Petroleum Engineering 324
Well Performance
Pseudosteady-State Flow in a Circular Reservoir

_Growth is the only evidence of life._
— John Henry Newman (1864)

**Topic:** Pseudosteady-State Flow in a Circular Reservoir

**Objectives:** (things you should know and/or be able to do)

- Be familiar with and be able to derive the single-phase, pseudosteady-state flow relations for compressible liquids in a radial flow system. In particular, you should be able to derive the following:

  - **$p_r - p_{wf}$ flow relation:**
    \[
    p_r = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{k h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left( \frac{r_e}{r_w} \right) - \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} + s \right]
    \]

  - **$\bar{p} - p_{wf}$ flow relation:**
    \[
    \bar{p} = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{k h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] \quad \text{(Well Centered in a Circular Reservoir)}
    \]
    \[
    \bar{p} = p_{wf} + \frac{1}{c_r} \frac{q B \mu}{k h} \left[ \frac{1}{2} \ln \left( \frac{4 A}{\gamma r_w^2 C_A} \right) + s \right] \quad \text{(General Formulation)}
    \]

where:
\[
\gamma = 0.577216... \quad \text{(Euler's Constant)}
\]
\[
C_A = \text{Dietz "shape factor" (e.g., } C_A = 31.62 \text{ for a well in a circular reservoir)}
\]

- **$p(r,t)$ solution for pseudosteady-state flow conditions:**

  \[
  p_r = p_i - \frac{q B \mu}{2 \pi k h} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \frac{(r_e^2 - r^2)}{(r_e^2 - r_w^2)} \frac{3}{4} \right] - \frac{q B}{V_p c_t} t \quad \text{(Darcy Units)}
  \]

  \[
  p_r = p_i - 141.2 \frac{q B \mu}{k h} \left[ \ln \left( \frac{r_e}{r} \right) + \frac{1}{2} \frac{(r_e^2 - r^2)}{(r_e^2 - r_w^2)} \frac{3}{4} \right] - 5.615 \frac{q B}{V_p c_t} t \quad \text{(Field Units)}
  \]

  where for field units we use $t$ in days, and $V_p$ in ft$^3$.

- **$p(r_w,t)$ solution for pseudosteady-state flow conditions:**

  \[
  p_{wf} = p_i - 141.2 \frac{q B \mu}{k h} \left[ \ln \left( \frac{r_e}{r_w} \right) - \frac{3}{4} + s \right] - 5.615 \frac{q B}{V_p c_t} t \quad \text{(Field Units)}
  \]

**Table of Units Conversions:** (for the equations given above)

<table>
<thead>
<tr>
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- Be able to derive the $r(\bar{p})$ result for a radial system and be able to relate this result to reservoir performance.
- Be able to sketch pressure distributions during steady-state and pseudosteady-state flow conditions for a radial system.
Lecture Outline:

- Development of pseudosteady-state flow relations (attached notes)
  - Material balance considerations.
  - $p - p_{wf}$ flow relation.
  - $\bar{p} - p_{wf}$ flow relation.
  - $p(r,t)$ solution for pseudosteady-state flow conditions.
- Review illustrative plots of pseudosteady-state flow performance.

Reading Assignment:

   - Chapter 6—Well Inflow Equations for Stabilized Flow Conditions.
- Review the attached notes.
  - Derivation of the Pseudosteady-State Flow Relations for a Radial System.
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Derivation of the Pseudosteady-State Flow Relations for a Radial System

- Physical Considerations
- Material Balance Considerations
- Pseudosteady-State Solutions of the Radial Flow Diffusivity Equation
  - $p_r - p_{wf}$ Formulation
  - $\bar{p} - p_{wf}$ Formulation
  - $\bar{p} - \bar{r}$ Concept (i.e., $r(\bar{p})$)
  - $p(r,t)$ Solution for Pseudosteady-State Flow Conditions

(from Petroleum Engineering 412 Course Notes -- 1997)
Pseudosteady-State Flow in a Radial System

Physical Considerations

The physical concept of pseudosteady-state is defined as the condition where the pressure at all points in the reservoir changes at the same rate. Mathematically, this condition is given by:

\[
\frac{d}{dt} \left[ p(r,t) \right] = \text{constant}
\]

