Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Math Lecture 1 — Review of Fundamentals and Introduction to Calculus

Quote du Jour:  *Work keeps us from the three great evils: boredom, vice, and want.*  
— Voltaire

**Topic:** Review of Fundamentals and Introduction to Calculus

**Objectives:** (things you should know and/or be able to do)

- Be familiar with and be able to apply the topics in Chapter 1 of the Schaum's text (by Spiegel):
  - Rules of algebra
  - Concepts of functions, types of functions (e.g. polynomials, logarithmic functions, trigonometric functions, etc.)
  - Simple calculus: rules for integration and differentiation
  - Series and convergence of series
  - Taylor series for functions of single and multiple dimensions
  - Partial differentiation
  - Matrix algebra
  - Leibnitz's rule for differentiation of an integral
  - Complex numbers and complex arithmetic
- Be familiar with the following calculus topics
  - Limits
  - Derivatives and differentiation formulas
  - Integrals and integration formulas
- Be familiar with simple numerical methods for differentiation and integration
- Be able to solve problems related to the review topics given above.

**Lecture Outline:**
- Discuss the attached handout materials that address the objectives given above.

**Reading Assignment:**
- Review the attached notes.
- Chapter 1 of the Schaum's Outline text, *Advanced Mathematics for Engineers and Scientists*, by Spiegel.

**Math Exercises:** For your own practice/skills building--do NOT turn in!

  - Functions: 1.5, 1.6, 1.7, 1.8, 1.9
  - Derivatives: 1.13, 1.15, 1.16, 1.18, 1.19, 1.20
  - Integrals: 1.23, 1.24, 1.25, 1.26, 1.28
  - Sequences and Series: 1.29, 1.31, 1.32, 1.33, 1.34
  - Taylor Series: 1.39, 1.40
  - Multi-Variable Functions: 1.41, 1.42, 1.43, 1.44, 1.45, 1.46
  - Linear Equations: 1.48, 1.49, 1.50, 1.51
  - Liebnitz Rule: 1.55
  - Complex Arithmetic: 1.56, 1.57, 1.58, 1.61
Math Exercises: (continued) For your own practice/skills building--do NOT turn in!


- Various problems in Chapter 32 of the Schaum's Outline text *Calculus*, by Ayres and Mendelson (4th edition) — Trigonometric Integrands and Trigonometric Substitutions. (Ex.=Example, SP=Solved Problems)
  - Problems: Ex. 3, Ex. 11, SP32.24, SP32.25, SP32.26, SP32.29

  - Problems: Ex. 5, Ex. 6, Ex. 9, Ex. 10

  - Problems: 35.1, 35.2, 35.3, 35.5, 35.6, 35.7, 35.9
Review of Fundamental Mathematical Concepts -- Selected Topics
(from Petroleum Engineering 620 Course Notes -- 1993 and 1995)

Petroleum Engineering 620
Fluid Flow in Reservoirs
Limits:

definition:
\[
\lim_{x \to a} f(x) = L \quad \text{for } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ that } \\
\text{if } 0 < |x-a| < \delta \text{ then } |f(x) - L| < \epsilon 
\]

theorems:

(a) \( \lim_{x \to a} c = c \) \ldots ... limit of a constant

(b) \( \lim_{x \to a} (mx+b) = ma + b \) \ldots "commutative" rule for limits

(c) \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \) \ldots limit of the sum is the sum of the limits

(d) \( \lim_{x \to a} f(x)g(x) = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] \) \ldots limit of the product is the product of the limits

(e) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) \ldots limit of the quotient is the quotient of the limits

(f) \( \lim_{x \to a} x^n = a^n \) and \( \lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \) \ldots power rule

Derivatives:

definitions:
\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}
\]

graphic representation:
Rules for Differentiation:

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{Chain Rule} \]

\[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \quad \text{Product Rule} \]

\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v (\frac{du}{dx}) - u (\frac{dv}{dx})}{v^2} = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx} \quad \text{Quotient Rule} \]

\[ \frac{d}{dx} (u^n) = n \cdot u^{n-1} \frac{du}{dx} \quad \text{Power Rule} \]

- You should be able to derive the product, quotient, and power rules from the definition of the derivative. See Schaum's Solved Problems 1.13 and 1.15.

