Petroleum Engineering 324  
Well Performance  
Dimensionless Variables and the Dimensionless Radial Flow Diffusivity Equation

*Any fool can tell the truth, but it requires a man of some wit to know how to tell a lie.*  
— Samuel Butler (1912)

**Topic:** Dimensionless Variables and the Dimensionless Radial Flow Diffusivity Equation  
**Objectives:** (things you should know and/or be able to do)

- Be able to develop the dimensionless form of the single-phase radial flow diffusivity equation—as well as the appropriate dimensionless forms of the initial and boundary conditions for the radial flow case, including the developments of the dimensionless radius, pressure, and time.

  - The Dimensionless Radial Flow Diffusivity Equation is given as
    \[
    \frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D}
    \]

  - Dimensionless Initial and Boundary Conditions (Radial Flow Case)
    - Dimensionless Initial Condition
      \[p_D(r_D, t_D < 0) = 0\]  
      (Uniform pressure in reservoir)
    - Dimensionless Inner Boundary Condition
      \[
      \left[ \frac{r_D}{\partial p_D} \right]_{r_D=1} = -1  
      \]  
      (constant rate at the well)
    - Dimensionless Outer Boundary Conditions
      a. "Infinite-Acting" Reservoir
      \[p_D(r_D \to \infty, t_D) = 0\]  
      (No reservoir boundary)
    b. "No-Flow" Boundary
      \[
      \left[ \frac{r_D}{\partial p_D} \right]_{r_D=r_eD} = 0
      \]  
      (No flux across the reservoir boundary)
    c. Constant Pressure Boundary
      \[p_D(r_eD, t_D) = 0\]  
      (Constant pressure at the reservoir boundary)

- Be able to derive the conversion factors for dimensionless pressure and time—for both SI and "Field" units.

**Lecture Outline:**

- Development of Dimensionless Variables
  - Attached notes -- development of dimensionless diffusivity equation
    - Derivation of dimensionless radius, pressure, and time:
      - Note the use of the initial and inner boundary conditions in the development of the dimensionless pressure function.
      - Note that the definitions of the dimensionless radius and time functions are essentially intuitive in that these functions contain "obvious" collections of dimensional variables.
    - Derivation of the dimensionless initial and boundary conditions:
      - Developed by substitution—note the "magic" numbers: 0, 1, and ∞.
  - Attached notes -- development of conversion factors for SI and field units
Lecture Outline:

- Development of the dimensionless diffusivity equation for radial flow
  - General introduction: radial flow forms of the diffusivity equation (liquid case)
  - Concept of the "equivalent" liquid case (for gases and compressible liquids)
    - Pseudopressure and pseudotime formulations
    - Discussion: Similarity of various formulations
- Development of the dimensionless diffusivity equation
  - General formulation in terms of "\(y\)" and "\(\tau\)" the pressure and time variables (respectively) that are used to model \(p, p^2\), or \(p_p\) (the \(y\)-function) and \(t\) or \(t_a\) (the \(\tau\)-function).
  - Use of the initial condition \([y(r, t \leq 0) = y_i]\) to establish the definition of the dimensionless pressure, \(p_D\).
  - Use of the inner boundary condition to complete the definition of the dimensionless pressure, \(p_D\). The constant rate inner boundary condition is given by
    \[
    q = \frac{2\pi kh}{Bu} \left( \frac{1}{(\frac{\partial y}{\partial p})} \right) \left[ r \frac{\partial y}{\partial r} \right]_{r_w} = \text{constant}
    \]
  - The definitions of the dimensionless radius and time functions are essentially intuitive in that these functions contain "obvious" collections of dimensional variables.
- Development of dimensionless initial and boundary conditions
  - Developed by substitution--note the "magic" numbers: 0,1, and \(\infty\).

Reading Assignment:

- Review the attached notes.
  - Development of the dimensionless diffusivity equation.
  - Development of conversion factors for SI and field units.
Derivation of the Dimensionless Radial Flow Diffusivity Equation

- Dimensionless Variables
- Dimensionless Initial and Boundary Conditions

(from Petroleum Engineering 620 Course Notes -- 1994)
Development of the Dimensionless Radial Flow Diffusivity Equation

General Introduction: Radial Flow Diffusivity Equation

In this section our objective is to derive a "dimensionless" radial flow diffusivity equation. We will begin with the diffusivity equation for a "slightly" compressible liquid and make extensions to gas and solution-gas drive systems using the appropriate pseudo-function transformations (i.e., pseudopressure and pseudotime). The rigorous form of the diffusivity equation for a "slightly" compressible liquid is given by:

\[ c (\nabla p)^2 + \nabla^2 p = \frac{\phi u c_l}{k} \frac{\partial p}{\partial t} \] .................................................. (1)

