Variable-Rate Reservoir Limits Testing
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ABSTRACT

This paper presents a new method of estimating drainage area size and shape from production data (bottom-hole pressures and flowrates). The method is a rigorously derived approximation for variable-rate flow in a closed reservoir. This method requires a graph of Δp/qm vs. the superposition plotting function (which is easily calculated by hand). The slope and intercept of the graph are used to provide the desired estimates of drainage area size and shape.

The method that we propose is an approximation, however it has been proved to be very accurate for the constant rate, constant pressure, exponential rate, log-logistic rate, hyperbolic rate, sinusoidal rate, and discrete rate cases. The method also gives acceptable results for square wave rate and random rate cases.

The new method is derived for the time after the initial pressure transient has reached the outer boundary. The changes in flowrate cause additional transients, but we assume that this effect is negligible when compared to the influence of the outer boundary. Therefore, if the change in flowrate does not dominate the influence of the outer boundary, the new method should give acceptable results. Also, at present, this method is only derived for single-phase flow of a liquid of small and constant compressibility.

INTRODUCTION

The purpose of this paper is to present a simple, but accurate method of predicting reservoir drainage area size and shape from variable-rate production data. Previous works have dealt with constant or cyclically constant rate and constant bottom-hole pressure production. A summary of these methods is shown graphically in Figure 1. Earlougher defined the cyclically constant or square wave rate case in Figure 2. Rather than focus on a particular rate scheme, we develop a general variable-rate approximation that should give accurate results for typical production situations.

Without a variable-rate solution we would have use the more tedious material balance methods that require average reservoir pressures to estimate reservoir pore volume. This would require the well to be shut-in, which results in lost revenue. However, with the new method, the reservoir pore volume and shape can be estimated directly from the production data, without shutting-in the well.

The problem of variable-rate flow in bounded systems is limited in the literature to the work by Earlougher and the "stabilized flow" methods (which use average reservoir pressures). Though both approaches give acceptable results for their specific application, Earlougher's case is not realistic and the "stabilized flow" methods again require the well to be shut-in for average reservoir pressure determinations. This suggests the need for a general solution for variable-rate flow in a bounded reservoir.

In the "Description of the New Method" section we will present the general variable-rate solution and the reservoir characteristics which can be derived from it. Also, we will verify the general variable-rate equation (Eq.(2)) using analytical and finite-difference simulation. Then a step-by-step procedure for applying our method and a complete example will be shown in the "Method of Application" section. Finally, we will present the derivation of the exact solution for variable-rate flow in a bounded circular reservoir and the approximate solution for variable-rate flow in any shape reservoir in the Appendix of this report.

DESCRIPTION OF THE NEW METHOD

In the Appendix, we derive an equation describing the behavior of the bottom-hole pressure, \( P_{bh} \), as a function of time for a single well producing in a bounded reservoir with a
variable rate. This solution assumes the following:

Radial flow into the well over the net pay thickness;
Homogeneous and isotropic porous medium;
Uniform net pay thickness;
Porosity and permeability constant (independent of pressure);
Fluid of small end constant compressibility;
Constant fluid viscosity;
Small pressure gradients;
Negligible gravity forces; and
Any rate schedule.

As shown in the Appendix, the complete solution for a circular drainage area is

\[
\Delta p = P_1 - P_w = 141.2 \frac{B u}{k h} \left[ q_m \ln \left( \frac{r}{r_w} \right) - \frac{3}{4} \right] + \frac{r_w^2}{2 \pi} \left( S - S_e \right) + 2 \pi \left( 0.0002637 \right) \frac{k}{\phi \mu e C_A} \frac{Q_m}{r_w}
\]

\[
- 2 \sum_{j=1}^{m} \left( \frac{q_j - q_{j-1}}{r_j} \frac{r_m}{r_{j-1}} \right) \left( \frac{1}{X_j} \right) \left( \frac{1}{X_{j-1}} \right) \exp \left( -\frac{X_j^2}{0.0002637} \frac{k}{\phi \mu e C_A} (t - t_{j-1}) \right)
\]

If we assume that the infinite series in eq. (1) is negligible then we call this "stabilized flow". Though each rate change introduces a new transient that keeps the well from reaching true pseudosteady-state flow, this does not keep the well from exhibiting a pseudosteady-state-like flow regime that we call "stabilized flow". Also, we use the term "start of stabilized flow" for the time to reach pseudosteady state for a constant rate. This gives us the lower bound for the start of stabilized flow. However, a large rate change will cause the infinite series to dominate eq. (1) and stabilized flow will not exist, though after some time the infinite series will become negligible and stabilized flow will again exist. What this means is that each new transient will eventually die off and stabilized flow will dominate, so there will be alternating periods of both transient and stabilized flow. Therefore, if we neglect the infinite series in eq. (1) (i.e., assume stabilized flow) and generalize the drainage area shape by using the Ditlev's shape factor, \( C_s \), the following variable-rate approximation can be written for stabilized flow:

\[
\Delta p = 70.6 \frac{B u}{k h} \ln \left( \frac{4A}{e C_A} \right) - 0.2339 \frac{B}{\phi h c T} \bar{\tilde{e}},
\]

where

\[
\bar{\tilde{e}} = \frac{q_m^{j-1} q_j (t_j - t_{j-1})}{q_m}
\]

Eq. 2 suggests that a graph of \( \Delta p/q_m \) vs. \( \bar{\tilde{e}} \) will be a straight line of slope

\[
p_{VT} = 0.2339 \frac{B}{\phi h c T} \bar{\tilde{e}} \quad \text{.................(4)}
\]

and intercept

\[
b_{VT} = 70.6 \frac{B u}{k h} \ln \left( \frac{4A}{e C_A} \right) \bar{\tilde{e}} \quad \text{.................(5)}
\]

will result. Since we know that transient and stabilized flow alternate for each rate change, we can suggest that for a small rate change stabilized flow will dominate. However for large rate changes the infinite series in Eq. (1) (i.e., transient flow) dominates and Eq. (2) is not valid. This means that transient data will not lie on the straight line predicted by Eq. (2). Therefore, these outlying points should not be used when obtaining the slope and intercept of the graph.

