Petroleum Engineering 324
Well Performance
Development of the Diffusivity Equations for Gas Flow

(Fanaticism) ... consists in redoubling your effort when you have forgotten your aim.
— Santayana (1905)

**Topic:** Development of the Diffusivity Equations for Gas Flow

**Objectives:** (things you should know and/or be able to do)

- Be able to develop the general form of the diffusivity equation for single-phase gas flow in terms of pressure (and p/z)--beginning with the density formulation. These relations are:

  \[
  \nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi p)}{\partial t} 
  \]

  \[
  \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \frac{\phi c_t}{k} \frac{\partial p}{\partial t}
  \]

- Be able to develop the diffusivity equation for single-phase gas flow in terms of the following: pseudopressure, pressure-squared \((p^2)\), and pressure \((p)\). All developments begin with the general form of the single-phase gas flow equation (shown above).

  - "Pseudopressure" \((p_p)\) Formulation:

    \[
    \nabla^2 p_p = \frac{\phi(\mu c_t)}{k} \frac{\partial p_p}{\partial t} \quad \text{where} \quad p_{pg} = \left[ \frac{\mu z}{P} \right] \int_{P_{base}}^{P} \frac{P}{\mu z} \, dp
    \]

  - "Pseudopressure-Pseudotime" \((p_p-t_a)\) Formulation:

    \[
    \nabla^2 p_p = \frac{\phi}{k} (\mu c_t)_{n} \frac{\partial p}{\partial t_a} \quad \text{where} \quad t_a = (\mu c_t)_{n} \int_{0}^{t} \frac{1}{\mu(p)c_t(p)} \, dt
    \]

  - "Pressure-Squared" \((p^2)\) Formulation:

    - General Form:

      \[
      \nabla^2 (p^2) - \frac{\partial}{\partial p^2} \left[ \ln(\mu z) \right] \nabla (p^2)^2 = \frac{\phi(\mu c_t)}{k} \frac{\partial (p^2)}{\partial t}
      \]

    - "Linearized" Form: (assumes \(\mu z\)=constant)

      \[
      \nabla^2 (p^2) = \frac{\phi(\mu c_t)}{k} \frac{\partial (p^2)}{\partial t}
      \]

      (assumed valid for \(p<2000\) psia)

  - "Pressure" \((p)\) Formulation:

    - General Form:

      \[
      \nabla^2 p - \frac{\partial}{\partial p} \left[ \ln \left( \frac{\mu z}{P} \right) \right] (\nabla p)^2 = \frac{\phi(\mu c_t)}{k} \frac{\partial p}{\partial t}
      \]

    - "Linearized" Form: (assuming \(p/\mu z\)=constant)

      \[
      \nabla^2 p = \frac{\phi(\mu c_t)}{k} \frac{\partial p}{\partial t}
      \]

      (probably never valid in practice)
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Lecture Outline:

- Derivation of the "general" form of the gas diffusivity equation:
  - Begin with the density formulation
  - Substitute the definition of gas density and expand
  - Isolate gas and rock compressibility terms (use total compressibility)

- Derivation of the "pseudopressure" forms of the gas diffusivity equation:
  - Derivative expansions (both sides of the equation)
  - Definitions of rock and gas compressibility, consolidate (total compressibility)
  - Incorporation of pseudopressure concept, (replace pressure)
  - Identification of "nonlinear" terms
  - Final, "linearized" partial differential equation, note applications and limitations
  - Pseudotime considerations

- Derivation of the "pressure-squared" form of the gas diffusivity equation:
  - Using the "pseudopressure" approach (begin with $p_p$ diffusivity equation)
  - Using the "general" approach (begin with fundamental gas diffusivity equation)
  - Validation using the $\mu z$ versus $p$ plots

- Derivation of the "pressure" form of the gas diffusivity equation:
  - Using the "pseudopressure" approach (begin with $p_p$ diffusivity equation)
  - Using the "general" approach (begin with fundamental gas diffusivity equation)
  - Validation using the $p/\mu z$ versus $p$ plots

- Conclusions and comments

Reading Assignment:

- Review the attached notes.
  - Derivation of the general form of the diffusivity equation for single-phase gas flow in terms of pressure (and $p/z$).
  - Derivation of the diffusivity equation for single-phase gas flow in terms of the following: pseudopressure ($p_p$), pressure-squared ($p^2$), and pressure ($p$)--with all developments starting with the general form of the single-phase gas flow equation.


