Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2b — Laplace Transform Solutions
of the Radial Flow Diffusivity Equation for a Bounded Circular Reservoir:
Infinite and Finite-Acting Reservoir Cases

It is bad to be oppressed by a minority, but it is worse to be oppressed by a majority.
— Lord Acton (1907)


Objectives: (things you should know and/or be able to do)

- Be able to derive the particular solutions (in Laplace domain) for a well produced at a constant flowrate in a homogeneous reservoir for the following initial condition, subject to the following inner and outer boundary conditions:
  - Initial Condition (Uniform Pressure Distribution)
    \[ p_D(r_D, t_D \leq 0) = 0 \]
  - Inner Boundary Condition (Constant Flowrate at the Well)
    \[ r_D \frac{\partial p_D}{\partial r_D} \bigg|_{r_D = 1} = -1 \]
  - Outer Boundary Conditions
    a. "Infinite-Acting" Reservoir
      \[ p_D(r_D \to \infty, t_D) = 0 \quad \text{(No reservoir boundary)} \]
    b. "Prescribed Flux" at the Boundary
      \[ r_D \frac{\partial p_D}{\partial r_D} \bigg|_{r_D = r_{eD}} = q_{\text{Dex}}(t_D) \quad \text{(Specified flux across the reservoir boundary)} \]
    c. Constant Pressure Boundary
      \[ p_D(r_{eD}, t_D) = 0 \quad \text{(Constant pressure at the reservoir boundary)} \]

Particular Solutions in the Laplace Domain

- "Infinite-acting" reservoir behavior: "cylindrical source" solution
  \[ \bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\sqrt{u} r_D)}{\sqrt{u} K_1(\sqrt{u})} \]
- "Infinite-acting" reservoir behavior: "line source" solution
  \[ \bar{p}_D(r_D, u) = \frac{1}{u} K_0(\sqrt{u} r_D) \quad \text{(where } \sqrt{u} K_1(\sqrt{u}) \to 1; \text{ for } \sqrt{u} \to 0) \]
- "Infinite-acting" reservoir behavior: "log approximation" solution
  \[ \bar{p}_D(r_D, u) = \frac{1}{u} K_0(\sqrt{u} r_D) = \frac{1}{2u} \ln \left[ \frac{4}{\epsilon^2 \gamma \beta^2 u} \right] (\gamma = 0.577216 \ldots \text{ Euler's Constant}) \]

- Bounded circular res. — "no-flow" at the outer boundary (i.e., \( q_{\text{Dex}}(t_D) = 0 \))
  \[ \bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\sqrt{u} r_D) I_1(\sqrt{u} r_{eD}) + K_1(\sqrt{u} r_{eD}) I_0(\sqrt{u} r_D)}{\sqrt{u} K_1(\sqrt{u}) I_1(\sqrt{u} r_{eD}) - \sqrt{u} I_1(\sqrt{u}) K_1(\sqrt{u} r_{eD})} \]

- Bounded circular reservoir — "constant pressure" at the outer boundary
  \[ \bar{p}_D(r_D, u) = \frac{1}{u} \frac{K_0(\sqrt{u} r_D) I_0(\sqrt{u} r_{eD}) - K_0(\sqrt{u} r_{eD}) I_0(\sqrt{u} r_D)}{\sqrt{u} K_1(\sqrt{u}) I_0(\sqrt{u} r_{eD}) + \sqrt{u} I_1(\sqrt{u}) K_0(\sqrt{u} r_{eD})} \]
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Objectives: (Continued)

- Particular Solutions in the Laplace Domain (Continued)
  - Bounded circular reservoir — "prescribed flux" at the outer boundary

\[
\tilde{p}_D(r_D,u) = \frac{1}{u} \cdot \bar{K}_0(\bar{v}r_D) I_1(\bar{v}r_D) + \frac{K_1(\bar{v}r_D)}{\bar{v}u} I_0(\bar{v}r_D) - \frac{1}{u} q_{Dext}(u) \left[ \frac{u}{\bar{v}r_D} \cdot \bar{K}_0(\bar{v}r_D) I_1(\bar{v}r_D) + I_0(\bar{v}r_D) \bar{v}u \cdot K_1(\bar{v}r_D) \right] \]

Lecture Outline: (Continued)

- General Approach to Laplace Transform Solutions:
  - Develop Bessel's modified differential equation from the Laplace transform of the diffusivity equation.

**Dimensionless Diffusivity Equation**

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial \tilde{p}_D}{\partial r_D} \right] = \frac{\partial \tilde{p}_D}{\partial r_D} + \frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{d\tilde{p}_D}{dr_D} \right] = u \tilde{p}_D
\]

**Transformed Diffusivity Equation:** (Modified Bessel Function form)

\[
z^2 \frac{d^2 \tilde{p}_D}{dz^2} + z \frac{d \tilde{p}_D}{dz} = z^2 \tilde{p}_D \text{ where } z = \bar{v}r_D.
\]

- Write the appropriate general solution and take the derivative with respect to \(r_D\) of the general solution. These results are:
  - General solution: (Bessel's modified differential equation)
    \[
    \tilde{p}_D(r_D,u) = A I_0(\bar{v}r_D) + B K_0(\bar{v}r_D)
    \]
  - Spatial (Radial) Derivative of the General Solution:
    \[
    \left[ r_D \frac{d \tilde{p}_D}{dr_D} \right]_{r_D} = A \bar{v}r_D I_1(\bar{v}r_D) - B \bar{v}r_D K_1(\bar{v}r_D)
    \]