Because Eq. (2) is an approximation and we know that stabilized flow will not be achieved for all rate changes, we must investigate empirically the accuracy and applicability of Eq. (2). Our investigation used the complete solution (Eq. (1)) and a finite-difference reservoir simulator. The finite-difference formulation was fully implicit. The simulator modeled radial, single-phase flow of a single-phase liquid of small and constant compressibility. The simulator was verified by comparison with the analytical solution for transient and pseudosteady-state flow in a bounded circular reservoir at a constant producing rate.

Earlougher developed a method to determine reservoir drainage area for wells with rate histories that are cyclically constant or square wave, as shown in Figure 2. He proved that the slope of a \( p_{VT} \) vs. time graph provided an estimate of pore volume. However, he did not give an explicit method for estimating reservoir drainage shape for this case. Earlougher used superposition to generate this data, therefore, to verify our method for the situation he studied we reproduced his examples using a similar approach. The main difference between our examples and Earlougher's is that he simulated a well centered in a square reservoir and we simulated a well in the center of a circular reservoir. This difference does not distort the interpretation or comparison of results.

Table 2 shows the reservoir data for the cases comparing results from the new method with results using Earlougher's method. Figure 3 is a graph of rate vs. time for the three cases that were simulated using Eq. (1). The first case is the square wave rate case, which serves as the basis for the Earlougher analysis technique. The second case is a sinusoidal rate case and though Earlougher did not model this exact case, he suggested that his method was applicable to seasonal rate changes. This case shows the rate demand schedule for approximately one year. The third and final case in this comparison is for a completely random rate schedule from a single well.
Earlougher gave a completely random rate case in his work, though it could not be realistically analyzed using his method.

The conventional $p_{wf}$ vs. time plot is shown in Figure 4 and we see that only the square wave rate case can realistically be analyzed using Earlougher’s method. Figure 5 shows our method with $\Delta p/\Delta t$ vs. $t$. Note that the square wave rate case gives two straight lines of the same slope and two intercepts. Since there is no theory for interpreting these intercepts we simply took an arithmetic average. The sinusoidal and random rate cases essentially line up on the same trend, which is what we expect since all results are for the same reservoir. Table 3 shows the relative error in slope and intercept for each of these three cases. The slope is directly proportional to pore volume while the intercept is exponentially proportional to the shape factor, $C_a$. Slopes and intercepts were determined from the best straight-line in a least squares sense passing through the linear portion of the data in each case. Table 3 shows our method is more accurate than Earlougher’s for determining slope and intercept, and thus for determining drainage area size and shape from variable-rate production data.

We next investigated the effect of the type of rate decline on the results obtained using our new method. We modeled six cases with our finite-difference simulator. Table 4 gives the reservoir properties used in the simulations. The types of rate decline schedules were constant rate, logarithmic decline, exponential decline, hyperbolic decline, and the pressure buildup (a step) decline, and the rate schedule resulting from constant bottom-hole pressure production. Figure 6 shows rate vs. time for each case while Figure 7 shows $p_{wf}$ vs. time for each of these cases. Only the constant rate cases can be analyzed using Figure 7. Figure 8 shows $\Delta p/\Delta t$ vs. $t$. Note that virtually all of the points fall on the same line, and again this is expected because all cases are for the same reservoir data. Error analysis results are given in Table 5. This table shows that the new method had small error in all cases, which suggests that the introduction of transients due to changes in rate has little effect on the stabilized flow solution (Eq. (2)).

APPLICATION OF METHOD

We suggest the following procedure to obtain the best results from our variable-rate test analysis technique.

1. Measure or estimate both bottom-hole pressures and flowrates as functions of time.
2. Calculate $\Delta p/\Delta t$ and the plotting function, $\bar{t}$ (Eq. (3)).
3. Plot $\Delta p/\Delta t$ vs. $\bar{t}$ on Cartesian coordinate graph paper.
4. Determine the slope, $m_{tr}$, and the intercept, $b_{tr}$, of the best straight line on the graph. A least-squares fit gives more accurate results than "eyeball" fits.
5. Estimate reservoir size from the slope, $m_{tr}$, of the graph using Eq. (4).
6. Estimate the reservoir shape factor, $C_a$, from the intercept, $b_{tr}$, of the graph and Eq. (5). Skin factor and reservoir permeability must be known (from a pressure buildup test, for example) to make this estimate.

Table 1 is a summary of plotting and analysis techniques for reservoir limits tests. Figure 1 shows the type of graph for each case. In each case, the slope provides an estimate of drainage area size, $A$, and the intercept provides an estimate of shape factor, $C_a$. A numerical value of shape factor allows us to estimate drainage area shape by using a table of shape factors and shapes, such as the one presented by Dietz. The new method for analyzing variable-rate production includes the other methods as special cases, similar results will be obtained whether the new general method or the older methods for special cases are used.

EXAMPLE—Random Rate Production

In this example we simulated a random rate decline in a bounded circular reservoir with the analytical solution. This is the same case shown earlier in Figures 3, 4 and 5. Also, the pertinent reservoir data is given in Table 2. The production data is given in Table 6.

