Petroleum Engineering 324  
Well Performance  
Solutions to the Radial Flow Diffusivity Equation

_The unexamined life is not worth living._  
— Socrates (399 B.C.)

**Topic:** Solutions to the Radial Flow Diffusivity Equation  
**Note:** The 2nd Edition of the Lee text, _Well Testing_, are available at the Copy Center in the Reed McDonald Building. Approximate cost is $50—this text is optional.  
**Objectives:** (things you should know and/or be able to do)

- Be able to recognize and apply the following solutions for an unfractured well produced at a constant flow rate in a homogeneous reservoir (with no skin or wellbore storage effects) for the following outer boundary conditions:
  
  - The dimensionless diffusivity equation for radial flow is given as
    \[
    \frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial P_D}{\partial r_D} = \frac{\partial P_D}{\partial t_D} \quad \text{(Fundamental identity)}
    \]
    
  - "Infinite-Acting" Reservoir Case: (line source solution)
    \[
    p_D(t_D, r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] \quad \text{for } t_D \geq 10 \quad \text{(the Exponential Integral Solution)}
    \]
    
  - Note that \( E_1(x) = -E_i(-x) \) — recall that the Lee texts use the \(-E_i(-x)\) form.
  
  - The "well testing" derivative function, \( p'_D = dP_D dt_d p_D(r_D,t_D) \) is given by
    \[
    p'_D(t_D, r_D) = \frac{1}{2} \exp \left[ -\frac{r_D^2}{4t_D} \right]
    \]
    
    - The so-called "log approximation" of the Exponential Integral solution is:
    \[
    p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4t_D}{\gamma r_D^2} \right] \quad \text{for } t_D \geq 25 \quad (\gamma=0.577216... \text{ Euler's Constant})
    \]
    
  - The "well testing" derivative function for the "log approximation" is given by
    \[
    p'_D(t_D, r_D) = \frac{1}{2}
    \]
    
  - "No-Flow" Outer Boundary: ("no-flow" at the outer boundary)
    \[
    p_D(t_D, r_D, r_{ed}) = \frac{1}{2} E_1 \left[ \frac{r_{ed}^2}{4t_D} \right] - \frac{1}{2} r_{ed} + \frac{1}{8t_D} \left[ \frac{r_{ed}^2}{4t_D} \right] \exp \left[ -\frac{r_{ed}^2}{4t_D} \right]
    \]
    
    - The "well testing" derivative function, \( p'_D = dP_D dt_d p_D(r_D,t_D) \) is given by
    \[
    p'_D(t_D, r_D, r_{ed}) = \frac{1}{2} \exp \left[ -\frac{r_{ed}^2}{4t_D} \right] + \frac{r_{ed}^2}{8t_D} \exp \left[ -\frac{r_{ed}^2}{4t_D} \right] - \frac{r_{ed}^2}{4t_D} \exp \left[ -\frac{r_{ed}^2}{4t_D} \right]
    \]
    
  - Constant Pressure Outer Boundary: "constant pressure" at the outer boundary
    \[
    p_D(t_D, r_D, r_{ed}) = \frac{1}{2} E_1 \left[ \frac{r_{ed}^2}{4t_D} \right] - \frac{1}{2} r_{ed} + \frac{1}{8t_D} \left( r_{ed}^2 - r_D^2 \right) \exp \left[ -\frac{r_{ed}^2}{4t_D} \right]
    \]
    
    - The "well testing" derivative function, \( p'_D = dP_D dt_d p_D(r_D,t_D) \) is given by
    \[
    p'_D(t_D, r_D, r_{ed}) = \frac{1}{2} \exp \left[ -\frac{r_{ed}^2}{4t_D} \right] - \frac{1}{2} \exp \left[ -\frac{r_{ed}^2}{4t_D} \right] + \frac{1}{8t_D} \left( r_{ed}^2 - r_D^2 \right) \exp \left[ -\frac{r_{ed}^2}{4t_D} \right]
    \]
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- Using the conversion factors given below, you should be able to "convert" the dimensionless solutions to Darcy, SI, or "field" units as necessary. The dimensionless variables and the appropriate conversions are given below.

