Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2a — Solutions of the Radial Flow Diffusivity Equation Using the Laplace Transform for a Well Produced at a Constant Rate in an Infinite-Acting Reservoir

A wise man will make more opportunities than he finds.
— Francis Bacon (1625)

**Topic:** Solutions of the Radial Flow Diffusivity Equation Using the Laplace Transform for a Well Produced at a Constant Rate in an Infinite-Acting Reservoir

**Objectives:** (things you should know and/or be able to do)

- Be able to recognize that the Laplace transform of the dimensionless form of the single-phase radial flow diffusivity equation is the modified Bessel differential equation. Also, be able to write the general solution for this transformed differential equation.

  **Dimensionless Diffusivity Equation**

  \[
  \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial p_D}{\partial r_D} \right] = \frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} = \frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{dp_D}{dr_D} \right] = \frac{u \bar{p_D}}{r_D}
  \]

  **Transformed Diffusivity Equation:** (Modified Bessel Function form)

  \[ z^2 \frac{d \bar{p_D}}{dz^2} + z \frac{d \bar{p_D}}{dz} = z^2 \bar{p_D} \quad (z = \nu r_D) \]

  **General Solution**

  \[ \bar{p_D}(r_D, u) = A I_0(\nu r_D) + B K_0(\nu r_D) \]

  **Derivative of the General Solution**

  \[ \frac{d \bar{p_D}}{dr_D} = A \nu I_1(\nu r_D) - B \nu K_1(\nu r_D) \]

- Be able to develop the particular solution (in the Laplace domain) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition. Also, be able to use the van Everdingen and Hurst result which "converts" the constant rate case to the constant wellbore pressure case.

  **Constant Rate Solution:** (infinite-acting reservoir)

  \[ \bar{p_D}(r_D, u) = \frac{1}{u} \frac{K_0(\nu r_D)}{\nu K_1(\nu)} \approx \frac{1}{u} K_0(\nu r_D) \]

  **Constant Rate-Constant Pressure Relation:** (from van Everdingen and Hurst)

  \[ \bar{q_D}(u) = \frac{1}{u^2} \bar{p_D}(u) \]

- Be able to develop the real domain (time) solution (i.e., the Exponential Integral solution) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition—using both the Laplace transform as well as the Boltzmann transform approaches. Also be able to develop the "log-approximation" for this solution.

  **Boltzmann Transform of the Diffusivity Equation:**

  \[ \frac{d^2 p_D}{d \varepsilon_D^2} + \left[ 1 + \frac{1}{\varepsilon_D} \right] \frac{dp_D}{d \varepsilon_D} = 0 \quad \text{(infinite-acting reservoir case only)} \]
Objectives: (Continued)

- "Exponential Integral" Solution for the Diffusivity Equation:

\[ P_D(t_D, r_D) = \frac{1}{2} E_1 \left( \frac{r_D^2}{4t_D} \right) \]

- "Log Approximation" Solution for the Diffusivity Equation:

\[ P_D(t_D, r_D) = \frac{1}{2} \ln \left( \frac{4 \cdot t_D}{e^2 \cdot r_D^2} \right) \]

Lecture Outline:

- Review of the Dimensionless Forms of the Single-Phase Radial Flow Diffusivity Equation and the Initial and Boundary Conditions
- Boltzmann Transform Solution
- Laplace Transform Solution
  - Laplace transform of the partial differential equation, application of the initial condition
  - Recast the Laplace transformed differential equation into an ordinary differential equation for a single independent variable. Note that the resulting form is the modified Bessel differential equation, and then give the general solution.
  - Differentiate the general solution with respect to the dimensionless radius, \( r_D \), for application of the inner boundary condition
  - Develop the particular solution, which requires application of both the inner and outer boundary conditions.
  - Attempt to invert the particular solution, give rationale for simplifications and discuss further approximations.

Reading Assignment:

- Review attached notes.
  - Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case:
    - Real domain solution via the Boltzmann transform.
    - Real domain solution via inversion of the Laplace transform solution.
  - Boltzmann Transform Identities

Exercises: For your own practice/skills building—do NOT turn in!

From the attached notes you are to rederive the following, and show all details.

- Derive the Boltzmann transform solution for the infinite-acting reservoir case.
- Derive the Laplace transform solutions for the infinite-acting reservoir case.
  - "Cylindrical source" solution (just show the result—you do not have to derive)
  - "Line source" solution (E_1 formulation)
  - "Line source" solution (log approximation)
- Plot the \( P_{wD}(t_D) \) solutions for \( 1 \leq t_D \leq 1 \times 10^5 \), on semilog (x-axis) and log-log scales.
Solution of the Dimensionless Radial Flow Diffusivity Equation:

- Infinite-Acting Reservoir Case: Boltzmann Transform and the Laplace Transform Approach

- Laplace Transform Solutions

- Real Domain Solutions via Inversion of the Laplace Transform Solutions

Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Solution via the Boltzmann Transform
- Boltzmann Transform Identities
- Real Domain Solution via Inversion of the Laplace Transform Solution

Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

Solution via the Boltzmann Transform

Radial Flow Solution for an Infinite-Acting Homogeneous Reservoir: Boltzmann Transform Approach

This method has been demonstrated by a variety of authors—the approach we choose was presented by J.L. Johnston in the 2nd edition of the Lee Well Testing text.

The basic partial differential equation is given in dimensionless form as

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial t}
\]  

(1)

where

\[
\phi = \frac{r}{r_w} \quad (2) \quad \phi = \phi_e \frac{k h (p_e - p_f)}{q h_c} \quad (3) \quad t_0 = t \frac{k}{\phi h_c \phi_c \phi_e} \quad (4)
\]

where \(t_c\) and \(\phi_c\) are given by

<table>
<thead>
<tr>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
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<tr>
<td>(t_c)</td>
<td>1</td>
<td>2.637 x 10^{-4}</td>
</tr>
<tr>
<td>(\phi_c)</td>
<td>(2\pi)</td>
<td>7.081 x 10^{-3}</td>
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<td></td>
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<td>5.557 x 10^{-6}</td>
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The "initial" condition is given as

\[
\phi (r_0, t_0 = 0) = 0 \quad \text{(uniform pressure distribution)} \quad (5)
\]

