Analysis of Simultaneously Measured Pressure and Sandface Flow Rate in Transient Well Testing

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Summary
New well test interpretation methods are presented that eliminate wellbore storage (afterflow) effects. These new methods use simultaneously measured sandface flow rate and wellbore pressure data. It is shown that formation behavior without storage effects (unit response or influence function) can be obtained from deconvolution of sandface flow rate and wellbore pressure data. The storage-free formation behavior can be analyzed to identify the system (reservoir flow pattern) that is under testing and to estimate its parameters. Convolution (radial multirate) methods for reservoir parameter estimation and a few synthetic examples for deconvolution and convolution also are presented.

Introduction
Well testing with measured sandface flow rate can be traced to the beginning of reservoir engineering. The rate must be measured over time to calculate and/or approximate constant rate to obtain even a single reservoir parameter from pressure measurements. This approximate constant rate has been sufficient for estimating permeability, skin, and initial formation pressure during the radial infinite-acting period. During this period, the well should produce at a constant rate at the sandface or at a zero rate if a buildup test is conducted. Because of compressible fluid in the production string (wellbore storage effects), it takes a long time to reach the radial infinite-acting period. The effect of outer boundaries also may start before the end of the wellbore storage effects.

In general, the storage capacity of the wellbore, wellbore geometry, near-wellbore complexities, and external boundaries affect transient behavior of a well. During the analysis of pressure-time data, each of these phenomena and its duration must be recognized for the application of semilog and type-curve techniques to determine formation flow capacity \( kh \), damage skin, and average formation pressure. The influence of these phenomena on transient behavior of a well progresses over time. For the sake of convenience, the test time can be divided into three periods according to which phenomenon is affecting the pressure. These periods are defined as follows.

Early-Time Period. The combined effects of wellbore storage, damage skin, and pseudoskin (which include partial penetration, perforation, acidizing, fractures, non-Darcy flow, and permeability reduction caused by gas saturation around the wellbore) dominate pressure behavior. The stratification and dual porosity also may affect wellbore pressure during this period.

Middle-Time Period. During this period, radial flow is established. Conventionally, semilog techniques are used to determine formation \( kh \) and initial pressure and skin.

Late-Time Period. During this period, outer boundary effects start to distort the semilog straight line. For example, the gas cap shows a curve-flattening effect on log-log and Horner plots. Sometimes the separation of these periods from each other is impossible; particularly, the effects of bottom-water influx and/or gas cap may start during the middle-time period. Thus, the semilog approach sometimes cannot be applied at all.

Furthermore, the drawdown or buildup tests as conducted today tend to homogenize the reservoir behavior. In other words, most of the reservoirs behave homogeneously during the storage-free radial infinite-acting period because most of the heterogeneous behavior takes place during the early-time period.

The type-curve approaches have been introduced to overcome some of these problems. The theories, applications, and elaborations of the type-curve methods, as well as many references, can be found in Ref. 1. In 1979, Gringarten et al. introduced new type-curves that use different parameterization than the earlier ones, namely Ramey, Agarwal et al., McKinley, and Earlougher and Kersch types. All the type curves presented by these authors, and many others, were developed under the assumption that the fluid compressibility (density) in the tubing and annulus remains constant during the test period. During the early time, particularly for buildup tests, shut-in pressure increases very rapidly; thus, the compressibility is usually higher than the compressibility of the fluid in the reservoir for producing wells. Since the pressure in the wellbore is a function of the depth, the compressibility of the fluid at the wellhead can be 10 or even 100 times greater than the compressibility of the fluid at the bottom. Thus, the assumption that the wellbore storage coefficient is constant during the drawdown, and particularly during buildups, may not be correct. A variable wellbore storage coefficient alone makes the application...
of type-curve methods almost impossible. The combination of variable or even constant wellbore storage with wellbore geometry further complicates the type-curve matching process. Moreover, wellbore pressure data alone may not indicate changing wellbore storage.

Some of the problems inherent with the use of type-curve methods can be eliminated by the simultaneous use of measured sandface flow rate and pressure data.

The purpose of this work is to study the use of the measured sandface flow rate in a broad sense with regard to transient well testing. Furthermore, we explore the use of convolution and deconvolution in the interpretation of pressure behavior of a well with afterflow (buildup case) or wellbore storage (drawdown case).

Background

The use of the sandface flow rate in transient testing is not new. To our knowledge, van Everdingen and Hurst were the first to estimate and use the sandface flow rate to calculate the wellbore pressure. To do this, they approximated the sandface flow rate by the formula

\[ q_s = q \left(1 - e^{-\beta t} \right), \quad \text{......(1)} \]

where \( \beta \) is a positive constant. These authors stated that the constant, \( \beta \), can be determined from well and reservoir parameters. Using the above formula and the convolution integral, van Everdingen and Hurst presented an expression for the wellbore pressure with a variable wellbore storage effect. Gladfelter et al. presented a method to determine the formation kh from pressure and afterflow data. The afterflow data were obtained by measuring the rise in the liquid level in the wellbore. Ramey applied the Gladfelter approach to gas well buildup tests.

A considerable amount of work also has been done on multirate (variable-rate) tests during the last 30 years. However, these are basically sequential constant-rate drawdowns; only transient pressure is measured and rate is assumed constant during each drawdown test. The techniques related to this type of multirate tests also can be found in Ref. 1. All the work mentioned so far deals with the direct problem. In other words, the constant-rate solution (the influence or the unit response function) is convolved (superimposed) with the time-dependent boundary condition to obtain solutions to the diffusivity equation. This process is called “convolution.”

Hutchison and Sikora, Katz et al., and Coates et al. presented methods for determining the influence function directly from field data for aquifers. The process of determining the influence function is called “deconvolution.”

Jargon and van Poolen were perhaps the first to use the deconvolution of variable rate and pressure data to compute the constant-rate pressure behavior (the influence function) of the formation in well testing. Bostic et al. used a deconvolution technique to obtain a constant-rate solution from a variable rate history with a known pressure history. They also extended the deconvolution technique to combine production and buildup data as a single test. Pascal also used deconvolution techniques to obtain a constant-rate solution from variable rate (measured at the surface) and pressure measurements of a drawdown test.