- You should also be familiar with, and be able to derive the derivative relations for \( e^x, \ln x, \sin x, \cos x, \ldots \), etc, as shown on p. 5 of Schaum's text.

**Integration**

**Definition:** Fundamental Theorem of Calculus

Given that \( f(x) = \frac{d}{dx} \int g(x) \), we can write

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} \frac{d}{dx} \left[ g(x) \right] \, dx = g(x) \bigg|_{a}^{b} = g(b) - g(a) \]

We also note that integration physically represents "the area under the curve" for \( f(x) \). Graphically we have

\[ \int_{a}^{b} f(x) \, dx \approx A_1 + A_2 + \ldots + A_n \]
generalizing this procedure we have

$$\int_a^b f(x) \, dx = \lim_{h \to 0} h \left[ f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h) \right]$$

where \( n-1 \) is the number of rectangles or “panels”

It would be inefficient to apply the above definition for \( h=0 \), so we have developed many different formulae for numerical integration. Some of these methods are

- Trapezoidal Rule
- Simpson’s Rule
- Quadrature
- Adaptive Procedures

We will discuss these methods later in the course and it is important to remember that “analytic integration is generally much more difficult than analytic differentiation, however numerical integration is more stable and accurate than numerical differentiation.” These concepts will become painfully apparent in later developments in this course.

**Rules for Integration:**

$$\int_a^b f(x) \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n-1} f(x_i) \Delta x = f(b) - f(a) \ldots \text{ base relation}$$

$$\int_a^b f(x) \, dx = -\int_a^b f(x) \, dx \quad (\text{for } b > a) \ldots \text{ reflection relation}$$

$$\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \ldots \text{ commutative rule}$$

$$\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx \ldots \ldots \ldots \text{ multiplication by a constant}$$

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

$$= \int_c^b f(x) \, dx - \int_c^a f(x) \, dx$$
and if \( f(x) \geq 0 \) and \( f(x) \) is integrable, then

\[
\int_a^b f(x) \, dx \geq 0 \quad \text{regardless of the sign of } a \text{ or } b
\]

\[
\int_a^b \left[ q(x) \right] f(x) \, dx = \int_q^b \left( \frac{f(m)}{m} \right) \, dm \quad (m=q(x)) \quad \cdots \text{change of variable formula}
\]

**Application of Change of Variable Formula**:

**given** \( \int_2^5 \frac{3}{\sqrt{5x-1}} \, dx \)

**method 1**: \( m = 5x-1 \); \( dm = 5 \, dx \)

at \( x = 2 \); \( m = 5(2)-1 = 9 \)

at \( x = 10 \); \( m = 5(10)-1 = 49 \)

\[
\int_q^b \frac{3}{\sqrt{m}} \, dm = \frac{3}{5} \int_q^b m^{-1/2} \, dm
\]

recall that \( \int z^p \, dz = \frac{1}{p+1} z^{p+1} \)

\[
\frac{3}{5} \int_9^{49} m^{-1/2} \, dm = \frac{3}{5} \left( \frac{1}{-\frac{1}{2}} \right) m^{1/2} \bigg|_9^{49} = \frac{6}{5} \cdot \sqrt{49} - \sqrt{9}
\]

or

\[
\int_2^5 \frac{3}{\sqrt{5x-1}} \, dx = \frac{6}{5} \left( \sqrt{49} - \sqrt{9} \right) = \frac{6}{5} \cdot 4 = \frac{24}{5}
\]

