Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Fundamental Flow Lecture 7 — The Diffusivity Equation for the Multiphase Flow

There are several good precautions against temptations, but the surest is cowardice.

— Mark Twain (1897)

**Topic:** The Diffusivity Equation for the Multiphase Flow

**Objectives:** (things you should know and/or be able to do)

- Be able to develop the continuity relations for the oil, gas, and water phases in terms of the fluid densities. Assume that the gas phase includes gas liberated from the oil and water phases.

**Oil Continuity Equation:**

$$\nabla \cdot (\rho_0 \nu_o) = - \frac{\partial}{\partial t} (\phi \rho_o)$$

**Water Continuity Equation:**

$$\nabla \cdot (\rho_w \nu_w) = - \frac{\partial}{\partial t} (\phi \rho_w)$$

**Gas Continuity Equation:**

$$\nabla \cdot (\rho_g \nu_g)_{tot} = \nabla \left[ \rho_g \nu_g + \frac{\nu_o}{B_o} R_{so} \rho_{gsc} + \frac{\nu_w}{B_w} R_{sw} \rho_{gsc} \right] = - \frac{\partial}{\partial t} [(\phi \rho_g)_{tot}]$$

- Be able to "convert" the density form of the continuity equation to the formation volume factor form. The "conversion" for the density/formation volume factor relations is given by:

$$\rho_i = \frac{\rho_{isc}}{B_i}$$

where $i$ = oil, gas, and water.

- Be able to write Darcy's law velocity relations for each phase. The general form is given by:

$$\nu_i = - \frac{k_i}{\mu_i} \nabla p_i$$

where $i$ = oil, gas, and water.

- Be able to develop the mass flux relations for the oil, gas, and water phases in terms of the fluid formation volume factors. Again, assume that the gas phase includes gas liberated from the oil and water phases.

**Oil Flux Equation:**

$$\rho_0 \nu_o = - \rho_{osc} \frac{k_o}{\mu_o B_o} \nabla p_o$$

**Water Flux Equation:**

$$\rho_w \nu_w = - \rho_{wsc} \frac{k_w}{\mu_w B_w} \nabla p_w$$

**Gas Flux Equation:**

$$\left( \rho_g \nu_g \right)_{tot} = - \rho_{gsc} \left[ \frac{k_g}{\mu_g B_g} \nabla p_g + R_{so} \frac{k_o}{\mu_o B_o} \nabla p_o + R_{sw} \frac{k_w}{\mu_w B_w} \nabla p_w \right]$$

- Be able to develop the mass relations for the oil, gas, and water phases in terms of the fluid formation volume factors. As before, assume that the gas phase includes gas liberated from the oil and water phases.

**Oil Mass Equation:**

$$(\phi \rho_o) = \phi \rho_o S_o = \phi \rho_{osc} \frac{S_o}{B_o}$$

**Water Mass Equation:**

$$(\phi \rho_w) = \phi \rho_w S_w = \phi \rho_{wsc} \frac{S_w}{B_w}$$

**Gas Mass Equation:**

$$(\phi \rho_g)_{tot} = \phi \rho_g S_g + \phi S_o \frac{R_{so}}{B_o} \rho_{gsc} + \phi S_w \frac{R_{sw}}{B_w} \rho_{gsc} = \phi \rho_{gsc} \left[ \frac{S_g}{B_g} + R_{so} \frac{S_o}{B_o} + R_{sw} \frac{S_w}{B_w} \right]$$
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Objectives: (Continued)

- Assuming no capillary pressure forces ($\nabla p = \nabla p_o = \nabla p_g = \nabla p_w$), be able to develop the generalized diffusivity relations for each phase. (Martin Eqs. 1-3)
  
  "Gas" Equation:
  
  $$\nabla \left[ \frac{k_g}{\mu_g B_g} + \frac{k_o}{\mu_o B_o} + R_{so} \frac{k_w}{\mu_w B_w} \right] \nabla p = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_o}{B_o} + R_{so} \frac{S_o}{B_o} + R_{sw} \frac{S_w}{B_w} \right) \right]$$  
  (Eq. 1)

  "Oil" Equation:
  
  $$\nabla \left[ \frac{k_o}{\mu_o B_o} \nabla p \right] = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_o}{B_o} \right) \right]$$  
  (Eq. 2)

  "Water" Equation:
  
  $$\nabla \left[ \frac{k_w}{\mu_w B_w} \nabla p \right] = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_w}{B_w} \right) \right]$$  
  (Eq. 3)

- NEGLECTING the $\nabla S_o \nabla p, \nabla S_w \nabla p$, and $\nabla p \nabla p = \nabla^2 p$ terms—be able to develop the diffusivity relations for each phase as shown by Martin (Eqs. 7-9)

  "Gas" Equation:
  
  $$\left( \frac{k_g}{\mu_g B_g} + \frac{k_o}{\mu_o B_o} + R_{so} \frac{k_w}{\mu_w B_w} \right) \nabla^2 p = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_o}{B_o} + R_{so} \frac{S_o}{B_o} + R_{sw} \frac{S_w}{B_w} \right) \right]$$  
  (Eq. 7)

  "Oil" Equation:
  
  $$\frac{k_o}{\mu_o B_o} \nabla^2 p = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_o}{B_o} \right) \right]$$  
  (Eq. 8)

  "Water" Equation:
  
  $$\frac{k_w}{\mu_w B_w} \nabla^2 p = \frac{\partial}{\partial t} \left[ \phi \left( \frac{S_w}{B_w} \right) \right]$$  
  (Eq. 9)

- Be familiar with and be able to derive the Martin relations for total compressibility and the associated saturation-pressure relations (Eqs. 10 and 11).