- Boundary Conditions in the Laplace Domain
  - Inner Boundary Condition (Constant Flowrate at the Well)
    \[
    \left[ r_D \frac{d \tilde{p}_D}{dr_D} \right]_{r_D=1} = \frac{1}{u}
    \]
  - Outer Boundary Conditions
    a. "Infinite-Acting" Reservoir
      \[
      \tilde{p}_D(r_D \to \infty, u) = 0 \quad \text{(No reservoir boundary)}
      \]
    b. "No Flow" at the Boundary
      \[
      \left[ r_D \frac{d \tilde{p}_D}{dr_D} \right]_{r_D=r_eD} = 0 \quad \text{(No flow at the reservoir boundary)}
      \]
Lecture Outline: (Continued)

- General Approach to Laplace Transform Solutions: (Continued)
  
  c. Constant Pressure Boundary
  \[ \overline{p}_{D}(r_{eD}, \mu) = 0 \]  
  (Constant pressure at the reservoir boundary)
  
  d. "Prescribed Flux" at the Boundary
  \[ \left[ r_{D} \frac{d\overline{p}_{D}}{dr_{D}} \right]_{r_{D}=r_{eD}} = \mathcal{L}[q_{ext}(t_{D})] = \overline{q}_{ext}(\mu) \]  
  (Specified flux at boundary)

- Establish the first "equation" by equating the inner boundary condition (constant rate at the well) and the radial derivative of the general solution.

- Establish the second "equation" by equating the desired outer boundary condition ("no-flow," "constant pressure," or "prescribed flux") and either the general solution or its radial derivative, as appropriate.

- Using the two equations/two unknowns approach, solve for the particular solution in the Laplace domain (i.e., the \( A \) and \( B \) parameters) and reduce to the most fundamental algebraic form.

Reading Assignment:

- Review attached notes.
- Solution of the Dimensionless Radial Flow Diffusivity Equation:
  - Laplace transform solutions.

Exercises: For your own practice/skills building—do NOT turn in!

Derivation of Solutions in the Laplace Domain:

From the attached notes you are to rederive the following, and show all details.

- Starting from the dimensionless diffusivity equation, derive the Laplace transform solutions for a well produced at a constant flowrate (inner boundary condition) in a homogeneous reservoir with the following outer boundary conditions:

  - "Infinite-acting" reservoir behavior
  - Bounded circular reservoir — "no-flow" at the outer boundary
  - Bounded circular reservoir — "constant pressure" at the outer boundary
  - Bounded circular reservoir — "prescribed flux" at the outer boundary
Exercises: For your own practice/skills building—do NOT turn in!

*Paper Reviews:*

- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- **Problem:**
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?

- **Assumptions and Limitations:**
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?

- **Practical Applications:**
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how could the solutions/results be used in practice?

- **Discussion:**
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.

- **Recommendations/Extensions:**
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Laplace Transform Solutions—Radial Flow Case (SPE 25479)

Log-log Plot: Error in Laplace Transform Solutions—Radial Flow Case (SPE 25479)
Solution of the Dimensionless Radial Flow Diffusivity Equation:

- Laplace Transform Solutions


Petroleum Engineering 620
Fluid Flow in Reservoirs
Solutions for Radial Flow in a Homogeneous Reservoir: Infinite-Acting,
No-Flow, and Constant Pressure Outer Boundaries - Laplace Transform Approach

The fundamental partial differential equation (the diffusivity equation)
is given in dimensionless form by,
\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{1}{\beta D} \frac{\partial^2 \phi}{\partial \tau^2} \tag{1}
\]
or
\[
\frac{1}{\beta D} \left[ \frac{\partial}{\partial \tau} \frac{\partial \phi}{\partial \tau} \right] = \frac{\partial \phi}{\partial \tau_D} \tag{2}
\]
where
\[
\beta = \frac{r}{\tau_D} \quad \phi = \frac{k}{2\mu c_w} \quad \tau = \frac{r^2 \phi}{k} \quad \frac{1}{\beta D} = \frac{2\pi}{\mu c_w r^2} \tag{3}
\]
where \( \tau_D \) and \( \beta \) are given by

<table>
<thead>
<tr>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_D )</td>
<td>( 2.637 \times 10^{-4} )</td>
<td>( 3.557 \times 10^{-6} )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( 7.081 \times 10^{-5} )</td>
<td>( 5.856 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

The initial condition is given as
\[
\phi(r, \tau_D = 0) = 0 \quad \text{(uniform pressure distribution)} \tag{4}
\]
The constant rate inner boundary condition is
\[
\left. \frac{\partial \phi}{\partial \tau_D} \right|_{\tau_D = 1} = -1 \quad \text{(constant flow rate at the well)} \tag{5}
\]
The outer boundary conditions are given by:

a. "Infinite-acting" outer boundary condition
\[
\phi(r = \infty, \tau_D) = 0 \tag{6}
\]
b. "No-flow" outer boundary condition
\[
\left. \frac{\partial \phi}{\partial \tau_D} \right|_{r = \tau_D} = \phi_{\text{ext}} \tag{7}
\]
c. "Constant pressure" outer boundary condition
\[
\phi(r = r_D, \tau_D) = \phi_{\text{ext}} = 0 \quad \text{(constant at initial pressure)} \tag{8}
\]
d. "Specified flux" outer boundary condition
\[
\left. \frac{\partial \phi}{\partial \tau_D} \right|_{r = r_D} = \phi_{\text{ext}}(\tau_D) \tag{9}
\]