**method 2**: \( m = \sqrt{5x-1} \); \( dm = \frac{1}{2 \sqrt{5x-1}} \, dx \)

\[
\int_2^5 \frac{1}{\sqrt{5x-1}} \, dx = \frac{5}{2} \int_2^5 \frac{1}{m} \, dm
\]

or \( \frac{dm}{dx} = \frac{1}{2 \sqrt{5x-1}} \)

\[
\frac{dx}{dm} = \frac{2 \sqrt{5x-1}}{5}
\]

transformed integral

\[
\int_q^b \frac{3}{\sqrt{m}} \, dm
\]

at \( x = 2 \); \( m = \sqrt{5(2)-1} = 3 \)

at \( x = 10 \); \( m = \sqrt{5(10)-1} = 7 \)

or

\[
\int_2^5 \frac{3}{\sqrt{5x-1}} \, dx = \frac{6}{5} \left( \sqrt{7} - \sqrt{3} \right) = \frac{24}{5}
\]


**Method 3:**

\[ m = \frac{1}{\sqrt{5x-1}} = (5x-1)^{-\frac{1}{2}} \]

\[ \frac{dm}{dx} = \frac{d}{dx} \left[ (5x-1)^{-\frac{1}{2}} \right] = \frac{-1}{2} (5x-1)^{-\frac{3}{2}} \]

or

\[ \frac{dm}{dx} = \frac{-5}{2} (5x-1)^{-\frac{3}{2}} = \frac{-5}{2} \left( \frac{1}{\sqrt{5x-1}} \right)^3 = \frac{-5}{2} \frac{m^3}{dx} \]

or

\[ dx = \frac{-5}{2} \frac{1}{m^3} dm \]

at \( x = 1 \), \( m = \frac{1}{\sqrt{5(1)-1}} = \frac{1}{2} \)

\[ \frac{1}{2} \int m \left( \frac{-2}{5} \frac{1}{m^3} \right) dm \]

or

\[ \int_2^3 \frac{3}{\sqrt{5x-1}} \ dm = \frac{6}{5} \left[ \frac{1}{\frac{1}{2}} \right] = \frac{6}{5} \left[ \frac{5}{2} \right] = \frac{15}{2} \]

**Integration-by-Parts**

Derivation of the integration-by-parts formula, recalling the product rule:

\[ \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

Multiplying by \( dx \):

\[ d(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

Integrating:

\[ uv = \int u \frac{dv}{dx} + \int v \frac{du}{dx} \]

Rearranging:

\[ \int u \frac{dv}{dx} = uv - \int v \frac{du}{dx} \]

**General rules:**

a. \( dv \) should be readily integrable (i.e., easy to integrate)

b. \( u \frac{du}{dx} \) should not be more complex than \( uv \) or integration-by-parts will not reduce your work.
Simply put, Taylor series are polynomial expansions of non-polynomial functions. For instance, say we wanted to model the behavior of \( \exp(x) \) near zero with a polynomial. We would simply apply the Taylor series formula using \( \exp(x) \), expanding this series at \( x = 0 \). We will illustrate this in a moment.

First we must derive the Taylor series from the following polynomial power series
\[
 f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n
\]
\[
 = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \ldots
\]

we immediately note that the series is differentiable and that by generating the derivatives we can arrive at a scheme for obtaining the series coefficients \( a_n(c) \). Differentiating the \( f(x) \) series term-by-term gives
\[
f'(x) = a_1 + 2a_2 (x-c) + 3a_3 (x-c)^2 + \ldots = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}
\]
\[
f''(x) = 2a_2 + 3(2a_3) (x-c) + 4(3a_4) (x-c)^2 + \ldots = \sum_{n=2}^{\infty} n(n-1) a_n (x-c)^{n-2}
\]
\[
f'''(x) = 3(2a_3) + 4(3(2a_4)) (x-c) + 5(4(3a_5)) (x-c)^2 + \ldots = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x-c)^{n-3}
\]

we note that the general progression yields
\[
f^{(n)}(x) = \sum_{n=k}^{\infty} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n!} a_n (x-c)^{n-k}
\]

