Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2c — Real Domain Solutions of the Radial Flow Diffusivity Equation for a Well Produced at a Constant Rate in a Bounded Circular Reservoir: Infinite and Finite-Acting Reservoir Cases

Resistance to tyrants is obedience to God.
— Thomas Jefferson (motto)

**Topic:** Real Domain Solutions of the Radial Flow Diffusivity Equation for a Well Produced at a Constant Rate in a Bounded Circular Reservoir: Infinite and Finite-Acting Reservoir Cases

**Objectives:** (things you should know and/or be able to do)
- Be able to derive the following particular solutions in the real domain using the appropriate Laplace transform solutions for an unfractured well produced at a constant flowrate in a homogeneous reservoir for the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior (line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] \]
  - "Infinite-acting" reservoir behavior (the so-called "log approximation," also a line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4}{\pi \cdot e} \right] \ln \left( \frac{r_D}{r_D^*} \right) \]
  - Bounded circular reservoir — "no-flow" at the outer boundary
    \[ p_D(t_D, r_D, r_{eD}) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{eD}^2}{4t_D} \right] + \frac{2t_D}{r_{eD}^2} \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] \exp \left[ -\frac{r_D^2}{4t_D} \right] \]
    and its "well testing" derivative function, \( p_D' = \frac{d}{dt_D} [p_D(r_D, t_D)] \) is given by
    \[ p_D'(t_D, r_D, r_{eD}) = \frac{1}{2} \exp \left[ -\frac{r_D^2}{4t_D} \right] + \frac{2t_D}{r_{eD}^2} \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] + \frac{1}{2t_D} \left[ r_D^2 - \frac{r_{eD}^2}{8} \right] \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] \]
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    \[ p_D(t_D, r_D, r_{eD}) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] - \frac{1}{2} E_1 \left[ \frac{r_{eD}^2}{4t_D} \right] + \frac{1}{8t_D} (r_{eD}^2 - r_D^2) \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] \]
    and its "well testing" derivative function, \( p_D' = \frac{d}{dt_D} [p_D(r_D, t_D)] \) is given by
    \[ p_D'(t_D, r_D, r_{eD}) = \frac{1}{2} \exp \left[ -\frac{r_D^2}{4t_D} \right] - \frac{1}{2} \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] + \frac{1}{8t_D} (r_{eD}^2 - r_D^2) \frac{r_{eD}^2}{4t_D} \exp \left[ -\frac{r_{eD}^2}{4t_D} \right] \]

**Lecture Outline:**
- Development of solutions in the real domain:
  - "Infinite-acting" reservoir behavior (line source solution)
    - Cylindrical source solution not directly invertable (in closed form).
  - Bounded circular reservoir — "no-flow" at the outer boundary
    - Inversion of line source solution using recursion relations and polynomial expansions for Bessel functions (for behavior near zero).
Lecture Outline: (Continued)

- Development of solutions in the real domain: (Continued)
  - Bounded circular reservoir — "no-flow" at the outer boundary (Continued)
    - Derivatives taken explicitly from real domain solution rather than Laplace transform solutions. Can check directly, term-by-term.
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    - Inversion of the line source solution using recursion relations and polynomial expansions for Bessel functions (for behavior near zero).
    - Derivatives taken explicitly from real domain solution rather than Laplace transform solutions. Can check directly, term-by-term.

- Discussion of Applications
  - Modelling of well performance (transient and pseudosteady-state performance, variable-rate superposition).
  - Development of short- and long-term analysis relations.

Reading Assignment:

- Review attached notes.
  - Solution of the Dimensionless Radial Flow Diffusivity Equation:
    - Real domain solutions via inversion of the Laplace transform solutions.

Exercises: For your own practice/skills building—do NOT turn in!

From the attached notes you are to rederive the following, and show all details.