(from Petroleum Engineering 620 Course Notes -- 1997)
Development of a Generalized Gas Diffusivity Equation (Any Geometry)

General Introduction

Our objective in this section is to develop the pressure, pressure-squared \((p^2)\), and pseudopressure \((p_p)\) forms of the diffusivity equation from the general form given below.

\[
\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi p)}{\partial t} \tag{1}
\]

Our approach will be completely general in that we will develop each relation separately, from fundamental principles.

Gas Pseudopressure-Pseudotime Formulation

Before proceeding into the following derivations, we need to establish the proper nomenclature and give the base definitions that we use for developing the pseudopressure-pseudotime formulations for gas reservoir systems. The definition of the gas formation volume factor, \(B_g\), is given as

\[
B_g = \frac{\frac{p_{sc}}{z_{sc} T_{sc}}}{\frac{z T}{p}} \tag{2}
\]

The general definition of the pseudopressure function is given as

\[
p_p = \left[ \frac{\mu B}{k} \right] \int_{p_{base}}^p \frac{k}{\mu B} \, dp \tag{3}
\]

As these efforts are intended for gas flow, we assume that the effective permeability, \(k\), remains constant at its initial value. This assumption reduces Eq. 3 to

\[
p_p = (\mu B) \int_{p_{base}}^p \frac{1}{\mu B} \, dp \tag{4}
\]

Substituting Eq. 2 into Eq. 4, we have

\[
p_{pg} = \left[ \frac{\mu z T}{p} \right] \int_{p_{base}}^p \frac{p}{\mu z T} \, dp
\]

If we assume isothermal flow conditions \((T = T_n)\), which is usually the case as cooling or heating are considered insignificant–then this expression reduces to

\[
p_{pg} = \left[ \frac{\mu z}{p} \right] \int_{p_{base}}^p \frac{p}{\mu z} \, dp \tag{5}
\]

The utility of the definition given by Eq. 5 is that this result allows us to make comments regarding the behavior of \(\mu, z, \) and \(p\) as functions of pressure. In particular, what are the particular forms of the diffusivity equation for a constant \(\mu z\), or a constant \(p/\mu z\) group? This will become apparent as we develop these results in later sections.

Recalling the generalized density form of the diffusivity equation (Eq. 1), we have

\[
\nabla \cdot \left[ \frac{\rho k}{\mu} \nabla p \right] = \frac{\partial (\phi p)}{\partial t} \tag{1}
\]
Recalling the definition of gas density, we have
\[ \rho_g = \frac{pM}{zRT} \]  

(6)

Substituting Eq. 6 into Eq. 1, and eliminating the \( \frac{M}{RT} \) terms, we obtain
\[ \nabla \cdot \left[ \frac{kP}{\mu z} \nabla p \right] = \frac{\partial}{\partial t} \left[ \frac{\phi P}{z} \right] \]

(7)

If we assume that the effective permeability, \( k \), is constant (which again, is a very reasonable assumption for gas reservoirs) and we expand the right-hand-side term using the product rule, then Eq. 7 becomes
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \frac{1}{k} \frac{\partial}{\partial t} \left[ \frac{P}{z} \right] = \frac{1}{k} \left( \frac{P}{z} \frac{\partial \phi}{\partial t} + \phi \frac{\partial}{\partial t} \left[ \frac{P}{z} \right] \right) \]

Expanding the time derivative terms using the chain rule yields
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \frac{1}{k} \left( \frac{P}{z} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial t} \left[ \frac{P}{z} \frac{\partial P}{\partial t} \right] \right) \]

(8)

Factoring out the porosity, we have
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \phi \frac{1}{k} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial t} \left[ \frac{P}{z} \frac{\partial P}{\partial t} \right] \]

(9)

Recalling the definition of pore-volume compressibility, \( c_f \), we have
\[ c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial P} \]

(10)

Substituting Eq. 9 into Eq. 8, we obtain
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \phi \frac{1}{k} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial t} \left[ \frac{P}{z} \frac{\partial P}{\partial t} \right] \]

(11)

Recalling the definition of isothermal gas compressibility, \( c_g \), we have
\[ c_g = \frac{1}{p} \left( \frac{\partial p}{\partial T} \right)_T \]

The alternative form of the definition of gas compressibility is (again for isothermal conditions, but dropping the \( T \) subscript)
\[ c_g = \frac{z}{p} \frac{\partial p}{\partial T} \]

(12)

Rearranging Eq. 12, we have
\[ \frac{\partial}{\partial P} \left[ \frac{P}{z} \right] = \frac{P}{z} c_g \]

(13)

Substituting Eq. 13 into Eq. 10, we obtain
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \phi \frac{1}{k} \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial t} \left[ \frac{P}{z} c_g \frac{\partial P}{\partial t} \right] \]