Using least squares on the straight-line portion of Figure 9, the following slope and intercept were obtained,

\[
m_{tr} = 1.269 \times 10^{-5} \text{ psi/STB/hr}
\]
\[
b_{tr} = 2.542 \times 10^{-4} \text{ psi/STB/D}
\]

The reservoir drainage area, $A$, is estimated from Eq. 4,

\[
m_{tr} = \frac{B}{\phi h c L A}
\]

therefore,

\[
A = \frac{B}{\phi h c m_{tr} L (1.0)} = \frac{2.039 \times 10^{-4}}{(0.15)(100)(5 \times 10^{-5})(1.269 \times 10^{-5})}
\]

\[
= 2.458 \times 10^6 \text{ ft}^2
\]

\[
= 5643 \text{ acres}
\]
The reservoir shape factor, $C_A$, is estimated from Eq. 5,

$$b_{vr} = \frac{70.6}{k h} \frac{B_u}{\ln} \frac{4A}{e^{C_A^2}}$$

therefore,

$$C_A = \frac{4 A}{e^{C_A^2}} \frac{b_{vr} k h}{\ln \left(\frac{70.6 B_u}{e^{C_A^2}}\right)}$$

$$= \frac{4 \left(2.458 \times 10^8\right)}{(1.781 \times 0.5)^2} \exp\left(\frac{2.542 \times 10^{-1}}{100 \times 100}\right) = 70.6 \times (1.0 \times 2.0)$$

$$C_A = 33.46$$

Our estimate for the reservoir shape factor, $C_A$, is slightly high, this is because if we exponentiate a small error in the intercept, $b_{vr}$, it becomes a large error in the reservoir shape factor, $C_A$. Therefore, a better comparison would be that of the input and calculated intercepts. This is shown in the summary table below.

**Summary of Results**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Input</th>
<th>This Work</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>5760 acres</td>
<td>5663</td>
<td>-2.30%</td>
</tr>
<tr>
<td>$C_A$</td>
<td>31.62</td>
<td>33.46</td>
<td>5.83%</td>
</tr>
<tr>
<td>$b_{vr}$</td>
<td>2.553 x 10^{-1}</td>
<td>2.542 x 10^{-1}</td>
<td>4.27%</td>
</tr>
</tbody>
</table>

**SUMMARY AND CONCLUSIONS**

We have developed a method of determining drainage area size and shape for wells with variable-rate production histories. This method is an approximation that gives very good results for cases where the rate changes are small. However, large rate changes only dominate the stabilized flow solution (Eq. (2)) until the transient introduced by that rate change becomes negligible. Therefore, the new method should be considered accurate for any production period so long as the reservoir outer boundary is being felt by the pressure response.

The condition of the reservoir can also prescribe the applicability of the new method in that if any of the assumptions concerning the reservoir are violated, the method may not work. Specifically, if water influx, solution gas evolution, multi-phase flow, or reservoir heterogeneities exist then this method should not be used.

**NOMENCLATURE**

$A$ = Reservoir drainage area, $\text{ft}^2$ ($\text{m}^2$)

$b_{cp}$ = $\frac{70.6}{k h} \frac{B_u}{\ln} \frac{4A}{e^{C_A^2}}$

intercept of log $q_w$ vs. $t$ graph for

constant pressure case, STB/Day
(standard $\text{m}^3$/d)

$b_{cr}$ = $P_i - 70.6 \frac{B_u}{k h} \ln \frac{4A}{e^{C_A^2}}$

intercept of $P_w$ vs. $t$ graph for constant rate case, psi (kPa)

$b_{vr}$ = $70.6 \frac{B_u}{k h} \ln \frac{4A}{e^{C_A^2}}$, intercept of $\Delta P/q_w$ vs. $t$ graph for variable-rate case, psi/STB/D (kPa/standard $\text{m}^3$/d)

$B$ = Liquid formation volume factor, RB/STB
(res $\text{m}^3$/standard $\text{m}^3$)

$C_A$ = Reservoir shape factor, dimensionless

$C_f$ = Pore space compressibility, psi$^{-1}$ (kPa$^{-1}$)

$C_g$ = Gas compressibility, psi$^{-1}$ (kPa$^{-1}$)

$C_o$ = Oil compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c_t$ = $c_s + c_o c_w + c_f + c_o$ total compressibility, psi$^{-1}$ (kPa$^{-1}$)

$c_w$ = Water compressibility, psi$^{-1}$ (kPa$^{-1}$)

$h$ = Net pay thickness, ft (m)

$J_o$ = Zero order Bessel function of the first kind

$J_1$ = First order Bessel function of the first kind

$k$ = Effective formation permeability, md

$m_{cp}$ = $-0.001439 \frac{k}{\phi u c_t A \ln \frac{4A}{e^{C_A^2}}}$

slope of log $q_w$ vs. $t$ Graph for constant pressure case, cycle/hour (cycle/hr)

$m_{cr}$ = $-0.2339 \frac{g_B}{\phi u c_t A}$, slope of $P_w$ vs. $t$ graph for constant rate case, psi/hour (kPa/hr)

$m_{aw}$ = $-0.2339 \frac{g_B}{\phi u c_t A}$, slope of $P_f$ vs. $t$ graph for square wave rate case, psi/hour (kPa/hr)

$m_{wr}$ = $0.2339 \frac{B}{\phi u c_t A}$, slope of $\Delta P/q_w$ vs. $t$ graph, psi/STB/D/hr (kPa/standard $\text{m}^3$/d/hr)