(1)

Physically, this condition is illustrated by

Where,

- \( p_{wf} \) = wellbore pressure at time \( t_i \)
- \( \bar{p} \) = average reservoir pressure at time \( t_i \)
- \( p_e \) = external boundary pressure at time \( t_i \)

Objectives: (for pseudosteady-state flow conditions)

1. Derive a pressure change relation (i.e., \( \frac{dp}{dt} \)) using the material balance relation.
2. Derive a relation between the average reservoir pressure, \( \bar{p} \), and the wellbore flowing pressure, \( p_{wf} \).
3. Derive a pressure-radius-time (i.e., \( p(r,t) \)) solution of the radial flow diffusivity equation.
Material Balance Considerations

Recalling the material balance relation for a slightly compressible liquid, we have

$$\bar{p} = \rho_i - \frac{B}{N B_i c_t} N p$$

or, noting that $N B_i = V_p$ we obtain

$$\bar{p} = \rho_i - \frac{B}{V_p c_t} N p$$

For a cylindrical reservoir, we have

$$N p = \phi h \pi (r_e^2 - r_w^2)$$

Substituting Eq. 4 into Eq. 3 gives us

$$\bar{p} = \rho_i - \frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} N p$$

Recalling the definition of the cumulative production, $N p$, we have

$$N p = \int_0^t q(t) \, dt$$

Therefore,

$$\frac{d N p}{dt} = q$$

Taking the derivative of Eq. 5 with respect to time

$$\frac{d \bar{p}}{dt} = -\frac{B}{\phi h \pi (r_e^2 - r_w^2) c_t} q$$

Note: all derivations are in "Darcy" Units unless otherwise noted.
Pseudosteady-State Flow Solutions for the Radial Flow Diffusivity Equation

The governing partial differential equation for flow in porous media is called the "diffusivity" equation. The diffusivity equation for a "slightly compressible liquid" is given (without derivation) as

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \frac{\phi u c t}{k} \frac{dp}{dt} \quad (9)$$

The significant assumptions made in Eq. 9 are:
- slightly compressible liquid (constant compressibility)
- constant fluid viscosity
- single-phase liquid flow
- gravity and capillary pressure are neglected
- constant permeability
- horizontal radial flow (no vertical flow)

If we assume that the flowrate, q, is constant then \( dp/dt \) is also constant—hence \( dp/dt \) is constant as well. Assuming \( q \) is constant, then

$$\frac{dp}{dt} = \frac{dp}{dt} = -\frac{B}{\phi \pi (r_e^2 - r_w^2) c_t} \quad q = \text{constant} \quad (10)$$

Substituting Eq. 10 into Eq. 9 (we note that partial derivatives are now expressed as ordinary derivatives), this gives

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = \left[ \frac{\phi u c t}{k} \right] \left[ -\frac{B}{\phi \pi (r_e^2 - r_w^2) c_t} \right]$$

or, reducing

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -\frac{q B m}{\pi k h (r_e^2 - r_w^2)} \quad (11)$$
Defining
\[ c = \frac{4\pi u}{k\gamma (r_e^2 - r_e^2)} \]  
(12)

Substituting Eq. 12 into Eq. 11 we have
\[ \frac{1}{r} \frac{d}{dr} \left[ r \frac{dp}{dr} \right] = -c \]  
(13)

Separating
\[ d \left[ r \frac{dp}{dr} \right] = -cr \, dr \]

Integrating (indefinite integration)
\[ \int d \left[ r \frac{dp}{dr} \right] = -c \int r \, dr \]

Completing
\[ r \frac{dp}{dr} = -c \frac{r^2}{2} + c_1 \]  
(14)

Multiplying through Eq. 14 by \( 1/r \) gives us
\[ \frac{dp}{dr} = -c \frac{r}{2} + c_1 \frac{1}{r} \]  
(15)

For pseudo-steady-state we assume a closed reservoir, that is
\[ \left[ \frac{dp}{dr} \right]_{r_e} = 0 \]
or
\[ \left[ \frac{dp}{dr} \right]_{r_e} = 0 = -c \frac{r_e}{2} + c_1 \frac{1}{r_e} \]

