To give a reason for anything is to breed a doubt of it.
— William Hazlitt (1826)

**Topic:** Variable-Rate Convolution (Superposition) and the Application to Pressure Buildup Tests

**Objectives:** (things you should know and/or be able to do)
- Be familiar with the convolution sums and integrals for the variable-rate and variable pressure drop cases.

**Variable-Rate Case:**

\[
p_i - p_{wf} = \frac{1}{P_{Dc}} \frac{B\mu}{kh} \sum_{j=1}^{n} \left( q_j - q_{j-1} \right) p_{sD}(t_D - t_{D,j-1})
\]

(discrete rate changes)

\[
p_i - p_{wf} = \frac{1}{P_{Dc}} \frac{B\mu}{kh} \int_0^{t_D} q'(\tau_D) p_{sD}(t_D - \tau_D) \, d\tau_D
\]

(continuous rate changes)

**Variable Pressure Drop Case:**

\[
q = P_{Dc} \frac{kh}{B\mu} \sum_{j=1}^{n} \left( p_i - p_{wf,j} \right) q_{Dcp}(t_D - t_{D,j-1})
\]

(discrete rate changes)

where,

- \( p_{sD} = p_{D} + s \) (constant rate \( p_D \) function plus the skin factor)
- \( q_{Dcp} \) = Dimensionless rate (constant wellbore pressure case)
- \( t_D \) = Dimensionless time
- \( \tau_D \) = Dummy variable of integration (dimensionless time formulation)

and, for the dimensionless variables, we have

\[
\text{Dimensionless Time} \quad \text{Dimensionless Pressure} \quad \text{Dimensionless Rate}
\]

\[
t_D = t_{Dc} \frac{kt}{\phi \mu c_r r_w^2} \quad p_D = p_{Dc} \frac{kh}{qB\mu} (p_i - p_{wf}) \quad q_{Dcp} = \frac{1}{P_{Dc}} \frac{B\mu}{kh(p_i - p_{wf})} q
\]

**Conversion Constants for the \( t_D, p_D, \) and \( q_{Dcp} \) Functions**

<table>
<thead>
<tr>
<th>Constant</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{Dc} )</td>
<td>1</td>
<td>2.637x10^{-4}</td>
<td>3.557x10^{-6}</td>
</tr>
<tr>
<td>( P_{Dc} )</td>
<td>2\pi</td>
<td>7.081x10^{-3}</td>
<td>5.356x10^{-4}</td>
</tr>
<tr>
<td>( P_{Dcr} = 1/P_{Dc} )</td>
<td>1/(2\pi)</td>
<td>141.2</td>
<td>1867.1</td>
</tr>
</tbody>
</table>
Be able to derive the general result for a variable-rate test sequence—assuming undistorted, transient radial flow. This result is given in Field Units as

\[ p_i - p_{wf} = 162.6 \frac{B \mu}{kh} \sum_{j=1}^{n} (q_j - q_{j-1}) \log(t - t_{j-1}) + 162.6 \frac{q_n B \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c r_w^2} \right) - 3.2275 + 0.8686s \right] \]

This expression can be "converted" into an analysis relation if we divide through by the \( q_n \) term. This relation is known as the "Odeh-Jones" result

\[ \frac{(p_i - p_{wf})}{q_n} = 162.6 \frac{B \mu}{kh} \sum_{j=1}^{n} (q_j - q_{j-1}) \frac{\log(t - t_{j-1})}{q_n} + 162.6 \frac{B \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c r_w^2} \right) - 3.2275 + 0.8686s \right] \]

where \( \frac{(p_i - p_{wf})}{q_n} \) is plotted versus \( \sum_{j=1}^{n} \frac{(q_j - q_{j-1})}{q_n} \log(t - t_{j-1}) \).

Be able to derive the "Horner" equation for a pressure "buildup" test sequence. The Horner equation is given in Field Units as

\[ p_i - p_{ws} = 162.6 \frac{qB \mu}{kh} \log \left( \frac{t_p + \Delta t}{\Delta t} \right) \]

(assumes a constant flowrate, \( q \))

where,

- \( p_{ws} \) = Shut-in pressure
- \( t_p \) = Total production time
- \( \Delta t \) = Shut-in time (elapsed time since \( t_p \))

Be able to derive the "effective time" relation for a pressure "buildup" test sequence. The "effective time" relation is given in Field Units as

\[ p_{ws} - p_{wf}(\Delta t=0) = 162.6 \frac{qB \mu}{kh} \log \left( \frac{t_p \Delta t}{t_p + \Delta t} \right) + 162.6 \frac{qB \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c r_w^2} \right) - 3.2275 + 0.8686s \right] \]

(this relation also assumes a constant flowrate, \( q \))

where,

- \( p_{wf}(\Delta t=0) \) = Flowing wellbore pressure at shut-in (\( i.e., \) at \( \Delta t=0 \) or \( t=t_p \))

Rewriting, we have

\[ p_{ws} - p_{wf}(\Delta t=0) = 162.6 \frac{qB \mu}{kh} \log(\Delta t_e) + 162.6 \frac{qB \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c r_w^2} \right) - 3.2275 + 0.8686s \right] \]

where,

\[ \Delta t_e = \frac{t_p \Delta t}{t_p + \Delta t} = \frac{\Delta t}{1 + \frac{\Delta t}{t_p}} \]

We note that the \( \Delta t_e \) form of the pressure buildup equation is exactly the same as that for the pressure drawdown case. This will be useful later in "type curve" matching, where we "match" pressure buildup test data to drawdown type curves (graphical solutions).
Petroleum Engineering 324  
Well Performance  
Variable-Rate Convolution (Superposition) and  
the Application to Pressure Buildup Tests