Laplace Transform Formulation: \( \mathcal{L}(\phi(r, \tau_D)) \), \( \mu = \) Laplace transform parameter
Taking the Laplace transform of Eq. 2 gives
\[
\frac{1}{\beta D} \left[ \frac{\partial}{\partial \tau} \frac{\partial \phi}{\partial \tau} \right] = \mathcal{L}(\phi(r, \tau_D = 0) \left[ \frac{\partial}{\partial \tau} \right] \mathcal{L}(d\tau) = \frac{d\mathcal{L}(\phi)}{d\tau_D} \tag{10}
\]
We recognize from Eq. 6 that \( y_b(r_b, t=0) = 0 \), combining Eqs. 6 and 12 we obtain
\[
\frac{d}{dr_b} \left[ \frac{v_b d^2 \bar{p}}{d r_b^2} \right] = u_c \bar{p} \tag{13}
\]
Taking the Laplace transform of the inner boundary condition gives
\[
\left[ \frac{v_b}{d r_b} \right]_{r_b=1} = \frac{-1}{u_c} \tag{14}
\]
Taking the Laplace transform of the outer boundary conditions,
a. Laplace transform of the "infinite-acting" outer boundary condition
\[
\bar{p}_b(r_b, \mu) = 0 \tag{15}
\]
b. Laplace transform of the "no-flow" outer boundary condition
\[
\left[ \frac{v_b d \bar{p}}{d r_b} \right]_{r_b=R_0} = 0 \tag{16}
\]
c. Laplace transform of the "constant pressure" outer boundary condition
\[
\frac{v_b}{r_b} \left\{ \Phi_r \left( R_0, \mu \right) \right\} = \frac{P_{ext}}{u_c} = 0 \text{ (constant at initial pressure)} \tag{17}
\]
d. Laplace transform of the "prescribed flux" outer boundary condition
\[
\left[ \frac{v_b d \bar{p}}{d r_b} \right]_{r_b=R_0} = \Phi_r \left( R_0, \mu \right) \tag{18}
\]
Multiplying through Eq. 18 by \( v_b^2 \) we have
\[
\frac{v_b}{r_b} \left[ \frac{v_b d \bar{p}}{d r_b} \right] = u_c \bar{p} \tag{19}
\]
Defining a variable of substitution, \( z \), as follows
\[
z = \sqrt{u_c} v_b \tag{20}
\]
or
\[
r_b = z / \sqrt{u_c} \tag{21}
\]
Applying the chain rule on the \( dv_b/dr_b \) terms in Eq. 19 we obtain
\[
\frac{v_b}{r_b} \frac{d}{dz} \left[ \frac{v_b}{r_b} \frac{d \bar{p}}{dz} \right] = u_c \bar{p} \tag{22}
\]
where
\[
\frac{dv_b}{dz} = \frac{dv_b}{d (\sqrt{u_c} r_b)} = \sqrt{u_c} \tag{23}
\]
Substituting Eqs. 21 and 23 into Eq. 22 we have
\[
\frac{z}{\sqrt{u_c}} \frac{d}{dz} \left[ \frac{z}{\sqrt{u_c}} \frac{d \bar{p}}{dz} \right] = \frac{z^2 \bar{p}}{u_c} \tag{24}
\]
Cancelling the $\sqrt{z}$ terms on the left-hand-side we obtain
\[ \frac{\partial}{\partial z} \left[ \frac{1}{r^2} \frac{\partial v}{\partial r} \right] = \frac{1}{r^2} \frac{\partial v}{\partial r} \]
\text{(24)}

Expanding the left-hand-side terms we have
\[ \frac{1}{r^2} \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{1}{r^2} \frac{\partial v}{\partial r} \]
\text{(25)}

From Abramowitz and Stegun, *Handbook of Mathematical Functions*, (p. 374, Eq. 9.6.1), the modified Bessel differential equation is given by
\[ \frac{d^2 w}{dz^2} + \frac{z}{d^2} dw = (z^2 + 4) w \]
\text{(26)}

The general solution of Eq. 26 is given by
\[ w = A I_0 (z) + B K_0 (z) \]
\text{(27)}

where the functions $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of the first and second kinds, respectively. By inspection, our general solution is
\[ \tilde{v}_0 (z) = A I_0 (z) + B K_0 (z) \]
\text{(28)}

or, substituting $z = \sqrt{v_0}$ (Eq. 20) into Eq. 28 we have
\[ \tilde{v}_0 (v_0, u) = A I_0 (\sqrt{u} v_0) + B K_0 (\sqrt{u} v_0) \]
\text{(29)}

In order to develop our particular solutions (i.e., to solve for the $A$ and $B$ parameters for each set of boundary conditions), we require the $\frac{d \tilde{v}_0}{dr_0}$ term. Using the chain rule we obtain
\[ \frac{d \tilde{v}_0}{dr_0} = \frac{xz}{d \rho} \frac{d \tilde{v}_0}{d \rho} \]
\text{(30)}

Substituting Eq. 23 into Eq. 30
\[ \frac{d \tilde{v}_0}{dr_0} = \sqrt{u} \frac{d \tilde{v}_0}{d \rho} \]
\text{(31)}

and the $\frac{d \tilde{v}_0}{dz}$ term is given by
\[ \frac{d \tilde{v}_0}{dz} = A \frac{d I_0 (z)}{dz} + B \frac{d K_0 (z)}{dz} \]
\text{(32)}