More recently, Meunier et al. have used sandface flow measurements with pressure data for buildup test analysis. This has been the first successful attempt to use direct measurements of sandface flow rate data in well testing. They showed that the Horner method can be modified to reach a semilog straight line earlier than type-curves or the \(1^{3/4}\)-cycle rule indicates.

Theoretical Developments

During the last 4 or 5 decades, many solutions have been developed for transient fluid flow through porous media. The superposition theorem (Duhamel’s theorem) has been used to derive solutions for time-dependent boundary conditions from time-independent boundary conditions. For example, the multiple-rate testing is a special application of the superposition theorem.

In their classic paper on unsteady-state flow problems, van Everdingen and Hurst presented the dimensionless wellbore pressure for a continuously varying flow rate as

\[ p_{wD}(t_D) = q_{D}(0) p_{sD}(t_D) + \int_0^{t_D} q_D(r) p_{sD}(t_D - r) \, dr. \quad \text{......(2a)} \]

An alternative form to Eq. 2a can be obtained by an integration by parts as

\[ p_{wD}(t_D) = q_D(t_D) p_{sD}(0) + \int_0^{t_D} q_D(r) p_{sD}(t_D - r) \, dr. \quad \text{......(2b)} \]

where

\[ p_{wD} = \frac{k h}{141.2 q B \mu} \left[ \rho_i - \rho_{wf}(t) \right], \]

\[ t_D = \frac{0.0002637 h}{\phi \mu c r_w^2}, \]

\[ p_{sD} = p_D + S, \]

\[ p_D(t_D) = \text{the dimensionless sandface pressure for the constant-rate case without wellbore storage and skin effects,} \]

\[ S = \text{steady-state skin factor,} \]

\[ p_D(t_D) = \frac{dp_D(t_D)}{dt_D}, \]

\[ q_s(t) = \frac{q_{D}(t)}{q_r}, \]

\[ q_s(t) = \text{variable sandface flow rate (flowmeter readings),} \]

\[ q_D(t) = \text{dimensionless sandface rate.} \]
Although traditionally the skin effect is considered a dimensionless quantity different from the dimensionless formation pressure, the skin effect will be treated here as part of the inner boundary condition for the solution of a unit rate production case. This boundary condition is known as the homogeneous boundary condition of the third kind.

It should be emphasized that Eqs. 2a and 2b can be applied for many reservoir engineering problems. The linearity of the diffusivity equation allows us to use Eqs. 2a and 2b for fractured, layered, anisotropic, and heterogeneous systems as long as the fluid in the reservoir is single phase. Eqs. 2a and 2b can be applied to both downhole and buildup tests if the initial conditions are known. For a reservoir with a constant and uniform pressure distribution \( p_D(0)=0 \), Eqs. 2a and 2b can be expressed as

\[
\frac{D}{D_t} p_D(t_D) = q_D(t_D) \quad \text{for} \quad t_D > 0
\]

Furthermore, Eq. 3a also can be expressed as

\[
P_wD(t_D) = \int_0^{t_D} q_D(t_D) p_xD(t_D - \tau) d\tau \quad \text{(3a)}
\]

\[
= S q_D(t_D) + \int_0^{t_D} q_D(t) p_xD(t_D - \tau) d\tau \quad \text{(3b)}
\]

In Eqs. 3a and 3c, it is assumed that \( q_D(t_D) \) exists. If \( q_D(t_D) \) is constant, then Eq. 3b must be used. Eqs. 3a and 3c are known as a Volterra integral equation of the first kind and the convolution type.

Although it is assumed that \( p_D \) is a constant-rate solution without storage effect, mathematically and physically it can be a solution of a constant-storage case. In practice, \( p_D \) always will be affected by the wellbore fluid that occupies the volume below the flowmeter unless the sandface rate is measured through perforations. However, the volume below the flowmeter will be small for most wells, since the flowmeter and the pressure gauge usually can be placed just above perforations. Therefore, throughout this paper, we will assume that \( p_xD \) is not affected by the fluid volume below the flowmeter.

The purpose of the test well interpretation, as stated by Gringarten et al., 2 is to identify the system and determine its governing parameters from measured data in the wellbore and at the wellhead. This problem is known as the inverse problem. The solution of the inverse problem usually is not unique. As Gringarten et al. 2 pointed out, if the number and the range of measurements increase, the nonuniqueness of the inverse problem will be reduced. Thus, combining sandface flow rate with pressure measurements will enhance the conventional (including type-curve) well test interpretation methods.

As an inverse problem, the sandface pressure, \( p_xD(t_D) \), has to be determined by the deconvolution of the integral in Eqs. 3a, 3b, and 3c. As stated previously, \( p_xD(t_D) \) is the solution for the constant-flow-rate (sandface) case. Taking the Laplace transform of Eq. 3a and solving for \( \tilde{p}_xD(s) \) yields

\[
\tilde{p}_xD(s) = \frac{\tilde{p}_wD(s)}{\tilde{s} q_D(s)} \quad \text{(4)}
\]

where \( s \) is the Laplace transform variable.

The Laplace transform of \( p_xD \) in Eqs. 3b and 3c will be the same as Eq. 4, keeping in mind that \( p_xD(0)=0 \). Thus, we have only one operational form of the convolution integral given by Eqs. 3a, 3b, and 3c. The superposition theorem in this case is nothing more than the convolution of \( p_xD(t_D) \) and \( q_D(t_D) \). The Laplace transform of the convolution integral allows us to express the convolution integral in many different forms. Furthermore, the kernel solution can be a solution of constant rate or constant-pressure case for the convolution integral.

If the wellbore storage is constant, the dimensionless sandface flow rate can be expressed as 4, 18

\[
q_D(t_D) = 1 - C_D \frac{dp_D(t_D)}{dt_D} \quad \text{(5)}
\]

where

\[
q_D(t_D) = \frac{q_D(t_D)}{q_r}
\]

As a special application of Eq. 4, the dimensionless wellbore pressure solution for the constant-storage case can be written directly from Eqs. 4 and 5 as

\[
\overline{p}_wD(s) = \frac{\overline{p}_xD(s)}{1 + C_D s^2 \overline{p}_D(s)} \quad \text{(6)}
\]

*Throughout this paper, the function \( F(s) \) will be called the Laplace transform of the function \( f(t) \).
where \( \tilde{p}_{sD}(s) \) is the dimensionless sandface pressure for the constant-rate case without storage effect but including skin.