- Be familiar with the Miller-Dyes-Hutchinson (MDH) relation which is valid for the analysis of pressure "buildup" test data where the well was produced to pseudosteady-state flow conditions. The utility of this result that it is independent of the producing time, $t_p$. The MDH equation is given in Field Units as

$$p_{ws} - p_{wf}(\Delta t=0) = 162.6 \frac{qB \mu}{kh} \log(\Delta t) + 162.6 \frac{qB \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu_c r_w^2} \right) - 3.2275 + 0.8686s \right]$$

(this relation also assumes a constant flowrate, $q$)

Lecture Outline:

- Discussion of the general variable-rate result  
  ■ General relation  
  ■ Odeh-Jones formulation--data analysis considerations  
    - Plotting functions.  
    - Results: Permeability and skin factor.  
- Development of the Horner equation for pressure buildup test analysis  
  ■ Derivation  
  ■ Application  
    - Plotting functions.  
    - Results: Permeability, skin factor, and the "extrapolated" pressure.  
    - Discussion: "extrapolated" pressure versus average reservoir pressure.  
- Development of the "effective time" relation for pressure buildup test analysis  
  ■ Derivation  
  ■ Application  
    - Plotting functions.  
    - Development of a skin factor result for use in "Horner" analysis.  
- Discussion of the MDH relation for pressure buildup test analysis  
  ■ Presentation of the MDH relation  
  ■ Application  
    - Does not require knowledge of the producing time, $t_p$.  
    - Limitation: Only rigorous for cases where production reaches the pseudosteady-state flow condition.

Reading Assignment:

Text Reading:

- Review Chapters 1 and 2 of the Lee, et al. Well Testing text, 2nd edition,  
  ■ Chapter 7—The Constant Terminal Rate Solution of the Diffusivity Equation and its Application to Oilwell Testing (Sections 7.5-7.7).  
  ■ Chapter 4—Oilwell Testing (Sections 4.9-4.13)
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the Application to Pressure Buildup Tests

Reading Assignment: (Continued)

Derivations:
- Review attached notes.
  - Analysis of Variable-Rate Well Performance
    - General Variable-Rate Case
    - Constant Rate Pressure Buildup Case
  - Development of Generalized Convolution (Superposition) Relations

Reference Articles:
Analysis of Variable-Rate Well Performance

- General Variable-Rate Case
- Constant Rate Pressure Buildup Case
  - "Horner" Analysis (Includes Rate History)
  - "MDH" Analysis (No Rate History)

(from Petroleum Engineering 412 Course Notes -- 1997)

Petroleum Engineering 324
Well Performance
Analysis of Variable Rate Test Data - Radial Flow

From convolution (also known as "superposition"), we have

\[ P_i - P_{rf} = \frac{1}{\phi \mu c} \sum_{j=1}^{n} \frac{Q_j}{kh} (q_{i,j} - q_{i-1,j}) \cdot P_{D0} (t_D - t_{D,j-1}) \]  \hspace{1cm} (1)

where,

\[ P_{D0} = P_0 + \delta \]  \hspace{1cm} (constant rate \( P_{D0} \) function)

and

\[ t_D = t_{DC} \frac{k}{\mu \phi c \omega^2} t \] \hspace{1cm} (2)

\[ P_{D} = P_{DC} \frac{kh}{\mu \phi c \omega^2} (P_i - P) \] \hspace{1cm} (3)

where \( t_{DC} \) and \( P_{DC} \) are given by

<table>
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<tr>
<th>Variable</th>
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<tr>
<td>( t_{DC} )</td>
<td>1</td>
<td>2.637 x 10^-4</td>
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<tr>
<td>( P_{DC} )</td>
<td>1</td>
<td>2.637 x 10^-4</td>
<td>2.557 x 10^-6</td>
</tr>
<tr>
<td>1/4 ( t_{DC} )</td>
<td>1/2\pi</td>
<td>141.2</td>
<td>18667.1</td>
</tr>
</tbody>
</table>

Using Field Units, Eq. 1 becomes

\[ P_i - P_{rf} = 141.2 \phi \mu c \sum_{j=1}^{n} \frac{Q_j}{kh} (q_{i,j} - q_{i-1,j}) \cdot P_{D0} (t_D - t_{D,j-1}) \]  \hspace{1cm} (4)

For radial flow, the \( P_{D0} \) function is given by

\[ P_{D0} = \frac{1}{2} \ln \left[ \frac{4}{e^2} \cdot t_D \right] + \delta \] \hspace{1cm} (5)

Recalling that \( \ln(x) = \ln(10) \cdot \log(x) \), we have

\[ P_{D0} = \frac{2.303}{2} \log \left[ \frac{4}{e^2} \cdot t_D \right] + \delta \] \hspace{1cm} (6)

Also, in Field Units, we have the following for \( t_D \)

\[ t_D = 0.0002637 \frac{k}{\phi \mu c \omega^2} \cdot t \] \hspace{1cm} (7)
Substituting Eq. 7 into Eq. 6, we have

\[ R_d = \frac{2.303}{Z} \log \left[ \frac{4}{e^2} \left( \frac{0.0002637}{k} \frac{k}{\phi u_c r_w^2} \right) t \right] + 5 \]

Expanding,

\[ R_d = \frac{2.303}{Z} \left[ \log(t) + \log\left( \frac{k}{\phi u_c r_w^2} \right) \right] + \log\left[ \frac{4}{e^2} \left( \frac{0.0002637}{k} \right) + 0.8686 \right] \]

Upon final reduction, we have

\[ R_d = \frac{2.303}{Z} \left[ \log(t) + \log\left( \frac{k}{\phi u_c r_w^2} \right) \right] - 3.2275 + 0.8686 \]