Neglecting the \(c(\nabla p)^2\) term in Eq. 1, we obtain our standard, linear form of the diffusivity equation for a slightly compressible liquid:

\[ \nabla^2 p = \frac{\phi u c_l}{k} \frac{\partial p}{\partial t} \] .................................................. (2)

Recall that for the case of a slightly compressible liquid we assume that \(\mu c_l\) is constant. While this condition may not always be met in practice, it is our governing assumption for the "liquid" flow case.

The \(\nabla \cdot \nabla \alpha = \nabla^2 \alpha\) operator terms are given by:

Linear Flow:

\[ \nabla \cdot \nabla \alpha = \nabla^2 \alpha = \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} \] .................................................. (3)

Radial Flow:

\[ \nabla \cdot \nabla \alpha = \nabla^2 \alpha = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \alpha}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \alpha}{\partial \theta^2} + \frac{\partial^2 \alpha}{\partial \phi^2} \] .................................................. (4)

For horizontal radial flow (i.e., flow with neither angular flow nor vertical flow effects), Eq. 4 becomes

\[ \nabla^2 \alpha = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \alpha}{\partial r} \right] = \frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{r} \frac{\partial \alpha}{\partial r} \] .................................................. (5)

Substituting Eq. 5 into Eq. 2, we obtain

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \alpha}{\partial r} \right] = \frac{\phi u c_l}{k} \frac{\partial p}{\partial t} \] .................................................. (6)

Expanding the derivative on the left-hand-side of Eq. 6, we have

\[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi u c_l}{k} \frac{\partial p}{\partial t} \] .................................................. (7)

We note that the \((\phi u c_l)/k\) term is typically assumed to be constant for liquid flow. Eqs. 6 and 7 represent the most basic forms of the horizontal radial flow diffusivity equation for a slightly compressible liquid and are given in terms of pressure. While these results are the most widely applied forms of the diffusivity equation, in terms of both solutions as well as analysis and interpretation methodologies, the assumption of constant viscosity and small and constant fluid compressibility are not universally applicable. As we noted earlier, non-constant viscosity-compressibility behavior is especially true for the flow of gases, but
is also true for the flow of "compressible" liquids (i.e., oil systems where the pressure is below the bubblepoint pressure).

**Concept of the "Equivalent" Liquid Case:**

The widespread application of analysis and interpretation methods based on the concept of a slightly compressible liquid makes the development of an "equivalent liquid" model quite desirable. Such an "equivalent liquid" model could be used when the assumptions for a slightly compressible liquid are not met, as in the case of gas flow or the flow of a compressible liquid.

Our goal is to develop a form of the radial flow diffusivity equation that can be solved analytically. If $\rho/\mu$ and $\mu_0$ vary significantly with pressure, then we must use linearizations (i.e., pseudopressure and pseudotime, as discussed in previous lectures) to account for this nonlinear behavior. The pseudopressure-time and pseudopressure-pseudotime forms of the horizontal, radial flow diffusivity equation are given (for the general oil or gas case) by:

$$
\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial p}{\partial r}\right] = \frac{\phi \mu_0}{k} \frac{\partial p}{\partial t} \quad \text{(pseudopressure-time formulation)} \quad \ldots \quad (8)
$$

$$
\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial p}{\partial r}\right] = \frac{\phi}{k} (\mu_0) \frac{\partial p}{\partial t_a} \quad \text{(pseudopressure-pseudotime formulation)} \quad \ldots \quad (9)
$$

The generalized pseudofunctions are given by:

**Pseudopressure:** (in terms of fluid density)

$$
pp = \left[ \frac{\mu}{\rho_0} \right] \int_{p_{base}}^{p} \frac{\rho}{\mu} dp \quad \ldots \quad (10)
$$

**Pseudopressure:** (in terms of the formation volume factor)

$$
pp = (\mu B_0) \int_{p_{base}}^{p} \frac{1}{\mu B} dp \quad \ldots \quad (11)
$$

**Pseudotime:**

$$
t_a = (\mu_0 c_0) \int_{0}^{t} \frac{1}{\mu(p)c(t)} dt \quad \ldots \quad (12)
$$

Eqs. 8 and 9 represent our "equivalent liquid" concept and can be used interchangeably with the radial flow diffusivity equation for a slightly compressible liquid (Eqs. 6 or 7). As a practical matter, we will use Eq. 6 (or Eq. 7) to define the dimensionless radial flow diffusivity equation, solve this relation for various boundary conditions, then substitute the $pp$ and $t_a$ variables as needed into the pressure and time $(p$ and $t)$ variables of a desired solution. Analysis is much the same in that we will use $pp$ and $t_a$ in place of pressure and time $(p$ and $t)$ in the appropriate "liquid" analysis relations.