(14)

Recalling the definition of total compressibility, \( c_t = c_g + c_f \), and substituting this identity into Eq. 14 gives us
\[ \nabla \cdot \left[ \frac{P}{\mu z} \nabla p \right] = \frac{\partial c_t}{\partial t} \frac{p}{k} \frac{\partial P}{\partial t} \]

(15)

Eq. 15 is the generalized diffusivity equation for gas flow. We will now use the pseudopressure and pseudotime concepts to "linearize" Eq. 15 into an equivalent liquid form (in other words, the pseudopressure/pseudotime result will look exactly like the result
for the slightly compressible liquid case). Using the $\nabla$ operator, the pseudopressure gradient, $\nabla p_p$, is given as

$$\nabla p_p = \frac{\partial p_{pg}}{\partial p} \nabla p$$

or, solving in terms of the pressure gradient, we have

$$\nabla p = \frac{\partial p}{\partial p_{pg}} \nabla p_p \tag{16}$$

The pressure derivative with respect to time for Eq. 15 can be expanded using the chain rule to yield the following pseudopressure relation

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial p_{pg}} \frac{\partial p_{pg}}{\partial t} \tag{17}$$

Substituting Eqs. 16 and 17 into Eq. 15, we obtain

$$\nabla \cdot \left[ \frac{p}{\mu z} \frac{\partial p_{pg}}{\partial p_{pg}} \nabla p_p \right] = \frac{\phi c_t}{k} \frac{p}{z} \frac{\partial p_{pg}}{\partial t} \tag{18}$$

Taking the derivative of the gas pseudopressure function (Eq. 5) with respect to pressure gives us

$$\frac{\partial p_{pg}}{\partial p} = \left[ \frac{\mu z}{p} \frac{1}{\mu z} \right] = \frac{p}{\mu z} \tag{19}$$

Substituting Eqs. 19 into Eq. 18, we have

$$\nabla \cdot \left[ \frac{p}{\mu z} \frac{1}{\mu z} \frac{p}{p} \nabla p_p \right] = \frac{\phi c_t}{k} \frac{1}{z} \left[ \frac{\mu z}{p} \frac{1}{\mu z} \right] \frac{\partial p_{pg}}{\partial t}$$

Factoring out like terms, we obtain

$$\nabla \cdot \nabla p_p = \frac{\phi c_t}{k} \frac{\partial p_{pg}}{\partial t}$$

Recalling our operator notation, $\nabla \cdot \nabla \alpha = \nabla^2 \alpha$, we have

$$\nabla^2 p_{pg} = \frac{\phi c_t}{k} \frac{\partial p_{pg}}{\partial t} \tag{20}$$

The use of pseudopressure completely "linearizes" the left-hand-side of the gas diffusivity equation as we note no nonlinear terms on the left-hand-side of Eq. 20. However, we immediately note that the $\mu c_t$ product in Eq. 20 is not constant, but is typically a strong function of pressure. In particular, performance data (rate and pressure data) during boundary-dominated flow (i.e., pseudosteady-state flow) as well as data from pressure buildup tests often show significant changes in the $\mu c_t$ product with pressure. This behavior is very significant, and it can cause analysis which are based on liquid solutions to be in considerable error.

As a means of accounting for changes in the $\mu c_t$ product with pressure, Agarwal\textsuperscript{1} defined an intuitive pseudotime function as

$$t_a = (\mu c_t)_n \int_0^t \frac{1}{\mu(p)c_t(p)} \, dt \tag{21}$$

Lee and Holditch\textsuperscript{2} gave detailed developments of the conditions for the applicability of Eq. 21 when applied to pressure buildup tests. Fraim and Wattenbarger\textsuperscript{3} noted that the
pseudotime integral must be evaluated at average reservoir pressure ($\bar{p}$) for the case of reservoir performance at boundary-dominated flow conditions.

Including pseudotime in the diffusivity formulation given by Eq. 21, we have

$$
\nabla^2 p_{pg} = \frac{\phi(\mu c_t)_n}{k} \frac{\partial p_{pg}}{\partial t_a}
$$

Review of $\mu c_t$ Behavior (Gas Case): $\mu c_t$ Versus Pressure Plots

In order to establish the behavior of the $\mu c_t$ product, we present plots of $\mu c_t$ versus pressure for various temperatures and gas compositions (gas gravities). This effort should help us to determine the applicability of assuming that the $\mu c_t$ product is approximately constant (as we do for certain forms of the gas diffusivity equation). These graphs are shown below--each plot is based on a particular temperature (100, 200 and 300°F), and each plot contains four curves--where each curve represents a different gas gravity ($\gamma_g = 0.6, 0.8, 1.0, \text{ and } 1.2, \text{ where air } = 1.0$). Both cartesian (linear) and log-log scales are used to establish the validity of the assumption that $\mu c_t$ is approximately constant.
Fig. 1 - Comparison of $\mu c_t$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Cartesian Format.