This is all good and fine, but how do we determine the \( a_n \)'s? Use \( x = c \) as the "point of expansion" in each \( f^{(n)}(x) \) expression. This gives
\[
f^{(0)}(c) = a_0, \quad f^{(1)}(c) = a_1, \quad f^{(2)}(c) = 2a_2, \quad f^{(3)}(c) = 3! \cdot 2a_3
\]
or
\[
f^{(n)}(c) = \frac{n!}{n!} a_n \quad \Rightarrow a_n = \frac{f^{(n)}(c)}{n!}
\]

therefore
\[
a_n = \frac{1}{n!} f^{(n)}(c)
\]
or
\[
a_0 = f(c), \quad a_1 = f'(c), \quad a_2 = \frac{1}{2} f''(c), \quad a_3 = \frac{1}{3!} f'''(c)
\]
Recombining these coefficients into the original power series gives us the "Taylor series" expansion about the point $c$ for the function $f(x)$. This series is written as

$$f(x) = f(c) + (x-c) f'(c) + \frac{(x-c)^2}{2!} f''(c) + \frac{(x-c)^3}{3!} f'''(c) + \ldots + R_n$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} f^{(n)}(c)$$

The remainder, $R_n$, results from the "residual" of using a polynomial to model the behavior of $f(x)$. This remainder is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \quad \text{where } z \text{ lies in between } x \text{ and } c$$

### Taylor Series for $\exp(x)$ about $c=0$

Given $f(x) = \exp(x)$,

- $f(0) = 1$
- $f'(0) = 1$
- $f''(0) = 1$
- $f'''(0) = 1$
- $\vdots$
- $f^{(n)}(0) = 1$

Therefore, the Taylor series becomes

$$\exp(x) = 1 + (x-0) (1) + \frac{(x-0)^2}{2!} (1) + \frac{(x-0)^3}{3!} (1) + \ldots + \frac{(x-0)^n}{n!} (1) + \ldots$$

or

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

or in series form we have

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

### Taylor Series for $\sin(x)$ about $c=0$

- $f(x) = \sin(x)$
- $f(0) = 0$
- $f'(x) = \cos(x)$
- $f'(0) = 1$
- $f''(x) = -\sin(x)$
- $f''(0) = 0$
- $f'''(x) = -\cos(x)$
- $f'''(0) = -1$
- $f^{(4)}(x) = \sin(x)$
- $f^{(4)}(0) = 0$
- $f^{(5)}(x) = \cos(x)$
- $f^{(5)}(0) = 1$

**Only odd terms are non-zero**
therefore the Taylor Series for \( \sin(x) \) becomes

\[
\sin(x) = (x-a) + \frac{(x-a)^2}{2!} (1) + \frac{(x-a)^3}{3!} (-1) + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots
\]

or

\[
\sin(x) = \frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots
\]

or in series form, we have

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

---

**Convergence of Series**

Consider the infinite series

\[
S = \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n
\]

The important theorems regarding series are

1. the **p-series test**: given the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \); This series
   a. converges if \( p > 1 \)
   b. diverges if \( p \leq 1 \)

2. Comparison test:
   a. if \( \sum |a_n| \) converges and \( 0 \leq |a_n| \leq |b_n| \) then \( \sum |b_n| \) converges
   b. conversely, if \( \sum |a_n| \) diverges and \( 0 \leq |a_n| \leq |b_n| \) then \( \sum |b_n| \) diverges

3. **Absolute convergence**: \( \sum |a_n| \) is "absolutely convergent" if \( \sum |a_n| \) converges

4. **Integral test**: for a positive-valued, continuous function that decreases for \( x \geq 1 \), we have
   a. converges if \( \int_{1}^{\infty} f(x) \, dx \) converges
   b. diverges if \( \int_{1}^{\infty} f(x) \, dx \) diverges

5. Test for divergence:
   \( \sum |a_n| \) diverges if \( \lim_{n \to \infty} |a_n| \neq 0 \)
6. Ratio Test: given \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \), then
   a. \( \sum_{n=1}^{\infty} |a_n| \) converges absolutely if \( r < 1 \)
   b. \( \sum_{n=1}^{\infty} |a_n| \) diverges if \( r > 1 \)
   c. if \( r = 1 \), no conclusion about convergence can be made using this test

7. Alternating Series Test:
   If \( a_n \geq a_{n+1} > 0 \) for every positive integer \( n \), and
   \( \lim_{n \to \infty} a_n = 0 \), then the alternating sign series \( \sum_{n=0}^{\infty} (-1)^n a_n \)
   is convergent.