**Development of the Dimensionless Radial Flow Diffusivity Equation:**

We use the general form of the radial flow diffusivity equation and its initial and inner boundary conditions to develop a unique set of dimensionless variables. In reservoir engineering we almost always assume a uniform pressure as our initial condition (i.e., $p(r,t=0)=p_i$, for all $r$ and $t=0$). Similarly, we almost always assume a constant flowrate, $q$, as our inner boundary condition—extensions can easily be made to variable-rate conditions using convolution (i.e., superposition), and the assumption of a constant flowrate is
generally the most straightforward approach to solving the diffusivity equation. "Darcy" units are used throughout these developments, and conversions to other units systems are provided in a later section.

Beginning with the generalized form of the diffusivity equation, we have

\[ \nabla^2 y = \frac{\phi u c}{k} \frac{\partial y}{\partial \tau} \]  
(13)

Where the \( y \)-function is a general function of pressure (e.g., \( p \), \( p^2 \), or \( p_d \)) and the \( r \)-function is a general time function (\( t \) or \( t_a \)). For horizontal radial flow, Eq. 13 becomes

\[ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial y}{\partial r} \right] = \frac{\phi u c t}{k} \frac{\partial y}{\partial \tau} \]  
(14)

Expanding the derivative on the left-hand-side of Eq. 14, we obtain

\[ \frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} = \frac{\phi u c t}{k} \frac{\partial y}{\partial \tau} \]  
(15)

Our objective is to make Eq. 15 "dimensionless," that is, define generalized variables that have no units. Such "dimensionless" variables are made up of combinations of reservoir and fluid properties (e.g., \( \phi \), \( \mu \), \( k \), \( h \), ...etc). Dimensionless variables are defined by the conditions of the problem (i.e., the initial and inner boundary conditions in our particular case), as well as by intuitive definitions (e.g., \( r_D = r/w \)).

In developing the dimensionless form of Eq. 15, we establish the following:

1. The dimensionless radius, \( r_D \), which is based intuitively on the wellbore radius, \( r_w \). We could have used another radii or characteristic length, but the wellbore radius is both logical and universal (i.e., \( r_w \) is always known).

2. By our requirement, the dimensionless pressure, \( p_D \), must satisfy the following mathematical conveniences:
   a. The initial condition: \( p_D(r_D, t_D=0) = 0 \)
   b. The constant rate inner boundary condition: \( \left[ r_D \frac{\partial p_D}{\partial r_D} \right] \bigg|_{r_D=1} = -1 \)

As noted above, the definition of the dimensionless radius is given intuitively as

\[ r_D = r/w \]  
(16)

or

\[ r = rw r_D \]  
(17)

Substituting Eq. 17 into Eq. 15 we have

\[ \frac{\partial}{\partial (r w r_D)} \left[ \frac{\partial y}{\partial (r w r_D)} \right] + \frac{1}{r w r_D} \frac{\partial y}{\partial (r w r_D)} = \frac{\phi u c t}{k} \frac{\partial y}{\partial \tau} \]

Factoring out the \( r_w \) terms from inside the derivatives we have

\[ \frac{1}{r_w^2} \frac{\partial^2 y}{\partial r_D^2} + \frac{1}{r_w^2} \frac{1}{r_D} \frac{\partial y}{\partial r_D} = \frac{\phi u c t}{k} \frac{\partial y}{\partial \tau} \]

Multiplying through this result by \( r_w^2 \) gives us

\[ \frac{\partial^2 y}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial y}{\partial r_D} = \frac{\phi u c r_w^2}{k} \frac{\partial y}{\partial \tau} \]  
(18)
We next need to develop a dimensionless pressure function, \( p_D \), which accounts for both the initial and inner boundary conditions. The initial condition is given by

\[
y(r, \tau \leq 0) = y_i
\]

We require a \( p_D \) definition that gives the following dimensionless initial condition

\[
p_D(r, \tau \leq 0) = 0
\]