Fig. 2 - Comparison of $\mu c_t$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Log-Log Format.
**Fig. 3** - Comparison of $\mu_c t$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Cartesian Format.

**Fig. 4** - Comparison of $\mu_c t$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Log-Log Format.
Fig. 5 - Comparison of $\mu c_t$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Cartesian Format.

Fig. 6 - Comparison of $\mu c_t$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Log-Log Format.
Development of the Pressure-Squared form of the Gas Diffusivity Equation

The application of the pressure-squared form of the gas diffusivity equation has been one of the most argued topics in gas reservoir engineering. We recognize that while the pressure-squared form provides a significant reduction in complexity from the rigorous pseudopressure form, the validity of applying the pressure-squared form of the diffusivity equation is often debated. This controversy is primarily due to two considerations. First, the pressure-squared formulation of the diffusivity equation is an approximation that has definable limits of applicability. Second, the application of the pressure-squared form has been extremely broad, from company policies to regulatory requirements.

The argument that most often arises is that of when the pressure-squared formulation is actually valid. We will show that the pressure-squared formulation of the diffusivity equation can be derived by two methods. First, by explicitly assuming that \( \mu z \) is constant, and second, from a completely general development that results in a condition where \( \mu z \) must be constant in order to obtain the pressure-squared form. Most references claim that \( \mu z \) is approximately constant for gas reservoirs where \( p \leq 2000 \) psia. We will demonstrate (using plots of \( \mu z \) versus \( p \) for various cases of temperature and gas gravity) that this condition \( (p \leq 2000 \) psia) is somewhat arbitrary, but is actually a fairly good criterion.

Specifically, we must both quantify as well as qualify the assumption that \( \mu z \) is approximately constant. This condition is specifically valid for an ideal gas--which has the following properties:

- Constant gas viscosity
- \( z = 1 \), (i.e., no volume correction)

For real gases the conditions where \( \mu z \) is approximately constant must be carefully investigated. As noted above, we attempt to verify this hypothesis by plotting \( \mu z \) versus pressure for various cases of temperature and gas composition (gas gravity).

Pressure-Squared Form of the Gas Diffusivity Equation: \( p_p \) Approach

In this approach we consider the behavior of the gas pseudopressure function, Eq. 5, by assuming that \( \mu z \) is constant with pressure, temperature, and gas composition. Recalling Eq. 5, we have

\[
p_{pg} = \left[ \frac{\mu z}{p} \right] \int_{P_{base}}^{p} \frac{P}{\mu z} \, dp \tag{5}
\]

If we assume that \( \mu z = (\mu z)_n = \text{constant} \), then Eq. 5 becomes

\[
p_{pg} = \frac{1}{p_n} \int_{P_{base}}^{p} p \, dp \tag{23}
\]

Completing the integration of Eq. 23, we obtain

\[
p_{pg} = \frac{1}{2p_n} (p^2 - p_{base}^2)
\]
Dividing through by $2p_n$, we have

\[ p_{pg} = \frac{p^2}{2p_n} - \frac{p^2}{2p_n} \text{base} \] .................................(24)

Recalling the pseudopressure-time form of the gas diffusivity equation (Eq. 20), we have

\[ \nabla^2 p_{pg} = \frac{\phi u_{ct}}{k} \frac{\partial p_{pg}}{\partial t} \] .................................(20)

Substituting Eq. 24 into Eq. 20, gives us

\[ \nabla^2 \left[ \frac{p^2}{2p_n} - \frac{p^2}{2p_n} \text{base} \right] = \frac{\phi u_{ct}}{k} \frac{\partial}{\partial t} \left[ \frac{p^2}{2p_n} - \frac{p^2}{2p_n} \text{base} \right] \]

Eliminating the constant $\frac{p^2}{2p_n}$ terms from the derivatives yields

\[ \nabla^2 \left[ \frac{p^2}{2p_n} \right] = \frac{\phi u_{ct}}{k} \frac{\partial}{\partial t} \left( \frac{p^2}{2p_n} \right) \]

and, finally eliminating the constant $\frac{1}{2p_n}$ factors, we are left with

\[ \nabla^2 (p^2) = \frac{\phi u_{ct}}{k} \frac{\partial}{\partial t} (p^2) \] .................................(25)

Eq. 25 is strictly valid for a constant $\mu z$ term in Eq. 5. It must be noted that no assumptions have been made regarding the $\mu c_t$ product on the right-hand-side of Eq. 25 (i.e., these terms can vary with pressure and are hence, non-linear). Unfortunately, the development of Eq. 25 starts with the assumption of a constant $\mu z$ term and using this approach it is not possible for us to develop the residual term (or terms) which explicitly account for the $\mu z$ product—in a general sense, without regard as to whether this term is constant or not.