- Starting from the Laplace transform solutions, derive the real domain solution(s) for an unfractured well produced at a constant flowrate (inner boundary) in a homogeneous reservoir with the following outer boundary condition(s):
  - Bounded circular reservoir — "no-flow" at the outer boundary
  - Bounded circular reservoir — "prescribed" at the outer boundary
  - Bounded circular reservoir — "constant pressure" at the outer boundary
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)

Semilog Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure Solutions—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir—Various $r_D$: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)
Log-log Plot: Constant Well Rate Solutions for a Constant Pressure Outer Boundary: Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)

Log-log Plot: Constant Wellbore Pressure Solutions for a Bounded Circular Reservoir: Dimensionless Rate Functions—Radial Flow Case (SPE 25479)
Solution of the Dimensionless Radial Flow Diffusivity Equation:

- Real Domain Solutions via Inversion of the Laplace Transform Solutions

Solutions for a Bounded Circular Reservoir: Infinite-Acting, No-Flow, and Constant Pressure Boundary Cases

The Laplace transform solutions under consideration are:

a. Infinite-Acting Reservoir Case:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} \left( \frac{k_D(vr_D)}{\sqrt{\mu} k_1(\mu r_D)} \right) \quad \text{(cylindrical source solution)} \]  
(1)

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} k_D(\mu r_D) \quad \text{(line source solution)} \]  
(2)

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{2\mu} \ln \left( \frac{4 + \frac{1}{\mu r_D^2}}{1} \right) \quad \text{(as \( \mu \to \infty \), log approximation)} \]  
(3)

b. No-Flow Boundary Case:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} \left( k_D(\mu r_D) I_0(\mu r_D) + k_1(\mu r_D) I_0(\mu r_D) \right) \]  
(4)

where for \( \mu \to \infty \), Eq. 4 reduces to:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} k_D(\mu r_D) + \frac{1}{M} k_1(\mu r_D) I_0(\mu r_D) \quad \text{(line source)} \]  
(5)

C. Constant Pressure Outer Boundary Case:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} \left( \frac{k_D(\mu r_D) I_0(\mu r_D) - k_0(\mu r_D) I_0(\mu r_D)}{I_1(\mu r_D) + I_0(\mu r_D)} \right) \]  
(6)

where for \( \mu \to \infty \), Eq. 6 reduces to:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{M} \frac{k_D(\mu r_D) - k_0(\mu r_D) I_0(\mu r_D)}{I_0(\mu r_D)} \quad \text{(line source)} \]  
(7)

Solutions for an Infinite-Acting Reservoir:

a. Cylindrical source solution:

Unfortunately, Eq. 1 cannot be inverted directly to yield a closed form, non-infinite series or integral solution. However, van Everdingen and Hurst give the following results:

\[ \bar{\phi}_D(r_D, \mu) = \frac{1}{\pi} \int_0^\infty \left[ 1 - e^{-\mu r_D^2} \right] \left[ J_0(\mu r_D) Y_0(\mu r_D) + Y_0(\mu r_D) J_0(\mu r_D) \right] \, d\mu \]  
(8)

and for the wellbore solution \( r_D = 1 \), Eq. 8 reduces to:

\[ \bar{\phi}_D(1, \mu) = \frac{4}{\pi^2} \int_0^\infty \left[ 1 - e^{-\mu r_D^2} \right] \, d\mu \]  
(9)
b. Line Source Solution:
Recalling Eq. 2 we have
\[
\phi_b(r, t_0, \omega) = \frac{1}{\mu} k_0(\omega r_0)
\]  
(2)

Multiplying through Eq. 2 by the Laplace transform parameter, \( \mu \), gives
\[
\mu \phi_b(r, \mu) = k_0(\omega r_0)
\]
\( \mu \)
(10)

Recalling the time derivative theorem for Laplace transforms we have
\[
L\left\{ \frac{d[f(t)]}{dt} \right\} = \mu \mathcal{F}(\mu) - f(t=0)
\]
(11)

assuming that \( f(t=0) \), which is true by our initial condition, we can similarly write
\[
\frac{d[f(t)]}{dt} = \mathcal{L}^{-1}\{\mu \mathcal{F}(\mu)\}
\]
(12)

or in terms of our problem we have
\[
\frac{d}{dt_0} \left[ \phi_b(r, t_0) \right] = \mathcal{L}^{-1}\{\mu \phi_b(r, \mu)\}
\]
(13)