Eq. 20 provides us with a mathematical convenience for developing analytical solutions of the dimensionless diffusivity equation. In fact, Eq. 20 suggests the following form of the dimensionless pressure function, \( p_D \)

\[
p_D = \frac{1}{y_{ch}} (y_i - y)
\]

where \( y_{ch} \) is a "characteristic" value of the pressure function, \( y \). The appropriate value of \( y_{ch} \) will be defined using the inner boundary condition. Rearranging Eq. 21 and solving for \( y \), we have

\[
y = y_i - y_{ch} p_D
\]

Substituting Eq. 22 into Eq. 18 we have

\[
\frac{\partial}{\partial r_D} \left[ \frac{\partial}{\partial r_D} (y_i - y_{ch} p_D) \right] + \frac{1}{r_D} \frac{\partial}{\partial r_D} (y_i - y_{ch} p_D) = \frac{\phi \mu c r_w^2}{k} \frac{\partial}{\partial \tau} (y_i - y_{ch} p_D)
\]

Expanding terms in Eq. 23 gives us

\[
\frac{\partial^2}{\partial r_D^2} (y_i - y_{ch} p_D) + \frac{1}{r_D} = \frac{1}{r_D} \frac{\partial}{\partial r_D} (y_i - y_{ch} p_D) = \frac{\phi \mu c r_w^2}{k} \frac{\partial}{\partial \tau} (y_i) - \frac{\phi \mu c r_w^2}{k} y_{ch} \frac{\partial p_D}{\partial r_D}
\]

Eliminating the derivative of the constant \( (y_i) \) terms from Eq. 24, we have

\[
(-y_{ch}) \frac{\partial^2 p_D}{\partial r_D^2} + (-y_{ch}) \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = (-y_{ch}) \frac{\phi \mu c r_w^2}{k} \frac{\partial p_D}{\partial \tau}
\]

Canceling the \( (-y_{ch}) \) terms in Eq. 25 gives

\[
\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu c r_w^2}{k} \frac{\partial p_D}{\partial \tau}
\]

As we strive to solve for the "characteristic pressure," \( y_{ch} \), we must establish an identity for doing so. We eventually work our way around to the inner boundary condition, which is based on Darcy's law. Writing the radial flow form of Darcy's law, we have

\[
v = \frac{k}{\mu} \frac{\partial p}{\partial \tau}
\]

Applying the chain rule to the gradient term \( \frac{\partial p}{\partial \tau} \) we can express Eq. 27 in terms of the \( y \)-function as follows

\[
v = \frac{k}{\mu} \frac{\partial y}{\partial \tau}
\]

Recalling the definition of velocity, we have

\[
v = \frac{q}{A}
\]
where \( A \) is the cross-sectional area for flow and \( B \) is the formation volume factor for the fluid--recall that flowrate is in "stock tank" units and velocity is in "reservoir" units, (the \( B \) factor provides for the proper conversion of these units). Substituting Eq. 29 into Eq. 28 and solving for the flowrate, \( q \), gives us

\[
q = \frac{kA}{\mu B} \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial r}
\]  

(30)

Recalling that the cross-sectional area for flow is a cylindrical shell (analogous to say, the wellbore) for the radial flow case. This gives us

\[
A = 2\pi rh
\]

(31)

Substituting Eq. 31 into Eq. 30, we have

\[
q = \frac{2\pi kh}{\mu B} \frac{\partial \psi}{\partial y} \left[ r \frac{\partial y}{\partial r} \right]
\]

(32)

Solving Eq. 32 for the gradient expression, \( \left[ r \frac{\partial y}{\partial r} \right] \), we obtain

\[
\left[ r \frac{\partial y}{\partial r} \right] = \frac{qB\mu}{2\pi kh} \frac{\partial \psi}{\partial \psi}
\]

(32)

But the question remains--How do we determine the "characteristic pressure," \( \psi_{ch} \)? Writing Darcy's law for the constant rate inner boundary condition (i.e., flowrate at the wellbore), using Eq. 32 gives us

\[
\left[ r \frac{\partial y}{\partial r} \right]_{r=r_w} = \frac{qB\mu}{2\pi kh} \frac{\partial \psi}{\partial \psi}
\]

(33)

Substituting the definitions of dimensionless radius, \( r_D \), and dimensionless pressure, \( p_D \), into Eq. 33 gives us

\[
\left[ (r_w r_D) \frac{\partial}{\partial (r_w r_D)} \left[ \psi_i - \psi_{ch} p_D \right] \right]_{r_D=r_w, r_w=1} = \frac{qB\mu}{2\pi kh} \frac{\partial \psi}{\partial \psi}
\]