$\Delta P$ = $P_i - P_w$, pressure drop, psi (kPa)
\[ \bar{P} = \text{Average reservoir pressure for outer boundary of reservoir, psi (kPa)} \]

\[ P_D = \frac{kh}{141.2 \mu q} (P_i - P_w), \text{ dimensionless pressure} \]

\[ P_i = \text{Original formation pressure, psi (kPa)} \]

\[ P_w = \text{Flowing bottom-hole pressure, psi (kPa)} \]

\[ P_r = \text{Pressure at radius } r \text{ from the center of the well, psi (kPa)} \]

\[ \bar{P}_r = \text{Average reservoir pressure for radius } r \text{ from the center of the well, psi (kPa)} \]

\[ q = \text{Liquid flowrate, STB/D (std m}^3/\text{d)} \]

\[ \bar{q} = \text{Average liquid flowrate, for square wave rate case, STB/D (std m}^3/\text{d)} \]

\[ q_m = \text{Liquid flowrate at time } t, \text{ variable-rate case, STB/D (std m}^3/\text{d)} \]

\[ Q_m = \text{Cumulative liquid production at time } t, \text{ variable-rate case, STB/D (std m}^3/\text{d)} \]

\[ r = \text{Distance from center of the well, ft (m)} \]

\[ r_D = \frac{r}{r_e}, \text{ dimensionless distance} \]

\[ r_e = \text{Drainage radius of the well, ft (m)} \]

\[ r_w = \text{Wellbore radius, ft (m)} \]

\[ r_w' = r_w - S, \text{ effective wellbore radius, ft (m)} \]

\[ S = \text{Skin factor, dimensionless} \]

\[ S_g = \text{Gas saturation, fraction} \]

\[ S_o = \text{Oil saturation, fraction} \]

\[ S_w = \text{Water saturation, fraction} \]

\[ t = \text{Flowing time, hr} \]

\[ t' = \text{Dummy variable of integration, hr} \]

\[ \bar{t} = \frac{\int_{t'}^t q(t')dt'}{q_m}, \text{ plotting function for variable-rate tests, hr} \]

\[ r_D = 0.0002637 \frac{kt}{\phi \mu c t_w^2}, \text{ dimensionless time} \]

\[ r_{DA} = 0.0002637 \frac{kt}{\phi \mu c t_w}, \text{ dimensionless time based on drainage area} \]

\[ V_r = \pi (r_w^2 - r_w'^2) \phi h, \text{ radial pore volume, ft}^3(\text{m}^3) \]

\[ X_n = \text{Root of first order Bessel function of the first kind (i.e., } J_1(x_n) = 0) \]

\[ Y = 0.577216, \text{ Euler's constant} \]

\[ \mu = \text{Liquid viscosity, cp (Pa-s)} \]

\[ \phi = \text{Porosity, Fraction} \]

**Integer Subscripts**

\[ j = \text{Rate counter} \]

\[ m = \text{Number of rates up to time } t \]

\[ n = \text{Infinite series counter in Muskat's equation} \]

**Other Subscripts**

\[ cp = \text{Constant pressure case} \]

\[ cr = \text{Constant rate case} \]

\[ sw = \text{Square wave rate case} \]

\[ vr = \text{General variable-rate case} \]

**REFERENCES**

Variable-Rate Reservoir Limits Testing SPE 15028


APPENDIX

In this appendix, we will derive the general variable-rate equation for stabilized flow in a bounded circular reservoir. We will then use the average reservoir pressure approach presented by Dietz and Templer-Lietz to derive a variable-rate equation for stabilized flow in any bounded reservoir.

When Duhem's theorem is applied to the constant rate solution for a continuously changing flow rate, the result is

$$P_4 - P_r = 141.2 \frac{B}{kh} \int_0^t q(t') \frac{dp}{dt'} \, dt' . \quad \text{(A-1)}$$

Applying the convolution theorem to Eq. (A-1) gives

$$P_4 - P_r = 141.2 \frac{B}{kh} \int_0^t \int_0^{t'} p_d(t-t') \frac{dp}{dt'} \, dt' . \quad \text{(A-2)}$$

To model discrete rate changes, we discretize Eq. (A-2) the result is

$$P_4 - P_r = 141.2 \frac{B}{kh} \sum_{q(t)} \int_0^{t-1} (q_3 - q_2) p_d(t-t_2) + ... \quad \text{(A-3)}$$

Eq. (A-3) suggests a general relation for rate changes. Writing this general relation in summation notation gives

$$P_4 - P_r = 141.2 \frac{B}{kh} \sum_{i=1}^n (q_i - q_{i-1}) p_d(t - t_{i-1}) \quad \text{(A-4)}$$

Eq. (A-4) is a general equation which models arbitrary rate changes in a producing well, specific flow regimes such as transient or pseudosteady-state can be modeled with the appropriate $p_d(t)$ function in Eq. (A-4). At this point we will develop a relation for variable-rate flow in a bounded reservoir, using Eq. (A-4). This requires that we know $p_d(t)$ for the specific reservoir geometry modeled. Matthews, Brons, and Haenbroek and Earlougher et al. give methods of determining $p_d(t)$ for various reservoir geometries. However, the most convenient geometry is a well centered in a bounded circular reservoir. Muskat give this solution as

$$P_0 = \frac{B}{kh} \sum_{i=1}^n \frac{x^2}{n} \frac{J_0(x)}{x} \exp\left(-x^2 \frac{2}{x} \frac{t_D}{x}ight) \quad \text{... (A-5)}$$

where

$$r_D = \frac{r}{r_e} \quad \text{... (A-6)}$$

$$x_n$$ are the positive roots of $J_1(x_n) = 0 \quad \text{... (A-7)}$$

$$t_{DA} = \frac{0.0002637}{\phi \mu c_l A} \quad \text{... (A-8)}$$

$$A = \pi r_e^2 \quad \text{... (A-9)}$$

Combining Eqns. A-5, A-6, A-8, and A-9, and letting $r = r_w$ gives

$$P_0 = \frac{B}{kh} \sum_{i=1}^n \frac{r_e}{r_w} \frac{x^2}{n} \frac{J_0(x)}{x} \exp\left(-x^2 \frac{2}{x} \frac{t_D}{x}ight) \quad \text{... (A-10)}$$