**Pressure-Squared Form of the Gas Diffusivity Equation: General Approach**

This approach is taken from the 1975 Energy Resources Conservation Board Manual. 4 We start with the general form of the gas diffusivity equation, Eq. 15, which is given as

\[ \nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{\phi c_t p}{k} \frac{\partial}{\partial t} \] .................................(15)

Applying the chain rule on the pressure-time derivative term in Eq. 15, we have

\[ \nabla \cdot \left[ \frac{p}{\mu z} \nabla p \right] = \frac{\phi c_t p}{k} \frac{1}{z} \frac{\partial}{\partial p} (p^2) \frac{\partial}{\partial t} (p^2) \] .................................(26)

In order to "convert" Eq. 26 into the pressure-squared (i.e., $p^2$) form, we must use the following identities:

\[ \frac{\partial}{\partial p} (p^2) = 2p \] .................................(27)

and

\[ \nabla (p^2) = 2p \nabla p \]

Rearranging, we have

\[ p \nabla p = \frac{1}{2} \nabla (p^2) \] .................................(28)
Substituting Eqs. 27 and 28 into Eq. 26, we obtain
\[ \nabla \cdot \left[ \frac{1}{\mu z} \frac{1}{2} \nabla (p^2) \right] = \frac{\phi c_t}{k} \frac{1}{z} \frac{1}{2p} \frac{\partial}{\partial t} (p^2) \]

Canceling the \( p \) terms on the right-hand-side, we have
\[ \nabla \cdot \left[ \frac{1}{\mu z} \frac{1}{2} \nabla (p^2) \right] = \frac{\phi c_t}{k} \frac{1}{2z} \frac{\partial}{\partial t} (p^2) \] ..........................(29)

Expanding the left-hand-side of Eq. 29 using the product rule, we have
\[ \frac{1}{2} \frac{1}{\mu z} \nabla \cdot \nabla (p^2) + \frac{1}{2} \nabla \cdot \left[ \frac{1}{\mu z} \right] \nabla (p^2) = \frac{\phi c_t}{k} \frac{1}{2z} \frac{\partial}{\partial t} (p^2) \]

Multiplying through by \( 2 \mu z \) and reducing the \( \nabla \cdot \nabla (p^2) \) term to \( \nabla^2 (p^2) \) gives
\[ \nabla^2 (p^2) + \mu z \nabla \cdot \left[ \frac{1}{\mu z} \right] \nabla (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial}{\partial t} (p^2) \] ..........................(30)

Expanding the \( \nabla \cdot \left[ \frac{1}{\mu z} \right] \nabla (p^2) \) term using \( \nabla \cdot \alpha = \frac{\partial \alpha}{\partial y} \nabla y \) with \( \alpha = \frac{1}{\mu z} \) and \( y = p^2 \) we have
\[ \nabla \cdot \left[ \frac{1}{\mu z} \right] \nabla (p^2) = \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu z} \right] \nabla (p^2) \nabla (p^2) \]

or
\[ \nabla \cdot \left[ \frac{1}{\mu z} \right] \nabla (p^2) = \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu z} \right] \nabla (p^2)^2 \] ..........................(31)

Substituting Eq. 31 into Eq. 30, we obtain
\[ \nabla^2 (p^2) + \mu z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu z} \right] \nabla (p^2)^2 = \frac{\phi \mu c_t}{k} \frac{\partial}{\partial t} (p^2) \]

Expanding the \( \mu z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu z} \right] \) term as an attempt to simplify the previous result, we have
\[ \mu z \frac{\partial}{\partial p^2} \left[ \frac{1}{\mu z} \right] = -\frac{\partial}{\partial p^2} \left[ \ln \left( \frac{1}{\mu z} \right) \right] = \frac{\partial}{\partial p^2} \left[ \ln(\mu z) \right] \] ..........................(32)

Substituting Eq. 32 into Eq. 31,
\[ \nabla^2 (p^2) + \frac{\partial}{\partial p^2} \left[ \ln \left( \frac{1}{\mu z} \right) \right] \nabla (p^2)^2 = \frac{\phi \mu c_t}{k} \frac{\partial}{\partial t} (p^2) \] ..........................(33)

or, we can obtain the alternative form, which is given as
\[ \nabla^2 (p^2) - \frac{\partial}{\partial p^2} \left[ \ln(\mu z) \right] \nabla (p^2)^2 = \frac{\phi \mu c_t}{k} \frac{\partial}{\partial t} (p^2) \] ..........................(34)