  Oil Saturation Equation
  
  $$\frac{dS_o}{dp} = \frac{S_o}{B_o} \frac{dB_o}{dp} + \frac{\lambda_o}{\lambda_t} c_t$$  
  (Eq. 10)

  Water Saturation Equation
  
  $$\frac{dS_w}{dp} = \frac{S_w}{B_w} \frac{dB_w}{dp} + \frac{\lambda_w}{\lambda_t} c_t$$  
  (Eq. 11)

Total Compressibility

$$c_t = -\frac{S_o}{B_o} \frac{dB_o}{dp} + \frac{S_o B_g}{B_o} \frac{dR_{so}}{dp} - \frac{S_w}{B_w} \frac{dB_w}{dp} + \frac{S_w B_g}{B_w} \frac{dR_{sw}}{dp} - \frac{S_g}{B_g} \frac{dB_g}{dp}$$

or

$$c_t = \left[ -\frac{1}{B_o} \frac{dB_o}{dp} + \frac{B_g}{B_o} \frac{dR_{so}}{dp} \right] S_o + \left[ -\frac{1}{B_w} \frac{dB_w}{dp} + \frac{B_g}{B_w} \frac{dR_{sw}}{dp} \right] S_w + \left[ -\frac{1}{B_g} \frac{dB_g}{dp} \right] S_g$$

or finally

$$c_t = c_o S_o + c_w S_w + c_g S_g + c_f$$  (note that $c_f = \frac{1}{\phi} \frac{d\phi}{dp}$)

where

$$c_o = -\frac{1}{B_o} \frac{dB_o}{dp} + \frac{B_g}{B_o} \frac{dR_{so}}{dp}, \quad c_w = -\frac{1}{B_w} \frac{dB_w}{dp} + \frac{B_g}{B_w} \frac{dR_{sw}}{dp}, \quad \text{and} \quad c_g = -\frac{1}{B_g} \frac{dB_g}{dp}.$$

Total Pressure Equation

$$\nabla^2 p = \phi \frac{c_t \frac{dp}{dt}}{\lambda_t}$$  (Martin Eq. 12) where $\lambda_t = \frac{k_o}{\mu_o} + \frac{k_g}{\mu_g} + \frac{k_w}{\mu_w}$
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Lecture Outline:
- Identify objectives and tasks:
  - Use continuity equations to couple mass and mass flux relations for each phase.
  - Determine the conditions for assuming that the left-hand-side terms are approximately constant to obtain Martin's starting point.
- Mass flux and mass expressions for each phase:
  - Velocities of the individual phases are controlled by Darcy's law—hence, so are the mass flux rates.
  - For the mass of each phase, simply account for the saturation of each phase and in the cases of liberated gas—we use the "gas-to-oil" and "gas-to-water" ratios and the appropriate saturations and densities.
- General diffusivity equations for each phase:
  - Once the mass flux and mass expressions are obtained, combine with the continuity equation for each individual phase.
- Conditions for applicability of Martin's equations:
  - Expand the left-hand-side (LHS) in terms of gradients.
  - Identify the conditions for reduction to forms which do not include saturation gradients or pressure gradients squared.
- Identify objectives and tasks:
  - Develop the Martin relations for total compressibility and the associated saturation-pressure relations (Martin's Eqs. 10 and 11).
- Starting Point:
  - Use Martin's Eqs. 7-9 (recall that these relations neglect saturation gradients and pressure gradients squared)
  - Be careful and consistent and use shorthand notation to keep track of different variables.
- End Point
  - Martin relations for total compressibility and the associated saturation pressure relations (Martin's Eqs. 10 and 11), discuss the applicability and limitations.
  - Comments on the use of pseudofunctions???

Reading Assignment:
- Review attached notes.
- Develop the mass and mass flux expressions for each phase.
- Develop the generalized diffusivity equation for each phase.
- Apply the "Martin" criteria—neglect the $\nabla S_o \nabla p, \nabla S_w \nabla p, \nabla p \nabla p=\nabla p^2$ terms.
- Develop the "Martin" saturation-mobility-compressibility relations.
Exercises: For your own practice/skills building—do NOT turn in!

- In each of these derivations/problems you are to work in complete detail, and you must show all work.
  - Derive the Martin relations (Martin Eqs. 1-9) in complete detail using the class derivations, fill in all steps and identify all assumptions.
  - Derive the Martin relations for total compressibility and the associated saturation-pressure relations (Martin Eqs. 10 and 11) in complete detail using the class derivations, fill in all steps and identify all assumptions. (use the attached notes as a guide, but do not assume that the formation compressibility is zero (i.e., $c_f=0$))
  - Derive the Muskat material balance relation (i.e., the $dS/dp$ relation) as shown in the attached appendix from Ariadji.

- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- Problem:
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?
- Assumptions and Limitations:
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?
- Practical Applications:
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how could the solutions/results be used in practice?
- Discussion:
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.
- Recommendations/Extensions:
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Derivation of the Perrine-Martin Diffusivity Equations for Individual Phases

(from Petroleum Engineering 620 Course Notes — 1997)

Petroleum Engineering 620
Fluid Flow in Reservoirs
Development of Perrine-Martini Diffusivity Equations for Individual DTMR 10/11/93 Flowing Phases

First we must establish the mass continuity relation
\[
\nabla \cdot \left( \frac{\phi, \psi_i}{\dot{m}_i} \right) = -\frac{1}{\dot{m}_i} \left( \phi \frac{\partial \psi_i}{\partial t} \right)
\]  

(1)

Now we need the individual mass fluxes, \( \phi, \psi_i \) and the mass of each phase, \( \phi, \psi_i \). The oil and water fluxes are straightforward but the mass flux of the gas must include "free" gas as well as gas that is dissolved in the oil and the water. The mass flux for the gas is given by:

\[
(\phi, \psi_i)_{\text{total}} = \frac{\psi_{igt} \frac{E_o}{E_o}}{\frac{E_w}{E_w}} + \frac{\psi_{igt} \frac{E_w}{E_w}}{\frac{E_w}{E_w}}
\]  

(2)

Recalling the general form of Darcy’s Law
\[
\psi_i = -k_i \frac{E_i}{E_i} \phi
\]  

where \( i = o, g, w \)  

(3)

Combining Eqs. 2 and 3 gives the mass flux of the gas as

\[
(\phi, \psi_i)_{\text{total}} = -\left[ \frac{E_i}{E_i} k_i \phi + \frac{E_o}{E_o} \frac{E_w}{E_w} \frac{k_o}{k_o} \phi + \frac{E_w}{E_w} \frac{E_w}{E_w} \frac{k_w}{k_w} \phi \right]
\]  

(4)

the mass flux of the oil is given by

\[
\phi \frac{\psi_o}{\phi} = -\frac{E_o}{E_o} \frac{k_o}{k_o} \phi
\]  