Combining Eqns. A-4 and A-10 gives

$$p_d = \frac{B}{kh} \frac{t_D}{x} \text{ln} \frac{r_e}{r_w} - \frac{3}{4} + \frac{2}{r_w} + S \quad \text{... (A-11)}$$

$$+ 2 \pi (0.0002637) \frac{kt}{\phi \mu c_l A}$$

$$\quad \text{... (A-12)}$$

$$- \sum_{i=1}^n \frac{x^2}{n} \frac{J_0(x)}{x} \exp\left(-x^2 \frac{2}{x} \frac{t_D}{x}ight) \quad \text{... (A-13)}$$

Combining Eqns. A-4 and A-10 gives

$$\Delta p = \frac{B}{kh} \text{ln} \frac{r_e}{r_w} - \frac{3}{4} + \frac{2}{r_w} + S \quad \text{... (A-14)}$$
\[ \frac{r^2}{2e} + S \right] + 2\pi(0.0002637) \frac{k}{\phi \mu c} Q_m \\

\text{where}

\[ Q_m = \int_{t_0}^{\infty} q(t) \, dt = \sum_{j=1}^{m} q_j (t_j - t_{j-1}) \] \hspace{1cm} (A-12)

We will now develop a plotting function from Eq. (A-11). Following the Odh and Jones' derivation of the plotting function for variable-rate transient flow, we divide both sides of Eq. (A-11) by the final flow rate, \( q_m \). Letting \( \frac{\bar{q}}{q_m} \) yields

\[ \Delta p = \frac{141.2}{kh} \ln \left( \frac{r_e}{r_w} \right) - \frac{r_w^2}{2e} + S \]

\[ + 0.2339 \left( \frac{B}{\phi c} \overline{\bar{q}} \right) \]

\[ - \frac{282.4}{kh} \sum_{j=1}^{m} \frac{q_j - q_{j-1}}{q_m} \sum_{n=1}^{j} \frac{J_0 \left( \frac{X n r}{r_e} \right)}{X_n^2 J_0^2 \left( X_n \right)} \]

\[ \exp \left( -\frac{2\pi(0.0002637) k}{\phi \mu c} \right) \left( t - t_{j-1} \right) \right] \hspace{1cm} (A-13)

For a usable plotting function to be developed, the infinite series in Eq. (A-13) must be negligible. This is true for the constant rate case at steady-state and was shown to be approximately true for the variable-rate case at bounded reservoir flow conditions using simulated examples in this paper. Therefore, we can approximate Eq. (A-13) by neglecting the infinite series.

\[ \frac{\Delta p}{q_m} \approx \frac{141.2}{kh} \ln \left( \frac{r_e}{r_w} \right) - \frac{r_w^2}{2e} + S \]

\[ + 0.2339 \left( \frac{B}{\phi c} \overline{\bar{q}} \right) \] \hspace{1cm} (A-14)

Eq. (A-14) is the variable-rate approximation for stabilized flow for a well centered in a bounded circular reservoir.

We will now use Dietz's \(^{11}\) approach for constant rate flow in a bounded reservoir of general shape. This requires \( \delta P / \delta t \) for both the constant rate and the variable-rate cases. If we combine Eqs. (A-5), (A-6), (A-8), (A-9), and the definition of dimensionless pressure we obtain the following equation relating pressure and time for constant rate flow:

\[ \delta P = P_1 - 141.2 \frac{B \mu}{kh} \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + \frac{r_w^2}{2e} + S \]

\[ + 2\pi(0.0002637) \frac{k}{\phi \mu c} Q_m \]

\[ \exp \left( -\frac{2\pi(0.0002637) k}{\phi \mu c} \right) \left( t - t_{j-1} \right) \right] \] \hspace{1cm} (A-15)

Solving Eq. (A-11) for \( \bar{p}_r \) (i.e., \( r_w = r \)) gives an equation relating pressure and time for variable rate flow:

\[ \delta P = P_1 - 141.2 \frac{B \mu}{kh} \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + \frac{r_w^2}{2e} + S \]

\[ + 2\pi(0.0002637) \frac{k}{\phi \mu c} Q_m \]

\[ \exp \left( -\frac{2\pi(0.0002637) k}{\phi \mu c} \right) \left( t - t_{j-1} \right) \right] \] \hspace{1cm} (A-16)

Taking the time derivative of each term in Eq. (A-15) yields:

\[ \frac{\delta P}{\delta t} \approx -0.2339 \frac{q \delta B}{\phi c} \left( 1 + \frac{r_w^2}{2e} \right) \]

\[ \exp \left( -\frac{2\pi(0.0002637) k}{\phi \mu c} \right) \left( t - t_{j-1} \right) \right] \] \hspace{1cm} (A-17)

\[ \frac{\delta P}{\delta t} \approx -0.2339 \frac{q \delta B}{\phi c} \left( 1 + \frac{r_w^2}{2e} \right) \sum_{j=1}^{m} \frac{J_0 \left( \frac{X n r}{r_e} \right)}{X_n^2 J_0^2 \left( X_n \right)} \]

\[ \exp \left( -\frac{2\pi(0.0002637) k}{\phi \mu c} \right) \left( t - t_{j-1} \right) \right] \] \hspace{1cm} (A-18)

When stabilized flow occurs (i.e., when the bounded reservoir terms dominate the transient terms) the infinite series in Eqs. (A-17) and (A-18) become negligible. Therefore, the two derivatives become

\[ \frac{\delta P}{\delta t} \approx -0.2339 \frac{q \delta B}{\phi c} \] \hspace{1cm} (A-19)
\[
\frac{\partial p_r}{\partial t} = -0.2339 \frac{q_m B}{\phi h c \tau} \quad (A-20)
\]

Only the rates in Eqns. (A-19) and (A-20) differ. Note that there is no dependence on shape for either equation.