The constant rate inner boundary condition is

\[
\left. \frac{\partial \phi}{\partial r} \right|_{r_0} = -1 \quad \text{(constant flow rate at the well)} \quad (6)
\]

The "infinite-acting" outer boundary condition is given by

\[
\phi (r \to \infty, t_0) = 0 \quad (7)
\]

Rewriting Eq. 1 we have

\[
\frac{1}{r_0} \left[ \frac{\partial}{\partial r} \frac{\partial \phi}{\partial r} \right] = \frac{\phi}{r_0^2}
\]  

(8)

The Boltzmann transform variable, \(\tau_0\), is defined as

\[
\tau_0 = \frac{r_0 b \tau_c}{t_0} \quad (9)
\]

where for our problem we have

\[
a = \frac{1}{4}, \quad b = 2, \quad c = -1
\]

which yields

\[
\tau_0 = \frac{r_0^2}{t_0} \quad (10)
\]
Expanding the $\frac{d}{dt_0} \frac{d^2}{dt_0^2}$ term we have
\[
\frac{1}{\frac{d}{dt_0}} \left[ \frac{d}{dt_0} \frac{d}{dt_0} \right] + \frac{1}{t_0} \frac{d}{dt_0} = \frac{d}{dt_0} = \frac{d}{dt_0} \frac{d}{dt_0}
\]

(11)

Applying the chain rule,
\[
\frac{d}{dt_0} = \frac{d}{dx} \frac{d}{dx}
\]

which combined with Eq. 11 gives
\[
\frac{d}{dt_0} \frac{d}{dt_0} \left[ \frac{d}{dt_0} \frac{d}{dt_0} \right] + \frac{1}{t_0} \frac{d}{dt_0} = \frac{d}{dt_0} \frac{d}{dt_0}
\]

Expanding
\[
\frac{d}{dt_0} \left[ \frac{d}{dt_0} \left( \frac{d}{dt_0} \right) \frac{d}{dt_0} \right] + \frac{1}{t_0} \frac{d}{dt_0} = \frac{d}{dt_0} \frac{d}{dt_0}
\]

(12)

Isolating terms
\[
\left( \frac{d}{dt_0} \right)^2 \frac{d^2}{dt_0^2} + \left[ \frac{d}{dt_0} \frac{d}{dt_0} \left( \frac{d}{dt_0} \right) \frac{d}{dt_0} \right] + \left[ \frac{1}{t_0} \frac{d}{dt_0} \right] \frac{d}{dt_0} = 0
\]

Dividing through by $(\frac{d}{dt_0})^2$ gives
\[
\frac{d^2}{dt_0^2} + \frac{1}{t_0} \frac{d}{dt_0} = 0
\]

(13)

Reducing the $\frac{1}{t_0}$ term we have
\[
\frac{d^2}{dt_0^2} + \frac{1}{(\frac{d}{dt_0})^2} \left[ \frac{d}{dt_0} \left( \frac{d}{dt_0} \right) + \frac{1}{t_0} \frac{d}{dt_0} \right] \frac{d}{dt_0} = 0
\]

which can be further reduced to
\[
\frac{d^2}{dt_0^2} + \frac{1}{(\frac{d}{dt_0})^2} \left[ \frac{d}{dt_0} \frac{d}{dt_0} \frac{d}{dt_0} + \frac{1}{t_0} \frac{d}{dt_0} \right] \frac{d}{dt_0} = 0
\]

Completing the factorization of the $(\frac{d}{dt_0})^2$ gives
\[
\frac{d^2}{dt_0^2} + \frac{1}{(\frac{d}{dt_0})^2} \left[ \frac{d}{dt_0} \frac{d}{dt_0} \frac{d}{dt_0} + \frac{1}{t_0} \frac{d}{dt_0} \right] \frac{d}{dt_0} = 0
\]

(14)

Using Eq. 10 we take the following derivatives
\[
\frac{d}{dt_0} = \frac{d}{dt_0} \left( \frac{d^2}{dt_0^2} \right) = \frac{d}{dt_0} \left( \frac{d}{dt_0} \frac{d}{dt_0} \right) = \frac{d}{dt_0} \frac{d}{dt_0} = \frac{d}{dt_0} \frac{d}{dt_0}
\]

(15)

\[
\frac{d^2}{dt_0^2} = \frac{d}{dt_0} \left[ \frac{d}{dt_0} \left( \frac{d^2}{dt_0^2} \right) \right] = \frac{d}{dt_0} \left( \frac{d}{dt_0} \frac{d}{dt_0} \right) = \frac{d}{dt_0} \frac{d}{dt_0} = \frac{d}{dt_0} \frac{d}{dt_0}
\]

(16)
Substituting Eqs. 13-15 into Eq. 12 gives
\[ \frac{d^2 r_D}{d \xi_D^2} + \left[ \frac{1}{(2 \xi_D / r_D)^2} \frac{\xi_D}{r_D} + \frac{1}{r_D (2 \xi_D / r_D)} - \frac{1}{(2 \xi_D / r_D)^2} \right] \frac{dr_D}{d \xi_D} = 0 \]
reducing
\[ \frac{d^2 r_D}{d \xi_D^2} + \left[ \frac{1}{2 \xi_D} + \frac{1}{2 \xi_D} + 1 \right] \frac{dr_D}{d \xi_D} = 0 \]
or finally we have
\[ \frac{d^2 r_D}{d \xi_D^2} + \left[ 1 + \frac{1}{\xi_D} \right] \frac{dr_D}{d \xi_D} = 0 \]  
(16)

Where Eq. 16 is our "Boltzmann" transformed differential equation.

We must now establish the initial and boundary conditions in terms of the Boltzmann transform variable, \( \xi_D \). Recalling the initial condition, Eq. 5, we have
\[ r_D (r_D, \xi_D = 0) = 0 \]  
(5)
where for \( r_D \to 0 \); \( \xi_D \to \infty \), which gives
\[ r_D (\xi_D \to \infty) = 0 \]  
(15)

Recalling the outer boundary condition, Eq. 7, we have
\[ r_D (r_D \to \infty, t_D) = 0 \]  
(7)
or as \( r_D \to \infty \); \( \xi_D \to \infty \), which yields
\[ r_D (\xi_D \to \infty) = 0 \]  
(16)

Where Eqs. 15 and 16 are the same, which illustrates that the Boltzmann transform "collapses" 2 conditions into 1. Combining this observation with the inner boundary condition, we have 2 "boundary" conditions. Coupling this observation with the fact that Eq. 16 is only a function of the Boltzmann variable, \( \xi_D \), we can solve Eq. 16 uniquely. Note that the "collapsing" of the initial and outer boundary conditions must occur or the Boltzmann transform is technically invalid.