We note that if \( \mu z = \) constant then Eqs. 33 and 34 reduce to
\[ \nabla^2 (p^2) = \frac{\phi \mu c_t}{k} \frac{\partial}{\partial t} (p^2) \] ..........................(25)

We also note that the \( \mu c_t \) product is not assumed constant on the right-hand-side of Eq. 25. We note that Eqs. 33 and 34 were developed without assuming any particular behavior of the \( \mu z \) product. On the other hand, Eq. 25 results directly from the assumption that the \( \mu z \) term is constant. While this condition is plausible, and perhaps even probable given the varied behavior of real gas systems, the assumption of the \( \mu z \) product being constant must be verified by inspection of plots of \( \mu z \) versus pressure.
Validation of the $p^2$ Form of the Gas Diffusivity Equation: $\mu z$ Versus Pressure Plots

Recall that the only case where $\mu z$ is exactly constant is for an ideal gas--where, by definition, viscosity is constant and $z=1$.

We present three plots of $\mu z$ versus pressure for various temperatures and gas compositions (gas gravities) to help us to recognize the applicability of the $\mu z$ product being approximately constant. These graphs are shown below, and each plot is based on a particular temperature (100, 200 and 300°F), and each plot contains four curves, where each curve represents a different gas gravity ($\gamma_g = 0.6, 0.8, 1.0, \text{and} 1.2$, where air = 1.0). Log-log scales are used to emphasize the duration of the regions where $\mu z$ is approximately constant.

Wattenbarger and Ramey\textsuperscript{5} suggest that, in general, $\mu z$ is constant for pressure values less than 2000 psia. It is clear from the figures below that the assumption of $\mu z=\text{constant}$ for that pressure range (i.e., $p<2000$ psia) is quite reasonable for all temperatures and gas compositions. This verifies, in principle, the use of the "pressure-squared" form of the diffusivity equation (Eq. 25) for $p<2000$ psia.
Fig. 7 - Comparison of $\mu z$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Cartesian Format.

Fig. 7 - Comparison of $\mu z$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Log-Log Format.
Fig. 9 - Comparison of $\mu z$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Cartesian Format.

Fig. 10 - Comparison of $\mu z$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Log-Log Format.
Fig. 11 - Comparison of $\mu z$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Cartesian Format.

Fig. 12 - Comparison of $\mu z$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Log-Log Format.
Derivation of the Pressure Form of the Gas Diffusivity Equation

In this case we assume that the \( p/\mu_z \) product remains approximately constant with pressure. This condition has somewhat questionable origins, but the approach yields a very convenient result—a gas diffusivity equation in terms of pressure. The validity of assuming \( p/\mu_z \) constant will be determined by observation of plots of \( p/\mu_z \) versus pressure for various cases of temperature and gas composition (gas gravity).

Pressure Form of the Gas Diffusivity Equation: \( p_p \) Approach

In this approach we will also start with the definition of the gas pseudopressure, Eq. 5. Recalling Eq. 5 we have

\[
p_{pg} = \left[ \frac{\mu_z}{\mu} \right]_n \int_{p_{base}}^{p} \frac{P}{\mu_z} \, dp \quad \text{........................................... (5)}
\]

Assuming that \( \frac{P}{\mu_z} = \left[ \frac{P}{\mu_z} \right]_n \) = constant, Eq. 5 becomes

\[
p_{pg} = \int_{p_{base}}^{p} \, dp \quad \text{........................................... (35)}
\]

Completing the integration, we obtain

\[
p_{pg} = P - p_{base} \quad \text{........................................... (36)}
\]

Recalling the pseudopressure form of the gas diffusivity equation (Eq. 20), we have

\[
\nabla^2 p_{pg} = \frac{\phi_{ UTC}}{k} \frac{\partial p_{pg}}{\partial t} \quad \text{........................................... (37)}
\]

Substituting Eq. 36 into Eq. 37 gives us

\[
\nabla^2 (P - p_{base}) = \frac{\phi_{ UTC}}{k} \frac{\partial}{\partial t} (P - p_{base})
\]

Eliminating the constant \( p_{base} \) terms from these derivatives we obtain

\[
\nabla^2 P = \frac{\phi_{ UTC}}{k} \frac{\partial P}{\partial t} \quad \text{........................................... (38)}
\]

where Eq. 38 is strictly valid for a constant \( p/\mu_z \) product. We again note that we have not assumed that the \( \mu_{TC} \) product is constant on the right-hand-side of Eq. 38. Unfortunately, the development of Eq. 38 begins with the assumption of a constant \( p/\mu_z \) term and, using this approach, we can not develop the residual term (or terms) which explicitly account for the \( p/\mu_z \) group—in a general sense, without regard as to whether this term is constant or not.

