1. (20 pts) Derivation of a Rate Relation for Pseudosteady-State Flow (constant $dp_{wf}/dt$)

**Given:**
You are given the following "liquid" relations for material balance and pseudosteady-state radial flow. Assume Darcy units (no need for conversion factors).

**Material Balance Equation:** (valid for all times)
\[
\bar{p} = p_i - \frac{B}{V_p c_t} N_p
\]

**Pseudosteady-State Flow Relation:** (only valid for pseudosteady-state flow)
\[
\bar{p} = p_{wf} + q b_{pss}
\]

Recall that the definition of the cumulative production, $N_p$, is given by:
\[
N_p = \int_0^t q(t) \, dt \text{ or, a more useful result is given by, } q = \frac{d}{dt}(N_p).
\]

**Required:**
You are to derive the flowrate relation for the case of $dp_{wf}/dt=$constant. This is accomplished by coupling the material balance equation and the pseudosteady-state flow equation to yield a first-order ordinary differential equation--then solving this differential equation as appropriate.

**Hints:**
1. At $t=0$, $c = q_i = \frac{1}{b_{pss}} (p_i - p_{wf})$.
2. The resulting differential equation is of the form:
\[
\frac{dy}{dx} + p(x)y = r(x)
\]
Which has the solution:
\[
y = \exp \left[ -\int p(x) \, dx \right] \int r(x) \exp \left[ \int p(x) \, dx \right] \, dx + c \exp \left[ -\int p(x) \, dx \right]
\]
where:
\[
\int p(x) \, dx \text{ is called the "integrating factor."}
\]

**Result:**
\[
q = \frac{1}{b_{pss}} (p_i - p_{wf}) \exp \left[ -\frac{B}{V_p c_t b_{pss}} t \right] + \frac{1}{b_{pss}} \frac{d}{dt} \frac{V_p c_t}{B} \left[ 1 - \exp \left[ \frac{B}{V_p c_t b_{pss}} t \right] \right]
\]

**Alternate Form:**
\[
q = \frac{1}{b_{pss}} (p_i - p_{wf}) \exp \left[ -\frac{B}{V_p c_t b_{pss}} t \right]
\]
Summary of Solutions

\[ q = e^{\left[ -\frac{B}{V_p C_t \text{ bps}^3} \right]} \left( \int \frac{V_p C_t}{B} \left[ 1 - e^{\left[ -\frac{B}{V_p C_t \text{ bps}^3} \right]} \right] \right. \\
+ \left. \frac{1}{\text{ bps}^3} \left( P_i - P_{wf} \right) e^{\left[ -\frac{B}{V_p C_t \text{ bps}^3} \right]} \right) \\
\]

which reduces to

\[ q = \left[ \frac{\text{ dP_{wf}} \ V_p C_t}{\text{ bps}^3} + \frac{1}{\text{ bps}^3} \left( P_i - P_{wf} \right) \right] e^{\left[ -\frac{B}{V_p C_t \text{ bps}^3} \right]} \\
- \frac{\text{ dP_{wf}} \ V_p C_t}{B} \]
MBE

\[ P = \rho_i - \frac{B}{\rho p_{ct}} \, \rho p \]  \hspace{1cm} (1)

Pseudosteady-state Flow Eq.

\[ \bar{P} = \rho_{st} + q \, \text{bps} \]  \hspace{1cm} (2)

Differentiation of Eq. 1 and Eq. 2 gives

\[ \frac{d\bar{P}}{dt} = -\frac{B}{\rho p_{ct}} \, q \]  \hspace{1cm} (3)

\[ \frac{d\bar{P}}{dt} = \frac{d\rho_{st}}{dt} + bps \, \frac{dq}{dt} \]  \hspace{1cm} (4)

Setting Eqs. 3 and 4 equal to each other, we obtain

\[ -\frac{B}{\rho p_{ct}} \, q = \frac{d\rho_{st}}{dt} + bps \, \frac{dq}{dt} \]  \hspace{1cm} (5)

Defining

\[ c = \frac{d\rho_{st}}{dt} = \text{constant} \]

Equation 5 becomes

\[ -\frac{B}{\rho p_{ct}} \, q = c + bps \, \frac{dq}{dt} \]  \hspace{1cm} (6)

Rearranging

\[ bps \, \frac{dq}{dt} + \frac{B}{\rho p_{ct}} \, q = c \]  \hspace{1cm} (7)

or

\[ \frac{dq}{dt} + \frac{B}{b ps} \, \frac{1}{\rho p_{ct}} \, q = -\frac{c}{bps} \]  \hspace{1cm} (8)
where Eq. 8 is of the form:

$$\frac{dy}{dx} + p(x) y = r(x) \tag{9}$$

where

$$y = q, \ x = t, \ p(x) = \frac{B}{V p c t \ b p s s}, \ r(x) = -\frac{C}{b p s s}$$

The solution of Eq. 9 is given by

$$y = e^{-\int p(x) \, dx} \int e^{\int p(x) \, dx} r(x) \, dx + c' e^{-\int p(x) \, dx} \tag{10}$$

The $\int p(x) \, dx$ is known as the "integrating factor."