Canceling like terms and eliminating the \( \partial \psi/\partial r_D \) term, we have

\[
\left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D=1} = - \frac{1}{\psi_{ch}} \frac{qB\mu}{2\pi kh} \frac{\partial \psi}{\partial \psi}
\]

(34)

Again as a mathematical convenience, we would like to express the dimensionless form of the inner boundary condition as

\[
\left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D=1} = -1
\]

(35)

We note that Eq. 35 is a mathematical convenience because this form will significantly reduce the tedium of the solution of the dimensionless diffusivity equation. Equating Eqs. 34 and 35, we obtain the "characteristic pressure," \( \psi_{ch} \), which is given as

\[
\psi_{ch} = \frac{qB\mu}{2\pi kh} \frac{\partial \psi}{\partial \psi}
\]

(36)

We now recall that the \( \psi \)-function can be defined as \( p \), \( p^2 \), or \( p_p \)--therefore, \( \partial \psi/\partial p \) can be readily obtained once the \( \psi \)-function is specified. Substituting Eq. 36 into Eq. 21, we obtain the final definition of the \( p_D \) function, which is given by

\[
p_D = \frac{2\pi kh}{qB\mu} \frac{1}{\left( \partial \psi/\partial p \right)} (\psi_i - \psi) \quad \text{(Darcy units system)}
\]

(37)
The various cases for $\partial y / \partial p$ are summarized in the table below:

<table>
<thead>
<tr>
<th>$y$-function</th>
<th>$\partial y / \partial p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$1 / \mu B$</td>
</tr>
<tr>
<td>$p^2$</td>
<td>$2p$</td>
</tr>
<tr>
<td>$p_p$ (general)*</td>
<td>$(\mu B)_n / \mu B$</td>
</tr>
</tbody>
</table>

We note that the "n" subscript represents the "normalizing" condition—we recommend that the initial condition (i.e., the initial reservoir pressure, $p_i$) be used as the normalizing condition for the pseudopressure ($p_p$) function.

* Recalling the definitions of the pseudopressure function (both the general case and the gas case), we have:

$$p_p = (\mu B)_n \int_{p_{base}}^{p} \frac{1}{\mu B} dp \quad \text{(General formulation)} \quad \cdots (38)$$

$$p_{pg} = \left[ \frac{\mu z}{p} \right]_n \int_{p_{base}}^{p} \frac{p}{\mu z} dp \quad \text{(Gas formulation)} \quad \cdots (39)$$

Either definition (Eq. 38 or Eq. 39) will work for gas because, once the constant terms are factored out, we note that these definitions are exactly identical.

Recalling our "intermediate" result for the dimensionless diffusivity equation for radial flow (Eq. 26), we have

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu c r_w^2}{k} \frac{\partial p_D}{\partial t_D} \quad \cdots (26)$$

From Eq. 26 we appear to be almost finished (note that all of our effort after developing Eq. 26 was to obtain the terms in the definition of the dimensionless pressure function, $p_D$). Observing Eq. 26 we note that the "leftover" term, $(\phi \mu c r_w^2)/k$, suggests an "intuitive" definition of the dimensionless time function, $t_D$. In short, $t_D$ should be defined by

$$t_D = \frac{\tau}{\tau_{ch}} \quad \cdots (40)$$

where $\tau_{ch}$ is a characteristic time (or time-like) function to be determined from Eq. 26. Rearranging Eq. 40 we have

$$\tau = \tau_{ch} t_D \quad \cdots (41)$$

Substituting Eq. 41 into Eq. 26, we obtain

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\phi \mu c r_w^2}{k} \frac{\partial p_D}{\partial (\tau_{ch} t_D)} = \frac{\phi \mu c r_w^2}{k} \frac{1}{\tau_{ch}} \frac{\partial p_D}{\partial t_D} \quad \cdots (42)$$

Defining $\tau_{ch}$ as

$$\tau_{ch} = \frac{\phi \mu c r_w^2}{k} \quad \cdots (43)$$

Substituting Eq. 43 into Eq. 42, we obtain the final form of the dimensionless diffusivity equation, which is given as

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \quad \cdots (44)$$
where,

\[ r_D = r/r_w \] ......................................................... (16)

\[ p_D = \frac{2\pi k h}{q B\mu} \frac{1}{(\partial y/\partial p)} (y_i - y) \] (Darcy units system) ............................................. (37)

\[ t_D = -\frac{k\tau}{\phi\mu c r_w^2} \] (Darcy units system) .................................................. (45)