Van Everdingen and Hurst\(^1\) presented an equation similar to Eq. 6, and Agarwal \textit{et al.}\(^4\) presented the same equation as an integro-differential form for radial systems. Cinco-Ley and Sananiego\(^1\) (for fractured reservoirs), and Kulkuk and Kirwan\(^2\) (for partially penetrated wells) presented the same expression for the dimensionless formation pressure as in Eq. 6.

On the other hand, the dimensionless sandface flow rate can be obtained directly from Eqs. 5 and 6 in terms of the dimensionless formation pressure as

\[
\tilde{q}_D(s) = \frac{1}{s[1 + C_D s^2 \tilde{p}_{sD}(s)]}.
\]  

Fig. 1 presents values of \( q_D \) calculated from Eqs. 1 and 7 as a function of real time for a buildup test using reservoir and fluid properties given in Table 1. As can be seen from this figure, for the exponential decline case, the sandface flow rate declines faster than the constant-wellbore-storage case. Ramey and Agarwal\(^2\) also presented values of \( q_D(t_D) \) as a function of \( t_D \) for various skin and storage constants.

Another important application of the convolution integral given in Eqs. 3a and 3c was presented by van Everdingen,\(^7\) Hurst,\(^8\) and Ramey\(^10\) for calculating the wellbore pressure by using Eq. 1 and the line source solution.

For a finite wellbore radius, the dimensionless wellbore pressure solution also can be written directly from Eqs. 1 and 4 for the exponential sandface rate decline case as

\[
\tilde{p}_{WB}(s) = \frac{\beta \tilde{p}_{sD}(s)}{\beta + s}.
\]  

The exponential constant, \( \beta \), is given by van Everdingen\(^7\) as

\[
\beta = \frac{\alpha \mu c_r w^{-2}}{0.000264} k,
\]

where \( \alpha \) can be determined from wellbore pressure data, such as the wellbore storage constant. Note that \( \beta \) is a dimensionless constant like \( C_D \).

As Ramey\(^10\) noted, \( \beta \) cannot be estimated as readily as the wellbore storage constant, \( C_D \). However, in principle, Eq. 9 has a much more immediate connection with the real systems. In fact, the pressure increases more rapidly during the early-time buildup tests; then it slows down during the transition period and builds up very slowly during the semilog period. Because of the rapid change of pressure, the wellbore storage will decrease continuously except in the case of phase redistribution.

In many cases, it is difficult to recognize changing wellbore storage effects because it is a gradual and continuous change. Furthermore, the finite closing time of the wellhead valve also will affect pressure at the same time.

Most of the work thus far in early-time analysis has been directed toward the construction of type curves from the solutions of Eq. 7 for a constant wellbore storage for different wellbore geometries, such as fractured wells, partial penetration, etc. Type curves for Eq. 8 for different values of \( \beta \) and skin also can be developed and used for gradually decreasing wellbore storage cases to determine skin and \( k_h \).

Fig. 2 presents a semilog plot of \( p_{wf}(\Delta t) \) vs. \( \Delta t \), calculated from Eqs. 7 and 8 by using reservoir and fluid parameters given in Table 1. As shown by this figure, the exponential decline case approaches a semilog straight line earlier than the constant-wellbore-storage case.

The most important point to be made from the above discussion and buildup data presented in Fig. 2 is that the principle limitation of the type-curve analysis stems from the lack of information about the sandface flow rate behavior. Thus, the type-curve analysis usually is used for qualitative answers and supported by the semilog analysis. For quantitative analyses of the early-time data, it is necessary to measure the sandface flow rate. Furthermore, the use of measured sandface flow rate also can improve the semilog analysis. In the following sections, the convolution and deconvolution of simultaneously measured sandface flow rate and pressure data will be shown to obtain the formation pressure and parameters.

Convolution (Superposition)

Continuous Multirate Method. The simplest approach to solving the convolution integral is to assume a \( p_{WD} \) function in Eq. 3a. This \( p_{WD} \) function could be a line-source solution, infinite conductivity vertical fractured solution, etc., for the constant-flow-rate case or the constant-pressure case. The chosen \( p_{WD} \) function can be convolved with \( q_D \) (sandface flow rate) by using the convolution integral (Eqs. 3a through 3c) to modify the time function or \( p_{WF} \). For example, the convolution (superposition) of variable rate with the log approximation for the \( p_{WD} \) function and wellbore pressure commonly is used for the analysis of multirate tests. The same technique also can be used for the analysis of buildup or drawdown tests with measured sandface flow rate data.

Using the log approximation for \( p_{WD} \) in Eq. 3a, the change in measured pressure in oilfield units can be expressed as

\[
\Delta p_{WF}(t) = m \int_0^t q_D(\tau)[\log(\tau - \tau) + \tilde{S}](\tau) d\tau,
\]  

\( \Delta p_{WF}(t) \) vs. \( \Delta t \) for \( \tilde{S} \) and \( \tilde{p}_{sD} \) presented in Fig. 2.
\[ \Delta p_{wf}(t) = p_i - p_{mf}(t), \]
\[ S = 0.875 + \log \left( \frac{k}{\phi \mu c r_w^2} \right) - 3.2275, \]
\[ q_D = \frac{q_f(t)}{q_B}, \]

and
\[ m = \frac{162.6 \mu B q}{kh}. \]

Eq. 9 can be rewritten as
\[ \frac{\Delta p_{wf}(t_n)}{q_D(t_n)} = \frac{m}{q_D(t_0)} \int_0^t q_D(t) \log(t - r) dr + b, \quad \ldots \ldots \ldots (10) \]

where \( b = \bar{S} m \).

Let us approximate the integral in Eq. 9 by the Riemann sum, which yields
\[ \frac{\Delta p_{wf}(t_n)}{q_D(t_n)} = m \sum_{i=1}^{n-1} \frac{[q_D(t_{i+1}) - q_D(t_i)] \log(t_{i+1} - t_i)}{q_D(t_n)} + b, \quad \ldots \ldots \ldots (11) \]

where \( t_n \) is the measured (discrete) time point. This equation has been presented elsewhere for multiple-rate analysis. Eq. 10 gives the continuous form of the multirate (variable-rate) equation. Any integration techniques, as well as the Riemann sum used in Eq. 11, can be used to evaluate the integral given in Eq. 10. A plot of the left side vs. the first term of the right side of Eqs. 10 or 11 will yield a straight line with a slope \( m \) and an intercept \( b \).