Recalling the constant rate inner boundary condition, Eq. 6,
\[ \left[ r_D \frac{d r_D}{d r_D} \right]_{r_D = 1} = -1 \quad \text{or} \quad \left[ r_D \frac{d r_D}{d r_D} \right]_{r_D \to 0} = -1 \]  
(line source condition) (6)
or
\[ \left[ r_D \frac{d r_D}{d r_D} \frac{d \xi_D}{d r_D} \right]_{r_D = 0} = \left[ r_D \left( \frac{2 \xi_D}{r_D} \right) \frac{d \xi_D}{d r_D} \right]_{r_D \to 0} = 2 \left[ \xi_D \frac{d \xi_D}{d r_D} \right]_{r_D \to 0} = -1 \]
which can be rearranged to yield
\[
\left[ \frac{dP}{d\epsilon_0} \right] = \frac{-1}{\epsilon_0 - \epsilon_0 \to 0}
\]

Making the following variable of substitution
\[
v = \frac{dP}{d\epsilon_0}
\]

Substituting Eq. 20 into Eq. 16, and noting the use of ordinary derivatives
\[
\frac{dv}{d\epsilon_0} + \left[ \frac{1}{1 + \frac{1}{\epsilon_0}} \right] v = 0
\]

\[
\frac{1}{v} dv = -\left[ \frac{1}{1 + \frac{1}{\epsilon_0}} \right] d\epsilon_0 = -\frac{d\epsilon_0}{\epsilon_0} - \frac{d\epsilon_0}{\epsilon_0}
\]

Integrating
\[
\ln(v) = -\epsilon_0 - \ln(\epsilon_0) + \beta \quad \beta = \text{constant of integration}
\]

Exponentiating
\[
v = \exp \left[ -\epsilon_0 - \ln(\epsilon_0) + \beta \right]
\]
or
\[
v = \exp \left[ -\epsilon_0 \right] \exp \left[ -\ln(\epsilon_0) \right] \exp [\beta]
\]

which reduces to
\[
v = \frac{\alpha}{\epsilon_0} \exp \left[ -\epsilon_0 \right]
\]

where \(\alpha = \exp[\beta]\), i.e., the constant of integration. Recalling Eq. 20 and combining gives
\[
\frac{dP}{d\epsilon_0} = \frac{\alpha}{\epsilon_0} \exp \left[ -\epsilon_0 \right]
\]

Multiplying through by \(\epsilon_0\) gives
\[
\epsilon_0 dP = \alpha \exp \left[ -\epsilon_0 \right]
\]

Substitution of Eq. 28 into Eq. 19 gives
\[
\lim_{\epsilon_0 \to 0} \left[ \exp \left[ -\epsilon_0 \right] \right] = \frac{-1}{2}
\]

or
\[
\alpha = \frac{-1}{2}
\]

Substitution of Eq. 24 into Eq. 22 gives
\[
\frac{dP}{d\epsilon_0} = \frac{-1}{2\epsilon_0} \exp \left[ -\epsilon_0 \right]
\]
Separating and integrating Eq. 25 gives
\[ \int_{p_0 = 0}^{p_0} dp_0 = -\frac{1}{2} \int_{\xi = 0}^{\xi} \frac{1}{\xi} e^{-\xi} d\xi \]
where we note that \( p_0 = 0 \) at \( \xi = \infty \) is the initial outer boundary condition. Completing the integration and reversing the limits we have
\[ p_0 = \frac{1}{2} \int_{\xi = 0}^{\infty} \frac{1}{\xi} e^{-\xi} d\xi \tag{26} \]
We note that the integral in Eq. 26 is the exponential integral, \( E_1(x) \), which is given by
\[ E_1(x) = \int_x^{\infty} \frac{1}{y} e^{-y} dy \tag{27} \]
Combining Eqs. 26 and 27 gives our final result
\[ p_0(r_0, t_0) = \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) \tag{28} \]
Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Boltzmann Transform Identities

Definition of Boltzmann Variable for Transient Radial Flow

Recall that the Boltzmann transform variable, $\xi$, is defined as

$$\xi = r^b t^c \quad (1)$$

But what are the $a, b, c$ constants? How do we define these constants?

Starting with the original form of the radial flow diffusivity equation, we have

$$\frac{k}{k_x} \left[ \frac{\partial \phi}{\partial r_x} \right] + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial t} \quad (2)$$

Combining Eqs. 1 and 2 we have (without details)

$$\frac{k^2}{k_x^2} \frac{\partial^2 \xi}{\partial r_x^2} + \frac{1}{(r_x \phi_x)^2} \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r_x} \frac{\partial \xi}{\partial r_x} - \frac{1}{(r_x \phi_x \phi_x')^2} \frac{\partial \xi}{\partial r} = 0 \quad (3)$$

Taking $\frac{\partial \xi}{\partial r_x}, \frac{\partial^2 \xi}{\partial r_x^2}, \frac{\partial^2 \xi}{\partial r^2}$ we have

$$\frac{\partial \xi}{\partial r_x} = \frac{\partial}{\partial r_x} (a r^b t^c) = a r_x^b \frac{\partial t^c}{\partial x} = a r_x^b t_x^c \frac{t}{t_x}$$

or

$$\frac{\partial \xi}{\partial r_x} = \frac{c \xi}{t_x} \quad (4)$$

$$\frac{\partial^2 \xi}{\partial r_x^2} = \frac{\partial}{\partial r_x} \left( a r_x^b \frac{\partial t^c}{\partial r_x} \right) = a t_x^c \frac{\partial}{\partial r_x} \left( b r_x^{b-1} \right) = a t_x^c b \frac{r_x^{b-1}}{r_x} \quad (5)$$

or

$$\frac{\partial^2 \xi}{\partial r_x^2} = \frac{b \xi}{r_x} \quad (6)$$

$$\frac{\partial^2 \xi}{\partial r^2} = \frac{\partial}{\partial r} \left( a r_x^b \frac{\partial t^c}{\partial r} \right) = a t_x^c b \frac{b-1}{r_x} \frac{r_x^{b-2}}{r_x^2}$$