Solving for \( c_1 \) gives
\[ c_1 = \frac{c r_e^2}{2} \]  
(16)
Substituting Eq. 16 into Eq. 15 gives

$$\frac{dp}{dr} = \frac{c}{2} \left[ \frac{r^2 - r}{r} \right]$$

(17)

Multiplying through Eq. 17 by $dr$ gives us

$$dp = \frac{c}{2} \left[ \frac{r^2 - r}{r} \right] dr$$

Integrating across the reservoir, we have

$$\int_{P_w}^{P_r} dp = \frac{c}{2} \int_{r_w}^{r} \left[ \frac{r^2 - r}{r} \right] dr$$

(18)

Completing the integration

$$P_r - P_{wf} = \frac{c}{2} \left[ \frac{r^2 \ln(r)}{r_w} \right]_{r_w}^{r} - \frac{r^2}{2}$$

or

$$P_r - P_{wf} = \frac{c}{2} \left[ \frac{r^2 \ln \left( \frac{r}{r_w} \right)}{r_w} - \frac{1}{2} (r^2 - r_w^2) \right]$$

(19)

Recalling Eq. 12

$$c = \frac{98\mu}{\pi kh (r_w^2 - r_w^2)}$$

(20)

Substituting Eq. 20 into Eq. 19, we obtain

$$P_r - P_{wf} = \frac{98\mu}{2\pi kh (r_w^2 - r_w^2)} \left[ \frac{r^2 \ln \left( \frac{r}{r_w} \right)}{r_w} - \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_w^2 - r_w^2)} \right]$$

Expanding through with the $1/(r^2 - r_w^2)$ term gives

$$P_r - P_{wf} = \frac{98\mu}{2\pi kh} \left[ \frac{r^2}{(r_w^2 - r_w^2)} \ln \left( \frac{r}{r_w} \right) - \frac{1}{2} \frac{(r^2 - r_w^2)}{(r_w^2 - r_w^2)} \right]$$

(21)

Eq. 21 is our final result (in “Darcy” units).
Development of a $\bar{p}$-$P_{wf}$ Relation - Pseudostate-State Flow

In this section we develop the relationship between the average reservoir pressure, $\bar{p}$, and the wellbore flowing pressure, $P_{wf}$. The definition of the average reservoir pressure is given as

$$\bar{p} = \frac{\int_{r_w}^{r} p_r \, dv}{\int_{r_w}^{r} dv} \quad (22)$$

and for a cylindrical reservoir, we have

$$v = \phi h \pi (r^2 - r_w^2) \quad (23)$$

$$dv = \phi h \pi (2r) \, dr \quad (24)$$

Substituting Eq. 24 into Eq. 22 gives

$$\bar{p} = \frac{\phi h 2\pi}{\phi h \pi (r^2 - r_w^2)} \int_{r_w}^{r} p_r \, r \, dr$$

which reduces to

$$\bar{p} = \frac{2}{(r^2 - r_w^2)} \int_{r_w}^{r} p_r \, r \, dr \quad (25)$$

Solving Eq. 21 for $p_r$ gives us

$$p_r = P_{wf} + \frac{4 \mu}{2 \pi k h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \frac{(r_e^2 - r_w^2)}{(r_e^2 - r_w^2)} \right] \quad (26)$$
Substituting Eq. 26 into Eq. 25 gives

$$
\bar{P}_r = \frac{Z}{(r^2-r_w^2)} \int_{r_w}^{r} \left[ \frac{1}{r} - \frac{g \rho u}{\pi k h} \frac{r_w^2}{(r^2-r_w^2)} \ln \left( \frac{r}{r_w} \right) - \frac{1}{Z} \frac{(r^2-r_w^2)}{(r^2-r_w^2)} \right] r \, dr
$$

(27)

Separating,

$$
\bar{P}_r = \frac{Z}{(r^2-r_w^2)} \left[ \frac{1}{r} \int_{r_w}^{r} r \, dr \right]

+ \frac{Z}{(r^2-r_w^2)} \frac{g \rho u}{\pi k h} \frac{r_w^2}{(r^2-r_w^2)} \int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr

