \right] \quad (26)$$
Substituting Eq. 26 into Eq. 25 gives

\[
\bar{R} = \frac{z}{(r^2 - r_w^2)} \int_{r_w}^{r} \left[ R_{\text{int}} + \frac{qB}{2\pi \hbar k} \left( \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_e^2 - r_w^2)} \right) \right] r \, dr
\]

Separating

\[
\bar{R} = \frac{z}{(r^2 - r_w^2)} \int_{r_w}^{r} R_{\text{int}} \, dr
\]

\[
+ \frac{z}{(r^2 - r_w^2)} \frac{qB}{2\pi \hbar k} \frac{r_e^2}{(r_e^2 - r_w^2)} \int_{r_w}^{r} r \ln \left[ \frac{r}{r_w} \right] \, dr
\]

\[- \frac{z}{(r^2 - r_w^2)} \frac{qB}{2\pi \hbar k} \frac{1}{z(r_e^2 - r_w^2)} \int_{r_w}^{r} r^3 \, dr
\]

\[
+ \frac{z}{(r^2 - r_w^2)} \frac{qB}{2\pi \hbar k} \frac{r_w^2}{z(r_e^2 - r_w^2)} \int_{r_w}^{r} r \, dr
\]

Isolating terms and evaluating each integral, we have

\[
\int_{r_w}^{r} r \, dr = \frac{1}{2} (r^2 - r_w^2)
\]

\[
\int_{r_w}^{r} r^3 \, dr = \frac{1}{4} (r^4 - r_w^4)
\]

\[
\int_{r_w}^{r} r \ln \left[ \frac{r}{r_w} \right] \, dr = ?
\]

Obviously, the integral of the logarithm term will require a little work to resolve; we could simply look up the appropriate result in a suitable text—but deriving the required result will be enlightening.
starting with the fundamental form of the logarithm integral, we have

\[ \int x \ln(x/c) \, dx \]

... integration by parts \( uv = uv - \int v du \)

\[ u = \ln(x/c) \quad dv = x \, dx \]

\[ du = \frac{1}{x} \, dx \quad v = \frac{1}{2} x^2 \]

Then

\[ \int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{2} \int x \, dx \]

Reducing

\[ \int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{4} x^2 \]

Therefore

\[ \int_{r_w}^{r} r \ln\left[ \frac{r}{r_w} \right] \, dr = \left[ \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} r^2 \right]_{r_w}^{r} \]

\[ = \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} r^2 - \left[ \frac{1}{2} r_w^2 \ln\left[ \frac{r_w}{r_w} \right] - \frac{1}{4} r_w^2 \right] \]

or finally, we have

\[ \int_{r_w}^{r} r \ln\left[ \frac{r}{r_w} \right] \, dr = \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} (r^2 - r_w^2) \quad (31) \]
Substituting Eqs. 29-31 into Eq. 28 gives

\[
\bar{\rho}_r = \frac{2}{(r^2 - r_w^2)} \rho_{wf} \frac{1}{Z} (r^2 - r_w^2)
\]

\[
+ \frac{Z}{(r^2 - r_w^2)} \frac{\gamma \delta \mu}{2 \pi k h} \left( \frac{r_e^2}{(r^2 - r_e^2)} \right) \left[ \frac{1}{Z} \frac{\ln \left[ \frac{r}{r_w} \right]}{r_e^2 - r_w^2} \right] - \frac{1}{4} \left( \frac{r^2 - r_w^2}{r_w^2} \right)
\]

\[
- \frac{Z}{(r^2 - r_w^2)} \frac{\gamma \delta \mu}{2 \pi k h} \frac{1}{Z} \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right)
\]

\[
+ \frac{Z}{(r^2 - r_w^2)} \frac{\gamma \delta \mu}{2 \pi k h} \frac{1}{Z} \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right)
\]