which gives

$$\frac{\partial^2 \xi}{\partial r^2} = \frac{a r_x^b t_x^c}{r_x} \frac{b(b-1)}{r_x^2} \quad (7)$$

or

$$\frac{\partial^2 \xi}{\partial r^2} = \frac{b(b-1) \xi}{r_x^2} \quad (8)$$

Recalling the multiplier term in Eq. 5 gives

$$\frac{1}{(r_x \phi_x \phi_x')^2} \frac{\partial \xi}{\partial r} = \frac{1}{r_x} \frac{\partial \xi}{\partial r_x} - \frac{1}{(r_x \phi_x)^2} \frac{\partial \xi}{\partial r} \quad (9)$$
Substituting Eqs. 4-6 into Eq. 7 we have
\[
\frac{1}{(b\epsilon_0/l_0)^2} \frac{b(b-1)\epsilon_0}{\epsilon_0} + \frac{1}{b \epsilon_0} \frac{1}{(b\epsilon_0/l_0)} = \frac{1}{(b\epsilon_0/l_0)^2} \frac{C}{\epsilon_0}
\]
or
\[
\frac{1}{(b^2\epsilon_0^2/l_0^2)} \frac{b(b-1)\epsilon_0}{\epsilon_0} + \frac{1}{b \epsilon_0} \frac{1}{(b\epsilon_0/l_0)} = \frac{1}{(b^2\epsilon_0^2/l_0^2)} \frac{C}{\epsilon_0}
\]
reducing
\[
\frac{b-1}{b} \frac{1}{\epsilon_0} + \frac{1}{b} \frac{1}{\epsilon_0} - \frac{c}{b^2 \epsilon_0} \frac{1}{\epsilon_0} \quad (8)
\]
Eq. 8 still remains in terms of \(r_0\) and \(t_0\), rather than only in terms of \(\epsilon_0\). In particular, the last term is of interest due to the \(r_0\) and \(t_0\) terms - and how do we "convert" these terms into an \(\epsilon_0\) term?

Looking at the last term in Eq. 8 we have
\[
-\frac{c}{b^2 \epsilon_0} \frac{1}{\epsilon_0} = -\frac{c}{b^2 \epsilon_0^2} \frac{1}{\epsilon_0} = \frac{-c}{ab^2} \frac{1}{\epsilon_0^2} \frac{1}{\epsilon_0} \quad (9)
\]
We will attempt to determine \(a\), \(b\), and \(c\) by setting the entire term equal to 1, then eliminating the \(r_0\) and \(t_0\) terms by establishing the constants \(a\) and \(c\). Systematically we have
\[
\frac{r_0^2}{b^2} = 1 \quad \text{if } b = 2 \quad \text{and} \quad \frac{t_0^{c^{-1}}}{t_0^{c}} = 1 \quad \text{if } c = -1
\]
Setting the entire term equal to unity, we obtain
\[
\frac{-c}{ab^2} \frac{1}{\epsilon_0^2} \frac{1}{\epsilon_0} = 1
\]
Assuming the following conditions
\[
b = 2 \quad \text{(10)} \quad c = -1 \quad \text{(11)}
\]
and substituting Eqs. 10 and 11 into Eq. 9
\[
\frac{1}{a(2)^2} \cdot \frac{r_0^{2}}{t_0^{c^{-1}}} \cdot \frac{1}{\epsilon_0^2} = 1
\]
or
\[
a = \frac{1}{4}
\]
And the final form of Eq. 8 is given by
\[
\frac{2-1}{2} \frac{1}{\epsilon_0} + \frac{1}{2} \frac{1}{\epsilon_0} + 1 = \left[ \frac{1}{\epsilon_0} + 1 \right]
\]
Making the final equality of Eqs. 7 and 13 we have
\[
\frac{1}{(\frac{de}{df})^2} \frac{d^2\epsilon_0}{df^2} + \frac{1}{e_0} \frac{1}{(\frac{de}{df})} \frac{d\epsilon_0}{df} - \frac{1}{(\frac{de}{df})^2} \frac{d^2\epsilon_0}{df^2} = 1 + \frac{1}{e_0}
\] (14)

Combining Eqs. 3 and 14 gives
\[
\frac{d^2\epsilon_0}{df^2} + \left[ \frac{1}{e_0} + \frac{1}{\epsilon_0} \right] \frac{d\epsilon_0}{df} = 0
\] (15)

where the following definitions are used:
\[
\epsilon_0 = a + b \epsilon_0 + c
\] (1)
\[
a = \frac{1}{4}
\] (12)
\[
b = 2
\] (10)
\[
c = -1
\] (11)

Substituting Eqs. 10-12 into Eq. 1, we have
\[
\epsilon_0 = \frac{r_0^2}{4\epsilon_0}
\] (16)

which is the basis for the application of the Boltzmann transform to the radial flow diffusivity equation.
Solution of the Dimensionless Radial Flow Diffusivity Equation—Infinite-Acting Reservoir Case

- Real Domain Solution via Inversion of the Laplace Transform Solution

Radial Flow Solution for an Infinite-Acting Homogeneous Reservoir: Laplace Transform Approach

The basic partial differential equation (i.e., the diffusivity equation) is given in dimensionless form by

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{d^2 \phi}{dr_0^2} \quad (1) \text{ or } \frac{1}{r_0} \int \frac{d}{dr_0} \left[ r_0 \frac{d \phi}{dr_0} \right] = \frac{d^2 \phi}{dr_0^2} \quad (2)$$

where

$$r_0 = \frac{r}{L} \quad (3) \quad \Phi = \frac{\Phi_c}{u} \left( \Phi_s - \Phi_p \right) \quad (4) \quad t_0 = \frac{t}{u} \frac{k}{\mu} \quad (5)$$

where \( t_0 \) and \( \Phi_c \) are given by

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<td>5.356 \times 10^{-4}</td>
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The initial condition is given as

$$\Phi (r_0, t_0 = 0) = 0 \quad \text{(uniform pressure distribution)} \quad (6)$$

The constant rate inner boundary condition is

$$\left[ r_0 \frac{d \Phi}{dr_0} \right]_{r_0 = 1} = 0 \quad \text{(constant flow rate at the well)} \quad (7)$$