- \frac{Z}{(r^2-r_w^2)} \frac{g \rho u}{\pi k h} \frac{1}{Z} \frac{1}{(r^2-r_w^2)} \int_{r_w}^{r} r^3 \, dr

+ \frac{Z}{(r^2-r_w^2)} \frac{g \rho u}{\pi k h} \frac{r_w^2}{Z} \left( \frac{1}{r^2-r_w^2} \right) \int_{r_w}^{r} r \, dr
$$

(28)

Isolating terms and evaluating each integral, we have

$$
\int_{r_w}^{r} r \, dr = \frac{1}{2} (r^2-r_w^2)
$$

(29)

$$
\int_{r_w}^{r} r^3 \, dr = \frac{1}{4} (r^4-r_w^4)
$$

(30)

$$
\int_{r_w}^{r} r \ln \left( \frac{r}{r_w} \right) \, dr = ?
$$

(31)

Obviously, the integral of the logarithm term will require a little work to resolve; we could simply look up the appropriate result in a suitable text—but deriving the required result will be enlightening.
Starting with the fundamental form of the logarithm integral, we have

$$\int x \ln(x/c) \, dx$$

... integration by parts

$$\int udv = uv - \int vdu$$

$$u = \ln(x/c), \quad dv = x \, dx$$

$$du = \frac{1}{x} \, dx, \quad v = \frac{1}{2} x^2$$

Then

$$\int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{2} \int x \, dx$$

Reducing

$$\int x \ln(x/c) \, dx = \frac{1}{2} x^2 \ln(x/c) - \frac{1}{4} x^2$$

Therefore

$$\int_{r_w}^{r} r \ln\left[ \frac{r}{r_w} \right] \, dr = \left[ \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} r^2 \right]_{r_w}^{r}$$

$$= \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} r^2 - \left[ \frac{1}{2} r_w^2 \ln\left[ \frac{r_w}{r_w} \right] - \frac{1}{4} r_w^2 \right]$$

or finally, we have

$$\int_{r_w}^{r} r \ln\left[ \frac{r}{r_w} \right] \, dr = \frac{1}{2} r^2 \ln\left[ \frac{r}{r_w} \right] - \frac{1}{4} \left( r^2 - r_w^2 \right) \quad (31)$$
Substituting Eqs. 29-31 into Eq. 28 gives

\[ Pr = \frac{Z}{(r^2 - r_w^2)} Pwf \frac{1}{Z} \left( r^2 - r_w^2 \right) \]

\[ + \frac{Z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{(r^2 - r_w^2)} \left[ \frac{1}{Z} r^2 \ln \left[ \frac{r}{r_w} \right] - \frac{1}{4} (r^2 - r_w^2) \right] \]

\[ - \frac{Z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{1}{Z} \left[ \frac{1}{Z} \right] \frac{1}{4} \left( r^4 - r_w^4 \right) \]

\[ + \frac{Z}{(r^2 - r_w^2)} \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{Z(r^2 - r_w^2)} \frac{1}{Z} \left( r^2 - r_w^2 \right) \]

Reducing

\[ Pr = Pwf + \frac{g_8 u}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{r_w^2}{(r^2 - r_w^2)} \left[ \frac{1}{Z} r^2 \ln \left[ \frac{r}{r_w} \right] - \frac{1}{4} (r^2 - r_w^2) \right] \]

\[ - \frac{g_8 u}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{1}{Z} \left[ \frac{1}{Z} \right] \frac{1}{4} \left( r^4 - r_w^4 \right) \left( r^2 + r_w^2 \right) \]

\[ + \frac{g_8 u}{2 \pi k h} \frac{Z}{(r^2 - r_w^2)} \frac{1}{Z} \left[ \frac{1}{Z} \right] \frac{r_w^2}{Z} \frac{(r^2 - r_w^2)}{Z} \]