Combining Eqs. 10 and 13
\[
\frac{d}{dt_0} \left[ \phi_b(r, t_0) \right] = \mathcal{L}^{-1}\{k_0(\omega r_0)\}
\]
(14)

Inversion of Eqs. 2 and 10 is accomplished by the use of Laplace transform tables, where the results of inversion are given below

<table>
<thead>
<tr>
<th>( \mathcal{F}(\mu) )</th>
<th>( f(t) )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\mu} k_0(\omega a) )</td>
<td>( \frac{1}{2} E_1 \left( \frac{a^2}{4t} \right) )</td>
<td>Carslaw and Jaeger: Conduction of Heat in Solids, Table V, Eq. 26, p. 495.</td>
</tr>
<tr>
<td>( k_0(\omega a) )</td>
<td>( \frac{1}{2} \exp \left( \frac{-a^2}{4t} \right) )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.120, p. 1028, and Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304.</td>
</tr>
</tbody>
</table>

Making the appropriate substitutions
\[
\phi_b(r, t_0) = \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right)
\]
(15)

and
\[
\frac{d}{dt_0} \left[ \phi_b(r, t_0) \right] = \frac{1}{2t_0} \exp \left( \frac{-r_0^2}{4t_0} \right)
\]
(16)
Defining the so-called "well testing derivative" we have
\[ p'(r_0, t_0) = t_0 \frac{d}{dt_0} \left[ p'(t_0, t_0) \right] \]  
(17)

Substituting Eq. 16 into Eq. 17 we have
\[ p'(r_0, t_0) = \frac{1}{2} \exp \left( -\frac{r_0^2}{4t_0} \right) \]  
(18)

C. Log-Approximation Solution:
Recalling Eq. 3 we have
\[ p'(r_0, u) = \frac{1}{2M} \ln \left( \frac{4}{e^{2x} r_0^2 u} \right) \]  
(19)

Expanding Eq. 3 into a more usable form, we have
\[ p'(r_0, u) = \frac{1}{2} \ln \left( -\ln(u) + \frac{1}{M} \ln \left( \frac{4}{e^{2x} r_0^2 u} \right) \right) \]  
(19)

Rather than attempt a derivative using \( u \), we will simply differentiate the inversion result of Eq. 19. The inverse Laplace transform of the \( \ln(u) \) and constant terms in Eq. 19 we have

<table>
<thead>
<tr>
<th>( \ln(u) )</th>
<th>( \ln(t) + \chi ) or ( \ln(e^{t_0}) )</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{1}{2} \ln(u) )</td>
<td>( \ln(t) + \chi ) or ( \ln(e^{t_0}) )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.98, p. 1027.</td>
</tr>
<tr>
<td>( \frac{1}{M} \ln \left( \frac{4}{e^{2x} r_0^2 u} \right) )</td>
<td>( \ln \left( \frac{4}{e^{2x} r_0^2 u} \right) )</td>
<td>(trivial) Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.91, p. 1021.</td>
</tr>
<tr>
<td>( \frac{1}{M} ) constant</td>
<td>( \frac{1}{M} ) constant</td>
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</table>

Writing the \( p'(r_0, t_0) \) inversion result for Eq. 19 is
\[ p'(r_0, t_0) = \frac{1}{2} \ln \left( e^{t_0} \right) + \ln \left( \frac{4}{e^{2x} r_0^2} \right) \]  
(19)

Collecting
\[ p'(r_0, t_0) = \frac{1}{2} \ln \left( \frac{4}{e^{t_0} r_0^2} \right) \]  
(20)
Isolating the $t_D$ term in Eq. 20 we have
\[
P_D(r_0, t_D) = \frac{1}{2} \left[ \frac{1}{t_D} \ln(t_D) + \frac{1}{2} \ln \left( \frac{4 \pi}{\varepsilon^2 r_0^2} \right) \right] (21)
\]
Substituting Eq. 21 into Eq. 17 to determine the well testing derivative we have
\[
P_D'(r_0, t_D) = t_D \left[ \frac{d}{dt_D} \left( \frac{1}{t_D} \ln(t_D) \right) \right] + t_D \left[ \frac{d}{dt_D} \left( \frac{1}{2} \ln \left( \frac{4 \pi}{\varepsilon^2 r_0^2} \right) \right) \right]
\]
which reduces to
\[
P_D'(r_0, t_D) = t_D \left[ \frac{1}{2t_D} \right] = \frac{1}{2}
\]