The "infinite-acting" outer boundary condition is given by

$$\Phi (r_0 \rightarrow \infty, t) = 0 \quad (8)$$

Laplace Transform Formulation: \( \bar{\Phi} = \mathcal{L} \left( \Phi (r_0, t_0) \right) \)

Taking the Laplace transform of Eq. 2 gives

$$\frac{1}{r_0} \int \frac{d}{dr_0} \left[ r_0 \frac{d \Phi}{dr_0} \right] = 5 \bar{\Phi} - \Phi (t_0 = 0) \quad \left[ \mathcal{L} \left( \frac{d (x(t))}{dt} \right) = \frac{d \Phi (s)}{dr_0} \right] \quad (9)$$

We recognize immediately from Eq. 6 that \( \Phi (t_0 = 0) = 0 \), combining Eqs. 6 and 9 we obtain

$$\frac{1}{r_0} \int \frac{d}{dr_0} \left[ r_0 \frac{d \bar{\Phi}}{dr_0} \right] = 5 \bar{\Phi} \quad (10)$$

Taking the Laplace transform of the inner boundary condition gives

$$\left[ r_0 \frac{d \bar{\Phi}}{dr_0} \right]_{r_0 = 1} = - \frac{1}{5} \quad (11)$$

Taking the Laplace transform of the outer boundary condition gives

$$\bar{\Phi} (r_0 \rightarrow \infty, s) = 0 \quad (12)$$
multiplying through Eq. 10 by $r_0^2$ we have
\[ r_0 \frac{d}{dr_0} \left[ r_0 \frac{dr_0}{ds} \right] = 5 r_0^2 \beta \]
we will define a variable of substitution, $\beta$, as
\[ \beta = \sqrt{r_0} \]
or
\[ r_0 = \frac{1}{\sqrt{\beta}} \]
Applying the chain rule on the $d(r)/dr_0$ terms in Eq. 13 we have
\[ r_0 \frac{ds}{dr_0} \frac{d}{ds} \left[ r_0 \frac{ds}{dr_0} \frac{dr_0}{ds} \right] = 5 r_0^2 \beta \]
where,
\[ \frac{ds}{dr_0} = \frac{1}{\beta} \]
Substituting Eqs. 15 and 17 into Eq. 16 we have
\[ \frac{\beta}{\sqrt{\beta}} \frac{d}{ds} \left[ \frac{\beta}{\sqrt{\beta}} \frac{dr_0}{ds} \right] = \beta \frac{dr_0}{ds} \]
Cancelling the $\sqrt{\beta}$ terms on the left-hand-side we have
\[ \beta \frac{d}{ds} \left[ \frac{d}{ds} \frac{dr_0}{ds} \right] = \beta \frac{dr_0}{ds} \]
Expanding the left-hand-side terms we have
\[ \beta \frac{d^2}{ds^2} \frac{dr_0}{ds} + \frac{d}{ds} \frac{dr_0}{ds} = \beta \frac{dr_0}{ds} \]
From Abramowitz & Stegun, Handbook of Mathematical Functions (p.374, Eq. 9.6.1) the modified Bessel differential equation is given by
\[ \beta \frac{d^2}{ds^2} \frac{dw}{ds} + \frac{d}{ds} \frac{dw}{ds} = (\beta^2 + \nu^2) \frac{dw}{ds} \]
The general solution of Eq. 20 is given by
\[ \frac{dw}{ds} = A J_\nu (\beta) + B K_\nu (\beta) \]
where $J_\nu (\beta)$ and $K_\nu (\beta)$ are the modified Bessel functions of the first and second kinds, respectively. Comparing Eqs. 19 and 20 we find that $\nu = 0$, which yields the general solution of our diffusivity equation.

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In order to develop the "particular" solution (i.e., solve for the $A$ and $B$ parameters), we need to determine $d\tilde{p}/dr$. Using the chain rule we have

$$\frac{d\tilde{p}}{dr} = \frac{dz}{dr} \frac{d\tilde{p}}{dz}$$  (24)

Substituting Eq. 17 into Eq. 24 we obtain

$$\frac{d\tilde{p}}{dr} = \sqrt{\lambda} \frac{d\tilde{p}}{dz}$$  (25)

and the $d\tilde{p}/dz$ term is given by

$$\frac{d\tilde{p}}{dz} = A \frac{dI_0(z)}{dz} + B \frac{dk_0(z)}{dz}$$  (26)

From Abramowitz and Stegun, Handbook of Mathematical Functions, we have

$$\frac{dI_0(z)}{dz} = I_1(z)$$  (Eq. 9.1.27, p. 376)  (27)

$$\frac{dk_0(z)}{dz} = -k_1(z)$$  (Eq. 9.1.27, p. 376)  (28)

Substituting Eqs. 27 and 28 into Eq. 26 we have

$$\frac{d\tilde{p}}{dz} = A I_1(z) - B k_1(z)$$  (29)

Combining Eqs. 25 and 29, and using $z = \sqrt{\lambda} r_0$, we obtain

$$\frac{d\tilde{p}}{dr} = A \sqrt{\lambda} I_1(\sqrt{\lambda} r_0) - B \sqrt{\lambda} k_1(\sqrt{\lambda} r_0)$$  (30)

Summarizing our efforts so far we have

**Laplace Transform of Inner Boundary Condition**

$$\left[ \begin{array}{c} r_0 \frac{d\tilde{p}}{dr} \\ \frac{d\tilde{p}}{dr} \end{array} \right] \bigg|_{r_0=1} = \frac{-1}{5}$$  (11)

**Laplace Transform of Outer Boundary Condition**

$$\tilde{p}_0(r_0 \to \infty, \lambda) = 0$$  (12)

**General Solution in Laplace Domain**

$$\tilde{p}_0(\lambda, r_0) = A I_0(\sqrt{\lambda} r_0) + B k_0(\sqrt{\lambda} r_0)$$  (23)

**Radial Derivative of the General Solution in Laplace Domain**

$$\frac{d\tilde{p}}{r_0} = A \sqrt{\lambda} I_1(\sqrt{\lambda} r_0) - B \sqrt{\lambda} k_1(\sqrt{\lambda} r_0)$$  (30)\text{or} \frac{d\tilde{p}}{dr_0} = A \sqrt{\lambda} r_0 I_1(\sqrt{\lambda} r_0) - B \sqrt{\lambda} r_0 k_1(\sqrt{\lambda} r_0)  (31)$$
Combining Eqs. 11 and 31 for the \textit{inner boundary condition} at \( r_0 \), we have
\[
A \sqrt{\gamma} I_0(\sqrt{\gamma} r_0) - B \sqrt{\gamma} k_0(\sqrt{\gamma} r_0) = -\frac{1}{5}
\]  
(32)