Dietsl developed an average reservoir pressure relation and a reservoir shape relation for constant rate flow based on Eq. (A-19). We will now derive similar relations for variable-rate flow using Eq. (A-20). The general equation of radial flow or the diffusivity equation was given by Muskat as

\[
L \frac{\partial^2 p_r}{\partial r^2} + 2 - 282.4 \frac{q_m B \mu r}{k h \tau} \phi \tau \quad (A-21)
\]

Combining Eqns. (A-20) and (A-21) yields

\[
\frac{\partial p_r}{\partial t} = -141.2 \frac{q_m B \mu r}{k h e} + \frac{C_1}{r} \quad (A-22)
\]

Integrating Eq. (A-22) yields

\[
\frac{\partial p_r}{\partial t} = -141.2 \frac{q_m B \mu r}{k h e} + C_1 \quad (A-23)
\]

Applying the outer boundary condition,

\[
\frac{\partial p_r}{\partial t} = 0 \text{ at } r = r_e \quad (A-24)
\]

Combining Eqns. (A-23) and (A-24)

\[
\frac{\partial p_r}{\partial t} = 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{1}{r_e} - \frac{r^2}{r_e^2} \right) \quad (A-25)
\]

Integrating Eq. (A-25) yields

\[
p_r = 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r}{r_e} - \frac{r^2}{r_e^2} \right) + C_2 \quad (A-26)
\]

Applying the inner boundary condition,

\[
p_r = p_{w0} \text{ at } r = r_w \quad (A-27)
\]

Combining Eqns. (A-26) and (A-27)

\[
p_r = p_{w0} + 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r}{r_e} - \frac{r^2}{r_e^2} \right) \quad (A-28)
\]

Introducing the average reservoir pressure, \( \bar{p}_r \), for any radius \( r \),

\[
\bar{p}_r = \frac{1}{V} \int_V p_r \, dV_r \quad (A-29)
\]

where

\[
V_r = \pi (r^2 - r_w^2) \phi h \quad (A-30)
\]

\[
dV_r = 2 \pi r \phi h \, dr \quad (A-31)
\]

Combining Eqns. (A-29) - (A-31) and solving for the average reservoir pressure, \( \bar{p}_r \),

\[
\bar{p}_r = \frac{1}{r (r^2 - r_w^2) \phi h} \left[ 2 \pi \phi h (141.2 \frac{q_m B \mu}{k h}) \frac{r}{r_w} \left( \ln \frac{r}{r_w} - \frac{r^2}{2r_w^2} \right) - \frac{(r^2 - r_w^2)}{8r_w^2} \right] \quad (A-32)
\]

Completing the integration in Eq. (A-32),

\[
\bar{p}_r = p_{w0} + 2(141.2 \frac{q_m B \mu}{k h}) \frac{r}{r_w} \left( \ln \frac{r}{r_w} - \frac{r^2}{2r_w^2} \right) - \frac{(r^2 - r_w^2)}{8r_w^2} \quad (A-33)
\]

Simplifying Eq. (A-33),

\[
\bar{p}_r = p_{w0} + 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r}{r_w} - \frac{3}{4} \right) \quad (A-34)
\]

If we let \( r = r_e \) and note that \( r_e^2 >> r_w^2 \) Eq. (A-34) becomes

\[
\bar{p}_r = p_{w0} + 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right) \quad (A-35)
\]

Eq. (A-35) gives us a relation for the average reservoir pressure, \( \bar{p}_r \), during stabilized flow.

Dietsl gave the following relation for the average reservoir pressure, \( \bar{p}_r \), for constant rate pseudosteady-state flow

\[
\bar{p}_r = p_{w0} + 141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right) \quad (A-36)
\]

Note that if we solve for the productivity index, \( J \), using Eqns. (A-35) and (A-36) we obtain the following equality

\[
J = \frac{q_m}{p_{w0} - \bar{p}_r} = \frac{q_m}{141.2 \frac{q_m B \mu}{k h} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} \right)} \quad (A-37)
\]

Eq. (A-37) proves that the productivity indices for constant rate pseudosteady-state and variable-rate stabilized flow are equal. Now we will use Eq. (A-35) and the approximate
variable-rate transient flow relation to develop a general reservoir shape factor, $C_A$. The approximate variable-rate transient solution is

$$\frac{q}{m} = \frac{q_{Bu}}{kh} \ln \left( \frac{A}{e^r D} \right) . \quad (A-38)$$

Combining Eqs. (A-35) and (A-38) for $p_1-p$ yields

$$p_1 - p = 70.6 \frac{q_{Bu}}{kh} \ln \left( C_A \frac{T_1}{T_A} \right) . \quad (A-39)$$

Where the reservoir shape factor, $C_A$, is defined for a circular reservoir as

$$C_A = \frac{\pi r^3}{4} \quad (A-40)$$

When $C_{r/A} \leq 1$, $p_1 - p = 0$, so $p_1 = p$ up to this time. Therefore, for $p_1 = p$, $T_{DA} > 1/C_A$. This means that, for $T_{DA} > 1/C_A$, $p$ can be predicted using Eq. (A-35). Also, since the constant rate and variable-rate solutions are essentially the same then the shape factors predicted for constant rate bare also valid for variable-rate flow. Therefore, we can generalize Eq. (A-14) for other shapes by using the Deitz shape factor, $C_A$.