From Abramowitz and Stegun, *Handbook of Mathematical Functions*, we have
\[ \frac{d I_0 (z)}{dz} = I_1 (z) \quad \text{(Eq. 9.6.27, p. 376)} \]
\text{(33)}

\[ \frac{d K_0 (z)}{dz} = -K_1 (z) \quad \text{(Eq. 9.6.27, p. 376)} \]
\text{(34)}
Substituting Eqs. 33 and 34 into Eq. 32 we have
\[
\frac{d\bar{P}}{d\bar{z}} = AI_1(\bar{z}) - BK_1(\bar{z})
\]  
(35)

Combining Eqs. 50 and 55, and substituting \( \bar{z} = \sqrt{\mu} \bar{r}_D \) (Eq. 28) into Eq. 55 we obtain
\[
\frac{d\bar{P}}{d\bar{r}_D} = A \sqrt{\mu} I_1(\sqrt{\mu} \bar{r}_D) - B \sqrt{\mu} K_1(\sqrt{\mu} \bar{r}_D)
\]  
(56)

Multiplying through by \( \bar{r}_D \) gives
\[
\left[ \bar{r}_D \frac{d\bar{P}}{d\bar{r}_D} \right] = A \sqrt{\mu} \bar{r}_D I_1(\sqrt{\mu} \bar{r}_D) - B \sqrt{\mu} \bar{r}_D K_1(\sqrt{\mu} \bar{r}_D)
\]  
(57)

//

Summarizing our efforts so far:

**General Solution in Laplace Domain**
\[
\bar{\Phi}_D(\bar{r}_D, \mu) = A I_0(\sqrt{\mu} \bar{r}_D) + B K_0(\sqrt{\mu} \bar{r}_D)
\]  
(29)

**Radial Derivative of the General Solution in Laplace Domain**
\[
\left[ \bar{r}_D \frac{d\bar{\Phi}_D}{d\bar{r}_D} \right] = A \sqrt{\mu} \bar{r}_D I_1(\sqrt{\mu} \bar{r}_D) - B \sqrt{\mu} \bar{r}_D K_1(\sqrt{\mu} \bar{r}_D)
\]  
(37)

Laplace transform of boundary conditions:

**Inner Boundary Condition** -
\[
\left[ \bar{r}_D \frac{d\bar{P}}{d\bar{r}_D} \right]_{\bar{r}_D = 1} = -\frac{1}{\mu} \quad \text{(constant rate at well)}
\]  
(14)

**Outer Boundary Conditions** -

a. "infinite-acting" reservoir
\[
\bar{P}_B(\bar{r}_D = \infty, \mu) = 0
\]  
(15)

b. "no-flow" outer boundary condition
\[
\left[ \bar{r}_D \frac{d\bar{P}}{d\bar{r}_D} \right]_{\bar{r}_D = \bar{r}_D} = 0
\]  
(16)

c. "constant pressure" outer boundary condition
\[
\bar{P}_B(\bar{r}_D = \bar{r}_D, \mu) = 0
\]  
(17)

d. "prescribed flux" outer boundary condition
\[
\left[ \bar{r}_D \frac{d\bar{P}}{d\bar{r}_D} \right]_{\bar{r}_D = \bar{r}_D} = \bar{\Phi}_{D,\text{ext}}(\mu)
\]  
(18)

Our goal is to use the boundary conditions to determine the A and B parameters. Our first step is to use the constant rate inner boundary condition (Eq. 14) as a starting point then combine this condition with each outer boundary condition.
condition in order to determine \( A \) and \( B \) for each case.

Starting with the inner boundary condition (Eq. 14) and the derivative of the general solution (Eq. 37) we have

\[
A \sqrt{\omega} I_1(\sqrt{\omega} l) - B \sqrt{\omega} K_1(\sqrt{\omega} l) = \frac{-1}{m}
\]

or

\[
A \sqrt{\omega} I_1(\sqrt{\omega} l) - B \sqrt{\omega} K_1(\sqrt{\omega} l) = \frac{-1}{m} \tag{38}
\]

**Outer Boundary Case 1: Infinite-acting reservoir**

Combining Eqs. 29 and 15 we have

\[
\lim_{r \to \infty} \left[ A I_0(\sqrt{\omega} r_o) + B K_0(\sqrt{\omega} r_o) \right] = 0 \tag{39}
\]

Given that we are taking the limit as \( r_o \to 0 \), we must establish the behavior of \( I_0(x \to \infty) \) and \( k_0(x \to \infty) \). Considering the behavior of \( I_0(x) \) and \( k_0(x) \) we have

\[
\lim_{x \to 0^+} I_0(x) = 0
\]

and

\[
\lim_{x \to 0^+} k_0(x) = 0
\]

Since \( I_0(x \to \infty) = \infty \), then \( A(\infty) + B(\infty) = 0 \); therefore \( A = 0 \). In order for the solution to be bounded. Setting \( A = 0 \) we solve Eq. 38 for \( B \), which gives

\[
B = \frac{1}{m} \frac{1}{\sqrt{\omega} K_1(\sqrt{\omega} l)} \tag{40}
\]

and of course

\[
A = 0 \tag{41}
\]