Combining Eqs. 12 and 28 for the \textit{outer boundary condition} we obtain
\[
\lim_{r_0 \to 0} \left[ A I_0(\sqrt{\gamma} r_0) + B k_0(\sqrt{\gamma} r_0) \right] = 0
\]  
(33)

The conventional approach would be to solve Eqs. 32 and 33 simultaneously to determine the \( A \) and \( B \) parameters; however, the \( r_0 \to \infty \) condition simplifies matters somewhat. In particular, we consider the behavior of the following terms
\[
\lim_{z \to \infty} I_0(z) = 0
\]
and
\[
\lim_{z \to \infty} k_0(z) = 0
\]

Considering the behavior illustrated above, if \( \lim_{z \to \infty} I_0(z) = 0 \) and \( \lim_{z \to \infty} k_0(z) = 0 \), then from Eq. 33 we find that \( A = 0 \) and by Eq. 33 \( B \) is indeterminate, i.e., \( B(0) = 0 \). Therefore, \( B \) is determined from the inner boundary condition. Using \( A = 0 \) and solving Eq. 32 for \( B \) we have
\[
B \sqrt{\gamma} k_0(\sqrt{\gamma}) = -\frac{1}{5}
\]
or
\[
B = \frac{1}{5} \frac{1}{\sqrt{\gamma} k_0(\sqrt{\gamma})}
\]  
(34)

and of course
\[
A = 0
\]  
(35)

Substituting Eqs. 34 and 35 into the general solution (Eq. 23) we obtain the particular solution which is given as
\[
I_0(\sqrt{\gamma} r_0) = -\frac{1}{5} \frac{k_0(\sqrt{\gamma} r_0)}{\sqrt{\gamma} k_0(\sqrt{\gamma})}
\]  
(36)

Eq. 36 is the so-called "cylindrical" source solution. Unfortunately, the quotient of \( k_0(\sqrt{\gamma} r_0) / (\sqrt{\gamma} k_0(\sqrt{\gamma})) \) cannot be inverted directly—except by residue methods as illustrated by van Everdingen and Hurst.
The van Everdingen and Hurst results are given as

\[ p_0 (r_0, t_0) = 2 \int_0^\infty \left[ 1 - \exp(-m^2 t_0) \right] \frac{J_0(m r_0) - Y_1(m r_0) J_0(m r_0)}{m^2 \left[ J_1^2(m) + Y_1^2(m) \right]} \, dm \]  

and for the wellbore solution (i.e., \( r_0 = 1 \)) we have

\[ p_0 (r_0=1, t_0) = 4 \int_0^\infty \left[ 1 - \exp(-m^2 t_0) \right] \, dm \]

However, neither Eq. 37 nor Eq. 38 is computationally efficient -- and for general applications we recommend numerical inversion of Eq. 36.

Line Source Solution:

Starting with the cylindrical source solution we will develop a line source solution by simplifying the denominator term. Recalling the cylindrical source solution we have

\[ p_0 (r_0, s) = \frac{1}{5} \frac{k_0 (s r_0)}{\sqrt{s} k_1 (s r_0)} \]

Looking at the denominator term we have

\[ \text{denom} = \sqrt{s} k_1 (s r_0) \]

And recalling that the Laplace transform parameter, \( s \), and the dimensionless time function, \( t_0 \), are inversely proportional we have

\[ s \propto \frac{1}{t_0} \]

As a matter of reducing the denominator term we will consider the "large" time (i.e., small \( s \)) behavior of the \( \sqrt{s} k_1 (s r_0) \) term. In particular,

\[ \lim_{s \to 0} \sqrt{s} k_1 (s r_0) \]

From Abramowitz and Stegun, Handbook of Mathematical Functions, p. 575, Eq. 9.6.9 (for \( \nu = 1 \)) we have

\[ k_1 (s) = \frac{1}{s} \quad \text{for} \quad s \to 0 \]

Multiplying through by \( s \) we have

\[ s k_1 (s) = 1 \quad \text{for} \quad s \to 0 \]

Which for our case is given by

\[ \text{denom} = \sqrt{s} k_1 (s r_0) = 1 \quad \text{for} \quad \sqrt{s} \ (\text{or} \ s) \to 0 \]
Applying this behavior to Eq. 36 we have
\[ \bar{\rho}_b(s, s) = \frac{1}{s} k_0(\sqrt{s} r_0) \] as \( s \to 0 \) (the source solution) \( (39) \)

However, this \( s \to 0 \) condition is not as restrictive as it might seem. For example, Eq. 39 has been shown to be equal to the cylindrical source solution for \( t_0 \geq 10 \).

But the question is how do we obtain the inverse of Eq. 39? The short answer is to simply use Laplace transform tables, but what if Eq. 39 or its exact form are not listed? Then we can proceed to the method of residues, which will yield an infinite integral or series solution, see van Everdingen and Hurst or Carslaw and Jaeger. Or we can simply use a variety of numerical methods to obtain \( \rho_b(r_0, t_0) \) values.

The fact is that Eq. 39 can be inverted analytically to yield a compact (non-infinite series), closed form solution. Our first effort will be a simple table lookup in a relatively obscure reference. Our second effort will also involve a table lookup for part of the inversion, but here we will use a fundamental theorem of the Laplace transform to develop the final result.