Reducing

\[
\bar{\rho}_r = \rho_{wf} + \frac{\gamma \delta \mu}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{r_e^2}{(r_e^2 - r_w^2)} \left[ \frac{1}{Z} \frac{r^2 \ln \left[ \frac{r}{r_w} \right]}{r_e^2 - r_w^2} - \frac{1}{4} \left( \frac{r^2 - r_w^2}{r_w^2} \right) \right]
\]

\[- \frac{\gamma \delta \mu}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{1}{Z} \frac{1}{2} \frac{1}{2} \left( \frac{r^2 - r_w^2}{r^2 + r_w^2} \right) \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right)
\]

\[+ \frac{\gamma \delta \mu}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{1}{Z} \frac{1}{2} \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right) \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right)
\]

Collecting

\[
\bar{\rho}_r = \rho_{wf} + \frac{\gamma \delta \mu}{2 \pi k h} \frac{r_e^2}{(r_e^2 - r_w^2)} \left[ \frac{r^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \right]
\]

\[- \frac{\gamma \delta \mu}{2 \pi k h} \frac{(r^2 + r_w^2)}{4 (r_e^2 - r_w^2)} + \frac{\gamma \delta \mu}{2 \pi k h} \frac{r_w^2}{2 (r_e^2 - r_w^2)}
\]

or "finally"

\[
\bar{\rho}_r = \rho_{wf}
\]

\[+ \frac{\gamma \delta \mu}{2 \pi k h} \frac{r_e^2}{(r_e^2 - r_w^2)} \left[ \frac{r^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \right] - \frac{(r^2 + r_w^2)}{4 (r_e^2 - r_w^2)} + \frac{r_w^2}{2 (r_e^2 - r_w^2)}
\]

\[= (32)\]
Eq. 32 (which is given in "Darcy" units) is our fundamental linking relation between the wellbore and average reservoir pressures during pseudosteady-state flow. However, \( \bar{p} \) (the average reservoir pressure at a given radius \( r \)) is of little use - except as a rigorous "linking" relation for pressures in the reservoir.

In contrast, if we consider \( \bar{\rho}_e \) (i.e., \( \rho \) at \( r = r_e \)) we obtain the average reservoir pressure based on the entire reservoir volume. Such a result can be directly coupled with the material balance equation to develop a time-pressure relation for pseudosteady-state flow.

Evaluating Eq. 32 at \( r = r_e \) we have
\[
\bar{p} = \bar{\rho}_e = \rho_w + \frac{4\Phi_0}{2\pi kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \left( \frac{r_e^2}{K_w} \ln \frac{r_e^2}{K_w} - \frac{1}{2} \right) - \frac{(r_e^2 + r_w^2)}{4(r_e^2 - r_w^2)} + \frac{r_w^2}{2(r_e^2 - r_w^2)} \right]
\]

Assuming that \( r_e \gg r_w \), then
\[
\frac{r_e^2}{(r_e^2 - r_w^2)} \approx 1; \quad \frac{(r_e^2 + r_w^2)}{(r_e^2 - r_w^2)} \approx 1; \quad \frac{r_w^2}{(r_e^2 - r_w^2)} \approx 0
\]

Substituting these expressions into Eq. 33, we obtain
\[
\bar{p} = \rho_w + \frac{4\Phi_0}{2\pi kh} \left[ \ln \frac{r_e^2}{K_w} - \frac{1}{2} - \frac{1}{4} \right]
\]
or
\[
\bar{p} = \rho_w + \frac{4\Phi_0}{2\pi kh} \left[ \ln \frac{r_e^2}{K_w} - \frac{3}{4} \right]
\]
Summarizing our results so far (using generalized units systems)

Pressure at any Radius:

\[ P_r = P_{wf} + \frac{q B_w}{c_r k h} \left[ \frac{r_e^2}{r_e^2 - r_w^2} \ln \left( \frac{r_e}{r_w} \right) - \frac{(r_e^2 - r_w^2)}{2} \left( \frac{r_e^2 + r_w^2}{r_e^2 - r_w^2} \right) \right] \]  

(35)

Average Reservoir Pressure at any Radius:

\[ \bar{P}_r = P_{wf} \]

\[ + \frac{q B_w}{c_r k h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \frac{r_e^2 + r_w^2}{4(r_e^2 - r_w^2)} + \frac{r_w^2}{2(r_e^2 - r_w^2)} \right] \]  

(36)