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**Solution for a No-Flow Outer Boundary**

It is not possible to invert the complete solution (Eq. 4) for this case so we will attempt an approximate solution of the line source form (Eq. 5). Recalling Eq. 5 we have
\[
\bar{P}(r_0, \infty) = \frac{1}{m} k_0(\varepsilon r_0) + \frac{1}{\mu} \frac{k_1(\varepsilon r_0)}{I_1(\varepsilon r_0)} I_0(\varepsilon r_0)
\]

We immediately recognize that the first term in Eq. 5 is the solution for an infinite-acting reservoir, and given the linearity of the inverse Laplace transform, we can invert Eq. 5 to yield
\[
P_D(r_0, t_D) = P_D(\varepsilon r_0, t_D) + \mathcal{L}^{-1} \left\{ \frac{1}{m} \frac{k_1(\varepsilon r_0)}{I_1(\varepsilon r_0)} I_0(\varepsilon r_0) \right\}
\]

where
\[
P_D(\varepsilon r_0, t_D) = \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_D} \right)
\]

So what is our strategy to invert the second term in Eq. 23? First we will use recursion relations to express the $k_n(z)$ Bessel functions then consider a two-term expansion of the resulting $I_0(z)/I_1(z)$ ratio. Recall that as $z \to 0$ that $I_0(z) \to 1$ and $I_1(z) \to 0$, which permits polynomial expansions.

From Abramowitz and Stegun, *Handbook of Mathematical Functions* (Eq. 9.6.15, p. 875) we have
\[
I_0(z) k_{n+1}(z) + I_{n+1}(z) k_n(z) = \frac{1}{z}
\]

Using $n = 0$
\[
I_0(z) k_1(z) + I_1(z) k_0(z) = \frac{1}{z}
\]
Using \( n = 1 \) we have
\[
I_1(z)k_2(z) + I_2(z)k_1(z) = \frac{1}{z}
\] (26)

Equate Eqs. 25 and 26 we have
\[
I_0(z)k_1(z) + I_1(z)k_0(z) = I_1(z)k_2(z) + I_2(z)k_1(z)
\]
\[
k_2(z)[I_0(z) - I_1(z)] = I_1(z)[k_1(z) - k_0(z)]
\]

or solving for \( k_1(z) \)
\[
k_1(z) = \frac{I_1(z)}{\frac{I_0(z) - I_1(z)}{k_2(z) - k_0(z)}}
\] (27)

Recalling the first recursion relation in Eq. 9.6.26, p. 376, Abramowitz and Stegun, Handbook of Mathematical Functions, in terms of \( I_n(z) \)
\[
I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z)
\]

for \( n = 1 \) we have
\[
I_0(z) - I_2(z) = \frac{2}{z} I_1(z)
\]

rearranging
\[
\frac{I_1(z)}{\frac{I_0(z) - I_2(z)}{2}} = \frac{z}{2}
\] (28)

Substituting Eq. 28 into Eq. 27 we have
\[
k_1(z) = \frac{z}{2} \left[ k_2(z) - k_0(z) \right]
\] (29)

Substituting Eq. 29 into Eq. 23 gives
\[
\varphi_\rho_\varphi_\rho_0 = \varphi_\rho_0 \varphi_\rho_0 + \sum \left[ \frac{1}{n} \frac{1}{z} \left( \sqrt{\varphi_\rho_\rho_0} - \varphi_\rho_\rho_0 \right) \right] I_0 \left( \sqrt{\varphi_\rho_\rho_0} \right)
\] (30)

Considering the \( I_0(b)/I_1(a) \) term, where \( b = \sqrt{\varphi_\rho_\rho_0} \) and \( a = \sqrt{\varphi_\rho_\rho_0} \), we will use the polynomial expansions for \( I_0(z) \) and \( I_1(z) \) taken from the general \( I_n(z) \) series given in Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.6.10, p. 375 we have
\[
I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{2304} + \ldots
\] (31)

and
\[
I_1(z) = \frac{z}{2} \left[ 1 + \frac{z^2}{8} + \frac{z^4}{192} + \frac{z^6}{9216} + \ldots + \right]
\] (32)