**Case I: Inversion of Eq. 39 by a single table lookup**

<table>
<thead>
<tr>
<th>( \bar{f}(s) )</th>
<th>( f(t) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{s} k_0(\sqrt{s} a) )</td>
<td>( \frac{1}{2} e^\left(\frac{-a^2}{4t}\right) )</td>
<td>Carslaw and Jaeger; Conduction of Heat in Solids, Table V, Eq. 26, p. 445</td>
</tr>
<tr>
<td>( k_0(\sqrt{s} a) )</td>
<td>( \frac{1}{2t} e^\left(-\frac{a^2}{4t}\right) )</td>
<td>Abramowitz and Stegun; Handbook of Mathematical Functions, Table 9.3, Eq. 29.3.120, p. 1028 and Roberts and Kaufman; Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304</td>
</tr>
</tbody>
</table>
Combining the Carslaw and Jeager result for \( \frac{1}{r} \text{e}^{-r^2} \) with the line source solution, Eq. 39, we obtain
\[
P_0(r, t) = \frac{1}{\pi} \frac{\text{e}^{-r^2}}{4t}
\] (40)

**Case 2:** Inversion of Eq. 39 by a table lookup and integration.

Recalling the integration theorem for Laplace transforms we have
\[
f(t) = \int_0^\infty q(s) ds = \mathcal{L}^{-1}\left\{ \frac{1}{t} q(s) \right\}
\] (41)

Combining Eqs. 39 and 41 we obtain
\[
P_0(r, t) = \int_0^\infty \frac{df}{dt} dt_d
\] (42)
or
\[
P_0(r, t) = \int_0^\infty \mathcal{L}^{-1}\left\{ \frac{1}{t} k_0(\sqrt{s} r_0) \right\} dt_d
\] (43)

But what is \( \mathcal{L}^{-1}\left\{ k_0(\sqrt{s} r_0) \right\} \)? From the table on the previous page we find
\[
\mathcal{L}^{-1}\left\{ k_0(\sqrt{s} r_0) \right\} = \frac{1}{2t_0} \exp\left( -\frac{r_0^2}{4t_0} \right) = \frac{df}{dt_0} \text{(note identity)}
\] (44)

Substituting Eq. 44 into Eq. 42 we have
\[
P_0(r, t) = \frac{1}{2} \int_0^\infty \frac{1}{t} \exp\left( -\frac{r_0^2}{4t} \right) dt \text{ (} t \text{ is a dummy variable)}
\] (45)

Introducing a variable of substitution for Eq. 45 we have
\[
m = \frac{r_0^2}{4t} \quad (46) \quad \frac{dm}{dt} = -\frac{r_0^2}{4t} \quad \frac{1}{t} \text{ or } \frac{dt}{m} = -\frac{1}{m} \frac{dm}{t}
\] (47)

where the limits are
- at \( t = 0; m = \infty \)
- at \( t = t_0; m = \frac{r_0^2}{4t_0} \)

These substitutions yield
\[
P_0(r, t) = \frac{1}{2} \int_0^\infty \frac{1}{m} e^{-m} dm
\]
or reversing the limits we have
\[
P_0(r, t) = \frac{1}{2} \int_0^\infty \frac{1}{m} e^{-m} dm
\] (48)
As with the Boltzmann solution, we note that the integral in Eq. 48
is the exponential integral, $E_i(x)$, which is given in Abramowitz and
Stegun, Handbook of Mathematical Functions, Eq. 5.1.1, p. 238. This
relation is
\[ E_i(x) = \int_x^{\infty} \frac{1}{m} e^{-m} \, dm \quad (49) \]
Combining Eqs. 48 and 49 we obtain
\[ P_D(x, t_D) = \frac{1}{z} E_i\left( \frac{4 \, t_D}{v_D^2} \right) \quad (50) \]

The "log" Approximation: Real Domain Approach
The application of Eq. 50 is often hampered by the tedious
computational nature of the exponential integral function, $E_i(x)$.
Consider the infinite series form of $E_i(x)$ given by
\[ E_i(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \, n!} \quad (51) \]
\[ \gamma = 0.577216 \ldots \text{ Euler's constant} \]
where for $x \leq 0.01$ we can neglect the infinite series term, which gives
\[ E_i(x) = -\gamma - \ln x = \ln \left( \frac{1}{e^x} \right) \quad \text{for } x \leq 0.01 \quad (52) \]
Substituting Eq. 52 into Eq. 50 we obtain
\[ P_D(x, t_D) = \frac{1}{z} \ln \left( \frac{4 \, t_D}{v_D^2} \right) \quad \text{for } t_D \geq 25 \text{ (or } \frac{4 \, t_D}{v_D^2} \geq 100) \quad (53) \]

The "log" Approximation: Laplace Domain Approach
Recalling the line source solution in the laplace domain, Eq. 34, we have
\[ P_D(x, s) = \frac{1}{s} k_0 \left( \frac{v_D^2 \, s}{5} \right) \quad (54) \]
We need an approximation for $k_0(s)$ as $s \rightarrow 0$. From Abramowitz and Stegun,
Handbook of Mathematical Functions, Eq. 9.6.13, p. 375 we have
\[ k_0(s) = -\left[ \ln \left( \frac{1}{s} \right) + \gamma \right] \, i_0(s) + \frac{1}{(1+1/2) \, (2s)^2} \, \left[ \frac{1}{4} \, \frac{1}{2} \right] \, \left[ \frac{1}{2} \right] \, i_0(s) + \ldots \]
where we note that the $\frac{1}{2}$ terms diminish as $s \rightarrow 0$ so we are left with
\[ k_0(s) \approx -\left[ \ln \left( \frac{v_D^2 \, s}{2} \right) \right] \, i_0(s) \]
which can also be written as
\[ k_0(z) \approx \ln\left(\frac{z}{e^z} \frac{1}{z}\right) I_0(z) \]

or multiplying and dividing by \( z \) we have
\[ k_0(z) \approx \frac{1}{z} \ln\left(\frac{z}{e^z} \frac{1}{z}\right) I_0(z) \]

using the rules of logarithms we have \( a \ln(x) = \ln(x^a) \), which gives
\[ k_0(z) \approx \frac{1}{z} \ln\left(\frac{4}{e^{2z}} \frac{1}{z^2}\right) I_0(z) \quad \text{as } z \to 0 \]  

(54)

But what is the behavior of \( I_0(z) \) as \( z \to 0 \)? Using the series representation from Abramowitz and Stegun, *Handbook of Mathematical Functions*, Eq. 9.6.12, p. 375, we have
\[ I_0(z) = 1 + \frac{1}{(1)!} \left[\frac{1}{4} \cdot \frac{z^2}{1}\right] + \frac{1}{(2)!} \left[\frac{1}{4} \cdot \frac{z^2}{2}\right]^2 + \frac{1}{(3)!} \left[\frac{1}{4} \cdot \frac{z^2}{3}\right]^3 + \ldots \]

where the behavior is \( I_0(z) \approx 1 \) as \( z \to 0 \)