Substituting Eqs. 40 and 41 into the general solution (Eq. 29) we obtain the particular solution for the infinite-acting reservoir case. This result is

\[
I_0(r_o, \omega) = \frac{1}{m} \frac{K_0(\sqrt{\omega} r_o)}{\sqrt{\omega} K_1(\sqrt{\omega} l)} \tag{42}
\]

Eq. 42 is called the cylindrical source solution.
Unfortunately, Eq. 42 is not readily invertible—therefore we will attempt to reduce Eq. 42 into a more usable form. From Abramowitz and Stegun, *Handbook of Mathematical Functions*, p. 375, Eq. 9.6.19 (for $v=1$) we have

$$k_v(x) = \frac{1}{x} \quad \text{for } x \rightarrow 0$$

or, multiplying through by $x$ we have

$$xk_v(x) = 1 \quad \text{as } x \rightarrow 0$$

for our case we have

$$\sqrt{\alpha} k_v(\sqrt{\alpha}) = 1 \quad \text{for } \sqrt{\alpha} \quad \text{(or } \mu) \rightarrow 0$$

Combining this result with Eq. 42 we obtain

$$P \Theta (\rho, \mu) = \frac{1}{\mu} k_v(\sqrt{\alpha} \rho) \quad \text{as } \mu \rightarrow 0 \quad (43)$$

Eq. 43 is called the line source solution and can be inverted directly.

Eq. 43 can be reduced further to yield a logarithmic relation that is commonly referred to as the "log approximation." In order to develop this result we require an approximation for $k_v(x)$ as $x \rightarrow 0$. From Abramowitz and Stegun, *Handbook of Mathematical Functions*, (Eq. 9.6.11, p. 375) we have

$$k_v(x) = \left[ -\frac{\ln \left( \frac{1}{x} \right) + 1}{1} \right] I_v(x) + \frac{1}{(11)^2} \left[ \frac{1}{4} x^2 \right] + \left[ \frac{1}{2} \right] \left[ \frac{1}{11} \right] \left[ \frac{1}{2} \right] \left[ \frac{1}{4} \right] x^2 + \ldots$$

where we note that as $x \rightarrow 0$, then $x^2 \rightarrow 0$, which reduces to

$$k_v(x) \approx \frac{\ln \left( \frac{1}{x} \right)}{e^{z x}} I_v(x) \quad \text{as } x \rightarrow 0$$

or multiplying and dividing by $z$ we have

$$k_v(x) \approx \frac{1}{z} \ln \left( \frac{4}{e^{2z} \frac{1}{x^2}} \right) I_v(x) \quad \text{as } x \rightarrow 0 \quad (44)$$

The behavior of $I_v(x)$ in the vicinity of $x \rightarrow 0$ is obtained using the series representation provided in Abramowitz and Stegun, *Handbook of Mathematical Functions*, (Eq. 9.6.12, p. 375). This expression is

$$I_v(x) = 1 + \frac{1}{(11)^2} \left[ \frac{1}{4} x^2 \right] + \frac{1}{(21)^2} \left[ \frac{1}{4} x^2 \right] + \frac{1}{(31)^2} \left[ \frac{1}{4} x^2 \right] + \ldots$$

where as $x \rightarrow 0$ we have

$$I_v(x) = 1 \quad \text{as } x \rightarrow 0 \quad (45)$$
Combining Eqs. 44 and 45
\[ k_0(x) \approx \frac{1}{2} \ln \left( \frac{4}{e^{2T} x^2} \right) \quad \text{as } x \to 0 \]  
(46)

Substituting Eq. 46 into Eq. 48 we obtain
\[ \bar{P}_0(r_0, \mu) = \frac{1}{2m} \ln \left( \frac{4}{e^{2T} r_0^2 \mu} \right) \quad \text{as } \mu \to 0 \]  
(47)

or in a form more amenable to inversion we have
\[ \bar{P}_0(r_0, \mu) = -\frac{1}{2m} \ln(\mu) + \frac{1}{2m} \ln \left( \frac{4}{e^{2T} r_0^2} \right) \]  
(48)

**Outer Boundary Case 2: No-Flow outer boundary**
Combining Eqs. 37 and 16 we obtain
\[ A \sqrt{r_0} I_1(\sqrt{r_0}) - B \sqrt{r_0} k_1(\sqrt{r_0}) = 0 \]  
(49)

Solving for the B parameter we obtain
\[ B = \frac{A I_1(\sqrt{r_0})}{k_1(\sqrt{r_0})} \]  
(50)

Recalling the inner boundary condition, Eq. 38, we have
\[ A \sqrt{r_0} I_1(\sqrt{r_0}) - B \sqrt{r_0} k_1(\sqrt{r_0}) = -1 \]  
(58)

Combining Eqs. 50 and 58 we obtain
\[ A \sqrt{r_0} I_1(\sqrt{r_0}) - A \sqrt{r_0} k_1(\sqrt{r_0}) I_1(\sqrt{r_0}) = -\frac{1}{m} \]  
or
\[ A \left[ \sqrt{r_0} I_1(\sqrt{r_0}) k_1(\sqrt{r_0}) - \sqrt{r_0} k_1(\sqrt{r_0}) I_1(\sqrt{r_0}) \right] = -\frac{1}{m} k_1(\sqrt{r_0}) \]

or solving for A we have
\[ A = \frac{k_1(\sqrt{r_0})}{m \left[ \sqrt{r_0} k_1(\sqrt{r_0}) I_1(\sqrt{r_0}) - \sqrt{r_0} I_1(\sqrt{r_0}) k_1(\sqrt{r_0}) \right]} \]  
(51)