For buildup tests, \( q_D(t) \) should be replaced by \( 1 - q_D(t) \) and \( \Delta p_{wf}(\Delta t) \) by \( \Delta p_{wf}(\Delta t) = p_{mf}(\Delta t) - p_{mf}(\Delta t = 0) \) in Eqs. 9 through 11.

The major advantage of continuous variable-rate test (rate measure just above the perforation) over the conventional multirate test is that the wellbore storage effects are minimized. The wellbore storage effect has not been discussed in the literature for the conventional multirate tests. The wellbore measured pressure used in the conventional analysis should be taken from the storage-free infinite-acting period. Thus, the end of the storage effect should be determined. In other words, the sandface rate should be equal to the constant-surface rate for each drawdown or buildup period.

The second problem with the conventional multirate test is the surface step rate change cannot be taken as a step rate change for the sandface. Neglecting the continuous rate change from one rate to another will affect the result of the conventional multirate test analysis.

As noted earlier, the continuous variable-rate test suggested here will not eliminate completely the effect of the wellbore storage, since there is a finite volume between the bottom of the well and the flow meter, but it will minimize the wellbore storage effect.

The method suggested by Gladfelter et al. further simplifies Eq. 12. However, it does not improve the multirate method. The disadvantage of the Gladfelter et al. method over the multirate method is that there is a possibility of the existence of one to three different straight lines.

Modified Horner Method. Using the measured sandface rate data, Meurier et al. presented a modification of the Horner time ratio, which they named "the rate-convolved buildup time function." They showed that the time required for the start of the semilog straight line can be reduced considerably by using the rate-convolved buildup time function instead of of the conventional Horner time ratio. Meurier et al. gave detailed explanations on how to modify the Horner time ratio if the sandface rate measurements are available. In this section, the fundamental nature of the Horner and modified Horner methods will be examined.

The buildup test is perhaps the most popular transient testing practiced by the oil industry over the last 3 decades. The Horner method used for the analysis of the buildup test is appealing for its simplicity, generality, and ease of application. The reliability of \( kh \), skin, and the extrapolated pressure estimated from the Horner method depends on the slope of the Horner semilog straight line.

The assumption required for the Horner method is that the sandface flow rate becomes zero during the semilog period. From a theoretical point of view, as long as the measured pressure increases at the wellbore, the sandface rate never will be zero unless the fluid in the wellbore is incompressible. Thus, the effect of the decaying sandface flow rate on the Horner semilog straight line will be investigated in this section.

The convolution integral given by Eq. 3b can be written for buildup tests as
\[ p_{Df}(t_{pD} + \Delta t_D) = S p_{D}(\Delta t_D) + P_D(t_{pD} + \Delta t_D) \]
\[ - \int_0^{\Delta t_D} [1 - q_D(r)] p_{D}(\Delta t_D - r) dr, \quad \ldots \ldots \ldots (12) \]

where
\[ p_{D} = \frac{kh}{141.2 \ q_B \mu} [p_i - p_{mf}(\Delta t)] \]
and \( q \) is the constant rate before shut-in.

For the sake of simplicity, let us assume that after shut-in, the afterflow rate declines exponentially, as suggested by van Everdingen and Hurst so that Eq. 1 becomes
\[ q_D(\Delta t) = e^{-\alpha \Delta t}, \quad \ldots \ldots \ldots (13) \]

where \( \alpha \) is determined from measured sandface flow rate data. For example, if sandface rate measurements are available, \( \alpha \) in Eq. 13 can be determined by the least-squares curve fitting of Eq. 13 to the sandface rate data.
Substitution of the exponential integral solution for \( p_D \) and Eq. 13 for \( q_D \) in Eq. 12 (the details of the derivation are given in Appendix A) yields

\[
p = p_D - p_{w_S}(\Delta t) = m \left[ \log \left( \frac{t_p + \Delta t}{\Delta t} \right) + \frac{1}{2.303\alpha \Delta t} \right].
\]  

(14)

The constant \( \alpha \) in Eq. 14 can be considered an afterflow parameter. The term \( 1/2.303\alpha \Delta t \) will modify the Horner time ratio, \( (t_p + \Delta t)/\Delta t \). A semilog plot of \( p_{w_S}(\Delta t) \) vs. \( \log((t_p + \Delta t)/\Delta t) + 1/2.303\alpha \Delta t \) will yield a straight line with a correct slope. This straight line starts much earlier than the Horner semilog straight line, as shown in Fig. 3 (approximately one-half cycle earlier). In fact, Chen and Brigham \(^{22} \) found that the correct semilog straight line on the Horner plot is obtained after an ex-

tremely long time period. It is obvious from Eq. 14 that as \( \Delta t \) increases, \( 1/2.303\alpha \Delta t \) becomes smaller, but \( \log [1/(t_p + \Delta t)/\Delta t] \) decreases as well. Thus, the Horner plot, as it is seen in Fig. 3, asymptotically approaches the correct semilog straight line.

If the term \( (t_p + \Delta t)/\Delta t \) is very large compared to the term \( 1/2.303\alpha \Delta t \), then the Horner and modified Horner straight lines are almost identical at large \( \Delta t \). In other words, if \( t_p \) is very large compared to the maximum shut-in time, then the correction caused by the afterflow becomes almost negligible as \( \Delta t \) becomes large.

Before determining an approximate formula for the start of the modified Horner semilog straight line, it will be interesting to determine an approximate formula for the start of the Horner semilog straight line. Theoretically speaking, the Horner semilog straight line never yields the correct straight line. However, we can define an error criterion between the correct and computed Horner slopes. Then, we take the time at which the error criterion is satisfied as the start of a Horner semilog straight line.