(55)

Combining Eqs. 54 and 55 we have
\[ k_0(z) \approx \frac{1}{z} \ln\left(\frac{4}{e^{2z}} \frac{1}{z^2}\right) \quad \text{as } z \to 0 \]  

(56)

Substituting Eq. 56 into Eq. 54 gives
\[ k_0(\rho_0, s) = \frac{1}{s} \ln\left(\frac{4}{e^{2\rho_0}} \frac{1}{\rho_0^2 s}\right) \quad \text{as } s \to 0 \ (\text{large } \rho_0) \]  

(57)

Rearranging we have
\[ \frac{1}{s} \ln\left(\frac{1}{\rho_0^2 s}\right) + \frac{1}{s} \ln\left(\frac{4}{e^{2\rho_0}} \frac{1}{\rho_0^2 s}\right) \]

or
\[ \frac{1}{s} \ln(s) + \frac{1}{s} \ln\left(\frac{4}{e^{2\rho_0}} \frac{1}{\rho_0^2 s}\right) \]

(58)

The inverse Laplace transform of the \( \ln(s) \) term is given by
\[ \frac{-1}{s} \ln(s) = \ln(s) \]

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; \ln(t)</td>
<td>Abramowitz and Stegun: <em>Handbook of Mathematical Functions</em>, Table 29.3, Eq. 29.3.98, p. 1027 and Roberts and Kaufman: <em>Table of Laplace Transforms</em>, Section 2, Eq. 4.1, p. 258</td>
</tr>
</tbody>
</table>
Term-by-term inversion of Eq. 58 gives
\[ \phi_b(r_0, t_0) = \frac{1}{2} \ln \left( e^{r_0} \right) + \frac{1}{2} \ln \left( \frac{4}{e^{2r_0}} \frac{1}{r_0^2} \right) \]
Collecting terms we have
\[ \phi_b(r_0, t_0) = \frac{1}{2z} \ln \left( \frac{4}{e^{2r_0}} \frac{t_0}{r_0^2} \right) \]  \hspace{1cm} (59)
where Eqs. 58 and 59 are exactly equivalent.

We can also establish a direct inversion formula for logarithmic functions by comparison of Eqs. 57 and 59. Recalling Eq. 57 we have
\[ \phi_b(r_0, s) = \frac{1}{2s} \ln \left( \frac{4}{e^{2r_0}} \frac{1}{r_0^2} \frac{1}{s} \right) \]  \hspace{1cm} (57)
Multiplying Eq. 57 by the Laplace transform parameter, \( s \), and rearranging the terms inside the logarithm we have
\[ s \phi_b(r_0, s) = \frac{1}{2} \ln \left( \frac{4}{e^{2r_0}} \frac{1}{r_0^2} \frac{1}{e^{s}} \right) \]  \hspace{1cm} (60)
Equating the right-hand-sides of Eqs. 59 and 60 we obtain
\[ t_0 = \frac{1}{e^{r_0}} \]
or more importantly, we can solve for \( s \) in terms of \( t_0 \) as
\[ s = \frac{1}{e^{r_0} t_0} \] \hspace{1cm} (61)
We also establish that
\[ \phi_0(r_0, t_0) \approx s \phi_b(r_0, s) \bigg|_{s = \frac{1}{e^{r_0} t_0}} \] \hspace{1cm} (62)
where Eq. 62 is called the "Schaperly" inversion formula for logarithms. This result may be of use for modelling and data analysis, but is only strictly valid for radial flow - so applications may be limited.
**Summary of Results:**

**Laplace Domain Solutions**

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{p}_0 ) general solution</td>
<td>( \tilde{p}_0 (r_0, s) = A \pi_0 (s^2 r_0^2) + B k_0 (s^2 r_0^2) )</td>
</tr>
<tr>
<td>( \rho_0 (d \tilde{p}_0 /d \nu_0) ) general solution</td>
<td>( \frac{d \tilde{p}_0}{d \nu_0} (r_0, s) = A \sqrt{s} r_0 I_1 (s r_0) + B \sqrt{s} r_0 k_1 (s r_0) )</td>
</tr>
</tbody>
</table>

**Cylindrical Source Solution**

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{p}_0 (r_0, s) = \frac{1}{s} k_0 (s^2 r_0^2) )</td>
<td></td>
</tr>
</tbody>
</table>

**Line Source Solution**

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{p}_0 (r_0, s) = \frac{1}{s} k_0 (s^2 r_0^2) \approx \frac{1}{2s} \left( \frac{1}{e^s} \frac{1}{s} \right) )</td>
<td></td>
</tr>
</tbody>
</table>

**Real Domain Solutions**

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical Source Solution</td>
<td>( \rho_0 (r_0, t_0) = \mathcal{L}^{-1} \left[ \frac{1}{s} \frac{k_0 (s^2 r_0^2)}{s^2 k_1 (s^2 r_0^2)} \right] )</td>
<td>not invertable to closed form</td>
</tr>
<tr>
<td>Line Source Solution</td>
<td>( \rho_0 (r_0, t_0) = \frac{1}{2} \frac{e^{-r_0^2/4t_0}}{4\pi t_0} )</td>
<td>( t_0 &gt; 10 )</td>
</tr>
<tr>
<td>Derivative of Line Source Solution</td>
<td>( \frac{d \rho_0 (r_0, t_0)}{dt_0} = \frac{1}{2t_0} \frac{e^{-r_0^2/4t_0}}{4\pi t_0} )</td>
<td>( t_0 &gt; 10 )</td>
</tr>
<tr>
<td>Log Approximation</td>
<td>( \rho_0 (r_0, t_0) = \frac{1}{2} \ln \left( \frac{4}{e^s} \frac{t_0}{r_0^2} \right) )</td>
<td>( \frac{t_0}{r_0^2} &gt; 25 )</td>
</tr>
</tbody>
</table>

"Schapery" Inversion Formula

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_0 (r_0, t_0) \approx \frac{1}{2} \tilde{p}_0 (r_0, s) \left</td>
<td>s = \frac{1}{e^s} \right. )</td>
</tr>
</tbody>
</table>

**Comments**

- not invertable to closed form
- \( t_0 > 10 \)
- \( t_0 > 10 \)
- \( \frac{t_0}{r_0^2} > 25 \)
- only for radial flow