Average Reservoir Pressure at Re (volumetric average Pressure):

\[ \bar{P} = P_{wf} + \frac{q B_w}{c_r k h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] \]  

(37)

For a general reservoir geometry, Eq. 38 becomes

\[ \bar{P} = P_{wf} + \frac{q B_w}{c_r k h} \left[ \frac{1}{2} \ln \left( \frac{4 \frac{A}{r_e^2}}{c_A} \right) \right] \]  

(38)

where

\[ \gamma = 0.5772159 \ldots \quad \text{Euler's Constant} \]

\[ c_A = \text{Dietz "shape factor" (e.g. } c_A = 31.62 \text{ for circular reservoir)} \]

Table of Units Conversion Factors

<table>
<thead>
<tr>
<th>Factor</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_r)</td>
<td>(2\pi)</td>
<td>(2\pi \times 1.127 \times 10^{-3})] or (7.081 \times 10^{-3})</td>
<td>(2\pi \times 8.527 \times 10^{-5}) or (5.358 \times 10^{-4})</td>
</tr>
</tbody>
</table>
An interesting (and possibly useful) result is the concept of \( \bar{r} \), which would be the location of the average reservoir pressure, \( \bar{p} \). This development is only rigorously valid for a vertical well centered in a bounded circular reservoir. A graphical illustration of this concept is shown below.

Mathematically, \( \bar{r} \) is defined by equating the \( p_r \) relation (Eq. 55) with the average reservoir pressure identity, \( \bar{p} \) (Eq. 37). Equating Eqns. 55 and 37 gives

\[
\frac{r_e^2}{(r_e^2 - r_w^2)} \frac{\ln \left[ \frac{\bar{r}}{r_w} \right]}{r_w} - \frac{(r_e^2 - r_w^2)}{2(r_e^2 - r_w^2)} = \frac{\ln \left[ \frac{r_e}{r_w} \right]}{r_w} - \frac{3}{4}
\]

We now assume \( \frac{r_e^2}{(r_e^2 - r_w^2)} \approx 1 \) and \( \frac{r_w^2}{(r_e^2 - r_w^2)} \approx 0 \) gives

\[
0 = \frac{\ln \left[ \frac{\bar{r}}{r_w} \right]}{r_w} - \ln \left[ \frac{r_e}{r_w} \right] - \frac{r_e^2}{2(r_e^2 - r_w^2)} + \frac{3}{4}
\]
Assuming that \( r_e \gg r_w \) (i.e., \((r_e^2 - r_w^2) \gg r_e^2\)) and rearranging, we have

\[
\ln \left[ \frac{r_e}{r_w} \right] - \frac{r_e^2}{2r_w^2} + \frac{3}{4} = 0
\]  

(39)

Defining a dimensionless radius, \( \bar{r}_D \), we obtain

\[
\bar{r}_D = \frac{r_e}{r_w}
\]  

(40)

Substituting Eq. 40 into Eq. 39 gives us

\[
\ln (\bar{r}_D) - \frac{1}{2} \bar{r}_D^2 + \frac{3}{4} = 0
\]  

(41)

Solving Eq. 41 for \( \bar{r}_D \), we obtain

\[
\bar{r}_D = 0.54928 \ldots \quad \text{(or } \bar{r} = 0.54928 \ldots r_e \text{)}
\]  

(42)

Development of a \( \rho(r,t) \) Relation for Pseudosteady-State Flow

Our last objective is to develop a \( \rho(r,t) \) relation for pseudosteady-state flow in a bounded circular reservoir. Recalling the material balance relation (Eq. 5) we have

\[
\bar{\rho} = \rho_i - \frac{8}{\phi h \pi (r_e^2 - r_w^2) c_e^2} Np
\]  

(5)

For a constant flow rate, \( q \), we have

\[
Np = \int_0^t q(t) \, dt = qt
\]  

(43)

Substituting Eq. 43 into Eq. 5,

\[
\bar{\rho} = \rho_i - \frac{4B}{\phi h \pi (r_e^2 - r_w^2) c_e^2} t \quad \text{(Darcy units)}
\]  