As developed in Appendix B, an approximate formula for the start of the Horner semilog straight line is given by

\[
2t_{pD} - \Delta t_D \left( t_p + \Delta t_D \right) \left\{ \beta e^{-\beta \Delta t_D \left( \ln \beta - 2\gamma \right)} + \ln \left( 4 + E(3\beta \Delta t_D + 2\gamma) - 1/\Delta t_D \right) \right\} = 0,
\]  

where

\[
e = \left| \frac{m_{cor} - m_{com}}{m_{cor}} \right|
\]

an error criterion,

\[m_{com} > m_{cor} \]

\[m_{cor} = \text{slope of correct Horner semilog straight line},\]

\[m_{com} = \text{slope of computed Horner semilog straight line},\]

\[t_{pD} = \text{dimensionless producing time}.\]

For a given \( \varepsilon \) (relative error), \( \beta \), skin, and producing time, zeros of Eq. 15 with respect to \( \Delta t_D \) will give the start of the Horner semilog straight line. It is interesting to
observe that the start of the semilog Horner straight line is a function of the producing time, as expected.

A simple formula cannot be derived for the start of the Horner semilog straight line from Eq. 15 because it is a transcendental function. Fig. 4 presents values of Eq. 15 as a function of dimensionless time for \( t_{PD} = 10^6 \), \( \beta = 10^{-4} \), \( S = 0.0 \), and \( \epsilon = 0.01 \). As can be seen from Fig. 4, Eq. 15 has two roots (zeros) for the values of \( t_{PD} \), \( \beta \), \( S \), and \( \epsilon \) given previously. The first zero results from the early-time period, which can be seen easily from Fig. 2. The second zero results from the radial infinite-acting period. The upper curve in Fig. 5 presents dimensionless time for the start of the Horner semilog straight line as a function of \( \beta \) for \( t_{PD} = 10^6 \), \( S = 0.0 \), and \( \epsilon = 0.1 \). As can be seen from Eq. 15, the dimensionless time for the start of the Horner semilog straight line is a very weak function of skin. It is basically a function of the production time and \( \epsilon \). This had been observed by Chen and Bingham for the constant-wellbore-storage case. Eq. 15 also can be used for the constant-wellbore-storage case by substituting \( 1/C_D \) for \( \beta \). However, \( 1/C_D \) is a very crude approximation for \( \beta \).

Eq. 15 can be quite useful for the design of buildup tests for an optimum value of a producing time to achieve a certain accuracy for the Horner semilog straight line. Particularly, Eq. 15 will be useful for drillstem tests, since production time is limited by production facilities.

For large producing time, \( t_{PD} \), Eq. 15 can be simplified to

\[
\Delta t_D = \frac{1}{2\epsilon \beta}.
\]

Eq. 16 also can be derived from drawdown solutions by using the same principle given in Appendix B.

If \( \beta \) is approximated by \( 1/C_D \), Eq. 16 then can be written as

\[
\Delta t_D = \frac{1}{2\epsilon} C_D.
\]

As noted previously, Eq. 17 will yield very optimistic values for the start of Horner semilog straight line if indeed the wellbore storage remains constant during the test. The dimensionless time for the start of the modified Horner semilog straight line (details of derivations are given in Appendix B) also can be given by

\[
e^{-\beta \Delta t_D} \cdot (-ln \beta - 2\gamma + ln 4 + 2S) = 0.
\]

As in the Horner case, the second root of Eq. 16 will give the dimensionless time for the start of the modified Horner semilog straight line as a function of \( \beta \). Fig. 4 presents values of Eq. 18 as a function of dimensionless time for \( t_{PD} = 10^6 \), \( \beta = 10^{-4} \), \( S = 0.0 \), and \( \epsilon = 0.01 \).

The lower curve in Fig. 5 presents dimensionless time for the start of the modified Horner semilog straight line as a function of \( \beta \) for \( t_{PD} = 10^6 \), \( S = 0.0 \), and \( \epsilon = 0.01 \). The start of modified Horner semilog straight line is also a weak function of skin.

For large producing time, \( t_{PD} \), Eq. 18 can be simplified as

\[
\Delta t_D = \frac{1}{4\epsilon \beta}.
\]

The modified Horner semilog straight line starts at least one-half cycle earlier than the Horner straight line. In other words, the time required for the start of the modified Horner straight line is half of the time required for the Horner straight line, as can be seen from Eqs. 16 and 19. Depending on the formation and fluid parameters, hours could be saved on the testing time.

Very simple \( q_D \) and \( P_D \) functions are used to explore the effect of afterflow on the Horner analysis. It must be recognized that the sandface flow rate (afterflow) has to be measured to obtain accurate and reliable results from the modified Horner analysis.

Even though the modified Horner analysis improves the semilog analysis, it cannot be applied to very early-time pressure data. The other methods, such as continuous multirate, require prequisites \( P_D \) functions. Thus, deconvolution methods will be used to obtain \( P_D \) functions (influence functions) and formation parameters in the following section.

Deconvolution

Determining wellbore geometries and reservoir types (fractured, layered, composite, etc.) is an important part of well testing. The reservoir engineer must have sufficient information about the system being analyzed. For example, if type curves for fully penetrated wells are used for partially penetrated wells, both \( kh \) and damage skin will be underestimated. Thus, the system identification becomes a very important part of well testing. For instance, identifying a one-half slope on a log-log plot of the pressure data will indicate a vertically fractured well, as two parallel straight lines on a Horner graph will indicate a fractured reservoir. However, either wellbore storage or afterflow usually dominates these characteristic behaviors of wells and reservoirs during the early-time period. Thus, the pressure behavior of formation without the wellbore storage effect must be calculated or the wellbore storage effect on the formation pressure should be minimized for the conventional identification of the system. This is merely the deconvolution of Eqs. 3a through 3c to calculate \( P_D(t_D) \) from \( P_wD(t_D) \) and \( q_D(t_D) \). Several graphs of \( P_D(t_D) \) vs. \( t_D \), such as linear, spherical, etc., will provide information about a given wellbore geometry and reservoir.

This approach to system identification is not general, but it uses our conventional knowledge about well and reservoir behaviors. In general, the problem of the system identification is much more complex because often we do not know the governing differential equation.

It is worth repeating that the fluid flow in the formation is described by the linear diffusivity equation for the deconvolution methods given next.

There are several methods for the deconvolution of Eqs. 3a through 3c. These methods will be discussed in the following section.