Substituting Eq. 51 into Eq. 50 we obtain
\[ B = \frac{k_1(\sqrt{r_0})}{I_1(\sqrt{r_0})} \]  
(52)

Substituting Eqs. 51 and 52 into the general solution Eq. 29, gives
\[ \bar{P}_0(r_0, \mu) = \frac{k_0(\sqrt{r_0}) I_1(\sqrt{r_0}) + k_1(\sqrt{r_0}) I_0(\sqrt{r_0})}{m \left[ \sqrt{r_0} k_1(\sqrt{r_0}) I_1(\sqrt{r_0}) - \sqrt{r_0} I_1(\sqrt{r_0}) k_1(\sqrt{r_0}) \right]} \]  
(53)
As before, in the case of an infinite-acting reservoir, we showed
\[ \int_0^\infty k(\mu) = 1 \quad \text{as } \mu \to 0 \]
Similarly from Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.7, p. 375) we have
\[ I_1(x) \approx \frac{1}{2} x \]
or
\[ xI_1(x) \approx \frac{1}{2} x^2 \]
where as \( x \to 0 \) we have
\[ xI_1(x) = 0 \quad \text{as } x \to 0 \]
or for our present problem we have
\[ \sqrt{\mu} I_1(\sqrt{\mu}) = 0 \quad \text{as } \mu \to 0 \]
Combining these relations with Eq. 58 we obtain
\[ \frac{\bar{p}_0}{p_0} = \frac{1}{m} k_0(\sqrt{\mu} p_0) + \frac{1}{m} k_1(\sqrt{\mu} p_0) I_0(\sqrt{\mu} p_0) \tag{54} \]
\[ \text{as } \mu \to 0 \]

**Outer Boundary Case 3: Constant pressure outer boundary**
Combining Eqs. 29 and 17 we have
\[ A I_0(\sqrt{\mu} p_0) + B k_0(\sqrt{\mu} p_0) = 0 \tag{55} \]
Solving for the \( B \) parameter we obtain
\[ B = -\frac{A I_0(\sqrt{\mu} p_0)}{k_0(\sqrt{\mu} p_0)} \tag{56} \]
Recalling the inner boundary condition, Eq. 38, gives us
\[ A\sqrt{\mu} I_1(\sqrt{\mu}) = B \sqrt{\mu} k_1(\sqrt{\mu}) = -\frac{1}{m} \tag{57} \]
Substituting Eq. 56 into Eq. 38 we have
\[ A\sqrt{\mu} I_1(\sqrt{\mu}) + A \sqrt{\mu} k_1(\sqrt{\mu}) I_0(\sqrt{\mu} p_0) = -\frac{1}{m} \frac{k_0(\sqrt{\mu} p_0)}{k_0(\sqrt{\mu} p_0)} \]
or
\[ A \left[ \sqrt{\mu} I_1(\sqrt{\mu}) k_0(\sqrt{\mu} p_0) + \sqrt{\mu} k_1(\sqrt{\mu}) I_0(\sqrt{\mu} p_0) \right] = -\frac{1}{m} \frac{k_0(\sqrt{\mu} p_0)}{k_0(\sqrt{\mu} p_0)} \]
or solving for \( A \) we have
\[ A = -\frac{k_0(\sqrt{\mu} p_0)}{m \left[ \sqrt{\mu} I_1(\sqrt{\mu}) k_0(\sqrt{\mu} p_0) + \sqrt{\mu} k_1(\sqrt{\mu}) I_0(\sqrt{\mu} p_0) \right]} \tag{57} \]
Substituting Eq. 57 into Eq. 56 gives
\[ B = \frac{I_0(\sqrt{\mu} p_0)}{m \left[ \sqrt{\mu} I_1(\sqrt{\mu}) k_0(\sqrt{\mu} p_0) + \sqrt{\mu} k_1(\sqrt{\mu}) I_0(\sqrt{\mu} p_0) \right]} \tag{58} \]
Substituting Eqs. 57 and 58 into the general solution, Eq. 29, we obtain

\[ P_0(r_0, \mu) = \frac{k_0(\mu r_0) I_0(\mu r_0) - k_2(\mu r_0) I_0(\mu r_0)}{m \left[ k_0(k_0 I_0(\mu r_0) + \sqrt{\mu I_0(\mu r_0) k_0(\mu r_0)} \right]} \]  

(57)

As in the previous cases, we want to consider the behavior as \( \mu \to 0 \) (large \( r_0 \)). As before, we have

\[ \sqrt{\mu I_0(\mu r)} = 1 \quad \text{as} \quad \mu \to 0 \]

\[ \sqrt{\mu I_0(\mu r)} = 0 \quad \text{as} \quad \mu \to 0 \]

Combining these relations with Eq. 59, we obtain

\[ P_0(r_0, \mu) = \frac{1}{m} \frac{k_0(\mu r_0)}{I_0(\mu r_0)} - \frac{1}{m} \frac{k_2(\mu r_0)}{I_0(\mu r_0)} I_0(\mu r_0) \]  

(60)

(as \( \mu \to 0 \))

Outer Boundary Case 4: "Prescribed flux" outer boundary

Combining Eqs. 37 and 18 we obtain

\[ A \sqrt{\mu r_0} I_0(\sqrt{\mu r_0}) - B \sqrt{\mu r_0} k_1(\sqrt{\mu r_0}) = \bar{P}_{0,\text{ext}} \]