\[
\text{Dimensionless Radius} \quad r_D = r/r_w
\]
\[
\text{Dimensionless Time} \quad t_D = t_{DC} \frac{kt}{\phi \mu_c r_w^2}
\]
\[
\text{Dimensionless Pressure} \quad p_D = p_{DC} \frac{k h}{q B \mu} (p_i - p_r)
\]

Conversion Constants for the \( t_{DC} \) and \( p_{DC} \) Functions

<table>
<thead>
<tr>
<th>Constant</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{DC} )</td>
<td>1</td>
<td>2.637 \times 10^{-4}</td>
<td>3.557 \times 10^{-6}</td>
</tr>
<tr>
<td>( p_{DC} )</td>
<td>2\pi</td>
<td>7.081 \times 10^{-3}</td>
<td>5.356 \times 10^{-4}</td>
</tr>
<tr>
<td>( p_{DCr} = 1/p_{DC} )</td>
<td>1/(2\pi)</td>
<td>141.2</td>
<td>1867.1</td>
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- Be able to develop the results given below for transient radial flow (infinite-acting homogeneous reservoir). Be able to show all steps and details.

- Exponential Integral solutions:
  - Reservoir Equation: (skin effects are NOT included)
    \[
    p_i - p_r = 70.6 \frac{q B \mu}{k h} E_1 \left( 948 \frac{\phi \mu_c r_w^2}{k t} \right)
    \]
  - Wellbore Equation: (skin effects are included)
    \[
    p_i - p_{wf} = 70.6 \frac{q B \mu}{k h} \left[ E_1 \left( 948 \frac{\phi \mu_c r_w^2}{k t} \right) + 2s \right]
    \]

- "Log approximation" solutions:
  - Reservoir Equation: (skin effects are NOT included)
    \[
    p_i - p_r = 70.6 \frac{q B \mu}{k h} \ln \frac{1}{1688 \frac{\phi \mu_c r_w^2}{k t}}
    \]
  - Wellbore Equation: (skin effects are included)
    \[
    p_i - p_{wf} = 70.6 \frac{q B \mu}{k h} \left[ \ln \frac{1}{1688 \frac{\phi \mu_c r_w^2}{k t}} + 2s \right]
    \]
  - Wellbore Equation: (log_{10} form [\ln(x)=\ln(10)\log(x)], skin effects are included)
    \[
    p_i - p_{wf} = 162.6 \frac{q B \mu}{k h} \left[ \log \frac{1}{1688 \frac{\phi \mu_c r_w^2}{k t}} + 0.8686s \right]
    \]
Lecture Outline:

- Development of solutions in the real domain
  - Discussion of solution methods
    - Boltzmann transform: Infinite-acting case only—application of the Boltzmann transform is problem specific, and cannot be generalized.
    - Laplace transform: (most general approach (see paper SPE 25479))
      a. "Infinite-Acting" Reservoir Case
      b. "No-Flow" Outer Boundary
      c. Constant Pressure Outer Boundary
  - Applications
    - Modelling of well performance
      a. Transient (infinite-acting) and pseudosteady-state well behavior (review the attached solution plots).
      b. Variable-rate superposition.
      c. Constant wellbore pressure performance.
    - Development of short and long-term analysis relations
      a. $E_1(x)$ and $\ln(x)$ solutions for the infinite-acting reservoir case.
      b. $p(r,t)$ for all times for the "no-flow" outer boundary case (not just pseudosteady-state performance as we previously derived).
      c. $p(r,t)$ for all times and late-time steady-state performance for the constant pressure outer boundary case.

- Development of analysis relations for transient radial flow (no wellbore storage)
  - $E_1(x)$ (line source) solution for $p_D$ and $\exp(x)$ solution for $p_D'$.
  - Log approximation of the line source solution—semilog analysis relations.