If we neglect $r^2/2r$ in Eq. (A-14) and use the effective wellboe radius to model the skin effect, we obtain,

$$\frac{\Delta p}{q_m} = 70.6 \frac{Bu}{kh} \ln \frac{r}{r_w} + 0.2339 \frac{B}{\phi h_c A} \tau . \quad (A-41)$$

Now we will include the reservoir shape factor, $C_A$, relation (eq. (A-40)) in the argument of the natural logarithm:

$$\frac{\Delta p}{q_m} = 70.6 \frac{Bu}{kh} \ln \frac{4A}{e^r C_A r_w^2} + 0.2339 \frac{B}{\phi h_c A} \tau . \quad (A-42)$$

Eq. (A-42) is the general stabilized flow equation which serves as the basis for our analysis technique. If we plot $p/cm$ vs. $\tau$ on Cartesian coordinate graph paper, Eq. (A-42) dictates the following slope and intercept.

$$m_{v_f} = 0.2339 \frac{B}{\phi h_c A_1} \quad (A-43)$$

$$b_{v_f} = 70.6 \frac{Bu}{kh} \ln \frac{4A}{e^r C_A r_w^2} \quad (A-44)$$

Solving Eq. (A-43) for the reservoir drainage area, $A$, yields

$$A = 0.2339 \frac{B}{\phi h_c m_{v_f}} \quad (A-43a)$$

Solving Eq. (A-44) for the reservoir shape factor, $C_A$, yields

$$C_A = \frac{2.246 A}{b_{v_f} \frac{C_A}{r_w} \ln \left( \frac{4A}{e^r C_A r_w^2} \right)} . \quad (A-44a)$$

Eq. (A-35) can also be put in a general form by use of the shape factor, $C_A$.

$$p = p_{wf} + 70.6 \frac{m_{v_f}}{k_{v_f}} \ln \frac{4A}{e^r C_A r_w^2} \quad (A-45)$$

And finally, we will express Eq. (A-42) in dimensionless variables as

$$p = \frac{1}{2} \ln \frac{4A}{e^r C_A r_w^2} + 2 \pi \tau \quad (A-46)$$

Where $\tau$ is substituted for $t$ in $T_{DA}$. Eq. (A-46) is the same as the constant rate $T_{DA}$ solution presented by Ramey and Cobb, except $\tau$ is substituted for $t$. 

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TABLE 1
SUMMARY OF RESERVOIR LIMITS TEST AND ANALYSIS TECHNIQUES

<table>
<thead>
<tr>
<th>TEST METHOD</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Rate (P_t vs. t Graph)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A = -0.2339 \frac{\log q}{Q_t}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C_A = \frac{2.246 A}{(h_k(p_e - P_w))^2} + \frac{1}{\exp\left(\frac{10.6 q A}{8u}\right)^2}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant Pressure (log q vs. t Graph)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A = -0.1016 \frac{b_p}{\mu_c}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C_A = \frac{2.246 A}{(h_k(p_e - P_w))^2} + \frac{1}{\exp\left(\frac{10.6 b_p A}{8u}\right)^2}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square Wave Rate^5 (P_t vs. t Graph) (Earlougher's Method)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A = -0.2339 \frac{b}{\mu_c \gamma_{Sw}}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C_A = \frac{2.246 A}{(h_k(p_e - P_w))^2} + \frac{1}{\exp\left(\frac{10.6 b A}{8u}\right)^2}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>General Variable-Rate (Q_t/\gamma vs. \gamma Graph)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A = -0.2339 \frac{b}{\mu_c \gamma_{Sw}}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C_A = \frac{2.246 A}{(h_k(p_e - P_w))^2} + \frac{1}{\exp\left(\frac{10.6 b A}{8u}\right)^2}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For all methods, use \gamma_w = \gamma_e + \delta for a non-zero skin factor.

TABLE 2
SYSTEM PROPERTIES FOR CASES SIMULATED ANALYTICALLY

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Single Well Centered in a Bounded Circular Reservoir</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, A</td>
<td>5760.0 Acres</td>
<td>9.9 mi^2</td>
<td></td>
</tr>
<tr>
<td>Reservoir drainage radius, r_e</td>
<td>8937 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net pay thickness, h</td>
<td>100.0 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wellbore radius, r_w</td>
<td>0.5 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir permeability, k</td>
<td>100.0 md</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir porosity, \phi</td>
<td>0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total pore volume, \phi A</td>
<td>3.76\times10^8 ft^3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fluid viscosity, \mu</td>
<td>2.0 cp</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total compressibility, \epsilon</td>
<td>5\times10^{-6} psia</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial Pressure, P_i</td>
<td>2000.0 psi</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Formation Volume Factor, B</td>
<td>1.0 RB/STB</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 3
ERROR ANALYSIS FOR CASES SIMULATED ANALYTICALLY

<table>
<thead>
<tr>
<th>Case</th>
<th>New Method</th>
<th>New Method</th>
<th>Earlougher Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>Intercept</td>
<td>Error, \gamma</td>
<td>Error, \gamma</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square wave rate</td>
<td>2.145x10^-2</td>
<td>2.678x10^-2</td>
<td>6.329x10^0</td>
</tr>
<tr>
<td>Sine Wave Rate</td>
<td>4.276x10^-1</td>
<td>-1.243x10^-2</td>
<td>&gt;&gt;100</td>
</tr>
<tr>
<td>Random rate</td>
<td>2.069x10^0</td>
<td>-4.270x10^-1</td>
<td>8.296x10^1</td>
</tr>
</tbody>
</table>