(Equation 8)

Substituting Eq. 8 into Eq. 4, we obtain

\[ R_{r-wf} = 141.2 \frac{B_m}{k h} \sum_{j=1}^{n} (q_j - q_{j-1}) \left[ \frac{2.303}{Z} \left[ \log(t - t_{j-1}) \right] 
+ \log\left( \frac{k}{\phi u_c r_w^2} \right) \right] - 3.2275 + 0.8686 \]

Reducing,

\[ R_{r-wf} = 162.6 \frac{B_m}{k h} \sum_{j=1}^{n} (q_j - q_{j-1}) \log(t - t_{j-1}) 
+ 162.6 \frac{B_m}{k h} \sum_{j=1}^{n} (q_j - q_{j-1}) \left[ \log\left( \frac{k}{\phi u_c r_w^2} \right) - 3.2275 + 0.8686 \right] \]

(Equation 9)
Upon final reduction, Eq. 9 becomes

\[ \pi_i - \pi_{wf} = 162.6 \frac{Bu}{kh} \sum_{j=1}^{n} (q_i - q_{i-1}) \log (t - t_{j-1}) \]

\[ + 162.6 \frac{BnBu}{kh} \left[ \log \left( \frac{k}{\phi \mu_c \chi_w} \right) - 3.2275 + 0.8686 s \right] \]  (10)

Eq. 10 is our general multirate solution for radial flow. Normalizing Eq. 10 by the current flowrate, \( q_n \), at the current time, \( t \) (or \( t_n \)), gives us

\[ \frac{\pi_i - \pi_{wf}}{q_n} = 162.6 \frac{Bu}{kh} \sum_{j=1}^{n} \frac{q_i}{q_n} (q_i - q_{i-1}) \log (t - t_{j-1}) \]

\[ + 162.6 \frac{BnBu}{kh} \left[ \log \left( \frac{k}{\phi \mu_c \chi_w} \right) - 3.2275 + 0.8686 s \right] \]  (11)

Eq. 11 is of the form

\[ y = mx + b \]

where,

\[ y = \frac{\pi_i - \pi_{wf}}{q_n} \]

\[ m = 162.6 \frac{Bu}{kh} \]

\[ x = \sum_{j=1}^{n} \frac{(q_i - q_{i-1})}{q_n} \log (t - t_{j-1}) \]

\[ b = m \left[ \log \left( \frac{k}{\phi \mu_c \chi_w} \right) - 3.2275 + 0.8686 s \right] \]

A plot of \( \frac{\pi_i - \pi_{wf}}{q_n} \) versus \( \sum_{j=1}^{n} \frac{(q_i - q_{i-1})}{q_n} \log (t - t_{j-1}) \) is called the "Odeh-Jones" plot and is used to estimate \( k \) and \( s \).
For this case, we have \( n = 2 \) and \( q_i = q \) and \( q_z = 0 \), for \( t > t_p \) Eq. 10 becomes

\[
P_i - P_{sw} = 162.6 \left( \frac{8u}{kh} \right) \left[ (q_i - q_i^0) \log(t) + (q_i^0 - q_i) \log(t - t_p) \right]
+ 162.6 \left( \frac{q_i^0}{4u} \right) \left[ \log \left( \frac{k}{\phi \mu c_t v_n^2} \right) - 3.2275 + 0.86665 \right]
\]
The previous relation reduces to

$$P_i - P_{ws} = 162.6 \frac{gBu}{kh} \left[ q_i \left[ \log(t) - \log(t-tp) \right] \right] \quad (t>tp) \quad (12)$$

Noting the $\Delta t$ notation, i.e.,

$$\Delta t = t-tp \quad (13)$$

$$t = tp + \Delta t \quad (14)$$

Substituting Eqs. 13 and 14 into Eq. 12, we have (using $q=q_i$)

$$P_i - P_{ws} = 162.6 \frac{gBu}{kh} \left[ \log(tp+\Delta t) - \log(\Delta t) \right]$$

Using the rules of logarithms $[\log(ab) = \log(a) - \log(b)]$, we have

$$P_i - P_{ws} = 162.6 \frac{gBu}{kh} \log \left[ \frac{tp+\Delta t}{\Delta t} \right] \quad (t>tp) \quad (15)$$

where $q=q_i$ and Eq. 15 is valid for $t>tp$. We note that Eq. 15 does not include the skin factor, $s_i$.

For $t \leq tp$, we write Eq. 10 for $n=1$. This gives

the “$f$” notation means “flowing” conditions

$$P_i - P_{ws} = 162.6 \frac{gBu}{kh} \log(t)$$

$$+ 162.6 \frac{gBu}{kh} \left[ \log \left[ \frac{k}{\phi u \phi^{\alpha} k_i} \right] - 3.2775 + 0.8686 s_i \right]$$

(16)
Writing Eq. 16 exactly at \( t = t_p \), we have

\[
\frac{P_i - P_{iw}(t = 0)}{kh} = 162.6 \frac{9.8 \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c_t v^2} \right) - 3.2775 + 0.86865 \right]
\]

(17)

where Eq. 17 is only valid for \( t = t_p \). Subtracting Eq. 15 from Eq. 17 gives

\[
\left[ \frac{P_i - P_{iw}(t = 0)}{kh} \right] - \left[ \frac{P_i - P_{iw}}{kh} \right] = 162.6 \frac{9.8 \mu}{kh} \left[ \log (t_p) - \log \left( \frac{t_p + \Delta t}{\Delta t} \right) \right] + 162.6 \frac{9.8 \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c_t v^2} \right) - 3.2775 + 0.86865 \right]
\]

which reduces to

\[
\frac{P_{iw} - P_{iw}(t = 0)}{kh} = 162.6 \frac{9.8 \mu}{kh} \log \left( \frac{t_p \Delta t}{t_p + \Delta t} \right) + 162.6 \frac{9.8 \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c_t v^2} \right) - 3.2775 + 0.86865 \right]
\]