Reading Assignment:

*Text Reading*:

- Review Chapters 1 and 2 of the Lee, et al. *Well Testing* text, 2nd edition, as well as Appendices A, B, and C. (these are similar to these handout notes)
  - Chapter 7—The Constant Terminal Rate Solution of the Diffusivity Equation and its Application to Oilwell Testing.
  - Chapter 4—Oilwell Testing (Sections 4.5-4.8)

*Derivations*:

- Review attached notes.
  - Solution of the Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case.
  - Development of the Analysis Relations for Infinite-Acting Radial Flow.
  - Example Pressure Drawdown Test Analysis: (Lee text (1st edition), Example 3.2).

*Reference Articles*:

Log-log Plot: Constant Well Rate Solutions for an "Infinite-Acting" Reservoir: Dimensionless Pressure Solutions—Radial Flow Case

Semilog Plot: Constant Well Rate Solutions for an "Infinite-Acting" Reservoir: Dimensionless Pressure Solutions—Radial Flow Case
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)

Semilog Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir—Various rd: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Constant Pressure Outer Boundary: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Wellbore Pressure Solutions for a Bounded Circular Reservoir: Dimensionless Rate Functions—Radial Flow Case (SPE 25479)
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**Fundamental Identities: Exponential Integral Function, E₁(x)**

- Functional definitions of $E_i(x)$ and $E_1(x) = -E_i(-x)$

$$E_i(x) = \int_{-\infty}^{\infty} \frac{e^{-t}}{t} \, dt \quad \text{and} \quad E_1(x) = \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt$$

- Series expansion for $E_1(x) = -E_i(-x)$:

$$E_1(x) = -\gamma - \ln(x) - \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k k!} \quad (x > 0) \quad (\gamma = 0.57721 56649 \ldots \text{Euler's constant})$$

- Derivative identity for $E_1(x)$:

$$\frac{d}{dx} E_1(x) = -\frac{e^{-x}}{x}$$

**Computational Formulas: Exponential Integral Function, E₁(x)**

- Computational Formulas for the Exponential Integral, $E_1(x)$: These relations are taken from Abramowitz and Stegun, *Handbook of Mathematical Functions*

  - **Eq. 5.1.53:** $(0 \leq x \leq 1)$

    $$E_1(x) + \ln x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \varepsilon(x)$$

    $$|\varepsilon(x)| < 2 \times 10^{-7}$$

    $$a_0 = -0.57721 566 \quad a_1 = 0.99999 193 \quad a_3 = 0.05519 968$$

    $$a_2 = -0.24991 055 \quad a_4 = -0.00976 004 \quad a_5 = 0.00107 857$$

  - **Eq. 5.1.56:** $(1 \leq x \leq \infty)$

    $$x e^x E_1(x) = \frac{x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 + \varepsilon(x)}{x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4}$$

    $$|\varepsilon(x)| < 2 \times 10^{-8}$$

    $$a_1 = 8.57332 87401 \quad b_1 = 9.57332 23454$$

    $$a_2 = 18.05901 69730 \quad b_2 = 25.63295 61486$$

    $$a_3 = 8.63476 08925 \quad b_3 = 21.09965 30827$$

    $$a_4 = 0.26777 37343 \quad b_4 = 3.95849 69228$$

- $|\varepsilon(x)|$ is the average error for a particular approximation, do **NOT** add this value to the computations.
Development of the Analysis Relations for Infinite-Acting Radial Flow

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Introduction: Cylindrical and Line Source Solutions

Our objective is to develop relations for the analysis of pressure and time data based on the constant rate pressure-time solutions for infinite-acting radial flow. While not necessarily obvious at this point, our focus is on the "log approximation" of the line source solution (i.e., the \( E_1(x) \) relation). From a practical standpoint, we can only analyze data graphically when the model (i.e., the solution) yields relations which can be interpreted on a particular plot (e.g., as a straight-line trend or as some other graphical feature).