**Initial and Boundary Conditions for Radial Flow:** (No Skin or Wellbore Storage Effects)

The initial and boundary conditions that we consider for the radial flow case are:

**Initial Condition:**

\[ y(r,t \leq 0) = y_i \] ......................................................... (46)

**Inner Boundary Condition:** (constant sandface flowrate at the well)

\[ q = \frac{2\pi k h}{B\mu} \frac{1}{(\partial y/\partial p)} \left[ r \frac{\partial y}{\partial r} \right]_{r_w} = \text{constant} \] ......................................................... (47)

**Outer Boundary Condition:**

Case 1: Infinite outer boundary

\[ y(r \rightarrow \infty, t) = y_i \] ......................................................... (48)

Case 2: "No-flow" outer boundary

\[ q = \frac{2\pi k h}{B\mu} \frac{1}{(\partial y/\partial p)} \left[ r \frac{\partial y}{\partial r} \right]_{r_e} = 0 \] ......................................................... (49)

Case 3: Constant pressure outer boundary (constant at initial pressure)

\[ y(r_e, t) = y_i \] ......................................................... (50)

Since we used the initial and inner boundary conditions to define the dimensionless pressure function, \( p_D \), we can simply write the dimensionless initial and boundary conditions directly from the standard forms given above (i.e., Eqs. 46-50). Summarizing, the dimensionless initial and boundary conditions are:

**Initial Condition:**

\[ p_D(r_D, t_D \leq 0) = 0 \] ......................................................... (51)

**Inner Boundary Condition:** (constant sandface flowrate at the well)

\[ \left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_D=1} = -1 \] ......................................................... (52)

**Outer Boundary Condition:**

Case 1: Infinite outer boundary

\[ p_D(r_D \rightarrow \infty, t_D) = 0 \] ......................................................... (53)

Case 2: "No-flow" outer boundary

\[ \left[ r_D \frac{\partial p_D}{\partial r_D} \right]_{r_eD} = 0 \] ......................................................... (54)

Case 3: Constant pressure outer boundary (constant at initial pressure)

\[ p_D(r_eD, t_D) = 0 \] ......................................................... (55)
Dimensionless Variables in Terms of Field and SI Units for the Radial Flow Diffusivity Equation

(from Petroleum Engineering 620 Course Notes -- 1994)
Dimensionless Variables in Terms of Field and SI Units for the Radial Flow Diffusivity Equation

Introduction and Definition of Variables:

As several unit systems are currently used in the petroleum industry, we must derive the appropriate conversions. The original definition of Darcy’s law lead to the "Darcy" units system where the fundamental definition of permeability is given by

$$k[1 \text{ darcy}] = \frac{q[1 \text{ cm}^3 \text{ sec}^{-1}] B [\text{ cm}^3/\text{cm}^3] \mu[cp]}{A[1 \text{ cm}^2] \frac{\partial p}{\partial r}[1 \text{ atm/1 cm}]}$$

or in words we have: (from Amyx, et al.--Petroleum Reservoir Engineering)

"A porous medium has permeability of one Darcy when a single-phase fluid of one centipoise viscosity that completely fills the voids of the medium will flow through it under conditions of viscous (laminar) flow at a rate of one cubic centimeter per second per square centimeter cross-sectional area under a pressure or potential gradient of one atmosphere per centimeter".

In order to work in a particular unit system, the $t_D$ and $p_D$ identities must be converted from the Darcy units system to the chosen "field" or SI units system. To facilitate this task, the units associated with each dimension are summarized in the table below for each unit system. However, we first note some useful conversions, these are:

<table>
<thead>
<tr>
<th>Unit</th>
<th>Conversion Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length: 1 ft</td>
<td>30.48 cm</td>
</tr>
<tr>
<td>Volume: 1 bbl</td>
<td>5.615 ft³</td>
</tr>
<tr>
<td>Pressure: 1 atm</td>
<td>14.696 psia</td>
</tr>
<tr>
<td>Viscosity: 1 cp</td>
<td>1 mPa·sec</td>
</tr>
</tbody>
</table>