\[ \text{Liouville's Rule} \]
(Differentiation of an integral)

Abramowitz & Stegun: Eq. 8.5.7, p. 11

\[ \frac{d}{dc} \left[ \int_{a(c)}^{b(c)} f(x,c) \, dx \right] = \int_{a(c)}^{b(c)} \frac{d}{dc} f(x,c) \, dx + f(b,c) \frac{db}{dc} - f(a,c) \frac{da}{dc} \]

\[ \text{Complex Numbers Operations} \]
Addition: \( (a+b+bi) + (c+di) = (a+c) + (b+d)i \)  \(\text{real to real, im to im}\)
Subtraction: \( (a+b+bi) - (c+di) = (a-c) + (b-d)i \)  \(\text{real to real, im to im}\)
Multiplication: \( (a+b+bi)(c+di) = (ac+ad+bc+bd)i^2 = (ac-bd)+(ad+bc)i \)
\[ i^2 = -1 \text{ so we have } (ac-bd)+(ad+bc)i \]
Division: \( \frac{(a+bi)}{(c+di)} \) multiply through by \( (c-di) \), this gives
\[ \frac{(a+b+bi)(c-di)}{(c+di)(c-di)} = \frac{ac-ad+bc-bd}{c^2+1} - \frac{bd+ac-ad}{c^2+1}i \]
or finally
\[ \frac{(a+b+bi)}{(c+di)} = \frac{ac+bd+bc-ad}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \]

Euler's Formula: \( e^{ix} = \cos(x) + i \sin(x) \)

DeMoivre's Theorem: \( \left[ \cos(z) + i \sin(z) \right]^n = \cos(nz) + i \sin(nz) \)
\[ \text{or } \left( f[\cos(z) + i \sin(z)] \right)^n = f^n[\cos(nz) + i \sin(nz)] \]
also
\[ \left[ \cosh(z) + \sinh(z) \right]^n = \cosh(nz) + \sinh(nz) \]
Binomial Series: From Abramowitz & Stegun: "Handbook of Mathematical Functions," Dover

Binomial Coefficients: Section 21.1.1, p. 822

Generating Functions

Integer form: (finite series)
\[(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]
where \(n\) is a positive integer (1, 2, ...)

Real exponent form: (infinite series)
\[(1+x)^d = \sum_{k=0}^{\infty} \binom{d}{k} x^k\]
d = any real number; \(1 < 1\)

Coefficients

Integer form:
\[\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n-1}{k} = \frac{n(n-1)...(n-k+1)}{k!}\]
\(n \geq k\)

Real exponent form: (from Section 6.1.21)
\[\binom{d}{k} = \frac{\Gamma(d+1)}{k! \Gamma(d-k+1)}\]
where \(\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} dt\) (Gamma Function)

Comments:

Binomial series often arise as attempts to expand functions as series. Sometimes in the expansion of a difficult to resolve denominator term, as well as other expansions such as the expansion of exponential terms in the Dover Laplace transform inversion algorithm.

Exponential and Logarithm Functions:

Exponential Functions: (Section 4.2 - Abramowitz & Stegun)

Definitions
\[\exp(x) = \lim_{m \to \infty} \left[ 1 + \frac{x^m}{m} \right]^m\]
\[\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\]

Special values
\[\exp(0) = 1\]
\[\exp(1) = 2.71828182845\ldots\]
\[\exp(\infty) = \infty\]
\[\exp(-\infty) = 0\]

or \[\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\]
Exponential Functions: (Continued)

- **Differentiation**
  \[
  \frac{d}{dx} [\exp(ax)] = a \exp(ax)
  \]

- **Integration**
  \[
  \int \exp(ax) \, dx = \frac{1}{a} \exp(ax)
  \]

- **Exponentiation**
  \[
  \exp(xy) = \exp(x)^y = \exp(y)^x
  \]

- **Exponential Multiplication**
  \[
  \exp(x) \exp(y) = \exp(x+y)
  \]

Logarithmic Functions: (Section 4.1 - Abramowitz & Stegun)