We begin by discussing the cylindrical source solution—which can not be reduced to a data analysis formulation. This is the truly exact solution for radial flow and should be used to model performance—either via tabulated solutions or using a computer program.

We next discuss the "line" source solutions (i.e., the \( E_1(x) \) relation and its "log approximation"), where we can use these results as starting points for the development of relations which can be used for data analysis.

Cylindrical Source Solution: (Exact Solution)

The cylindrical source solution is given by van Everdingen and Hurst as

\[
p_D(t_D, r_D = 1) = \frac{4}{\pi^2} \int_0^\infty \frac{1-e^{-u^2t_D}}{u^3}\frac{1}{J_1^2(u)+Y_1^2(u)} \, du \tag{1}
\]

Obviously the form given by Eq. 1 is not convenient for computations, let alone graphical analysis of data. In short, Eq. 1 can be used in a computer algorithm to predict or match well performance, but this result cannot be used for graphical data analysis. We must probe further for a viable data analysis solution.

Line Source Solution: (1st Approximate Solution)

The line source solution is derived in the attached handout notes and the results are given below. The dimensionless pressure result \( p_D \) for the line source solution is given as

\[
p_D(t_D, r_D) = \frac{1}{2} \left[ \frac{r_D^2}{4t_D} \right] \quad \text{for } t_D \geq 10 \quad \text{(The Exponential Integral Solution)} \tag{2}
\]

And the "well testing" derivative of the line source solution is given by

\[
p_D'(t_D, r_D) = \frac{1}{2} \exp \left[ -\frac{r_D^2}{4t_D} \right] \quad \text{(Derivative of the Exponential Integral Solution)} \tag{3}
\]

The form of the \( p_D \) result given by Eq. 2 (the Exponential Integral solution) is "closed form", but the Exponential Integral is at best expressed with a truncated infinite series given by a single logarithm term (i.e., the log approximation solution, which is given later). In other words, the form given by Eq. 2 is unlikely to yield any sort of graphical analysis relations (except for the single-term logarithmic expansion, which is discussed below).

However, Eq. 3, the derivative of Eq. 2 (with respect to the logarithm of \( t_D \)), has been proposed as a possible analysis relation for pressure derivative data. While this may sound encouraging, we believe that the utility of the pressure derivative lies mainly in the interpretation of particular flow regimes (e.g., wellbore storage, radial flow, and boundary effects). In short, the analysis of pressure derivative data using Eq. 3 would be severely limited by wellbore storage and skin effects, which are at best interpreted, not quantified, using early-time well test data.
"Log Approximation" Solution: (2nd Approximate Solution)

The "log approximation" solution is derived in the attached notes and the results are given below. The dimensionless pressure \( p_D \) result for the log approximation solution is

\[
p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4}{e^{\gamma \frac{t_D}{r_D^2}}} \right] \text{ for } t_D \geq 25 \quad \text{(The "Log Approximation" Solution)} \quad (4)
\]

where \( \gamma = 0.577216 \ldots \text{ Euler's Constant.} \) In addition, the "well testing" derivative of the "log approximation" solution is given by

\[
p'_D(t_D, r_D) = \frac{1}{2} \quad \text{(Derivative of the "Log Approximation" Solution)} \quad (5)
\]

The so-called "log approximation" (Eq. 4) serves as the starting point for all modern well test analysis and this result clearly shows a pressure versus logarithm of time relationship—which can be used as a basis for plotting data and determining reservoir properties. We will dimensionalize Eq. 4 and develop explicit relations for analyzing well test data which exhibit radial flow behavior.