(5)

and the mass flux of the water is given by

\[
\phi \frac{\psi_w}{\phi} = -\frac{E_w}{E_w} \frac{k_w}{k_w} \phi
\]  

(6)

Using the concept of the formation volume factor, \( \beta_i \), to replace density we have

\[
\beta_i = \frac{\psi_i}{\frac{E_i}{E_i}}
\]  

(7)

or

\[
\beta_i = \frac{\psi_i}{\frac{E_i}{E_i}}
\]  

(8)

Substituting Eq. 8 as needed into Eqs. 4-6 we have
\[
(s, y)_{\text{total}} = -s_{\text{sc}} \left[ \frac{k_p}{\mu_p \phi_p} \frac{\partial p}{\partial \phi} + \frac{k_o}{\mu_o \phi_o} \frac{\partial p}{\partial \phi_o} + \frac{k_w}{\mu_w \phi_w} \frac{\partial p}{\partial \phi_w} \right] 
\] (9)

\[
\gamma_o \phi_o = -s_{\text{sc}} \frac{k_o}{\mu_o \phi_o} \frac{\partial p}{\partial \phi_o} 
\] (10)

\[
\gamma_w \phi_w = -s_{\text{sc}} \frac{k_w}{\mu_w \phi_w} \frac{\partial p}{\partial \phi_w} 
\] (11)

Although it is not really obvious from Eqs. 9-11, we must assume that the pressures in the oil, water, and gas phases are equal. This means that there are no capillary pressure forces in the system. That is
\[
\phi_o = \phi_w = \phi_g 
\]
or
\[
\gamma_p = \gamma_{p_o} = \gamma_{p_w} = \gamma_{p_g} 
\]

Using Eq. 12 in Eqs. 9-11 gives
\[
(s, y)_{\text{total}} = -s_{\text{sc}} \left[ \frac{k_p}{\mu_p \phi_p} + \frac{k_o}{\mu_o \phi_o} + \frac{k_w}{\mu_w \phi_w} \right] \frac{\partial p}{\partial \phi} 
\] (12)

\[
\gamma_o \phi_o = -s_{\text{sc}} \frac{k_o}{\mu_o \phi_o} \frac{\partial p}{\partial \phi_o} 
\] (14)

\[
\gamma_w \phi_w = -s_{\text{sc}} \frac{k_w}{\mu_w \phi_w} \frac{\partial p}{\partial \phi_w} 
\] (15)

As required by the right-hand-side (RHS) of the continuity equation (Eq.1), we must establish the mass fractions for each phase. Again, we must account for the total mass of gas in the system, including the masses of gas dissolved in the oil and water phases. Starting with the gas phase we have
\[
(s, p)_{\text{total}} = s_{\text{sc}} \frac{\partial p}{\partial \phi} + s_o \frac{\partial p_o\phi_o}{\phi_o} + s_w \frac{\partial p_w\phi_w}{\phi_w} 
\]

\[
= s_{\text{sc}} \left[ \frac{\partial p}{\partial \phi} + \frac{k_o}{\mu_o \phi_o} \frac{\partial p}{\partial \phi_o} + \frac{k_w}{\mu_w \phi_w} \frac{\partial p}{\partial \phi_w} \right] 
\] (16)
oil phase:

\[
\phi_{o} = \phi_{o_{0}} \frac{S_{o}}{S_{o_{0}}} = \phi_{o_{sc}} \frac{S_{o}}{S_{o_{sc}}}
\]

(17)

water phase:

\[
\phi_{w} = \phi_{w_{sc}} \frac{S_{w}}{S_{w_{sc}}}
\]

(18)

Now we simply couple the mass fluxes and mass fractions for each phase using the continuity equation, Eq. 1.

For the gas phase we combine Eqs. 1, 13, and 16, which gives

\[
\nabla \left[ - \phi_{g_{sc}} \left( \frac{k_{g_{sc}}}{\mu_{g_{sc}} \mu_{p}} + \frac{R_{o_{sc}} k_{o_{sc}}}{\mu_{o_{sc}} \mu_{o_{sc}} k_{o}} + \frac{R_{w_{sc}} k_{w_{sc}}}{\mu_{w_{sc}} \mu_{w_{sc}} k_{w}} \right) \right] \nabla p \]

\[
= - \frac{1}{\mu_{p}} \left[ \phi_{g_{sc}} \left( \frac{k_{g_{sc}}}{\mu_{g_{sc}} \mu_{p}} + \frac{R_{o_{sc}} S_{o}}{S_{o_{sc}}} + \frac{R_{w_{sc}} S_{w}}{S_{w_{sc}}} \right) \right]
\]

Eliminating the \( \phi_{g_{sc}} \) terms and cancelling the negative signs gives

\[
\nabla \left[ \left( \frac{k_{g_{sc}}}{\mu_{g_{sc}} \mu_{p}} + \frac{R_{o_{sc}} k_{o_{sc}}}{\mu_{o_{sc}} \mu_{o_{sc}} k_{o}} + \frac{R_{w_{sc}} k_{w_{sc}}}{\mu_{w_{sc}} \mu_{w_{sc}} k_{w}} \right) \phi_{g_{sc}} \right] = \frac{1}{\mu_{p}} \left[ \phi_{g_{sc}} \left( \frac{k_{g_{sc}}}{\mu_{g_{sc}} \mu_{p}} + \frac{R_{o_{sc}} S_{o}}{S_{o_{sc}}} + \frac{R_{w_{sc}} S_{w}}{S_{w_{sc}}} \right) \right]
\]

(19)

For the oil phase we combine Eqs. 1, 14, and 17, which gives

\[
\nabla \left[ - \phi_{o_{sc}} \frac{k_{o_{sc}}}{\mu_{o_{sc}} \mu_{o}} \nabla p \right] = - \frac{1}{\mu_{o}} \left[ \phi_{o_{sc}} \phi_{o_{sc}} \right]
\]

Eliminating the \( \phi_{o_{sc}} \) terms and cancelling the negative signs gives

\[
\nabla \left[ \frac{k_{o_{sc}}}{\mu_{o_{sc}} \mu_{o}} \nabla p \right] = \frac{1}{\mu_{o}} \left[ \phi_{o_{sc}} \phi_{o_{sc}} \right]
\]

(20)