Linearization of the Convolution Integral. In this section, the convolution equation, Eq. 3c, will be solved directly by using the linearization method. Eq. 3c can be discretized as

\[
P_wD(t_{Dn+1}) = \sum_{j=0}^{n} \int_{\tau_j}^{t_{Dn+1}} q_D(t_D' - \tau) \cdot P_D(\tau) d\tau.
\]
Fig. 6—Calculated formation pressure drop using linearization method and wellbore pressure drop.

By using the trapezoidal rule for integration in Eq. 20,

\[
P_{wd}(t_{Dn+1}) = \sum_{i=0}^{n} \left( P_{sd}(t_{Di}) q_{D}(t_{Di+1} - t_{Di+1}) + P_{sd}(t_{Di+1}) q_{D}(t_{Di+1} - t_{Di}) \right) \frac{t_{Di} + t_{Di+1}}{2} \ldots (21)
\]

Eq. 21 gives a system of linear algebraic equations. The coefficient matrix of this system of equations is a lower triangular matrix; that is, all its nonzero elements are in the lower right of the matrix. The system of equations can be solved easily by forward substitution.

For field data, \( P_{wd} \) should be replaced by \( P_{ws} - P_{wf} \) and \( P_{ws} - P_{wf} \) for drawdown and buildup tests, respectively; and \( P_{sd} \) should be replaced by \( P_{1} - P_{wf} \) and \( P_{1} - P_{wf} \) which are sandface pressure differences without storage effects (formation pressure drop). \( q_{D} \) for drawdown and \( (1 - q_{D}) \) for buildup must be used to replace \( q_{D} \) in Eq. 21 for field case data. If the flow is radial, formation \( k_{h} \) can be determined from the slope of the straight line of the \( (P_{ws} - P_{wf}) \) vs. \( \log \Delta t \) plot. Skin factor also can be determined from the conventional skin formula. However, the trapezoidal method used in Eq. 21 gives oscillatory results. As can be seen in Fig. 6, at very early times, the calculated values of \( (P_{ws} - P_{wf}) \) oscillates. Furthermore, higher-order methods, including the Simpson rule, yield divergent results for the integral in Eq. 20.

Hamming\(^{23}\) suggested a stable integration scheme for evaluation of the convolution integrals. He also showed that direct integration (discussed as the linearization method) of convolution integrals usually will result in an oscillation.

The integral in Eq. 20 can be approximated as

\[
P_{sd}(t_{Di+1}) = \sum_{i=0}^{n} \left( P_{sd}(t_{Di}) q_{D}(t_{Di+1} - t_{Di+1}) + P_{sd}(t_{Di+1}) q_{D}(t_{Di+1} - t_{Di}) \right) \frac{t_{Di} + t_{Di+1}}{2} \ldots (22)
\]

The right side of Eq. 22 can be integrated directly. Substitution of the integration results in Eq. 20, and solving for \( P_{sd} \) yields

\[
P_{sd}(t_{Di+1}) = P_{sd}(t_{Di}) - \text{sum} \frac{P_{ws}(t_{Di})}{q_{D}(t_{Di+1} - t_{Di})} \ldots (23)
\]

where ‘‘sum’’ is equal to

\[
\sum_{i=0}^{n} P_{sd}(t_{Di}) q_{D}(t_{Di+1} - t_{Di}) \ldots \ldots (24)
\]

As can be seen from Fig. 7, Eq. 23 gives a stable and nonoscillatory integration scheme for the convolution integral given in Eq. 20. Furthermore, the derivative of the sandface flow rate data, \( q_{D}(t_{D}) \), is not needed in Eq. 23. However, for Eq. 21, \( q_{D}(t) \) must be known. A finite difference approximation also can be used for \( q_{D}(t) \), but some accuracy would be lost.

The semilog plot of \( (P_{ws} - P_{wf}) \) vs. \( \Delta t \), given in Fig. 7 for radial synthetic buildup test data, yields a straight line starting from a very early time (\( \Delta t = 0.05 \) hours) with a correct slope. The lower curve in the same figure is the plot of \( (P_{ws} - P_{wf}) \), which includes the wellbore storage effect.

Laplace Transform Deconvolution. The convolution of Eq. 4 yields

\[
P_{sd}(t_{D}) = \int_{0}^{t_{D}} K(r) q_{wd}(t_{D} - r) \, dr \ldots (25)
\]

where

\[
K(t_{D}) = \mathcal{L}^{-1} \left[ \frac{1}{q_{D}(s)} \right] \ldots (26)
\]

\( K(t_{D}) \) can be computed either from the Laplace transforms of \( q_{D}(t_{D}) \) data or a curve-fitted equation of \( q_{D}(t_{D}) \) data. However, it would be time consuming to invert all the \( q_{D}(t_{D}) \) data in Laplace space and transform it back to real space in accordance with Eq. 26. Thus, approximation functions must be used for \( q_{D}(t_{D}) \) data. Once an approximation is obtained, it will be easy to compute \( K(t_{D}) \) and integrate Eq. 25 to determine \( p_{sd}(t_{D}) \).

A few types of approximation functions can be used to approximate \( q_{D}(t_{D}) \). We have tried rational functions, power series, and exponential functions. Exponential functions give a good representation of \( q_{D}(t_{D}) \) data. Ex-

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The resulting expression in Eq. 26 yields...

The success of this method depends on how well the approximation function represents the behavior of $P_{D}$. As noted earlier, the log approximation was used to approximate $P_{D}(t_D)$. In this case, the natural choice will be the power series of $\ln t_D$ such that...

Substitution of Eq. 31 into Eq. 20 yields $n$ number of equations and $m$ number of unknown parameters, $c = (c_1, c_2, ..., c_m)^T$. In fact, obtaining these parameters, $c_i$, becomes an unconstrained optimization problem. We have applied a fourth-degree polynomial, Eq. 31, to wellbore pressure and rate data (synthetic) that were obtained from a fractured reservoir for the reservoir and fluid parameters given in Table 2. The second column of Table 3 presents the calculated values of $\Delta P_{sf}$ from curve-fit approximation for the fluid and reservoir data given in Table 2, while the third column of Table 3 presents the values of $\Delta P_{sf}$ calculated directly from an analytical solution without storage effect. The fourth column in Table 3 presents $\Delta P_{w}$ with the wellbore storage effect.

As seen from Table 3, differences in $\Delta P_{sf}$ from the curve-fit approximation and the analytical solutions are almost identical. The relative error decreases for large times.