(61)

Recalling the inner boundary condition, Eq. 58, we have

\[ A \sqrt{\mu r} I_0(\sqrt{\mu r}) - B \sqrt{\mu r} k_1(\sqrt{\mu r}) = -1 \]  

(62)

We will solve Eqs. 61 and 58 simultaneously to determine \( A \) and \( B \). The algebra becomes a bit tedious, but we will show all steps. Solving for the \( A \) parameter we divide through Eq. 61 by \( \sqrt{\mu r_0} k_1(\sqrt{\mu r_0}) \), then we divide through Eq. 58 by \( \sqrt{\mu r} k_1(\sqrt{\mu r}) \). These operations give

\[ A \frac{\sqrt{\mu r_0} I_0(\sqrt{\mu r_0})}{\sqrt{\mu r} k_1(\sqrt{\mu r})} - B = \bar{P}_{0,\text{ext}} \frac{1}{\sqrt{\mu r_0} k_1(\sqrt{\mu r_0})} \]

(63)

Subtracting Eq. 63 from Eq. 62 we have

\[ A \left[ \sqrt{\mu r_0} I_0(\sqrt{\mu r_0}) - \sqrt{\mu r} I_0(\sqrt{\mu r}) \right] = \bar{P}_{0,\text{ext}} \frac{1}{\sqrt{\mu r_0} k_1(\sqrt{\mu r_0})} + \frac{1}{m} \frac{1}{\sqrt{\mu r} k_1(\sqrt{\mu r})} \]

Expanding to yield a uniform denominator on both sides

\[ A \frac{\sqrt{\mu r_0} I_0(\sqrt{\mu r_0}) - \sqrt{\mu r} I_0(\sqrt{\mu r})}{\sqrt{\mu r_0} k_1(\sqrt{\mu r_0}) \sqrt{\mu r} k_1(\sqrt{\mu r})} = \frac{\bar{P}_{0,\text{ext}} \sqrt{\mu r_0} k_1(\sqrt{\mu r_0}) + \sqrt{\mu r_0} k_1(\sqrt{\mu r_0})}{\sqrt{\mu r_0} k_1(\sqrt{\mu r_0}) \sqrt{\mu r} k_1(\sqrt{\mu r})} \]
Solving for \( A \) we have
\[
A = \frac{1}{\mu} \sqrt{k_1 (\text{ref})} \left[ \nu k_{\text{rep}} + \bar{q}_{\text{ext}} \sqrt{\mu} k_1 (\text{ref}) \right] \left( \sqrt{I_1 (\text{ref})} \sqrt{I_{\text{rep}}} - \sqrt{I_1 (\text{ref})} \sqrt{V_{\text{rep}}} k_1 (\text{ref}) \right)
\]

Factoring out the \( \sqrt{V_{\text{rep}}} \) terms and bringing out the \( \nu \) factor
\[
A = \frac{1}{\mu} \frac{k_1 (\text{ref})}{\sqrt{k_1 (\text{ref})} I_1 (\text{ref}) - \sqrt{I_1 (\text{ref})} k_1 (\text{ref})}
\]

Comparing Eq. 64 to the result for the no-flow boundary case (Eq. 51) we have
\[
A_f = \frac{1}{\mu} \frac{k_1 (\text{ref})}{\sqrt{k_1 (\text{ref})} I_1 (\text{ref}) - \sqrt{I_1 (\text{ref})} k_1 (\text{ref})}
\]

where we find that Eq. 64 is identical to Eq. 51 for the \( q_{\text{ext}} = 0 \) case.

Solving for the \( B \) parameter we divide through Eq. 61 by \( \sqrt{V_{\text{rep}}} I_1 (\text{ref}) \) and we divide through Eq. 68 by \( \sqrt{I_1 (\text{ref})} \), which gives
\[
A = \frac{-B \sqrt{V_{\text{rep}}} k_1 (\text{ref})}{\sqrt{V_{\text{rep}}} I_1 (\text{ref})} = \bar{q}_{\text{ext}} \frac{1}{\sqrt{V_{\text{rep}}} I_1 (\text{ref})}
\]

\[
A = \frac{-B \sqrt{V_{\text{rep}}} k_1 (\text{ref})}{\sqrt{I_1 (\text{ref})}} = -\frac{1}{\mu} \frac{1}{\sqrt{V_{\text{rep}}} I_1 (\text{ref})}
\]

Subtracting Eq. 66 from Eq. 65 we have
\[
B \left[ -\sqrt{V_{\text{rep}}} k_1 (\text{ref}) + \sqrt{V_{\text{rep}}} k_1 (\text{ref}) \right] = \bar{q}_{\text{ext}} \frac{1}{\sqrt{V_{\text{rep}}} I_1 (\text{ref})} + \frac{1}{\mu} \frac{1}{\sqrt{I_1 (\text{ref})}}
\]

Expanding to yield a uniform denominator on both sides gives
\[
B \left[ \frac{-\sqrt{V_{\text{rep}}} k_1 (\text{ref}) + \sqrt{V_{\text{rep}}} k_1 (\text{ref}) \sqrt{V_{\text{rep}}} I_1 (\text{ref})}{\sqrt{V_{\text{rep}}} I_1 (\text{ref})} \right] = \left[ \bar{q}_{\text{ext}} \sqrt{V_{\text{rep}}} I_1 (\text{ref}) + \frac{1}{\mu} \sqrt{V_{\text{rep}}} I_1 (\text{ref}) \right]
\]