Summary of Variables Associated with the Darcy, Field, and SI Unit Systems

<table>
<thead>
<tr>
<th>Variable</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permeability, $k$</td>
<td>d</td>
<td>md</td>
<td>md</td>
</tr>
<tr>
<td>Net pay thickness, $h$</td>
<td>cm</td>
<td>ft</td>
<td>m</td>
</tr>
<tr>
<td>$y (p \text{ or } p_p)$</td>
<td>atm</td>
<td>psia</td>
<td>kPa</td>
</tr>
<tr>
<td>Flowrate, $q$</td>
<td>cm³/sec</td>
<td>STB/D (MSCF/D)</td>
<td>m³/D</td>
</tr>
<tr>
<td>Form. volume factor, $B$</td>
<td>rcm³/stand cm³</td>
<td>RB/STB(or RB/MSCF)</td>
<td>rm³/stand m³</td>
</tr>
<tr>
<td>Fluid viscosity, $\mu$</td>
<td>cp</td>
<td>cp</td>
<td>mPa·sec</td>
</tr>
<tr>
<td>$\partial y/\partial p$ ($y = p$ or $p_p$)</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$\partial y/\partial \rho$ ($y = p^2$)</td>
<td>atm</td>
<td>psia</td>
<td>kPa</td>
</tr>
<tr>
<td>Time function $\tau$ ($t \text{ or } t_a$)</td>
<td>sec</td>
<td>hr</td>
<td>hr</td>
</tr>
<tr>
<td>Total compressibility, $c_t$</td>
<td>atm⁻¹</td>
<td>psia⁻¹</td>
<td>kPa⁻¹</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$</td>
<td>cm</td>
<td>ft</td>
<td>m</td>
</tr>
</tbody>
</table>
Adjustment for the $p^2$ Form of the Gas Diffusivity Equation:

The conversion of the $p_D$ function requires a minor adjustment to put the $p^2$ relation into an equivalent form with the results for the $p$ and $p_p$ cases. This "adjustment" is defined as

$$B_p^2 = \frac{\partial (p^2)}{\partial p} \frac{B}{2pB}$$

(56)

The use of $B_p^2$ in place of the $\frac{\partial (p^2)}{\partial p} \frac{B}{2pB}$ term makes the $p^2$ form of the $p_D$ function the same as the $p$ and $p_p$ forms—and gives a single unit conversion for all three cases ($p, p^2, p_p$).

Conversion to Field Units:

The dimensionless pressure and time functions, $p_D$ and $t_D$, for radial flow, are given by

$$p_D = \frac{2\pi kh}{qB\mu} \frac{1}{(\partial y/\partial p)} (y_i - y)$$  \hspace{1cm} \text{(Darcy units system)} \hspace{1cm} (37)

and

$$t_D = \frac{k\tau}{\phi \mu_c r_w^2}$$  \hspace{1cm} \text{(Darcy units system)} \hspace{1cm} (45)

Substituting the appropriate unit conversions into Eq. 37 (the $p_D$ function) to convert from Darcy units to field units, we have:

$$p_D = \frac{2\pi [md \frac{14dni}{1000md} h[ft] \frac{30.48cm}{1ft} (y_i - y)[psia] \frac{1atm}{14.696 \text{psia}}}{q[STB/D] \frac{5.615 \text{bbl}}{1[30.48cm]^3} \frac{1[D]}{24hr} \frac{1[ \text{sec}]}{3600} B[RB] \frac{\mu[cp]}{STB} \frac{\frac{\partial y}{\partial p}}{[psi]} \frac{\frac{\partial}{\partial p}[psi]}{}}$$

$$p_D = 2\pi \frac{[1127 \times 10^{-3}]}{D} \frac{[ \text{d cm atm bbl sec}]}{[\text{md ft psia cm}^3]} \frac{kh}{qB\mu (\partial y/\partial p)} \frac{1}{(y_i - y)}$$

Dropping the units on the conversion constant and reducing, this result gives us

$$p_D = 7.081 \times 10^{-3} \frac{kh}{qB\mu (\partial y/\partial p)} (y_i - y)$$

or in the common form used for well testing, we have

$$p_D = \frac{1}{141.2} \frac{kh}{qB\mu (\partial y/\partial p)} (y_i - y)$$

(57)

Substituting the appropriate unit conversions into Eq. 45 (the $t_D$ function) to convert from Darcy units to field units, we have:

$$t_D = \frac{k[md] \frac{1d}{1000md} \frac{\tau[hr]}{1hr} \frac{3600 \text{sec}}{1hr}}{\phi \text{[fraction]} \frac{1}{\mu[cp]} \frac{c_1}{[1atm]} \frac{14.696 \text{psia}}{1atm} \frac{r_w^2[ft]}{30.48 \text{cm}^2}}$$

Reducing this result gives us

$$t_D = 2.637 \times 10^{-3} \frac{[d \text{ sec atm ft}^2]}{[md \text{ hr psia cm}^2]} \frac{k\tau}{\phi \mu_c r_w^2}$$

Dropping the units on the conversion constant, we have

$$t_D = 2.637 \times 10^{-3} \frac{k\tau}{\phi \mu_c r_w^2}$$

(58)
We should recall that the "y" variable in Eq. 57 represents any pressure-type of function (i.e., pressure, pressure-squared, or pseudopressure). Similarly, in Eq. 58 the variable "t" represents any time-type of function (i.e., time or pseudotime).