(18)

where we will define an effective time function, \( \Delta t_e \), as

\[
\Delta t_e = \frac{t_p \Delta t}{t_p + \Delta t}
\]

(19)

Substituting Eq. 19 into Eq. 18

\[
\frac{P_{iw} - P_{iw}(t = 0)}{kh} = 162.6 \frac{9.8 \mu}{kh} \log (\Delta t_e) + 162.6 \frac{9.8 \mu}{kh} \left[ \log \left( \frac{k}{\phi \mu c_t v^2} \right) - 3.2775 + 0.86865 \right]
\]

(20)
We note that Eq. 20 (the buildup equation in \( \Delta t \)) is exactly the same form as Eq. 16 (the drawdown equation in \( t \)), hence \( \Delta t \) is "effective" time.

At \( \Delta t = 1 \text{hr} \), Eq. 18 becomes

\[
\frac{P_{ws, 1 \text{hr}} - P_{wf} (\Delta t=0)}{P_{ws, 1 \text{hr}} - P_{wf} (\Delta t=0)} = \frac{162.6 \, \frac{q_{Rw}}{kh} \left[ \log \left( \frac{tp}{tp+1} \right) + \log \left( \frac{k}{\phi \mu c_t r_w^2} \right) \right] - 3.2275 + 0.86865 \, \frac{\Delta t}{m} }{1.1513 \left[ \log \left( \frac{tp}{tp+1} \right) - \log \left( \frac{k}{\phi \mu c_t r_w^2} \right) \right] + 3.2275} \]  

(21)

Solving Eq. 21 for the skin factor gives,

\[
m = 162.6 \, \frac{q_{Rw}}{kh} \]  

(22)

\[
S = 1.1513 \left[ \frac{P_{ws, 1 \text{hr}} - P_{wf} (\Delta t=0)}{m} - \log \left( \frac{tp}{tp+1} \right) - \log \left( \frac{k}{\phi \mu c_t r_w^2} \right) \right] + 3.2275 \]  

(23)

where \( P_{ws, 1 \text{hr}} \) is taken from the straight-line (or its extrapolation) on the \( P_{ws} \) versus \( \log \left[ \frac{tp+\Delta t}{\Delta t} \right] \) plot.

The \( P_{ws} \) versus \( \log \left[ \frac{tp+\Delta t}{\Delta t} \right] \) plot is known as the Hornor Plot.

A schematic Hornor plot is shown below:

![Hornor Plot Diagram](image)
Using the straight-line analysis on the Horner plot we have

**Graphical Results**

\[ m = \Delta p, \log_{\text{cycle}} = \text{Difference between two pressures } (p_1 \text{ and } p_2) \text{ taken } 1 \text{ log cycle apart on the straight-line trend, or its extrapolation.} \]

\[ \bar{p}_{\text{mhr}} = \bar{p}_{\text{s}} \text{ taken at } \Delta t = 1 \text{ hr from the straight-line trend, or its extrapolation.} \]

\[ \bar{p}^* = \text{Extrapolated pressure at } \frac{tp + \Delta t}{\Delta t} = 1, \text{ related to } \bar{p} \text{ but must be "corrected" using the Matthews-Brons-Hazebroek (MBH) technique.} \]

**Analysis**

**Permeability**

\[ k = \frac{162.6 \text{ GBm}}{\text{mh}} \quad (24) \]

**Skin Factor**

\[ s = 1.1513 \left[ \frac{\bar{p}_{\text{mhr}} - \bar{p}_{\text{s}}(\Delta t=0)}{m} - \log \left[ \frac{tp}{tp+1} \right] - \log \left[ \frac{k}{\phi \mu_0 \gamma_0 f_2} \right] + 3.2375 \right] \quad (23) \]

**Average Reservoir Pressure**

\( \bar{p} \) is estimated using \( \bar{p}^* \) and the MBH technique, which requires prior knowledge of reservoir shape and volume.
MDH Approach

In practice, the production (or injection) history prior to shut-in is often unknown. Rather than try to reproduce the production history, Miller, Dyes, and Hutchinson chose simply to ignore the production history. This results in analysis equations which are exactly the same as those for the drawdown case.

In theory, the MDH plot is valid if the reservoir is produced to pseudosteady-state, flow conditions. Recall that the Horner plot is rigorously derived and hence, is always valid. Given proper care, Horner and MDH analysis should be essentially identical (unless, the well produced erratically, and the production history invalidates both analyses).

One "problem" with the MDH analysis is it provides no measure of $\bar{p}$ (the average reservoir pressure). However, recent efforts have lead to a semi-analytical result for $\bar{p}$ using the Rectangular Hyperbola Method (RHM). This will be discussed later.

The pressure relation for the MDH plot is given by

\[ P_{w5} - P_{wf}(At=0) = 162.6 \frac{q_8 u}{k h} \log(At) \]

\[ + 162.6 \frac{q_8 u}{k h} \left[ \log \left( \frac{k}{\Phi u_2 \gamma_w} \right) - 3.2275 + 0.8656 \right] \] \hspace{1cm} (25)
A schematic MDH plot is shown below.