- **Definitions**
  \[
  \ln(x) = \int_1^x \frac{1}{t} \, dt
  \]

- **Special Values**
  \[
  \ln(0) = -\infty, \quad \ln(1) = 0, \quad \ln(\infty) = \infty
  \]

- **Series Expansion**
  \[
  \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad |x| < 1
  \]

- **Logarithm of the Function**
  \[
  \ln[\exp(x)] = x, \quad \ln[\exp(1)] = 1, \quad \ln(-\infty) = \text{undefined}
  \]

- **Differentiation**
  \[
  \frac{d}{dx} [\ln(x)] = \frac{1}{x}
  \]

- **Multiple Differentiation**
  \[
  \frac{d^n}{dx^n} [\ln(x)] = (-1)^{n-1} (n-1)! x^{-n}
  \]

- **Arbitrary Base of the Logarithm**
  \[
  \log_b(xy) = \log_b(x) + \log_b(y) \quad (\text{logarithmic addition})
  \]
  \[
  \log_b(xy) = \log_b(x) - \log_b(y) \quad (\text{logarithmic subtraction})
  \]
  \[
  \log_b(x^n) = n \log_b(x) \quad (\text{logarithmic exponentiation})
  \]
Logarithmic Functions: (continued)

Translation of base

\[ \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \]

or

\[ \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \]

(Abramowitz and Stegun, 6.1.19, p. 67)

typical conventions on bases of logarithms

\[ \log = \log_{10} \] (base 10 - "engineering" logarithm)

\[ \ln = \log_e \] (base e - "natural" logarithm)

\[ \log(x) = \frac{1}{\ln(10)} \ln(x) \] or \[ \ln(x) = \ln(10) \log(x) \]

trick for memory: \[ \log(10) = 1; \text{ therefore, } \log(10) = \frac{1}{\ln(10)} \ln(10) = 1 \]

Comments:

Exponential and logarithmic functions arise in virtually all aspects of engineering mathematics - especially in solutions where terms have a "vanishing" behavior (i.e., the solutions decay rapidly). These functions are also popular as models for data as well as "kernel" functions for distributions of data. You are encouraged to be fluent in these functions.
Numerical Integration - A Graphical Approach

Introduction:

We will study detailed developments of numerical integration and differentiation - but it is important that we establish a graphical basis for integration as we did on p. 1 of this handout. This concept, integration by graphical methods, was briefly discussed on p. 2. Recalling this work we can begin.

Graphical Presentation:

**Increasing Function**

![Graph of an increasing function showing points at x, x+h, x+2h, ..., x+n and approximating the area under the curve with rectangles.]

**Decreasing Function**

![Graph of a decreasing function showing points at x, x+h, x+2h, ..., x+n and approximating the area under the curve with rectangles.]

Recall that the area under a given curve is estimated by using rectangular strips of width \( h \) and height \( f(x_i) \). We immediately note that this approach will underestimate the area for an increasing function and overestimate the area for a decreasing function. Recall that the defining relations are:

\[
\int_{a}^{b} f(x) \, dx = \sum_{i=1}^{n} A_i \quad \text{where} \quad A_i = h f(x_i)
\]

and

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} h \left[ f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h) \right]
\]

\[
= \lim_{h \to 0} h \sum_{i=1}^{n} f(a+(i-1)h)
\]
A much better approach than $h \to 0$, which is inefficient at best, and computationally unstable (possibly) at worst, is to use the geometry of the problem to our advantage. Recomposing the graphical setup,

where a general panel or trapezoid is given by

$$A_p = h f(x_i)$$
$$A_T = \frac{1}{2} h [f(x_{i+1}) - f(x_i)]$$
$$A_i' = h f(x_i) + \frac{1}{2} h [f(x_{i+1}) - f(x_i)]$$
or
$$A_i' = h \frac{f(x_i) + f(x_{i+1})}{2}$$

which gives,

$$\int_a^b f(x) \, dx \approx \sum_{i=1}^{n-1} A_i' = \frac{h}{2} \sum_{i=1}^{n-1} [f(x_i) + f(x_{i+1})]$$

which is known as the **trapezoidal rule**.