Pressure Form of the Gas Diffusivity Equation: General Approach

As with the pressure-squared case, we again borrow this "general" approach from the 1975 ERCB Manual for Gas Well Testing.\(^5\) This development starts with the general form of the gas diffusivity equation, Eq. 15.

\[
\nabla \cdot \left[ \frac{P}{\mu_z} \nabla P \right] = \frac{\phi_{ UTC} P}{k} \frac{\partial P}{\partial t} \quad \text{........................................... (15)}
\]
Expanding the left-hand-side of Eq. 15 using the product rule gives us
\[ \nabla \cdot \left( \frac{p}{\mu z} \right) \nabla p + \frac{p}{\mu z} \nabla \cdot \nabla p = \frac{\phi c_t}{k} \frac{\partial p}{\partial t} \]
or, writing the \( \nabla \cdot \nabla p \) term in \( \nabla^2 \) notation we have
\[ \nabla \cdot \left( \frac{p}{\mu z} \right) \nabla p + \frac{p}{\mu z} \nabla^2 p = \frac{\phi c_t}{k} \frac{\partial p}{\partial t} \]
Multiplying through by \( \frac{\mu z}{p} \) we have
\[ \nabla^2 p + \frac{\mu z}{p} \nabla \cdot \left( \frac{p}{\mu z} \right) \nabla p = \frac{\phi c_t \mu z}{k} \frac{\partial p}{\partial t} \]  
(39)
Using the identity \( \nabla \cdot \alpha = \frac{\partial \alpha}{\partial y} \cdot \nabla y \), we can write the \( \nabla \cdot \left( \frac{p}{\mu z} \right) \nabla p \) term as
\[ \nabla \cdot \left( \frac{p}{\mu z} \right) \nabla p = \frac{\partial}{\partial p} \left( \frac{p}{\mu z} \right) \nabla p \nabla p = \frac{\partial}{\partial p} \left( \frac{p}{\mu z} \right) \nabla p^2 \]  
(40)
Substituting Eq. 40 into Eq. 39, we obtain
\[ \nabla^2 p + \frac{\mu z}{p} \frac{\partial}{\partial p} \left( \frac{p}{\mu z} \right) \nabla^2 p = \frac{\phi c_t \mu z}{k} \frac{\partial p}{\partial t} \]  
(41)
In addition, the \( \frac{\mu z}{p} \frac{\partial}{\partial p} \left( \frac{p}{\mu z} \right) \) term in Eq. 41 can be written as
\[ \frac{\mu z}{p} \frac{\partial}{\partial p} \left( \frac{p}{\mu z} \right) = \frac{\partial}{\partial p} \left( \ln \left( \frac{p}{\mu z} \right) \right) = -\frac{\partial}{\partial p} \left[ \ln \left( \frac{\mu z}{p} \right) \right] \]  
(42)
Substituting Eq. 42 into Eq. 41 gives us
\[ \nabla^2 p + \frac{\partial}{\partial p} \left[ \ln \left( \frac{p}{\mu z} \right) \right] \nabla^2 p = \frac{\phi c_t \mu z}{k} \frac{\partial p}{\partial t} \]  
(43)
And the alternative form of Eq. 43 is given by
\[ \nabla^2 p - \frac{\partial}{\partial p} \left[ \ln \left( \frac{\mu z}{p} \right) \right] \nabla^2 p \nabla^2 p = \frac{\phi c_t \mu z}{k} \frac{\partial p}{\partial t} \]  
(44)
From Eqs. 43 and 44 it is clear that if \( p/\mu z \) is constant, then we have
\[ \nabla^2 p = \frac{\phi c_t \mu z}{k} \frac{\partial p}{\partial t} \]  
(38)
Where Eq. 38 is only rigorously valid for a constant \( p/\mu z \) term. Note that we have not assumed the \( \mu c_t \) product to be constant on the right-hand-side of Eq. 38—which is yet another nonlinear term.