Solving for \( B \) we have
\[
B = \frac{1}{\mu} \frac{\sqrt{V_{\text{rep}}} I_1 (\text{ref}) + \bar{q}_{\text{ext}} \sqrt{V_{\text{rep}}} I_1 (\text{ref})}{\sqrt{k_1 (\text{ref})} \sqrt{V_{\text{rep}}} I_1 (\text{ref}) - \sqrt{I_1 (\text{ref})} \sqrt{V_{\text{rep}}} k_1 (\text{ref})}
\]

As before, factoring out the \( \sqrt{V_{\text{rep}}} \) terms and bringing out the \( \mu \) factor
\[
B = \frac{1}{\mu} \frac{I_1 (\text{ref})}{\sqrt{k_1 (\text{ref})} I_1 (\text{ref}) - \sqrt{I_1 (\text{ref})} k_1 (\text{ref})}
\]

Comparing Eq. 67 with the result for the no-flow boundary case we recall Eq. 52
\[
B_f = \frac{1}{\mu} \frac{I_1 (\text{ref})}{\sqrt{k_1 (\text{ref})} I_1 (\text{ref}) - \sqrt{I_1 (\text{ref})} k_1 (\text{ref})}
\]
where we find that Eq. 67 is identical to Eq. 52 for \( q_{\text{ext}} = 0 \). Having shown this for both A and B we have verified these results.

In order to determine the particular solution for this case, we substitute Eqs. 64 and 67 into the general solution (Eq. 29). This gives

\[
\bar{\alpha}_d (\hat{r}, \hat{m}) = \frac{1}{m} \frac{k_0 (\sqrt{I}_d \hat{r}_d) I_1 (\sqrt{I}_d \hat{r}_{d0}) + I_0 (\sqrt{I}_d \hat{r}_d) k_1 (\sqrt{I}_d \hat{r}_{d0})}{\sqrt{I}_d k_1 (\sqrt{I}_d \hat{r}_d) I_1 (\sqrt{I}_d \hat{r}_{d0}) - \sqrt{I}_d I_0 (\sqrt{I}_d \hat{r}_d) k_1 (\sqrt{I}_d \hat{r}_{d0})}
\]

\[+ \frac{1}{m} \bar{\alpha}_{d, \text{ext}} \left( \frac{\hat{m}}{\sqrt{I}_{d0}} \right) \frac{k_0 (\sqrt{I}_d \hat{r}_d) \sqrt{I}_1 (\sqrt{I}_d \hat{r}_d) + I_0 (\sqrt{I}_d \hat{r}_d) \sqrt{k}_1 (\sqrt{I}_d \hat{r}_d)}{\sqrt{I}_d k_1 (\sqrt{I}_d \hat{r}_d) I_1 (\sqrt{I}_d \hat{r}_{d0}) - \sqrt{I}_d I_0 (\sqrt{I}_d \hat{r}_d) k_1 (\sqrt{I}_d \hat{r}_{d0})}
\]

where the first part of Eq. 68 is exactly Eq. 53, the solution for the no-flow boundary case (i.e., \( \bar{\alpha}_d = 0 \)). Note that \( \bar{\alpha}_{d, \text{ext}} = \bar{\alpha} (\hat{r}, \hat{m}) \).

As with the previous cases we can consider the behavior of Eq. 68 as \( \hat{m} \to 0 \). As we saw before

\[\sqrt{I}_d k_1 (\sqrt{I}_d \hat{r}_d) \to 1 \text{ as } \hat{m} \to 0\]

\[\sqrt{I}_d I_0 (\sqrt{I}_d \hat{r}_d) \to 0 \text{ as } \hat{m} \to 0\]

Combining these relations with Eq. 68 gives

\[
\bar{\alpha}_d (\hat{r}, \hat{m}) = \frac{1}{m} \frac{k_0 (\sqrt{I}_d \hat{r}_d)}{I_1 (\sqrt{I}_d \hat{r}_{d0})}
\]

\[+ \frac{1}{m} \bar{\alpha}_{d, \text{ext}} \left( \frac{\hat{m}}{\sqrt{I}_{d0}} \right) \frac{k_0 (\sqrt{I}_d \hat{r}_d) \sqrt{I}_1 (\sqrt{I}_d \hat{r}_d) + I_0 (\sqrt{I}_d \hat{r}_d) \sqrt{k}_1 (\sqrt{I}_d \hat{r}_d)}{\sqrt{I}_d k_1 (\sqrt{I}_d \hat{r}_d) I_1 (\sqrt{I}_d \hat{r}_{d0}) - \sqrt{I}_d I_0 (\sqrt{I}_d \hat{r}_d) k_1 (\sqrt{I}_d \hat{r}_{d0})}
\]

The second term in Eq. 68 (or 69) should not be arbitrarily reduced without a comprehensive study of the interplay of individual terms. For example, reduction using the behavior of \( I_1 (x) \) and \( k_1 (x) \) as \( x \to 0 \) yields \( I_0 (\sqrt{I}_d \hat{r}_d) I_1 (\sqrt{I}_d \hat{r}_{d0}) \) which tends to \( 1 / 0 \to \infty \) as \( \hat{m} \to 0 \). Considerable care must be exercised when making such reductions.