**Graphical Results**

\[ m = \Delta P_{\text{cycle}} = \text{Difference between two pressures} \]

\[ (P_1 \text{ and } P_2) \text{ taken 1 log cycle apart on the straight-line trend, or its extrapolation.} \]

\[ P_{\text{us, 1hr}} = P_{\text{us}} \text{ taken at } \Delta t = 1 \text{ hr from the straight-line trend, or its extrapolation.} \]

\[ \bar{P} = \text{Estimate from rectangular hyperbola} \]

\[ P_{\text{us}} = \bar{P} - \frac{a}{b + \Delta t} \]

**Analysis**

**Permeability**

\[ k = \frac{162.6 \text{ sq ft}}{\text{m} \cdot \text{h}} \] (24)

**Skin Factor**

\[ \delta = 1.1518 \left[ \frac{P_{\text{us, 1hr}} - P_0 (\Delta t = 0)}{m} - \log \left( \frac{k}{\phi \mu c \rho_{\text{w}} d_i} \right) + 3.2275 \right] \] (26)
Development of Convolution (Superposition) Relations for the Analysis, Interpretation, and Modelling of Pressure and Rate Performance

- Variable-Rate Case
- Variable Pressure Drop Case
- Constant Rate/Constant Pressure Identity

(from Petroleum Engineering 620 Course Notes -- 1994)
Variable Rate Case: (Discrete Constant Rate Steps)

Given a series of discrete (stair-step) changes in flowrate, with each rate acting over a particular interval, we have the following tabular and graphical representations of data:

<table>
<thead>
<tr>
<th></th>
<th>( t )</th>
<th>( q )</th>
<th>( P_{\text{ref}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( t_0 )</td>
<td>( q_0 )</td>
<td>( P_{\text{ref},0} = P_1 )</td>
</tr>
<tr>
<td>1</td>
<td>( t_1 )</td>
<td>( q_1 )</td>
<td>( P_{\text{ref},1} )</td>
</tr>
<tr>
<td>2</td>
<td>( t_2 )</td>
<td>( q_2 )</td>
<td>( P_{\text{ref},2} )</td>
</tr>
<tr>
<td>( n )</td>
<td>( t_n )</td>
<td>( q_n )</td>
<td>( P_{\text{ref},n} = P_{\text{ref},n+1} )</td>
</tr>
</tbody>
</table>

Considering these data graphically, we note the discrete or "step" changes in flowrate and the effect of these rate changes on the wellbore pressure profile.

We also provide a plot of \((q_i - q_{i-1})\) versus time where \((q_i - q_{i-1})\) establishes the pressure change over a particular step. Physically, we see that the "total" pressure change, \(P_i - P_{\text{ref},n}\), is the sum of the previous pressure drops.

Quantitatively, this "total" pressure drop is given as

\[
\Delta P_t = (q_i - P_{\text{ref},n}) = (q_i - q_{i-1}) + (P_{\text{ref},i-1} - P_{\text{ref},i}) + (P_{\text{ref},i-1} - P_{\text{ref},i}) + \ldots + (P_{\text{ref},n-1} - P_{\text{ref},n}) \tag{1}
\]

The "incremental" pressure drops are computed using a ratio of the pressure drop for the constant rate solution, \(\Delta P_{\text{ref}}(t-t_{i-1})\), and some reference rate, \(q_r\), to the incremental pressure drop \((P_{\text{ref},i-1} - P_{\text{ref},i})\) divided by the rate change over the interval \((q_i - q_{i-1})\). This relation is expressed as
\[
\frac{(\eta_{w1,j-1} - \eta_{w1,j})}{(q_j - q_{j-1})} = \frac{1}{q_{r}} \Delta P_{cr} (t - t_{j-1})
\]  

The physical translation of Eq. 2 would be that the pressure drop over a particular interval \((\eta_{w1,j-1} - \eta_{w1,j})\) is equal to the pressure drop for a constant rate, \(\Delta P_{cr}(t-t_{j-1})\), multiplied by the rate difference over the interval, \((q_j - q_{j-1})\), divided (or normalized) by the constant reference rate, \(q_r\). The rate ratio term, \((q_j - q_{j-1})/q_{r}\), represents the fraction of the total constant rate pressure drop, \(\Delta P_{cr}(t-t_{j-1})\). Rearranging Eq. 2 we obtain

\[
\frac{(\eta_{w1,j-1} - \eta_{w1,j})}{q_{r}} = \frac{1}{q_{r}} \Delta P_{cr} (t - t_{j-1})
\]

where Eq. 3 is the fundamental basis of our efforts in the development of convolution relations. Combining Eqs. 1 and 3 we obtain

\[
\Delta p_j = (p_j - p_{w1,n}) = \frac{1}{q_{r}} \left[ \sum_{i=0}^{n} \left( (q_j - q_{j-1}) \Delta P_{cr} (t - t_{i}) + (q_{j-1} - q_i) \Delta P_{cr} (t - t_{j-1}) \right) \right]
\]

or

\[
\frac{p_j - p_{w1,n}}{q_{r}} = \frac{1}{q_{r}} \sum_{i=0}^{n} \left( (q_j - q_{j-1}) \Delta P_{cr} (t - t_{j-1}) \right)
\]

where in Eq. 4, \(t_0 = 0\) and \(q_0 = 0\), and \(\Delta P_{cr}(t-t_{j-1})\) is the constant rate pressure drop solution. Recalling the definitions of dimensionless time, pressure, and normalized flow rate, we have

\[
t = t_{dc} \frac{k}{\delta w_{cr} r_{w}^2} t \quad \beta = \beta_{dc} \frac{k h (p_i - p_f)}{q_{B} m} \quad \beta_{0} = \frac{\beta}{q_{r}}
\]

where \(t_{dc}\) and \(\beta_{dc}\) are given by

<table>
<thead>
<tr>
<th>Carly Units</th>
<th>Field Units</th>
<th>ST Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_{dc})</td>
<td>1</td>
<td>2.637 \times 10^{-4}</td>
</tr>
<tr>
<td>(\beta_{dc})</td>
<td>(2\pi)</td>
<td>7.081 \times 10^{-3}</td>
</tr>
<tr>
<td>(1/\beta_{dc})</td>
<td>(1/2\pi)</td>
<td>141.2</td>
</tr>
</tbody>
</table>