TABLE 4
SYSTEM PROPERTIES FOR CASES MODELED WITH THE FINITE-DIFFERENCE SIMULATOR

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Single Well Centered in a Bounded Circular Reservoir</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, A</td>
<td>40.0 Acres</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir drainage radius, r_e</td>
<td>745 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net pay thickness, h</td>
<td>30.0 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wellbore radius, r_w</td>
<td>0.2 ft</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir permeability, k</td>
<td>1.0 md</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir porosity, \phi</td>
<td>0.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total pore volume, \phi A</td>
<td>15.7x10^6 ft^3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Oil viscosity, \mu</td>
<td>0.4 cp</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total compressibility, \epsilon</td>
<td>15x10^{-6} psia</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial pressure, P_i</td>
<td>4800.0 psi</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 5
ERROR ANALYSIS FOR CASES MODELED WITH THE FINITE-DIFFERENCE SIMULATOR

<table>
<thead>
<tr>
<th>Case</th>
<th>Slope Error, ( \beta )</th>
<th>Intercept Error, ( \alpha )</th>
<th>Time, ( \text{hrs} )</th>
<th>Flowrate, ( \text{STB/D} )</th>
<th>Cum. Prod., ( \text{STB} )</th>
<th>Pressure, ( \text{psia} )</th>
<th>( \Delta p/\Delta q ), ( \text{psi/STB/D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Rate</td>
<td>-1.893x10^{-4}</td>
<td>-1.388x10^{-2}</td>
<td>0.7200E 03</td>
<td>0.1500E 04</td>
<td>0.4500E 05</td>
<td>0.1607E 04</td>
<td>0.7200E 03</td>
</tr>
<tr>
<td>Logarithmic Rate Decline</td>
<td>-9.218x10^{-3}</td>
<td>2.000x10^{-2}</td>
<td>0.1440E 04</td>
<td>0.7000E 03</td>
<td>0.4600E 05</td>
<td>0.1800E 03</td>
<td>0.2263E 02</td>
</tr>
<tr>
<td>Exponential Rate Decline</td>
<td>-3.232x10^{-2}</td>
<td>9.377x10^{-1}</td>
<td>0.2880E 04</td>
<td>0.1200E 04</td>
<td>0.1590E 06</td>
<td>0.1644E 06</td>
<td>0.3180E 04</td>
</tr>
<tr>
<td>Discrete (Step-step) Rate</td>
<td></td>
<td></td>
<td>0.3600E 04</td>
<td>0.25200E 04</td>
<td>0.2234E 06</td>
<td>0.1294E 04</td>
<td>0.2264E 04</td>
</tr>
<tr>
<td>Rate Decline</td>
<td>9.910x10^{-3}</td>
<td>-1.706x10^{-2}</td>
<td>0.4320E 04</td>
<td>0.4500E 03</td>
<td>0.3690E 06</td>
<td>0.7459E 03</td>
<td>0.1968E 04</td>
</tr>
<tr>
<td>Hyperbolic Rate Decline</td>
<td>-1.697x10^{-2}</td>
<td>5.263x10^{-7}</td>
<td>0.5060E 04</td>
<td>0.4000E 03</td>
<td>0.3810E 06</td>
<td>0.1775E 03</td>
<td>0.2286E 03</td>
</tr>
<tr>
<td>Constant Pressure</td>
<td>-1.908x10^{-6}</td>
<td>2.515x10^{-1}</td>
<td>0.6440E 04</td>
<td>0.2100E 04</td>
<td>0.7935E 06</td>
<td>0.7073E 06</td>
<td>0.4252E 03</td>
</tr>
</tbody>
</table>

### Special Case Results:

<table>
<thead>
<tr>
<th>Case</th>
<th>Slope Error, ( \beta )</th>
<th>Intercept Error, ( \alpha )</th>
<th>Time, ( \text{hrs} )</th>
<th>Flowrate, ( \text{STB/D} )</th>
<th>Cum. Prod., ( \text{STB} )</th>
<th>Pressure, ( \text{psia} )</th>
<th>( \Delta p/\Delta q ), ( \text{psi/STB/D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Rate (( P_{wf} ) vs. ( t ) Graph)</td>
<td>5.128x10^{-4}</td>
<td>5.717x10^{-3}</td>
<td>0.1522E 03</td>
<td>0.4000E 04</td>
<td>0.1146E 07</td>
<td>0.1320E 03</td>
<td>0.5796E 04</td>
</tr>
<tr>
<td>Constant Pressure (( \log q_m ) vs. ( t ) Graph)</td>
<td>-9.935x10^{-1}</td>
<td>-7.327x10^{-1}</td>
<td>0.5336E 03</td>
<td>0.3530E 04</td>
<td>0.1693E 07</td>
<td>0.1845E 03</td>
<td>0.3354E 04</td>
</tr>
</tbody>
</table>

### Table 6
SIMULATED PRODUCTION DATA

<table>
<thead>
<tr>
<th>( P_{wf} )</th>
<th>( \log ) ( q_m )</th>
<th>( \Delta p/\Delta q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{psi} )</td>
<td>( \text{psi/STB/D} )</td>
<td>( \text{psi/STB/D} )</td>
</tr>
</tbody>
</table>

---

![Fig. 1—Typical reservoir limits test performance.](image-url)
Fig. 6—Rate histories for finite-difference simulation cases.

Fig. 7—Pressure drawdown data for finite-difference simulation cases.

Fig. 8—$\alpha p_{in}$ vs. $\Gamma$ curves for finite-difference simulation cases.

Fig. 9—Example graph of $\alpha p_{in}$ vs. $\Gamma$ for random rate case simulated analytically.