Conversion to SI Units:
Substituting the appropriate unit conversions into Eq. 37 (the \( p_D \) function) to convert from Darcy units to SI units, we have:

\[
p_D = \frac{2\pi k [\text{md}] \left[ \frac{1 \text{d}}{1000 \text{md}} \right] h [\text{m}] \left[ \frac{100 \text{cm}}{\text{m}} \right] (y_i - y) [\text{kPa}] \left[ \frac{1 \text{atm}}{101.33 \text{kPa}} \right]}{q \left[ \text{m}^3/\text{D} \right]} \left[ \frac{100 \text{cm}}{1 \text{m}} \right] \left[ \frac{1 \text{D}}{24 \text{hr}} \right] \left[ \frac{1 \text{hr}}{3600 \text{ sec}} \right] B \left[ \frac{\text{m}^3}{\text{std} \text{ m}^3} \right] \mu [\text{mPa}\cdot\text{sec}] \left[ \frac{1 \text{cp}}{1 \text{ mPa}\cdot\text{sec}} \right] \frac{\partial y}{\partial p} \left[ \frac{\text{atm}}{\text{atm}} \right]
\]

\[
p_D = 2\pi \left[ 8.527 \times 10^{-5} \right] \left[ \frac{d \text{ cm atm m}^3}{\text{sec mPa}\cdot\text{sec}} \right] \frac{k h}{q B \mu} \left( \frac{1}{\partial y/\partial p} \right) (y_i - y)
\]

Dropping the units on the conversion constant, we have

\[
p_D = 5.3574 \times 10^{-4} \frac{k h}{q B \mu} \left( \frac{1}{\partial y/\partial p} \right) (y_i - y) \tag{59}
\]

Substituting the appropriate unit conversions into Eq. 45 (the \( t_D \) function) to convert from Darcy units to SI units, we have:

\[
t_D = \frac{k [\text{md}] \left[ \frac{1 \text{d}}{1000 \text{md}} \right] \tau [\text{hr}] \left[ \frac{3600 \text{sec}}{\text{hr}} \right]}{\phi [\text{fraction}] \mu [\text{mPa}\cdot\text{sec}] \left[ \frac{1 \text{cp}}{1 \text{ mPa}\cdot\text{sec}} \right] c_t [\frac{1}{\text{kPa}}] \left[ \frac{101.33 \text{kPa}}{1 \text{atm}} \right] r_w^2 \left[ \frac{\text{m}^2}{1 \text{m}} \right]}
\]

Reducing this result gives us

\[
t_D = 3.557 \times 10^{-6} \left[ \frac{d \text{ sec mPa}\cdot\text{sec atm m}^2}{\text{md hr cp kPa cm}^2} \right] \frac{k \tau}{\phi c_t r_w^2}
\]

Dropping the units on the conversion constant, we obtain

\[
t_D = 3.557 \times 10^{-6} \frac{k \tau}{\phi c_t r_w^2} \tag{60}
\]

Summary of Unit Conversion Developments:
A summary of the definitions presented in this section can be generalized by writing \( t_D \) and \( p_D \) in the following general form

\[
t_D = t_{Dc} \frac{k \tau}{\phi c_t r_w^2} \tag{61}
\]

and

\[
p_D = p_{Dc} \frac{k h}{q B \mu} \left( \frac{1}{\partial y/\partial p} \right) (y_i - y) \tag{62}
\]

The following table summarizes the values of the dimensionless time constant, \( t_{Dc} \), and the dimensionless pressure constant, \( p_{Dc} \):

<table>
<thead>
<tr>
<th>Conversion Constants for the ( t_D ) and ( p_D ) Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constant</strong></td>
</tr>
<tr>
<td>( t_{Dc} )</td>
</tr>
<tr>
<td>( p_{Dc} )</td>
</tr>
<tr>
<td>( p_{Dc} = \frac{1}{2\pi} p_{Dc} )</td>
</tr>
</tbody>
</table>
References