Conclusions

1. The convolution integral (superposition theorem) is used for the analysis of continuously varying wellbore flow rate and pressure. This analysis is very similar to the conventional multirate methods.

2. Wellbore storage (afterflow) effects can be present to a significant degree in the Horner semilog straight line. The Horner analysis is modified by using measured sandface flow rate data to obtain a correct semilog straight line. The modified Horner semilog straight line starts at least one-half cycle earlier than the conventional Horner.
straight line. Approximate formulas are presented for the
start of the modified Horner and Horner semilog straight
lines as a function of sandface rate decline, production
time, and the relative error between the correct and com-
puted slopes.

3. The formation pressure (influence function) can be
calculated from the deconvolution of measured wellbore
pressure and sandface flow rate data. Some new decon-
volution techniques are introduced to compute the for-
mation pressure (influence function) without wellbore storage
(afterflow) effects. Wellbore and reservoir geometries can
be identified from this computed formation pressure. Fur-
thermore, the conventional methods can be used to analyze
this computed formation pressure to determine formation
parameters.

4. Deconvolution of synthetic data from a homogeneous
and fractured reservoir shows that it is possible to com-
pute the formation pressure from the beginning of the test.
This computed pressure reveals the characteristic behavior of
homogeneous and fractured reservoirs.

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also thank H.J. Ramey Jr. for providing a solution for
the integral in Appendix A.

Nomenclature

- \( B \) = oil formation volume factor, RB/STB
- \( c_t \) = system total compressibility, psi\(^{-1} \)
- \( C \) = wellbore storage coefficient, bbl/psi
- \( C_D \) = wellbore storage constant, dimensionless
- \( h \) = formation thickness, ft [m]
- \( k \) = formation permeability, md
- \( K \) = kernel of the convolution integral
- \( p_D \) = differential of \( p_D \)
- \( p_{sh} \) = shut-in pressure drop, dimensionless
- \( P_D(t_D) \) = formation pressure, dimensionless
- \( p_i \) = initial pressure, psi [kPa]
- \( P_{PD} \) = \( p_D + S \) = formation pressure including
  skin, dimensionless
- \( P_{PD} \) = pressure drawdown, dimensionless
- \( p_{sh} \) = bottomhole flowing pressure, psi [kPa]
- \( p_{shf} \) = bottomhole shut-in pressure, psi [kPa]
- \( q \) = stabilized constant rate, STB/D
- \( q_{sd} \) = sandface flow rate, dimensionless
- \( q_{rf} \) = reference flow rate, B/D [m\(^3\)/d]
- \( q_R \) = reservoir flow rate, B/D [m\(^3\)/d]
- \( r_w \) = wellbore radius, ft [m]
- \( s \) = Laplace transform variable
- \( S \) = skin factor
- \( t \) = time, hours
- \( t_D \) = time, dimensionless
- \( t_p \) = production time, hours
- \( t_H \) = dimensionless production time
- \( T \) = transpose
- \( \Delta t \) = running testing time, hours
- \( \Delta_D \) = shut-in time, dimensionless
- \( \alpha = 0.000264k\beta /\mu c_p r_w^2 \)
- \( \beta = \) a positive constant
- \( \gamma = 0.57722... = Euler's constant \)
- \( \mu = \) viscosity, cp [Pa\cdot s]
- \( \xi = \) rate-pressure convolved time function
- \( \tau = \) dummy integration variable
- \( \phi = \) porosity, fraction
- \( \psi = \) Laplace transform of

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APPENDIX A

Substitution of the exponential integral solution for \( p_D \) and Eq. 13 for \( q_D \) in Eq. 12 yields

\[
p_D[(t_P + \Delta t)_D] = -\frac{1}{2} E\left[-\frac{1}{4(t_P + \Delta t)_D}\right]
\]

\[
-\frac{1}{2} E\left[-\frac{1}{4(t_P + \Delta t)_D}\right] - \int_{0}^{\Delta t_D} e^{-\delta \tau} \exp\left[-\frac{1}{4(\Delta t_D - \tau)}\right] \frac{d\tau}{2(\Delta t_D - \tau)} + S q_D(\Delta t_D), \quad \ldots \quad (A-1)
\]

where

\[
q_D = e^{-\alpha \Delta t} = e^{-\beta \Delta t_D}.
\]

Thus, the two forms of \( q_D \) will be used interchangeably.

The integral in Eq. A-1 cannot be integrated readily and, unfortunately, van Everdingen\(^7\) and Ramey\(^10\) did not present the details of integration. Ramey\(^*\) provided me a heuristic derivation that was edited out of his paper.\(^*\) Ramey’s integration method is given below. The Laplace transform of the integral given in Eq. A-1 is

\[
\overline{K}(s) = \int_{0}^{\infty} e^{-st} dt = \frac{1}{s+\beta}.
\]

The long-time approximation for \( K_0(\sqrt{s}) \) is

\[
K_0(\sqrt{s}) = -\left(\ln \frac{\sqrt{s}}{2} + \gamma\right).
\]

Substitution of Eq. A-4 in Eq. A-3 yields

\[
\overline{\xi}(s) = \frac{1}{2(s+\beta)} (\ln s - \ln 4 + 2\gamma).
\]

The inverse Laplace transform of Eq. A-5 yields

\[
\xi(\Delta t_D) = -\frac{1}{2} e^{-\beta \Delta t_D} \left[-\ln \beta - \pi^2 / 2 - 2\gamma + \ln 4 + Ei(\beta \Delta t_D)\right].
\]

\[
+ Ei(\beta \Delta t_D), \quad \ldots \quad (A-6)
\]

The values of integral \( \xi(\Delta t_D) \) from Eq. A-7 were compared with the values computed from the Stehfest\(^25\) numerical Laplace transform inversion technique using Eq. A-3. When \( \Delta t_D > 30 \), the difference between two values of \( \xi(\Delta t_D) \) becomes less than 1%. The difference becomes smaller as \( \Delta t_D \) increases. Substitution of the log approximations for the exponential integrals given in Eqs. A-1 and A-7 for the integral yields (in practical units)

\[
p_i - p_w(\Delta t) = m \left[ \log \left( \frac{t_P + \Delta t}{\Delta t} \right) + \xi(\Delta t) \right], \quad \ldots \quad (A-8)
\]

where

\[
\gamma = 0.5772
\]

and

\[
Ei(\alpha \Delta t) = \int_{-\infty}^{\alpha \Delta t} \frac{e^u}{u} \, du.
\]