And for the water phase we combine Eqs. 1, 15, and 18, which give

\[
\nabla \left[ - \phi_{w_{sc}} \frac{k_{w_{sc}}}{\mu_{w_{sc}} \mu_{w}} \nabla p \right] = - \frac{1}{\mu_{w}} \left[ \phi_{w_{sc}} \phi_{w_{sc}} \right]
\]

Eliminating the \( \phi_{w_{sc}} \) terms and cancelling the negative signs gives

\[
\nabla \left[ \frac{k_{w_{sc}}}{\mu_{w_{sc}} \mu_{w}} \nabla p \right] = \frac{1}{\mu_{w}} \left[ \phi_{w_{sc}} \phi_{w_{sc}} \right]
\]

(21)
Summarizing our results so far

\[
\nabla \left[ \frac{(k_s + k_w) E_s + k_w E_w}{k_s k_w} \right] = \nabla \left[ \phi \left( \frac{S_o}{E_s} + \frac{E_o S_o}{E_p} + \frac{E_w S_w}{E_w} \right) \right]
\]

(19)

\[
\nabla \left[ \frac{k_s E_s}{k_s + k_w} \right] = \nabla \left[ \phi \frac{S_o}{E_p} \right]
\]

(20)

\[
\nabla \left[ \frac{k_w E_w}{k_s + k_w} \right] = \nabla \left[ \phi \frac{S_w}{E_w} \right]
\]

(21)

where the only simplifying assumptions so far are that gravity and capillary pressure effects are neglected. Now we need to consider the left-hand-side gradients, which are of the form

\[ \nabla \left[ a \phi \right] \]

where \(a = a(S_o, S_w, p)\), no \(S_p\) term is required because

\[ S_p = 1 - S_o - S_w \]

or

\[ -dS_p = dS_o + dS_w \]

Using the product rule, we have

\[ \nabla \left[ a \phi \right] = \nabla a \phi + a \nabla \phi \]

or

\[ \nabla \left[ a \phi \right] = \nabla a \phi + a \nabla^2 \phi \]

(22)

By inspection we note that \( \nabla a \phi \) is non-linear and must either be accounted for or neglected. For this derivation we will neglect the \( \nabla a \phi \) term, but not before establishing its components for each phase.

From Schaum's outline, we have that the "differential" of a 2-d function, \( f(x, y) \), is given by

\[ df = \frac{df}{dx} dx + \frac{df}{dy} dy \]

or, in terms of gradients

\[ df = \frac{df}{dx} dx + \frac{df}{dy} dy \]

(23)

Using this identity for \( a(S_o, S_w, p) \) we have

\[ a = \frac{\partial a}{\partial S_o} S_o + \frac{\partial a}{\partial S_w} S_w + \frac{\partial a}{\partial p} \phi \]

(22)

Combining Eqs. 22 and 23

\[ \nabla \left[ a \phi \right] = a \nabla^2 \phi + \left( \frac{\partial a}{\partial S_o} S_o + \frac{\partial a}{\partial S_w} S_w + \frac{\partial a}{\partial p} \phi \right) \nabla \phi \]
Now in order to develop Martini's Eqs. 4, 5, and 6 we simply combine Eqs. 19, 20, and 21, respectively with Eq. 24. This gives

\[
\text{gas Eq. } \left( \frac{k_g}{m_g} + \frac{R_g}{m_g} \right) \frac{\partial^2 p}{\partial t^2} + \frac{\partial}{\partial \psi} \left( \frac{k_g}{m_g} + \frac{R_g}{m_g} \right) \frac{\partial p}{\partial \psi} + \frac{1}{\psi} \left( \frac{k_g}{m_g} + \frac{R_g}{m_g} \right) \frac{\partial^2 p}{\partial \psi^2} + \frac{1}{\psi} \left( \frac{k_g}{m_g} + \frac{R_g}{m_g} \right) \frac{\partial p}{\partial \psi} \left( \frac{\partial^2 p}{\partial \psi^2} \right)^2 = \frac{1}{m_g} \left[ \phi \left( \frac{S_g + R_g}{m_g} \right) \right] \]
\]

\[
\text{oil Eq. } \left( \frac{k_o}{m_o} \right) \frac{\partial^2 p}{\partial t^2} + \frac{\partial}{\partial \psi} \left( \frac{k_o}{m_o} \right) \frac{\partial p}{\partial \psi} + \frac{1}{\psi} \left( \frac{k_o}{m_o} \right) \frac{\partial^2 p}{\partial \psi^2} + \frac{1}{\psi} \left( \frac{k_o}{m_o} \right) \frac{\partial p}{\partial \psi} \left( \frac{\partial^2 p}{\partial \psi^2} \right)^2 = \frac{1}{m_o} \left[ \phi \frac{S_o}{m_o} \right] \]
\]

\[
\text{water Eq. } \left( \frac{k_w}{m_w} \right) \frac{\partial^2 p}{\partial t^2} + \frac{\partial}{\partial \psi} \left( \frac{k_w}{m_w} \right) \frac{\partial p}{\partial \psi} + \frac{1}{\psi} \left( \frac{k_w}{m_w} \right) \frac{\partial^2 p}{\partial \psi^2} + \frac{1}{\psi} \left( \frac{k_w}{m_w} \right) \frac{\partial p}{\partial \psi} \left( \frac{\partial^2 p}{\partial \psi^2} \right)^2 = \frac{1}{m_w} \left[ \phi \frac{S_w}{m_w} \right] \]
\]

\[
\text{NEGLIGENCE the } \frac{\partial^2 p}{\partial \psi^2}, \frac{\partial p}{\partial \psi}, \text{ and } \frac{\partial^2 p}{\partial \psi^2} \text{ terms gives}
\]

\[
\text{gas Eq. } \left( \frac{k_g}{m_g} + \frac{R_g}{m_g} \right) \frac{\partial^2 p}{\partial t^2} = \frac{1}{m_g} \left[ \phi \left( \frac{S_g + R_g}{m_g} \right) \right] \]
\]

\[
\text{oil Eq. } \frac{k_o}{m_o} \frac{\partial^2 p}{\partial t^2} = \frac{1}{m_o} \left[ \phi \frac{S_o}{m_o} \right] \]
\]

\[
\text{water Eq. } \frac{k_w}{m_w} \frac{\partial^2 p}{\partial t^2} = \frac{1}{m_w} \left[ \phi \frac{S_w}{m_w} \right] \]
Derivation of the Simplified Diffusivity Equation for Multiphase Flow ("Martin" Equations)