Solving Eq. 6 for \(\Delta P_{cr}(t-t_{j-1})\) we have

\[
\Delta P_{cr} = \frac{1}{\beta_{dc}} \frac{q_{B} m}{k h} \beta_{0} \quad (8) \quad \text{where} \quad \beta_{0} = \beta + \epsilon
\]
where $j$ is the skin factor, which represents a dimensionless pressure due to damage or stimulation in the near well region. Substituting Eqs. 5, 7, and 8 into Eq. 4 we obtain

$$P_i - P_{w,i} = \frac{1}{\frac{q_i}{kh}} \sum_{j=1}^{m} \left( q_{D,j} - q_{D,j-1} \right) \rho_{sd} (t - t_{D,j-1})$$ (10)

or

$$P_{wd} (t) = \frac{q}{\frac{q_i}{kh}} \sum_{j=1}^{m} \left( q_{D,j} - q_{D,j-1} \right) \rho_{sd} (t - t_{D,j-1})$$ (11)

Returning to Eq. 4 our goal is to develop a continuous or integral form of Eq. 4. Recalling Eq. 4

$$P_i - P_{w,i} = \frac{1}{q_i} \sum_{j=1}^{m} \left( q_j - q_{j-1} \right) \Delta P_{cr} \left( t - t_{j-1} \right)$$ (4)

Defining $\Delta \tau = t_j - t_{j-1} = t_j - \Delta \tau$ or $\Delta \tau = t_j - t_{j-1}$ and multiplying through the summation term in Eq. 4 by $\Delta \tau / \Delta \tau$ gives

$$P_i - P_{w,i} = \frac{1}{q_i} \sum_{j=1}^{m} \frac{\Delta \tau}{\Delta \tau} \Delta P_{cr} \left( t - \tau \right) \Delta \tau$$

and defining $\Delta q_j = q_j - q_{j-1}$ we have

$$P_i - P_{w,i} = \frac{1}{q_i} \sum_{j=1}^{m} \frac{\Delta q_j}{\Delta \tau} \Delta P_{cr} \left( t - \tau \right) \Delta \tau$$ (12)

Taking the limit of the RHS of Eq. 12 as $\Delta \tau \to 0$, we have

$$P_i - P_{w,i} = \lim_{\Delta \tau \to 0} \left[ \frac{1}{q_i} \sum_{j=1}^{m} \frac{\Delta q_j}{\Delta \tau} \Delta P_{cr} \left( t - \tau \right) \Delta \tau \right]$$

or

$$P_i - P_{w,i} (t) = \frac{1}{q_i} \int_{\tau}^{t} \delta (\tau) \Delta P_{cr} \left( t - \tau \right) d\tau$$ (13)

Applying the same concepts to the dimensionless form (Eq. 11) we obtain

$$P_{wd} (t) = \int_{0}^{t} q' (\tau) \rho_{sd} (t - \tau) d\tau$$ (14)
Variable-Rate Case: Summary of Results

Discrete (Stair-Step) Rate Changes

\[ p_i - p_{ref, n} = \frac{i}{q_r} (q_i - q_{i-1}) \Delta P_{th} (t - t_{i-1}) \]  
(4)

\[ P_0 (t) = \sum_{i=1}^{n} (q_{D,i} - q_{D,i-1}) P_{0i} (t - t_{D,i-1}) \]  
(11)

Continuous Rate Changes

\[ P_i - P_{ref}(t) = \frac{1}{q_r} \int_{0}^{t} \alpha(t') \Delta P_{th} (t - t') dt' \]  
(13)

\[ P_{ref}(t) = \int_{0}^{t} q_i (t') P_{D}(t - t') dt' \]  
(14)

Variable Pressure Drop Case: (Discrete Constant Pressure Case)

As in the variable-rate case, we consider discrete changes in wellbore pressure over a particular time interval. As we considered a "total" pressure drop for the variable-rate case, we consider a "total" rate given by

\[ q_{tot} = q_1 + q_2 + q_3 + \ldots + q_n \]  
(15)

Similar to the discrete rate change case, the case for discrete pressure changes considers an equivalent total rate, \( q_{tot} \), that models the rate behavior had a constant pressure been maintained throughout the production history. The mathematical form is given by