(44)
Recalling the average reservoir pressure identity for a well centered in a bounded circular reservoir, we have

\[ \bar{p} = p_{wf} + \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] \quad \text{(Darcy Units)} \quad (45) \]

Substituting Eq. 45 into Eq. 44 gives

\[ p_{wf} + \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] = p_i - \frac{98}{\phi h \pi (r_e^2 - r_w^2) \Delta t} t \]

Rearranging

\[ p_i - p_{wf} = \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] + \frac{98}{\phi h \pi (r_e^2 - r_w^2) \Delta t} t \quad (46) \]

or

\[ p_i - p_{wf} = \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} \right] + \frac{98}{V_{pc} \Delta t} t \quad (47) \]

Recalling the wellbore-reservoir pressure relation (Eq. 26), we have \( \text{(upon slight rearranging)} \)

\[ p_r - p_{wf} = \frac{98u}{2\pi kh} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} \right] \quad (48) \]

Subtracting Eq. 48 from Eq. 47 and solving for \( p_r \) gives us

\[ p_r = p_i - \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} - \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r_e}{r_w} \right) + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} \right] \]

\[ - \frac{98}{V_{pc} \Delta t} t \quad (49) \]

Assuming that \( r_e >> r_w \) (i.e., \( r_e^2 - r_w^2 \approx r_e^2 \)) gives us

\[ p_r = p_i - \frac{98u}{2\pi kh} \left[ \ln \left( \frac{r_e}{r_w} \right) + \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} - \frac{3}{4} \right] - \frac{98}{V_{pc} \Delta t} t \quad (50) \]
Summarizing, we have the following relations in Darcy units:

\[
\frac{P_r - P_i}{\frac{q B u}{2 \pi k h}} = \frac{ln \left[ \frac{r_e}{r_w} \right]}{4} - \frac{r_e^2}{2 (r_e^2 - r_w^2)} \times \frac{ln \left[ \frac{r}{r_w} \right]}{4} + \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{q B}{V_p C t} \quad (49)
\]

and

\[
\frac{P_r - P_i}{\frac{q B u}{2 \pi k h}} = \frac{ln \left[ \frac{r_e^2}{r_w} \right]}{2} + \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{3}{4} - \frac{q B}{V_p C t} \quad (50)
\]

In Field Units, we have

\[
\frac{P_r - P_i}{\frac{141.2 q B u}{k h}} = \frac{ln \left[ \frac{r_e}{r_w} \right]}{4} - \frac{r_e^2}{2 (r_e^2 - r_w^2)} \times \frac{ln \left[ \frac{r}{r_w} \right]}{4} + \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right)
\]

\[
- 5.615 \frac{q B}{V_p C t} \quad (t \text{ in days, } V_p \text{ in ft}^3) \quad (51)
\]

and

\[
\frac{P_r - P_i}{\frac{141.2 q B u}{k h}} = \frac{ln \left[ \frac{r_e^2}{r_w} \right]}{2} + \frac{1}{2} \left( \frac{r^2 - r_w^2}{r_e^2 - r_w^2} \right) - \frac{3}{4} - 5.615 \frac{q B}{V_p C t} \quad (52)
\]

For \( t \) in hours we use \( 5.615/24 = 0.23395 \).

Finally, for conditions at the well, we have

Darcy Units:

\[
\frac{P_{w f} - P_i}{\frac{q B u}{2 \pi k h}} = \frac{ln \left[ \frac{r_e}{r_w} \right]}{4} - \frac{q B}{V_p C t} \quad (53)
\]

Field Units:

\[
\frac{P_{w f} - P_i}{\frac{141.2 q B u}{k h}} = \frac{ln \left[ \frac{r_e^2}{r_w} \right]}{4} - 5.615 \frac{q B}{V_p C t} \quad (54)
\]

\[
\frac{P_{w f} - P_i}{\frac{141.2 q B u}{k h}} = \frac{ln \left[ \frac{r_e^2}{r_w} \right]}{4} - 0.23395 \frac{q B}{V_p C t} \quad (55)
\]

Recall that the pore volume, \( V_p \), is given by

\[
V_p = \phi h A (r_e^2 - r_w^2) = \phi h A \quad (4)
\]