Neglecting the imaginary term \( \pi i \), the van Everdingen\(^7\) and Hurst\(^8\) forms are obtained as

\[
\xi(\Delta t_D) = -\frac{1}{2} e^{-\beta \Delta t_D} \left[-\ln \beta - 2\gamma + \ln 4 + Ei(\beta \Delta t_D)\right].
\]

\[
+ Ei(\beta \Delta t_D), \quad \ldots \quad (A-7)
\]

The long-time approximation for \( \xi(\Delta t) \) may be obtained by substituting limiting forms of certain terms in Eq. A-7 for long times. For larger values of \( \Delta t \), the term \( e^{-\beta \Delta t}(-\ln \beta - 2\gamma + \ln 4 + Ei(\beta \Delta t_D) \) in Eq. A-9 approaches zero. The term \( e^{-\alpha \Delta t} Ei(\alpha \Delta t) \) can be approximated as

\[
e^{-\alpha \Delta t} Ei(\alpha \Delta t) = \frac{1}{\alpha \Delta t}, \quad \ldots \quad (A-10)
\]

when \( \Delta t > 1 \).

Substituting Eq. A-10 in Eq. A-9 yields

\[
\xi(\Delta t) = \frac{1}{2.3026 \alpha \Delta t}.
\]

Further substitution of Eq. A-11 in Eq. A-8 gives Eq. 14 in the main text.

APPENDIX B

The dimensionless form of Eqs. A-8 and A-9 can be written as

\[
p_{Dw}(\Delta t) = 0.5 \left[ \log \left( \frac{t_P + \Delta t}{\Delta t} \right) + \xi(\Delta t_D) \right], \quad \ldots \quad (B-1)
\]

\[
\gamma = 0.5772
\]
where
\[
\xi(\Delta t_D) = \frac{1}{2} e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 + Ei(\beta \Delta t_D) \right] + 2S. \tag{B-2}
\]
The differentiation of Eq. B-1 with respect to 
\[
\ln \left( \frac{(t_P + \Delta t)_D}{\Delta t_D} \right)
\]
\[m_{com} = 0.5 + \frac{d}{2dx} \xi(\Delta t_D), \tag{B-3}
\]
where \(m_{com}\) is the computed dimensionless slope of the Horner semilog straight line and \(x\) equals \(\ln \left( \frac{(t_P + \Delta t)_D}{\Delta t_D} \right)\).

Let us define a relative error for the computed slope as
\[
e = \frac{0.5 - m_{com}}{0.5}, \tag{B-4}
\]
where
\[m_{com} > 0.5.
\]
Substitution of Eq. B-4 in Eq. B-3 yields
\[
e = \frac{\Delta t_D}{2t_P D} \left( \beta e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 \right] + Ei(\beta \Delta t_D) + 2S \right). \tag{B-5}
\]
Taking the derivative of \(\xi(\Delta t_D)\) with respect to \(x\) and substituting in Eq. B-5 yields
\[
e = \frac{\Delta t_D}{2t_P D} \left( \beta e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 \right] + Ei(\beta \Delta t_D) + 2S \right) - \frac{1}{\Delta t_D}. \tag{B-6}
\]
Eq. B-6 gives the dimensionless time for the start of the Horner semilog straight line as a function of \(\varepsilon, \beta, t_P D\), and \(S\).

A formula for the dimensionless time for the start of the modified Horner semilog straight line can be found from Eqs. A-1 and A-2 as follows. Let us rewrite Eq. A-1 in terms of the modified Horner time:
\[
P_D(\Delta t_D) = 0.5 \ln \left( \frac{(t_P + \Delta t)_D}{\Delta t_D} \right) + \frac{1}{2\beta \Delta t_D} + \frac{1}{2} \xi(\Delta t_D), \tag{B-7}
\]
where
\[
\xi(\Delta t_D) = \frac{1}{2} e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 + 2S \right]. \tag{B-8}
\]
\[
\frac{1}{2} \xi(\Delta t_D) \text{ in Eq. A-7 is the only remaining term that is not included in the modified Horner time. Thus, the relative error of slope of the computed modified Horner semilog straight line (as in the Horner case) can be expressed as}
\[
e = \frac{1}{2\beta \Delta t_D} \left( \beta e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 + 2S \right] \right). \tag{B-9}
\]
}\[
\frac{1}{2\beta \Delta t_D} \left( \beta e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 + 2S \right] \right) \cdot \left( \frac{t_P + \Delta t}{\Delta t} \right)_D \tag{B-10}
\]
Differentiation of Eq. B-8 with respect to \(x\) and substitution of the result in Eq. B-9 yields
\[
e = \frac{\Delta t_D}{2t_P D} \left( \beta e^{-\beta \Delta t_D} \left[ - \ln \beta - 2 \gamma + \ln 4 + 2S \right] \right) - \frac{1}{\Delta t_D} \tag{B-11}
\]
Eq. B-11 gives the dimensionless time for the start of the modified Horner semilog straight line as a function of \(\varepsilon, \beta, t_P D\), and \(S\).

SI Metric Conversion Factors

<table>
<thead>
<tr>
<th>Unit</th>
<th>Conversion Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>bbl</td>
<td>(1.589873 \times 10^{-1} ) m³</td>
</tr>
<tr>
<td>cp</td>
<td>(1.0 \times 10^{9} ) (\text{Pas} )</td>
</tr>
<tr>
<td>ft</td>
<td>(3.048 \times 10^{-2} ) m</td>
</tr>
<tr>
<td>psi</td>
<td>(6.894757 \times 10^{6} ) kPa</td>
</tr>
</tbody>
</table>

*Conversion factor is exact.*

Fig. 3—Modified Horner and Horner plots for exponential decline sandface rate.
Fig. 4—Calculated formation pressure drop using linearization method and wellbore pressure drop.
Fig. 5—Calculated formation pressure drop using Hamming method and wellbore pressure drop.
Fig. 6—Calculated formation pressure drop using polynomial approximation and wellbore pressure for a fractured reservoir.