5.2.3 Diffusivity Equation for Multiphase Flow. To develop the diffusivity equation for multiphase flow, we begin with principles and then develop continuity equations for each phase flowing in the reservoir. Consider a unit control volume that contains oil, water, and gas with saturations $S_o$, $S_w$, and $S_g$, respectively. The mass of oil in the reservoir in the unit reservoir volume is

$$m_o = \frac{\phi S_o}{B_o} \rho_o.$$  \hspace{1cm} (5.60)

Similarly, the mass of water in the reservoir in the unit reservoir volume is

$$m_w = \frac{\phi S_w}{B_w} \rho_w.$$  \hspace{1cm} (5.61)

Gas exists in two phases: as free gas and as gas dissolved in the oil and water. The mass of the free gas is

$$m_{g,f} = \frac{\phi S_g}{B_g} \rho_{g,f}.$$  

the mass of the dissolved gas is

$$m_{g,d} = \frac{\phi R_o S_o}{B_o} \rho_{g,d} + \frac{\phi R_w S_w}{B_w} \rho_{g,d}.$$  \hspace{1cm} (5.62)

Therefore, the total mass of gas in the unit control volume is

$$m_g = \frac{\phi S_g}{B_g} \rho_{g,f} + \frac{\phi R_o S_o}{B_o} \rho_{g,d} + \frac{\phi R_w S_w}{B_w} \rho_{g,d}.$$  \hspace{1cm} (5.63)

Eq. 5.62 contains terms representing both free gas (terms with the subscript $g$) and gas dissolved in oil and water (terms with the subscripts $o$ and $w$, respectively).

We can use Darcy's law for an equation of motion into the control volume for each phase. For the oil, water, and gas phases,

$$\rho_o \frac{\partial}{\partial t} \frac{k_o}{\mu_o} = -0.0002537 \frac{k_o}{\rho_o B_o} \frac{\partial \rho_o}{\partial r}.$$  \hspace{1cm} (5.63)
\[ \rho_w \mu_{rw} = -0.0002536 \frac{k_w \frac{\partial p}{\partial r}}{\mu_w B_w \mu_{rw}} \] .......... (5.64) \\

and \( \rho_w^2 \rho_R = -0.0002637 \) .......... (5.65).

A continuity equation can now be written for each phase because the rate of mass into the control volume less the rate of mass out is \((1/\partial)(\partial\rho/\partial t)(\text{area})\), and the rate of change of mass in the control volume is \(\partial m/\partial t\). Therefore, the continuity equation for any Phase \( x \) is

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho_x \frac{\partial m_x}{\partial r} \right) = \frac{\partial}{\partial t} \left( r \rho_x m_x \right) \] ............ (5.70)

Similarly, for water and gas, we can write

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_w \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_w B_w} \frac{\partial}{\partial t} \left( S_w \right) \] .......... (5.71)

Now, we expand the partial derivatives with respect to time for Eqs. 5.66 through 5.68. Because saturations, FVF's, and solubilities are functions of pressure, the oil equation becomes

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_o \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_o B_o} \frac{\partial}{\partial t} \left( S_o \right) \] .......... (5.72)

For convenience, we let all partial derivatives with respect to pressure be denoted by \( B' \), \( S' \), etc., and we use the fact that

\[ \frac{\partial}{\partial p} \left( \frac{1}{B_o} \right) = \frac{1}{B_o^2} \frac{\partial B_o}{\partial p} \] ............ (5.73)

Therefore,

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_o \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_o B_o} \frac{\partial}{\partial t} \left( S'_o \right) \] .......... (5.74)

Similarly, for water,

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_w \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_w B_w} \frac{\partial}{\partial t} \left( S'_w \right) \] .......... (5.75)

The gas equation is slightly more difficult because it contains solubility terms that also are functions of pressure. Expanding the right side of Eq. 5.66 with the chain rule gives

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_g \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_g B_g} \frac{\partial}{\partial t} \left( S'_g \right) \] .......... (5.76)

Or we can write

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_g \frac{\partial p}{\partial r} \right) = \frac{\phi}{\mu_g B_g} \frac{\partial}{\partial t} \left( S'_g \right) \] .......... (5.77)

We now define mobility as the ratio of the effective permeability of the porous medium to fluid viscosity. Eqs. 5.75 through 5.77 define the oil, water, and gas mobilities, respectively:
\[
\lambda_w = k_w/\mu_w, \quad \lambda_o = k_o/\mu_o, \quad \lambda_g = k_g/\mu_g \tag{5.75, 5.76, 5.77}
\]

After we substitute Eq. 5.75 into Eq. 5.72, the oil equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \phi}{\partial r} \right) = \frac{\phi}{0.0002637 \lambda_w} \left( S_o' - S_o B_o' \right) \frac{\partial p}{\partial t}. \tag{5.78}
\]

Similarly, combining Eqs. 5.76 and 5.73 for water and Eqs. 5.77 and 5.74 for gas gives

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \phi}{\partial r} \right) = \frac{\phi}{0.0002637 \lambda_w} \left( S_w' - S_w B_w' \right) \frac{\partial p}{\partial t}. \tag{5.79}
\]

and

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \phi}{\partial r} \right) = \frac{\phi}{0.0002637 \lambda_w} \left( R_w \frac{S_w' - S_w B_w'}{B_w} \right) \frac{\partial p}{\partial t}. \tag{5.80}
\]

respectively.