Development of Computational and Analysis Relations: (Field Units)

The definition of dimensionless radius, \( r_D \), is given by

\[
r_D = r / r_w \quad \text{.......................................................... (6)}
\]

The definition of dimensionless time, \( t_D \), for the field units case, is given by

\[
t_D = 0.0002637 \frac{kt}{\phi \mu c r_w^2} \quad \text{................................. (7)}
\]

The definition of dimensionless pressure, \( p_D \), for the field units case, is given by

\[
p_D = \frac{1}{141.2} \frac{kh}{qB \mu} (p_i - p_r) \quad \text{.................................................. (8)}
\]

Substituting Eqs. 6-8 into Eq. 2, we have

\[
p_i - p_r = \frac{141.2}{2} \frac{qB \mu}{kh} E_1 \left[ \frac{\phi \mu c t}{4(2.637 \times 10^{-4})k} \right] \quad \text{.................................................. (9)}
\]

Reducing Eq. 9, we obtain

\[
p_i - p_r = 70.6 \frac{qB \mu}{kh} E_1 \left[ 948 \frac{\phi \mu c r_w^2}{k} \frac{r^2}{t} \right] \quad \text{.................................................. (10)}
\]

Eq. 10 is useful for calculating pressure distributions within the reservoir and for "interference testing," where an "observation well" is used to monitor the influence of an "active well" that is under production or injection. As such, Eq. 2 (i.e., the dimensionless form of Eq. 10) is used to generate a "type curve" solution for interference test analysis, but we cannot use Eq. 10 to develop a data analysis plot such as a semilog or cartesian plot.

Evaluating Eq. 10 at \( r = r_w \), and including the skin factor term, \( s \), (which we consider to be a wellbore effect), we have

\[
p_i - p_{wf} = 70.6 \frac{qB \mu}{kh} E_1 \left[ 948 \frac{\phi \mu c r_w^2}{k} \frac{1}{t} \right] + 2s \quad \text{.................................................. (11)}
\]

Eq. 11 cannot be used directly to evaluate data as the \( E_1 \) function cannot be written explicitly (e.g., like a logarithm, sine, or cosine function)—therefore, Eq. 11 remains a "modelling" relation and can be used as a computational tool (like Eq. 10). We will use the "log approximation" to reduce the \( E_1 \) function to a logarithm, and hence, to develop data analysis relations.
Substituting Eqs. 6-8 into Eq. 4, we have

\[ p_i - p_r = \frac{141.2}{2} \frac{q B \mu}{k h} \ln \left[ \frac{4 (2.637 \times 10^4)}{\exp(0.577216...)} k \frac{t}{\phi \mu c_t r^2} \right] \] ..........................(12)

Reducing Eq. 9, we obtain

\[ p_i - p_r = 70.6 \frac{q B \mu}{k h} \ln \left[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r^2} \right] \] ..........................(13)

Eq. 13 is the "approximate" form of Eq. 10. However, this logarithm formulation does not accurately represent the pressure behavior at significant distances away from the wellbore.

Using Eq. 13 we can however develop the concept of the "radius of investigation," by setting \( p_r = p_i \) and solving the resulting relation for radius. Setting \( p_r = p_i \), and reducing, we obtain

\[ \ln \left[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r^2} \right] = 0 \]

Exponentiating both sides of this relation, we have

\[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r^2} = 1 \]

Solving for the radius of investigation, \( r_{inv} \), gives us

\[ r_{inv} = \sqrt{\frac{k}{1688 \phi \mu c_t}} t = 2.434 \times 10^{-2} \sqrt{\frac{k}{\phi \mu c_t}} t \] ..........................(14)

Returning to our development of analysis relation, we evaluate Eq. 13 at \( r = r_w \), and include the skin factor term, \( s \), (where \( s \) is only a wellbore effect). This gives us

\[ p_i - p_{wf} = 70.6 \frac{q B \mu}{k h} \left[ \ln \left[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r_w^2} \right] + 2s \right] \] ..........................(15)

Converting Eq. 15 to "common" logarithms (i.e., the logarithm to base 10, \( \log = \log_{10} \)), we note that \( \ln(x) = \ln(10) \log(x) \). Substituting the \( \ln(x) - \log(x) \) identity into Eq. 15 gives us

\[ p_i - p_{wf} = 70.6 (2.303) \frac{q B \mu}{k h} \left[ \log \left[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r_w^2} \right] + \frac{2}{(2.303)} s \right] \]