\[ \frac{q_i}{(P_i - P_{refI})} = \frac{q_{tot}}{(P_i - P_r)} \]  
(16)
where \( p_r \) is the "reference" pressure (maintained constant throughout the production history) and \( q_{ep}(t-t_i,t_{i-1}) \) is the rate response to \( p_r \). Solving Eq. 16 for \( q_i \) gives
\[
q_i = \frac{(p_i-p_{rw,i})}{(p_i-p_r)} q_{ep}(t-t_{i-1}) \tag{17}
\]
Substituting Eq. 17 into Eq. 15 we have
\[
q_i = \frac{1}{(p_i-p_r)} \left[ (p_i-p_{rw,1}) q_{ep}(t-t_1) + (p_i-p_{rw,2}) q_{ep}(t-t_2) + \ldots + (p_i-p_{rw,n}) q_{ep}(t-t_{n-1}) \right] \tag{18}
\]
or
\[
q_i = \frac{1}{(p_i-p_r)} \sum_{j=1}^{n} (p_i-p_{rw,j}) q_{ep}(t-t_{j-1}) \tag{19}
\]
where \( t_0 = 0 \) and \( q_{ep}(t-t_{i-1}) \) is the constant pressure rate solution over a particular time interval, \( t_i-t_{i-1} \). If we consider the cumulative production function we have
\[
q_i = \frac{1}{(p_i-p_r)} \sum_{j=1}^{n} (p_i-p_{rw,j}) q_{ep}(t-t_{j-1}) \tag{20}
\]
where \( q_{ep}(t-t_{i-1}) \) is the constant pressure cumulative production solution over a particular time interval, \( t_i-t_{i-1} \). Recalling the definition of dimensionless pressure, \( p_b \), we have
\[
p_b = p_r \frac{k h}{q_{ep} \mu} \tag{6}
\]
We define the dimensionless rate, \( q_{d, ep} \), as the reciprocal of the dimensionless pressure. This is given as
\[
q_{d, ep} = \frac{1}{p_b} \frac{\mu}{k h (p_i-p_r)} q_{ep} \tag{21}
\]
where the "ep" notation is used to indicate the constant pressure solution, where \( p_r \) is the constant reference pressure. Substituting Eq. 21 into Eq. 19 we have
\[
q_i (t) = p_b \frac{k h}{\mu} \sum_{j=1}^{n} (p_i-p_{rw,j}) q_{d, ep}(t-t_{j-1}) \tag{22}
\]
where \( q_{d, ep} \) can be written in terms of \( t-t_{i-1} \) or \( t_0-t_{0,n-1} \).
Multiplying through Eq. 22 by \( \frac{1}{Tc c \frac{k}{h(p_i-p_r)}} \) and using \( t \) rather than \( t_D \), we have
\[
q_{tD}(t) = \sum_{j=1}^{i} \left( \frac{p_i - p_{w},j}{p_i - p_r} \right) q_{c p}(t) \left( t - t_D,j - 1 \right)
\]  (23)

Defining a dimensionless pressure ratio, \( \beta_{c p} \), for the pressure drop groups in Eq. 23. This ratio is
\[
\beta_{c p,j} = \left( \frac{p_i - p_{w},j}{p_i - p_r} \right)
\]  (24)

Substituting Eq. 24 into Eq. 23 we obtain
\[
q_{tD}(t) = \sum_{j=1}^{i} \beta_{c p,j} q_{c p}(t - t_D,j - 1)
\]  (25)

Similarly, for the cumulative production function we have
\[
Q_{tD}(t) = \sum_{j=1}^{i} \beta_{c p,j} Q_{c p}(t - t_D,j - 1)
\]  (26)

The definition of cumulative production is given by
\[
Q(t) = \int_0^t q(t) dt \quad (t \text{ in hours by convention})
\]  (27)

Recalling the definition of dimensionless time (Eq. 5)
\[
t_D = \frac{t_D c}{k} \frac{k}{\mu c_t r_i} t
\]  (5)

Defining a variable of substitution, \( \tau \), based on dimensionless time, \( t_D \), we have
\[
\tau_D = \frac{t_D c}{k} \frac{k}{\mu c_t r_i^2} \tau
\]  (28)

Solving Eq. 28 for \( \tau \) we obtain
\[
\tau = \frac{1}{t_D c} \frac{k}{\mu c_t r_i^2} \tau_D
\]  (29)

and we can differentiate Eq. 29 to yield
\[
d\tau = \frac{1}{t_D c} \frac{k}{\mu c_t r_i^2} d\tau_D
\]  (30)

Recalling the dimensionless rate (Eq. 21) we have
\[
q_{c p} = \frac{1}{Tc c \frac{k}{h(p_i-p_r)}} q_{c p}(t)
\]  (6)
Solving for \( q_{ep}(t) \) we have
\[
q_{ep}(t) = \frac{Bc}{kh(p_i-p_r)} q_{Dcp}
\]
Recalling Eq. 28 and substituting the limits of Eq. 27 we have
\[
\text{at } t=0, \quad \varphi_0 = 0
\]
\[
\text{at } t=t_c, \quad \varphi_D = t_c
\]
Substituting Eqs. 30 and 31, along with the transformed limits into Eq. 27 we have
\[
\varphi(t) = 24 \int_0^{t_c} \left( \frac{Bc}{kh(p_i-p_r)} q_{Dcp} \left( \frac{1}{t_{dc}} \frac{\partial w_t u^2}{k} \right) \right) dt_c
\]
Isolating the dimensionless terms on the right-hand-side and cancelling like terms we have
\[
24 \frac{t_{dc}}{Bc} \frac{E}{\phi h c_t w^2 (p_i-p_r)} \varphi(t) = \int_0^{t_c} q_{Dcp}(t) dt_c
\]
Defining the dimensionless cumulative production, \( Q_D(t_D) \), we have
\[
Q_D(t_D) = 24 \frac{t_{dc}}{Bc} \frac{E}{\phi h c_t w^2 (p_i-p_r)} Q(t)
\]
Van Everdingen and Hurst Identity in Laplace Domain for Relating the Constant Pressure and Constant Rate Solutions

Recalling the convolution relation for a continuously changing rate history, we have

\[ p_i(t) - p_w(t) = \frac{1}{q_r} \int_0^t q(\tau) \Delta p_{scr}(t - \tau) d\tau \]  

(3)

We want to illustrate that the derivative inside the convolution integral can be moved from the rate term to the pressure drop terms. This can be obtained by integrating Eq. 13 by parts. This gives

\[ \frac{d}{dt} \left[ \Delta p_{scr}(t - \tau) \right] \]

\[ \frac{d}{dt} [\Delta p_{scr}(t - \tau)] \\
\]

\[ v = q(t) \]