Collecting

\[ Pr = Pwf + \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{(r^2 - r_w^2)} \left[ \frac{r^2}{Z} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \right] \]

\[ - \frac{g_8 u}{2 \pi k h} \frac{(r^2 + r_w^2)}{4 (r^2 - r_w^2)} + \frac{g_8 u}{2 \pi k h} \frac{r_w^2}{Z (r^2 - r_w^2)} \]

or "finally"

\[ Pr = Pwf \]

\[ + \frac{g_8 u}{2 \pi k h} \left[ \frac{r_w^2}{(r^2 - r_w^2)} \left[ \frac{r^2}{Z} \ln \left[ \frac{r}{r_w} \right] - \frac{1}{2} \right] - \frac{(r^2 + r_w^2)}{4 (r^2 - r_w^2)} + \frac{r_w^2}{Z (r^2 - r_w^2)} \right] \]

(32)
Eq. 32 (which is given in "Darcy" units) is our fundamental linking relation between the wellbore and average reservoir pressures during pseudosteady-state flow. However, \( \bar{p} \) (the average reservoir pressure at a given radius, \( r \)) is of little use—except as a rigorous "linking" relation for pressures in the reservoir.

In contrast, if we consider \( \bar{p}_r \) (i.e., \( p \) at \( r=r_e \)) we obtain the average reservoir pressure based on the entire reservoir volume. Such a result can be directly coupled with the material balance equation to develop a time-pressure relation for pseudosteady-state flow.

Evaluating Eq. 32 at \( r=r_e \) we have

\[
\bar{p} = \bar{p}_r = \bar{p}_{wf} + \frac{4\beta u}{2\pi k h} \left[ \frac{r_e^2}{(r_e^2-r_w^2)} \left[ \frac{r_e^2}{r_w^2} \ln \frac{r_e}{r_w} - \frac{1}{2} \right] - \frac{(r_e^2+r_w^2)}{4(r_e^2-r_w^2)} + \frac{r_w^2}{2(r_e^2-r_w^2)} \right] \tag{33}
\]

Assuming that \( r_e \gg r_w \), then

\[
\frac{r_e^2}{(r_e^2-r_w^2)} \approx 1 ; \quad \frac{(r_e^2+r_w^2)}{(r_e^2-r_w^2)} \approx 1 ; \quad \frac{r_w^2}{(r_e^2-r_w^2)} \approx 0
\]

Substituting these expressions into Eq. 33, we obtain

\[
\bar{p} = \bar{p}_{wf} + \frac{4\beta u}{2\pi k h} \left[ \ln \frac{r_e}{r_w} - \frac{1}{2} - \frac{1}{4} \right]
\]

or

\[
\bar{p} = \bar{p}_{wf} + \frac{4\beta u}{2\pi k h} \left[ \ln \frac{r_e}{r_w} - \frac{3}{4} \right] \tag{34}
\]
Summarizing our results so far (using generalized units systems)

Pressure at any Radius:

\[ P_r = P_{wf} + \frac{qB_u}{c_r k_h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r_e}{r_w} \right] - \frac{(r_e^2 - r_w^2)}{2(r_e^2 - r_w^2)} \right] \]  

Average Reservoir Pressure at any Radius:

\[ P_r = P_{wf} + \frac{qB_u}{c_r k_h} \left[ \frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{r_e}{r_w} \right] - \frac{1}{2} \right] - \frac{(r_e^2 + r_w^2)}{4(r_e^2 - r_w^2)} + \frac{r_w^2}{2(r_e^2 - r_w^2)} \]  

Average Reservoir Pressure at \( r_e \) (Volumetric Average Pressure):

\[ \bar{P} = P_{wf} + \frac{qB_u}{c_r k_h} \left[ \ln \left[ \frac{r_e}{r_w} \right] - \frac{3}{4} \right] \]  

For a general reservoir geometry, Eq. 38 becomes

\[ \bar{P} = P_{wf} + \frac{qB_u}{c_r k_h} \left[ \frac{1}{2} \ln \left[ \frac{4}{e} \frac{A_A}{r_w^2 C_A} \right] \right] \]  

where

\[ \gamma = 0.577216 \ldots \text{ Euler's Constant} \]

\[ C_A = \text{Dietz "shape factor" (e.g., } C_A = 31.62 \text{ for circular reservoir)} \]

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**φ-φ Concept**

An interesting (and possibly useful) result is the concept of \( \bar{p} \), which would be the location of the average reservoir pressure, \( \bar{p} \). This development is only rigorously valid for a vertical well centered in a bounded circular reservoir. A graphical illustration of this concept is shown below.