Reducing the numerical constants, we have

\[ p_i - p_{wf} = 162.6 \frac{q B \mu}{k h} \left[ \log \left[ \frac{1}{1688} k \frac{t}{\phi \mu c_t r_w^2} \right] + 0.8686s \right] \] ..........................(16)

We note that Eq. 16 is the basis for radial flow analysis, where this analysis approach uses a pressure versus logarithm of time plot to analyze data (note that we use \( \log(x) \) rather than \( \ln(x) \) for our "engineering" analysis). Our remaining work focuses on the reduction of Eq. 16 into analysis components, in particular, relations for estimating the formation permeability, \( k \), and the near-well skin factor, \( s \), from the slope and intercept (respectively) of the data on the semi-log plot.

Expanding the logarithms in Eq. 16, we obtain

\[ p_i - p_{wf} = 162.6 \frac{q B \mu}{k h} \left[ \log(t) + \log \left[ k \frac{t}{\phi \mu c_t r_w^2} \right] + \log \left[ \frac{1}{1688} \right] + 0.8686s \right] \]

Reducing the remaining numerical constants gives us

\[ p_i - p_{wf} = 162.6 \frac{q B \mu}{k h} \left[ \log(t) + \log \left[ k \frac{t}{\phi \mu c_t r_w^2} \right] - 3.2275 + 0.8686s \right] \] ..........................(17)
Rearranging Eq. 17 into a more convenient form, we have

\[ p_{wf} = \left( p_i - 162.6 \frac{q B \mu}{k h} \left( \log \frac{k}{\phi \mu c r_w^2} - 3.2275 + 0.8686 s \right) \right) - 162.6 \frac{q B \mu}{k h} \log(t) \] .......(18)

Eq. 18 is of the form:

\[ p_{wf} = p_{1hr} - m \log(t) \] .......(19)

where

\[ p_{1hr} = p_i - m \left( \log \frac{k}{\phi \mu c r_w^2} - 3.2275 + 0.8686 s \right) \] .......(20)

and

\[ m = 162.6 \frac{q B \mu}{k h} \] .......(21)

Solving Eq. 21 for the formation permeability, \( k \), we obtain

\[ k = 162.6 \frac{q B \mu}{m h} \] .......(22)

Solving Eq. 20 for the skin factor, \( s \), we have

\[ s = 1.1513 \left[ \frac{(p_i - p_{1hr})}{m} - \log \frac{k}{\phi \mu c r_w^2} + 3.2275 \right] \] .......(23)

In addition to the analysis of the pressure-time profile, the log approximation solution also provides a useful means for relating the slope of the semilog straight line and the well testing derivative function, \( \Delta p' \). We do not use \( \Delta p' \) as an "analysis" relation, but we can use \( \Delta p' \) to directly couple the semilog and log-log data analysis.

Substituting Eqs. 7 and 8 into Eq. 5, and reducing terms, we have

\[ \Delta p' = t \frac{d(\Delta p)}{dt} = 70.6 \frac{q B \mu}{k h} \] .......(24)

We immediately note from Eq. 24 (as well as Eq. 5) that the pressure derivative function is constant during undistorted transient radial flow. "Undistorted" is typically used to denote that there are no wellbore storage effects, or that wellbore storage effects have diminished. This constant behavior of the pressure derivative function will be exploited later when we discuss type curve analysis, where we use the radial flow regime (if it exists) to anchor our analysis and interpretations.

Substituting Eq. 21 into Eq. 24, and solving for the slope term, \( m \), we obtain

\[ m = 2.303 \Delta p' \] (for undistorted transient radial flow) .......(25)

Eq. 25 is only valid for undistorted transient radial flow, but this result is potentially very useful—the constant value of \( \Delta p' \) could be read from the log-log (type curve) plot, then used to guide and validate the semilog pressure-time analysis. Note that the 2.303 factor arises from the use of \( \log(x) \) in the semilog analysis relations (2.303 = \( \ln(10) \)).