Next, we derive a single equation to describe multiphase flow through a porous medium. First, we equate the right sides of the oil and water equations (Eqs. 5.78 and 5.79), realizing that the left sides of both are equivalent:

\[
\frac{\phi}{\lambda_w} \left( S_o' - S_o B_o' \right) \frac{\partial p}{\partial t} = \frac{\phi}{\lambda_o} \left( S_o' - S_o B_o' \right) \frac{\partial p}{\partial t}. \tag{5.81}
\]

Simplifying gives

\[
\lambda_w \left( S_o' - S_o B_o' \right) = \lambda_o \left( S_o' - S_o B_o' \right). \tag{5.82}
\]

Similarly, we equate the right sides of the oil and gas equations (Eqs. 5.78 and 5.80), again realizing that the left sides are equivalent:

\[
\frac{\phi}{\lambda_w} \left( S_o' - S_o B_o' \right) \frac{\partial p}{\partial t} = \frac{R_w \frac{S_w' - S_w B_w'}{B_w}}{\lambda_w} \frac{\partial p}{\partial t}. \tag{5.83}
\]

Grouping terms on the right side gives

\[
1 \left( S_o' - S_o B_o' \right) \left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g' \tag{5.84}
\]

Substituting Eq. 5.81 into Eq. 5.82 gives

\[
1 \left( S_o' - S_o B_o' \right) \left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g' \tag{5.85}
\]

Substituting Eq. 5.81 into Eq. 5.83 gives

\[
\left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) \left( S_o' - S_o B_o' \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.86}
\]

Substituting Eq. 5.81 into Eq. 5.85 gives

\[
\left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) \left( S_o' - S_o B_o' \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.87}
\]

Factoring all terms on the left side containing \([S_o' - (S_o B_o')/B_o]\), we obtain

\[
\left( S_o' - S_o B_o' \right) \left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.88}
\]

We know that, for a three-phase system,

\[
S_g + S_w + S_o = 1. \tag{5.89}
\]

Differentiating Eq. 5.84 with respect to pressure gives

\[
S_o' + S_w' + S_g' = 0. \tag{5.90}
\]

Also, the total mobility of a three-phase system is the sum of the individual mobilities:

\[
\lambda_t = \lambda_o + \lambda_w + \lambda_g. \tag{5.91}
\]

Substituting Eqs. 5.85 and 5.86 into Eq. 5.83 for \(S_g\) and \(\lambda_g\), respectively, gives

\[
\left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) \left( S_o' - S_o B_o' \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.92}
\]

Substituting Eqs. 5.85 and 5.86 into Eq. 5.83 for \(S_g\) and \(\lambda_g\), respectively, gives

\[
\left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) \left( S_o' - S_o B_o' \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.93}
\]

Substituting Eqs. 5.85 and 5.86 into Eq. 5.83 for \(S_g\) and \(\lambda_g\), respectively, gives

\[
\left( \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right) \left( S_o' - S_o B_o' \right) = \left[ \frac{R_w \lambda_o + R_w \lambda_w + \lambda_g}{B_o} \right] \left( S_o' - S_o B_o' \right) + \frac{R_w \lambda_w}{B_w} S_w' + \frac{R_w \lambda_g}{B_g} S_g'. \tag{5.94}
\]

We add the terms \((S_o B_o')/B_o\) and \((S_w B_w')/B_w\) to each side of the equation, we can again group terms that contain \([S_o' - (S_o B_o')/B_o]\):
\[
\left( S'_o - \frac{S_o B'_o}{B_o} \right) \left( \frac{\lambda_t - \lambda_o - \lambda_w}{\lambda_o} \right) + \frac{S_o B'_o}{B_o} + \frac{S_w B'_w}{B_w} = \frac{R'_i S_o B'_o}{B_o} + \frac{R'_w S_o B'_o}{B_o} - \frac{S'_o}{B'_o} + \frac{S'_o B'_o}{B_o} - \frac{S'_w B'_w}{B_w} - \frac{S'_w B'_w}{B_w} - \frac{S'_g B'_g}{B_g} \tag{5.87}
\]

Recall Eq. 5.81:
\[
\frac{\lambda_w}{\lambda_o} \left( S'_o - \frac{S_o B'_o}{B_o} \right) = \left( S'_w - \frac{S_w B'_w}{B_w} \right).
\]

Substituting Eq. 5.81 into Eq. 5.87 yields
\[
\left( S'_o - \frac{S_o B'_o}{B_o} \right) \left( \frac{\lambda_t - \lambda_o - \lambda_w}{\lambda_o} \right) + \frac{S_o B'_o}{B_o} + \frac{S_w B'_w}{B_w} = \frac{R'_i S_o B'_o}{B_o} + \frac{R'_w S_o B'_o}{B_o} - \left( S'_w - \frac{S_w B'_w}{B_w} \right) \frac{\lambda_w}{\lambda_o} \left( S'_w - \frac{S_w B'_w}{B_w} \right)
\]

\[
\frac{S'_w B'_w}{B_w}
\]

Again, factoring all terms containing \( [S'_o - (S_o B'_o)/B_o] \) on the left side gives
\[
\left( S'_o - \frac{S_o B'_o}{B_o} \right) \left( \frac{\lambda_t - \lambda_o - \lambda_w}{\lambda_o} + \frac{\lambda_w}{\lambda_o} \right) = \frac{R'_i S_o B'_o}{B_o} + \frac{R'_w S_o B'_o}{B_o} \frac{S_o B'_o}{B_o} + \frac{S_w B'_w}{B_w} - \frac{S'_w B'_w}{B_w} - \frac{S'_g B'_g}{B_g},
\]
or finally,
\[
\left( S'_o - \frac{S_o B'_o}{B_o} \right) \left( \frac{\lambda_t}{\lambda_o} \right) = \frac{R'_i S_o B'_o}{B_o} + \frac{R'_w S_o B'_o}{B_o} - \frac{S'_o}{B'_o} + \frac{R'_w S_w B'_w}{B_w} - \frac{S'_w B'_w}{B_w}
\]

\[
\frac{S'_w B'_w}{B_w} - \frac{S'_g B'_g}{B_g} \tag{5.88}
\]

For a system with multiple phases,
\[
c_i = s_i c_o + s_w c_w + s_g c_g \tag{5.89}
\]
where, below the bubblepoint,
\[
c_o = \frac{1}{B_o} (B'_o - R'_i B'_i) \tag{5.90}
\]

Similarly, water compressibility, \( c_w \), below the bubblepoint is
\[
c_w = \frac{1}{B_w} (B'_w - R'_w B'_w) \tag{5.91}
\]
while gas compressibility, \( c_g \), is
\[
c_g = -\frac{B'_g}{B_g} \tag{5.92}
\]