Computing the integrating factor, we have

$$\int_{0}^{t} \left[ \frac{B}{V p c t \ b p s s} \right] dt = \frac{B}{V p c t \ b p s s} \int_{0}^{t} dt$$

or

$$\int p(x) \, dx = \frac{B}{V p c t \ b p s s} t = (B t) \tag{11}$$

The integral $\int e^{\int p(x) \, dx} r(x) \, dx$

$$\int e^{\int p(x) \, dx} r(x) \, dx = \int_{0}^{t} e^{B t} \left[ \frac{-C}{b p s s} \right] dt$$

$$= -\frac{C}{b p s s} \int_{0}^{t} e^{B t} \, dt = \frac{C}{b p s s} \frac{1}{B} (1 - e^{B t})$$
or substituting $B = \frac{B}{v_{pc} \cdot b_{ps}}$, we have

\[ \int e^{\int p(x)dx} = \int e^{\int v(x)dx} = \frac{c}{b_{ps}} \times \frac{v_{pc} \cdot b_{ps}}{B} \left[ 1 - e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \right] \]

(12)

And

\[ -\int e^{\int p(x)dx} = e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \]

(13)

Therefore

\[ q = e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot \frac{dt}{dt}} \times \frac{v_{pc} \cdot b_{ps}}{B} \left[ 1 - e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \right] + c \cdot e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \]

(14)

**Erroneous Approach**: As given, $c' = \frac{1}{b_{ps}} (q_i - P_{in})$

\[ q = e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot \frac{dt}{dt}} \times \frac{v_{pc} \cdot b_{ps}}{B} \left[ 1 - e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \right] + \frac{1}{b_{ps}} (q_i - P_{in}) \left[ e^{\frac{-8}{v_{pc} \cdot b_{ps}} \cdot t} \right] \]

(15)

Which is erroneous, but ok...
Correct Approach: \[ q_i = q(t=0) = \frac{1}{b_{ps}} (p_i - p_{mf}) \]

Solving Eq. 14 for \( q(t=0) \) gives

\[
\frac{1}{b_{ps}} (p_i - p_{mf}) = \exp[\sigma] \frac{d p_{mf}}{d t} \frac{v_{pc} t}{B} \left[ 1 - \exp[\sigma] \right] \\
+ c' \exp[\sigma]
\]

or \[ c' = \frac{1}{b_{ps}} (p_i - p_{mf}) \ldots \text{which gives} \]

Final Reduction

for convenience, substitute \[ B = \frac{B}{v_{pc} t} \frac{1}{b_{ps}} \] into Eq. 15

\[
q = \exp[-Bt] \frac{d p_{mf}}{d t} \frac{v_{pc} t}{B} \left[ 1 - \exp[Bt] \right] \\
+ \frac{1}{b_{ps}} (p_i - p_{mf}) \exp[-Bt] \\
= \frac{d p_{mf}}{d t} \frac{v_{pc} t}{B} \left[ \exp[-Bt] - \exp[-Bt] \exp[Bt] \right] \\
+ \frac{1}{b_{ps}} (p_i - p_{mf}) \exp[-Bt]
\]
Continuing

\[ q = \left[ \frac{dpwf}{dt} \frac{V_p C_t}{B} + \frac{1}{b_{pss}} (p_i - p_{wf}) \right] \exp \left[ -Bt \right] \]

\[ - \frac{dpwf}{dt} \frac{V_p C_t}{B} \]

Substituting \( B = \frac{B}{V_p C_t b_{pss}} \), we obtain

\[ q = \left[ \frac{dpwf}{dt} \frac{V_p C_t}{B} + \frac{1}{b_{pss}} (p_i - p_{wf}) \right] \exp \left[ -B \frac{1}{V_p C_t b_{pss}} t \right] \]

\[- \frac{dpwf}{dt} \frac{V_p C_t}{B} \]

(16)