Mathematically, \( \bar{p} \) is defined by equating the \( \bar{p} \) relation (Eq. 35) with the average reservoir pressure identity, \( \bar{p} \), (Eq. 37). Equating Eqs. 35 and 37 gives

\[
\frac{r_e^2}{(r_e^2 - r_w^2)} \ln \left[ \frac{\bar{r}}{r_w} \right] - \frac{(r_e^2 - r_w^2)}{2(r_e^2 - r_w^2)} = \ln \left[ \frac{r_e}{r_w} \right] - \frac{3}{4}
\]

We now assume \( \frac{r_e^2}{(r_e^2 - r_w^2)} \approx 1 \) and \( \frac{r_w^2}{(r_e^2 - r_w^2)} \approx 0 \) gives

\[
0 = \ln \left[ \frac{\bar{r}}{r_w} \right] - \ln \left[ \frac{r_e}{r_w} \right] - \frac{\bar{r}^2}{2(r_e^2 - r_w^2)} + \frac{3}{4}
\]
Assuming that \( \Ree \gg \Rez \) (i.e., \( \Ree - \Rez \approx \Ree^2 \)) and rearranging, we have

\[
\ln \left( \frac{\Ree}{\Rez} \right) - \frac{\Ree^2}{2 \Rez^2} + \frac{3}{4} = 0
\]  

Defining a dimensionless radius, \( \Re_D \), we obtain

\[
\Re_D = \frac{\Ree}{\Rez}
\]  

Substituting Eq. 40 into Eq. 39 gives us

\[
\ln (\Re_D) - \frac{1}{2} \Re_D^2 + \frac{3}{4} = 0
\]  

Solving Eq. 41 for \( \Re_D \), we obtain

\[
\Re_D = 0.54928 \ldots \quad \text{(or } \Re = 0.54928 \ldots \Ree \text{)}
\]

**Development of a \( p(r, t) \) Relation for Pseudosteady-State Flow**

Our last objective is to develop a \( p(r, t) \) relation for pseudosteady-state flow in a bounded circular reservoir. Recalling the material balance relation (Eq. 5) we have

\[
\Ree = \Ree - \frac{8}{\phi \Pi (\Ree^2 - \Rez^2) c_t} \cdot Np
\]

For a constant flowrate, \( q \), we have

\[
Np = \int_0^t q(t) \, dt = qt
\]  

Substituting Eq. 43 into Eq. 5,

\[
\Ree = \Ree - \frac{48}{\phi \Pi (\Ree^2 - \Rez^2) c_t} \cdot t \quad \text{(Darcy units)}
\]
Recalling the average reservoir pressure identity for a well centered in a bounded circular reservoir, we have

\[ p = p_{wf} + \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) - \frac{3}{4} \right] \] (Darcy Units) \hspace{1cm} (45)

Substituting Eq. 45 into Eq. 44 gives

\[ p_{wf} + \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) - \frac{3}{4} \right] = p_{i} - \frac{q_{s}}{\phi h \pi (r_{e}^{2} - r_{w}^{2}) c_{t}} t \]

Rearranging

\[ p_{i} - p_{wf} = \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) - \frac{3}{4} \right] + \frac{q_{s}}{\phi h \pi (r_{e}^{2} - r_{w}^{2}) c_{t}} t \hspace{1cm} (46) \]

or

\[ p_{i} - p_{wf} = \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) - \frac{3}{4} \right] + \frac{q_{s}}{\phi h \pi (r_{e}^{2} - r_{w}^{2}) c_{t}} t \hspace{1cm} (47) \]

Recalling the wellbore-reservoir pressure relation (Eq. 26), we have (upon slight rearranging)

\[ p_{r} - p_{wf} = \frac{q_{s} \mu}{2\pi kh} \left[ \frac{r_{e}^{2}}{(r_{e}^{2} - r_{w}^{2})} \ln \left( \frac{r}{r_{w}} \right) - \frac{1}{2} \frac{(r^{2} - r_{w}^{2})}{(r_{e}^{2} - r_{w}^{2})} \right] \hspace{1cm} (48) \]