Substituting Eqs. 5.90 through 5.92 into Eq. 5.89 gives the expression for calculating total compressibility:
\[
c_i = \frac{R'_i S_o B'_o}{B_o} - \frac{S_o B'_o}{B_o} + \frac{R'_w S_o B'_o}{B_o} - \frac{S'_o}{B'_o} + \frac{R'_w S_w B'_w}{B_w} - \frac{S'_w B'_w}{B_w} - \frac{S'_g B'_g}{B_g} \tag{5.93}
\]

Substituting Eq. 5.93 into Eq. 5.88 yields
\[
\left( S'_o - \frac{S_o B'_o}{B_o} \right) \left( \frac{\lambda_t}{\lambda_o} \right) = c_i \tag{5.94}
\]

We can now substitute Eq. 5.94 into the original oil, gas, or water equation to obtain the diffusivity equation for multiphase flow. For the oil phase,
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial p}{\partial r} \right) = \frac{\phi}{0.0002637 \lambda_o} \left( S'_o - \frac{S_o B'_o}{B_o} \right) \frac{\partial p}{\partial t} \tag{5.78}
\]

becomes
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial p}{\partial r} \right) = \frac{\phi c_i}{0.0002637 \lambda_o} \frac{\partial p}{\partial t} \tag{5.95}
\]

Eq. 5.95 is similar to the single-phase diffusivity equation for slightly compressible liquids:
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial p}{\partial r} \right) = \frac{\phi \mu c_i}{0.0002637 k_o} \frac{\partial p}{\partial t}, \tag{5.96}
\]

which implies that the solutions to the single-phase diffusivity equation presented later in this chapter apply to multiphase flow as long as \( c_i \) is defined by Eq. 5.93 and we use \( \lambda_t \) instead of \( k_o/\mu_o \).
Derivation of the Modified Muskat Material Balance Equation

(from Tutuka Ariadji Ph.D. Dissertation (Texas A&M U. — 1996))
MODELING SIMULTANEOUS OIL-GAS FLOW
USING SINGLE-PHASE ANALYTICAL SOLUTIONS

A Dissertation
by
TUTUKA ARIADJI

Submitted to the Office of Graduate Studies of
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in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

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APPENDIX B
DERIVATION OF MODIFIED MUSKAT MATERIAL BALANCE

The stock tank barrels of oil, \( N_r \), remaining in the reservoir at any pressure is given by

\[
N_r = \frac{S_o V_p}{B_o}. \quad \text{.................................................................(B-1)}
\]

Differentiating with respect to pressure yields

\[
\frac{dN_r}{dp} = \frac{B_o (S_o V_p' + S_o V_p') - S_o V_p B_o'}{B_o^2} \nonumber
\]

\[
= \frac{1}{B_o} S_o V_p' + \frac{1}{B_o} S_o V_p - \frac{1}{B_o^2} S_o V_p B_o' \nonumber
\]

\[
= V_p \left( \frac{S_o V_p'}{B_o} + \frac{S_o}{B_o} - \frac{B_o'}{B_o^2} S_o \right) \nonumber
\]

\[
= V_p \left( c_f \frac{S_o}{B_o} + \frac{S_o'}{B_o} - \frac{B_o'}{B_o^2} S_o \right). \quad \text{.................................................................(B-2)}
\]

The SCF of gas remaining in the reservoir, including free and dissolved gas both in the oil and water phase is

\[
G_r = \frac{R_{so} V_p S_o}{B_o} + \frac{R_{sw} V_p S_w}{B_w} + \frac{S_g V_p}{B_g} \quad \text{.................................................................(B-3)}
\]

or

\[
G_r = V_p \left( \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} + \frac{(1 - S_o - S_w)}{B_g} \right). \quad \text{.................................................................(B-4)}
\]
Differentiating with respect to pressure yields

\[
\frac{dG_r}{dp} = V_p \left( \frac{B_o (R_{so} S_o'+R_{so} S_o) - R_{so} S_o B_o'}{B_o^2} + \frac{B_w (R_{sw} S_w'+R_{sw} S_w) - R_{sw} S_w B_w'}{B_w^2} \right) + \frac{B_g (-S_o' - S_w') - (1 - S_o - S_w) B_g'}{B_g^2} + \left( \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} + \frac{(1 - S_o - S_w)}{B_g} \right) V_p',
\]

or

\[
\frac{dG_r}{dp} = V_p \left( \frac{B_o (R_{so} S_o'+R_{so} S_o) - R_{so} S_o B_o'}{B_o^2} + \frac{B_w (R_{sw} S_w'+R_{sw} S_w) - R_{sw} S_w B_w'}{B_w^2} \right) + \frac{B_g (-S_o' - S_w') - (1 - S_o - S_w) B_g'}{B_g^2} + \left( \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} + \frac{(1 - S_o - S_w)}{B_g} \right) \frac{V_p'}{V_p},
\]

or

\[
\frac{dG_r}{dp} = V_p \left( \frac{B_o (R_{so} S_o'+R_{so} S_o) - R_{so} S_o B_o'}{B_o^2} + \frac{B_w (R_{sw} S_w'+R_{sw} S_w) - R_{sw} S_w B_w'}{B_w^2} \right) + \frac{B_g (-S_o' - (1 - S_o - S_w) B_g')}{B_g^2} + c_f \left( \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} + \frac{(1 - S_o - S_w)}{B_g} \right).
\] \hspace{1cm} \text{(B-5)}

We can define the producing GOR as follows:

\[
R = \frac{dG_r}{dp} = \frac{dN_r}{dp}.
\] \hspace{1cm} \text{(B-6)}

Substituting (B-2) and (B-5) into (B-6) yields
\[
R = \frac{B_o \left( R_{3o} S_o + R_{3w} S_w \right) - R_{3o} S_o B_o}{B_o^2} + \frac{B_w \left( R_{3w} S_w + R_{3w} S_w \right) - R_{3w} S_w B_w}{B_w^2} + \frac{B_g \left( -S_o \right) - \left( 1 - S_o - S_w \right) B_g}{B_g^2} + c_f \left( \frac{R_{3o} S_o}{B_o} + \frac{R_{3w} S_w}{B_w} + \left( 1 - S_o - S_w \right) \right) \frac{B_o}{B_g} \frac{S_o}{B_o} \frac{S_o}{B_o} - \frac{S_w}{B_o} \frac{S_o}{B_o} \\
\]