References

Example Pressure Drawdown Test Analysis:
(Lee text (1st edition), Example 3.2)
Example Pressure Drawdown Test Analysis:
(Lee text (1st edition), Example 3.2)

**Given Data:** (Lee text (1st edition), Example 3.2)
These data are taken from Example 3.2 in the Lee text, *Well Testing*. These data are for a constant rate pressure "drawdown" test run on an oil (liquid) well.

**Reservoir properties:**
\( \phi = 0.039 \quad r_w = 0.198 \text{ ft} \quad c_f = 17 \times 10^{-6} \text{ psia}^{-1} \quad h = 69 \text{ ft} \)

**Oil properties:**
\( B_o = 1.136 \text{ RB/STB} \quad \mu_o = 0.8 \text{ cp} \)

**Production parameters:**
\( p_i = 4412 \text{ psia} \quad q_o = 250 \text{ STB/D} \)

**Test Data and Data Functions:**

<table>
<thead>
<tr>
<th>( t ), hr</th>
<th>( p_{wsi}, \text{ psia} )</th>
<th>( \Delta p, \text{ psi} )</th>
<th>( \Delta p', \text{ psi} )</th>
<th>( p_{wsi}, \text{ psia} )</th>
<th>( \Delta p_i, \text{ psi} )</th>
<th>( \Delta p_i', \text{ psi} )</th>
</tr>
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Semilog Plot: Pressure Data

Log-Log Plot: Pressure and Pressure Derivative Data
Graphical Analysis (Semilog and Log-log Data Plots): (Lee text (1st edition), Example 3.2)

We now focus our attention on the analysis of the data shown on the semilog and log-log plots (on the previous pages). In terms of interpretation, we must first look at the log-log plot and determine whether or not we actually see a horizontal trend in the "well testing" pressure derivative function, $\Delta p'$, and if we do see such a trend, then we must "mark" the start and end of the trend.

We do note a horizontal trend in $\Delta p'$ and we then construct a horizontal line at approximately 30 psi, as well as place the "start" and "end" markers for radial flow at 12 and 120 hours, respectively. The placement of these "markers" is somewhat subjective, so you should be as consistent as possible.

Therefore, our first "results" are:

$$\Delta p_{\text{radial flow}} = 30 \text{ psi} \quad (\text{with } t_{\text{start}}=12 \text{ hours and } t_{\text{end}}=120 \text{ hours})$$

At this point we only use the log-log plot to verify the existence of a horizontal trend in the $\Delta p'$ function, and if such a trend exists, we then estimate the magnitude and duration of the constant $\Delta p'$ trend. We then "convert" the $\Delta p'$ value to an equivalent slope, $m$, on the semilog plot (recall that the sign convention on $m$ is always positive). Recalling the appropriate result, we have

$$m = 2.303 \Delta p'$$

Computing, we obtain

$$m = 2.303 (30 \text{ psi}) = 69.1 \text{ psia/cycle} \quad (m \text{ result from the log-log plot})$$

We now have an estimate of $m$ from the log-log plot, and for consistency, we must check the semilog data and compare to see if these data exhibit a similar slope, $m$. The slope on a semilog plot is obtained using the difference of two pressure values which are taken one log cycle apart, where these pressure points must lie on the straight-line trend, (i.e., the straight line that you draw through the data), or on the extrapolation of the straight-line trend. From the semilog plot we have:

$$p_{10\text{hr}} = 3580 \text{ psia}$$
$$p_{100\text{hr}} = 3510 \text{ psia}$$

Computing the slope from these data, we have

$$m = (3580 \text{ psia}) - (3510 \text{ psia}) = 70 \text{ psia/cycle} \quad (m \text{ result from the semilog plot})$$

We compare the slope values ($m$'s) from the semilog and log-log analyses (70 and 69.1, psia/cycle) and conclude that we have good (actually very good) agreement. In practice, you are encouraged to "force" these results to be the same, by determining the most representative estimate of the horizontal $\Delta p'$ trend (log-log plot) and the best estimate of the slope of the pressure versus logarithm of time plot, then averaging or judiciously choosing the "forced" $m$ value.