Substituting

\[ \Delta p_i = p_i(t) - p_w(t) = \frac{1}{q_r} \int_0^t q(\tau) \frac{d}{dt} [\Delta p_{scr}(t - \tau)] d\tau \]

where \( q(0) \leq 0 \), and \( \Delta p_{scr}(0) = 0 \), which gives

\[ \Delta p_i = p_i(t) - p_w(t) = \frac{1}{q_r} \int_0^t q(\tau) \frac{d}{dt} [\Delta p_{scr}(t - \tau)] d\tau \]  

(34)

Taking the Laplace transform of Eq. 34 we obtain

\[ \tilde{\Delta p}_i(s) = \frac{1}{q_r} \int_0^t q(\tau) \frac{d}{dt} [\Delta p_{scr}(t - \tau)] d\tau \]

(35)

From Roberts and Kaufman, "Table of Laplace Transforms," p. 7, Part I, Eq. 40 we have the following "convolution" identity

\[ \frac{f(t)}{\int_0^t f(t) f(t - \tau) d\tau} \frac{F(s)}{f(t) f(t)} \]

Substituting these results into Eq. 35 gives

\[ \Delta p_i(s) = \frac{1}{q_r} \tilde{q}(s) \tilde{p}(s) \tilde{L} \left\{ \frac{d}{dt} [\Delta p_{scr}(t)] \right\} \]

or

\[ \Delta p_i(s) = \frac{1}{q_r} \tilde{q}(s) \left[ m \Delta p_{scr}(m) - \Delta p_{sc}(0) \right] \]
which reduces to
\[ \Delta \bar{p}_{r}(m) = \frac{m}{\bar{q}_{r}} \Delta \bar{p}_{scr}(m) \]  
(36)

if we consider that the pressure is maintained constant
we have
\[ \Delta \bar{p}_{r}(m) = \frac{\Delta \bar{p}_{p}}{m} \]  
(37)

Taking the laplace transform of Eq. 37 gives
\[ \Delta \bar{p}_{r}(m) = \frac{\Delta \bar{p}_{p}}{m} \]  
(38)

Substituting Eq. 38 into Eq. 36 gives (where \( q(t) = \bar{q}_{cp} \))
\[ \frac{1}{m} \Delta \bar{p}_{p} = \frac{m}{\bar{q}_{cp}} \Delta \bar{p}_{scr}(m) \]  
(39)

Rearranging
\[ \frac{\bar{q}_{cp}(m)}{\Delta \bar{p}_{p}} = \frac{\Delta \bar{p}_{scr}(m)}{\bar{q}_{r}} = \frac{1}{m^2} \]  
(40)

Recalling the definitions of dimensionless rate and pressure
\[ \bar{q}_{cp}(t) = \frac{1}{\bar{q}_{cp}} \frac{\bar{q}_{cp}(t)}{q_{cp}(t)} \quad \text{or} \quad \frac{\bar{q}_{cp}}{\Delta \bar{p}_{p}} = \frac{\bar{q}_{cp}}{\bar{q}_{r}} \]  
(41)

and
\[ \frac{\bar{q}_{cp}(t)}{\Delta \bar{p}_{p}} = \frac{\bar{q}_{cp}}{\bar{q}_{r}} \frac{\Delta \bar{p}_{scr}(m)}{\bar{q}_{r}} \]  
(42)

Taking the laplace transform of Eqs. 41 and 42
\[ \frac{\bar{q}_{cp}(m)}{\Delta \bar{p}_{p}} = \frac{\bar{q}_{cp}}{\bar{q}_{r}} \frac{\Delta \bar{p}_{scr}(m)}{\bar{q}_{r}} \]  
(43)

and
\[ \frac{\Delta \bar{p}_{scr}(m)}{\bar{q}_{r}} = \frac{1}{m^2} \]  
(44)

Substituting Eqs. 43 and 44 into Eq. 40
\[ \left[ \frac{\bar{q}_{cp}}{\bar{q}_{r}} \right] \left[ \frac{1}{m^2} \right] = \frac{1}{m^2} \]  
Cancelling like terms
\[ \frac{\bar{q}_{cp}(m)}{\bar{q}_{r} \bar{p}_{so}(m)} = \frac{1}{m^2} \]  
(45)
The result given by Eq. 45 clearly shows that the constant rate and constant pressure solutions can easily be related in the Laplace domain. This result means that all we need ever derive is either the constant rate or pressure solution in the Laplace domain, then establish the other solution using Eq. 45.

As an aside, we also note that Eq. 45 can be inverted using the convolution theorem to yield:

\[
\int_0^T \tilde{q}(t) \tilde{p}_0(t-T) \, dt = T
\]

or

\[
\int_0^T \tilde{p}_0(t) \tilde{q}(t-T) \, dt = T
\]

The utility of Eqs. 46 and 47 is not obvious, but a likely application would be the determination of the desired solution from values of the other solution tabulated in the real \( t_0 \) domain.

**Laplace Domain Convolution: Summary of Results**

**Continuous Variable Rate Convolution**

\[
\Delta p = P_0 - P_{out}(t) = \frac{1}{q_r} \int_0^T \tilde{q}(t) \Delta p_{in}(t-T) \, dt
\]

or

\[
\Delta P = P_0 - P_{out}(t) = \frac{1}{q_r} \int_0^T \tilde{q}(t-T) \, \left[ \Delta p_{in}(t-T) \right] \, dt
\]

**Laplace Domain Convolution: Constant Pressure Rate Relation**

\[
\tilde{q}_{out}(s) \tilde{p}_0(s) = \frac{1}{s^2}
\]