**Validation of the p-Form of the Gas Diffusivity Equation: p/μz Versus Pressure Plots**

As an attempt to validate our hypothesis of a constant \( p/\mu z \) term, we provide plots of \( p/\mu z \) versus pressure using the same temperatures and gas compositions that we considered in the validation of the pressure-squared (\( \mu z \) = constant) case. Wattenbarger and Ramey\(^5\) suggest that in general, the \( p/\mu z \) function is approximately constant for pressures greater than 4000 psia. However, we immediately note from Figs. 4-6 that \( p/\mu z \) is never truly constant. Only the \( T_g = 0.6 \) cases appear to be constant for pressures greater than 5000 psia—which would seem to agree with Wattenbarger and Ramey’s criterion, but is certainly not a generalization.
On the other hand, the cases for $\gamma_g=0.8, 1.0, \text{ and } 1.2$ indicate only a short "constant" region near the point where the $p/\mu L$ versus pressure trend has a maximum and begins to decrease with increasing pressures. In light of this behavior, it would be ill-advised to recommend the use of the pressure diffusivity equation for gas (Eq. 38), even at high pressures (i.e., $p>5000$ psia).
Fig. 13 - Comparison of $\frac{P}{\mu g}$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Cartesian Format.

Fig. 14 - Comparison of $\frac{P}{\mu g}$ versus Pressure for a Temperature of 100°F and Various Gas Compositions, Log-Log Format.
Fig. 15 - Comparison of $\frac{P}{\mu_z}$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Cartesian Format.

Fig. 16 - Comparison of $\frac{P}{\mu_z}$ versus Pressure for a Temperature of 200°F and Various Gas Compositions, Log-Log Format.
Fig. 17 - Comparison of $\frac{P}{\mu z}$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Cartesian Format.

Fig. 18 - Comparison of $\frac{P}{\mu z}$ versus Pressure for a Temperature of 300°F and Various Gas Compositions, Log-Log Format.
Conclusions and General Comments

Recalling the diffusivity equation for a slightly compressible liquid, we have

$$\nabla^2 p = \frac{\phi \mu_c} k \frac{\partial p}{\partial t}$$

($\mu_c$ is constant by definition). ………………………………..(45)

The pseudopressure ($p_p$) form of the gas diffusivity equation is

$$\nabla^2 p_p = \frac{\phi \mu_c} k \frac{\partial p_p}{\partial t}$$

(no assumptions are made regarding $\mu_c$ ) ……………..(20)

The pressure-squared ($p^2$) form of the gas diffusivity equation is

$$\nabla^2 (p^2) = \frac{\phi \mu_c} k \frac{\partial (p^2)}{\partial t}$$

(no assumptions are made regarding $\mu_c$ ) ……………..(25)

The pressure ($p$) form of the gas diffusivity equation is

$$\nabla^2 p = \frac{\phi \mu_c} k \frac{\partial p}{\partial t}$$

(no assumptions are made regarding $\mu_c$ ) ……………..(38)

We note that although the slightly compressible liquid relation (Eq. 45) is of exactly the same in form as the pseudopressure, pressure-squared, and pressure relations for gas flow (Eqs. 20, 25, and 38, respectively), there are critical differences, not the least of which is the presumed behavior of the $\mu_c$ product. These differences are illustrated by the governing assumptions for each relation, which are cited below.

Assumptions common to all relations (Eqs. 25, 38, and 45):
- Laminar flow (non-laminar flow effects are not considered)
- Homogeneous and isotropic porous media (uniform permeability)
- Isothermal flow
- Negligible gravity forces (horizontal flow only)

Assumptions exclusive to Eq. 45: "Slightly compressible liquid" case
- Slightly compressible liquid (liquid of small and constant compressibility)
- Constant fluid viscosity

Assumptions exclusive to Eq. 25: Gas flow case; "pressure-squared" equation
- $\mu z$ = constant
- $\mu c$ is assumed to vary with pressure

Assumptions exclusive to Eq. 38: Gas flow case; "pressure" equation
- $p/\mu z$ = constant
- $\mu c$ is assumed to vary with pressure

Given these assumptions, as well as our observations of the behavior of the $\mu z$ and $p/\mu z$ functions with respect to pressure, the following conclusions are made:

- General applications for gas flow case (analysis, modelling, and prediction) must use the pseudopressure form of the diffusivity equation, Eq. 20. In addition, cases where the $\mu c$ product varies significantly must use the pseudotime transform.

- The "pressure-squared" form of the diffusivity equation (Eq. 25) may be used for gas flow cases where all pressures are less than 2000 psia. However--we strongly advise that the pseudopressure form, Eq. 20, be always used in practice.

- The "pressure" form of the gas diffusivity equation (Eq. 38) should not be used under any circumstances for the modelling of gas flow behavior, or for the analysis of gas well performance data.
References


References for Fluid Property Correlations