Subtracting Eq. 48 from Eq. 47 and solving for \( p_{r} \) gives us

\[ p_{r} = p_{i} - \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) - \frac{3}{4} - \frac{r_{e}^{2}}{(r_{e}^{2} - r_{w}^{2})} \ln \left( \frac{r}{r_{w}} \right) + \frac{1}{2} \frac{(r^{2} - r_{w}^{2})}{(r_{e}^{2} - r_{w}^{2})} \right] \]

\[ - \frac{q_{s}}{\phi h \pi (r_{e}^{2} - r_{w}^{2}) c_{t}} t \hspace{1cm} (49) \]

Assuming that \( r_{e} \gg r_{w} \) (i.e., \( (r_{e}^{2} - r_{w}^{2}) \approx r_{e}^{2} \)) gives us

\[ p_{r} = p_{i} - \frac{q_{s} \mu}{2\pi kh} \left[ \ln \left( \frac{r_{e}}{r_{w}} \right) + \frac{1}{2} \frac{(r^{2} - r_{w}^{2})}{(r_{e}^{2} - r_{w}^{2})} - \frac{3}{4} \right] - \frac{q_{s}}{\phi h \pi (r_{e}^{2} - r_{w}^{2})} t \hspace{1cm} (50) \]
Summarizing, we have the following relations in Darcy units:

\[ p_r = p_i - \frac{98}{2\pi kh} \left[ \ln \left( \frac{R_e}{R_W} \right) - \frac{3}{4} - \frac{R_e^2}{(R_e^2 - R_W^2)} \ln \left( \frac{R}{R_W} \right) + \frac{1}{2} \frac{(R_e^2 - R_W^2)}{(R_e^2 - R_W^2)} \right] - \frac{98}{V_{pc_t}} t \]  

(49)

and

\[ p_r = p_i - \frac{98}{2\pi kh} \left[ \ln \left( \frac{R_e}{R_W} \right) + \frac{1}{2} \frac{(R_e^2 - R_W^2)}{(R_e^2 - R_W^2)} - \frac{3}{4} \right] - \frac{98}{V_{pc_t}} t \]  

(50)

In Field Units we have

\[ p_r = p_i - 141.2 \frac{98}{kh} \left[ \ln \left( \frac{R_e}{R_W} \right) - \frac{3}{4} - \frac{R_e^2}{(R_e^2 - R_W^2)} \ln \left( \frac{R}{R_W} \right) + \frac{1}{2} \frac{(R_e^2 - R_W^2)}{(R_e^2 - R_W^2)} \right] \]

\[- \frac{5615}{V_{pc_t}} \frac{98}{t} \]  

(t in days, Vp in ft³)  

(51)

and

\[ p_r = p_i - 141.2 \frac{98}{kh} \left[ \ln \left( \frac{R_e}{R_W} \right) + \frac{1}{2} \frac{(R_e^2 - R_W^2)}{(R_e^2 - R_W^2)} - \frac{3}{4} \right] - \frac{5615}{V_{pc_t}} \frac{98}{t} \]  

(t in days, Vp in ft³)  

(52)

For t in hours we use 5615/24 = 0.23395.

Finally for conditions at the well, we have

**Darcy Units:**

\[ p_w = p_i - \frac{98}{2\pi kh} \left[ \ln \left( \frac{R_e}{R_W} \right) - \frac{3}{4} \right] - \frac{98}{V_{pc_t}} t \]  

(53)

**Field Units:**

\[ \frac{p_w}{(t, \text{days}; V_p, \text{ft}^3)} = p_i - 141.2 \frac{98}{kh} \left[ \ln \left( \frac{R_e}{R_W} \right) - \frac{3}{4} \right] - \frac{5615}{V_{pc_t}} \frac{98}{t} \]  

(54)

\[ \frac{p_w}{(t, \text{hours}; V_p, \text{ft}^3)} = p_i - 141.2 \frac{98}{kh} \left[ \ln \left( \frac{R_e}{R_W} \right) - \frac{3}{4} \right] - 0.23395 \frac{98}{V_{pc_t}} t \]  

(55)

Recall that the pore volume, \( V_p \), is given by

\[ V_p = \phi h A (R_e^2 - R_W^2) = \phi h A \]  

(4)