What must now be plain is that if we cannot confirm the existence of a horizontal $\Delta p'$ trend (i.e., radial flow behavior), then it is possible (or even probable) that semilog analysis is not applicable. Note that the $\Delta p'$ data are of higher resolution than the pressure data and if radial flow is not indicated in the $\Delta p'$ trend, then this flow regime probably does not exist.

In addition to obtaining the slope of the pressure data on the semilog plot, we also extrapolate the straight-line trend to 1 hour to obtain the pressure at 1 hour, $p_{1\text{hr}}$—which is then used to estimate the skin factor. Extrapolating the straight-line trend on the semilog plot, we have

$$p_{1\text{hr}} = 3652 \text{ psia} \quad (p_{1\text{hr}} \text{ result from the semilog plot})$$
Pressure-Semilog Time Analysis Relations: (Lee text (1st edition), Example 3.2)

From our earlier efforts, we developed the following analysis relations, which are based on
the log approximation solution for the radial flow diffusivity equation. These relations are:

Formation Permeability:

\[ k = 162.6 \frac{qB \mu}{m h} \] \hspace{2cm} (1)

Near-Well Skin Factor:

\[ s = 1.1513 \left[ \frac{(p_1 - p_{1hr})}{m} \right] \log \left\{ \frac{k}{\phi \mu_c r_w^2} \right\} + 3.2275 \] \hspace{2cm} (2)

Radius of Investigation:

\[ r_{inv} = \sqrt{\frac{1}{1688} \frac{k}{\phi \mu_c} t} = 2.434 \times 10^{-2} \sqrt{\frac{k}{\phi \mu_c} t} \] \hspace{2cm} (3)

Analysis Results: (Lee text (1st edition), Example 3.2)

Formation Permeability:

Using Eq. 1 and our graphical analysis results, we have

\[ k = 162.6 \frac{250 \text{ STB/D}(1.136 \text{ RB/STB})(0.8 \text{ cp})}{(70 \text{ psia/cycle})(69 \text{ ft})} \]

or

\[ k = 7.65 \text{ md} \]

Near-Well Skin Factor:

Using Eq. 2 and our graphical analysis results, we have

\[ s = 1.1513 \left[ \frac{4412 \text{ psia} - 3652 \text{ psia}}{70 \text{ psia/cycle}} \right] \log \left\{ \frac{7.65 \text{ md}}{(0.039)(0.8 \text{ cp})(17 \times 10^{-6} \text{ psia}^{-1})(0.198 \text{ ft})^2} \right\} + 3.2275 \]

or

\[ s = 6.35 \]

Radius of Investigation:

Using Eq. 3 and our graphical analysis results (as well as the computed value of the
formation permeability), we now estimate the "radius of investigation" at \( t = 120 \text{ hrs} \) (the
end of transient radial flow behavior). This gives

\[ r_{inv} = \sqrt{\frac{1}{1688} \frac{(7.65 \text{ md})}{(0.039)(0.8 \text{ cp})(17 \times 10^{-6} \text{ psia}^{-1})(120 \text{ hr})}} \]

or

\[ r_{inv} = 1013 \text{ ft} \]

Using the radius of investigation, we can roughly approximate reservoir size (area).
However, what we are really estimating in this particular case is the distance to the onset
of boundary effects, which is not necessarily the same as the response for a fully closed
reservoir boundary. With these limitations noted, we have

\[ A = \pi r^2 \]

or

\[ A = \pi (1013 \text{ ft})^2 = 74.0 \text{ acres} \text{ (where 1 acre = 43,560 ft}^2 \)