Now recall the Darcy's Law for the oil phase

\[
q_o = \alpha_1 \frac{k h \left( \frac{k_{ro}}{\mu_o B_o} \right) \frac{\partial P}{\partial r}}{\mu_o B_o} \] ........................... (B-8)

and gas phase

\[
q_g = \alpha_1 k h \left( \frac{k_{rg}}{\mu_g B_g} + \frac{R_{3o} \frac{k_{rg}}{\mu_o B_o} + R_{3w} k_{rw}}{\mu_w B_w} \right) \frac{\partial P}{\partial r} \] ........................... (B-9)

Thus, the producing GOR also can be written as

\[
R = \frac{q_g}{q_o} \left( \frac{k_{rg}}{\mu_g B_g} + \frac{R_{3o} k_{ro}}{\mu_o B_o} \right) = R_{3o} + \frac{k_{rg}}{k_{ro}} \frac{\mu_o B_o}{\mu_g B_g} \] ........................... (B-10)

Combining (B-7) and (B-10) yields
\[
\begin{align*}
\frac{R_{so}S_o'}{B_o} &+ \frac{R_{so}'S_o}{B_o} - \frac{R_{sw}S_wB_o'}{B_w^2} + \frac{R_{sw}S_wB_w'}{B_w^2} + \left(-\frac{S_o'}{B_g}\right) - \left(1-S_o-S_w\right)B_g' + \\
c_f\left(\frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w} + \left(1-S_o-S_w\right)\right) = \\
\frac{S_o}{B_o} + \frac{S_o'}{B_o} - \frac{S_oB_0'}{B_o^2} \\
R_{so} + \frac{k_{rg} \mu_o B_o}{k_{ro} \mu_g B_g}
\end{align*}
\]

\[(B-11)\]

Rearranging and expanding yields

\[
\begin{align*}
\frac{R_{so}S_o'}{B_o} &+ \frac{R_{so}'S_o}{B_o} - \frac{R_{sw}S_wB_o'}{B_w^2} + \frac{R_{sw}S_wB_w'}{B_w^2} - \frac{S_o'}{B_g} - \left(1-S_o-S_w\right)B_g' + \\
c_f\left(\frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w} + \left(1-S_o-S_w\right)\right) = \\
\frac{S_o}{B_o} \frac{R_{so}'}{B_o} + \frac{S_o'}{B_o} - \frac{S_oB_0'}{B_o^2} + c_f \frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o} + c_f \frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o} - \frac{S_oB_0'}{B_o^2} \frac{k_{rg} \mu_o B_o}{B_o}.
\end{align*}
\]

\[(B-12)\]

Collecting the like terms, the derivative of fluid properties and the derivatives of saturations

\[
\begin{align*}
\frac{R_{so}S_o'}{B_o} &+ \frac{R_{sw}'S_w}{B_w} - \frac{R_{sw}S_wB_w'}{B_w^2} - \left(1-S_o-S_w\right)B_g' + c_f\left(\frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w} + \left(1-S_o-S_w\right)\right) - \\
\frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o} + \frac{S_oB_0'}{B_o^2} \frac{k_{rg} \mu_o B_o}{B_o} + \frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o} + \frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o} = \\
\frac{R_{so}S_o'}{B_o} + \frac{S_o'}{B_o} - \frac{S_oB_0'}{B_o^2} + \frac{S_o}{B_o} \frac{k_{rg} \mu_o B_o}{B_o}.
\end{align*}
\]

\[(B-13)\]

Simplifying (B-13) yields
\[
\frac{R_{s_o} S_o}{B_o} + \frac{R_{s_w} S_w}{B_w} - \frac{R_{s_w} S_w B_w}{B_o^2 B_w^2} \frac{1-S_o}{B_o} - \frac{1-S_w}{B_w} B_o + \frac{c_f}{B_g} \left( \frac{R_{s_w} S_w}{B_w} B_g + (1-S_o-S_w) \right) \\
-S_o \frac{k_{rg} \mu_o}{k_{ro} \mu_g} + \frac{S_o}{B_g} \frac{B_o}{B_o} \frac{k_{rg} \mu_o}{k_{ro} \mu_g} = S_o' (1+ \frac{k_{rg} \mu_o}{k_{ro} \mu_g}). \quad \text{(B-14)}
\]

Thus

\[
\frac{R_{s_o} S_o}{B_o} B_g + \frac{R_{s_w} S_w}{B_w} B_g - \frac{R_{s_w} S_w B_w}{B_o^2 B_w^2} B_g - \frac{1-S_o}{B_g B_w} B_g + \frac{c_f}{B_g} \left( \frac{R_{s_w} S_w}{B_w} B_g + (1-S_o-S_w) \right) \\
(1-S_o-S_w)^2 - S_o \frac{k_{rg} \mu_o}{k_{ro} \mu_g} + \frac{S_o}{B_o} \frac{B_o}{B_g} \frac{k_{rg} \mu_o}{k_{ro} \mu_g} \\
S_o' = \frac{1 - \frac{k_{rg} \mu_o}{k_{ro} \mu_g} + \frac{k_{rg} \mu_o}{k_{ro} \mu_g}}{1+ \frac{k_{rg} \mu_o}{k_{ro} \mu_g}}. \quad \text{(B-15)}
\]
Performance of the Saturation/Pressure Calculations for Solution-Gas-Drive Reservoirs

From:

Fig. 4—Oil saturation at wellbore vs. dimensionless time, $t_{ID}$: Case 1 with $q_o = 125.8$ STB/D.

Fig. 6—Oil saturation at the wellbore vs. dimensionless shut-in time, $\Delta t_{ID}$: Case 2 with $q_o = 125.8$ STB/D.

Fig. 7—Saturation profiles vs. dimensionless radius, $r_D$, at the end of 5-hour drawdown and after 5-minute buildup: Case 2 with $q_o = 125.8$ STB/D.

Fig. 5—Semilog plot of $S_o$ vs. $t_D$; $s = 0$.

Fig. 10—Semilog plot of $S_o$ vs. $t_D$; $s = 5$.

Fig. 3—Approximation of sandface $S_o$ for Case 3.

Fig. 4—Approximation of saturation/pressure relation for Case 2.

Fig. 6—Approximation of saturation/pressure relation for Case 1.

Fig. 7—Approximation of sandface saturation/pressure relation for Case 3.