Using two term expansions for \( I_0(b) \) and \( I_1(a) \) we have
\[
I_0(b) = 1 + \frac{b^2}{4}
\] (33)

and
\[
I_1(a) = \frac{a}{z} \left( 1 + \frac{a^2}{8} \right)
\] (34)
Establishing the $I_0(b)/I_1(a)$ ratio using Eqs. 33 and 34 we have

$$\frac{I_0(b)}{I_1(a)} = \frac{z}{a} \frac{1 + b^2/4}{(1 + a^2/8)} \tag{35}$$

assuming $a^2/8 < 1$ we can express $(1 + a^2/8)^{-1}$ as a binomial series of the form $(1 + x)^{-1}$ from Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 3.6.10, p. 15, we have

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots \quad (1 \leq |x| < 1)$$

using a two term expansion of $(1 + a^2/8)^{-1}$ we have

$$(1 + a^2/8)^{-1} = 1 - a^2/8 \tag{36}$$

Substituting Eq. 36 into Eq. 35 we have

$$\frac{I_0(b)}{I_1(a)} = \frac{z}{a} \frac{1 + b^2/4}{(1 - a^2/8)}$$

Expanding

$$\frac{I_0(b)}{I_1(a)} = \frac{z}{a} \left(1 + \frac{b^2}{4} - \frac{a^2}{8} - \frac{a^2 b^2}{32} \right)$$

neglecting the $a^2 b^2 / 32$ term we have

$$\frac{I_0(b)}{I_1(a)} = \frac{z}{a} \left(1 + \frac{b^2}{4} - \frac{a^2}{8} \right) \tag{37}$$

Recalling that $b = \sqrt{\alpha_0}$ and $a = \sqrt{\alpha_0 r_0}$ and substituting Eq. 37 into Eq. 30

$$\rho_b(r_0, t_0) = \rho_{b, \inf}(r_0, t_0) + \mathcal{L}^{-1} \left\{ \frac{1}{a} \left[ k_z(r_0) - k_0(r_0) \right] \frac{z}{4} \left(1 - \frac{a^2}{8} + \frac{b^2}{4} \right) \right\}$$

Cancelling the $a/4$ terms and using $a = \sqrt{\alpha_0 r_0}$ and $b = \sqrt{\alpha_0}$

$$\rho_b(r_0, t_0) = \rho_{b, \inf}(r_0, t_0) + \mathcal{L}^{-1} \left\{ k_z(r_0) - k_0(r_0) \right\} \left(1 - \frac{\alpha_0 r_0^2}{4} + \frac{\alpha_0 z^2}{4} \right)$$

Continuing the expansion

$$\rho_b(r_0, t_0) = \rho_{b, \inf}(r_0, t_0) + \mathcal{L}^{-1} \left\{ \left(\frac{\alpha_0 r_0^2}{4} - \frac{\alpha_0 z^2}{8} \right) k_z(r_0) - \left(\frac{\alpha_0 z^2}{4} - \frac{\alpha_0 r_0^2}{8} \right) k_0(r_0) \right\} \tag{38}$$

For reference, we note the Laplace transform of Eq. 38

$$\rho_b(r_0, t_0) = \frac{1}{\mathcal{M}} k_0(\sqrt{\alpha_0} r_0) + \frac{1}{\mathcal{M}} k_z(\sqrt{\alpha_0} r_0) - \frac{1}{\mathcal{M}} k_0(\sqrt{\alpha_0} r_0) + \left(\frac{\alpha_0 z^2}{4} - \frac{\alpha_0 r_0^2}{8} \right) k_z(\sqrt{\alpha_0} r_0) - \left(\frac{\alpha_0 z^2}{4} - \frac{\alpha_0 r_0^2}{8} \right) k_0(\sqrt{\alpha_0} r_0) \tag{39}$$
Multiplying through Eq. 39 by the laplace transform parameter, \( \mu \), gives

\[
\mu \mathcal{F} \left( F_0 \right) = \mathcal{F} \left( k_0 (\nabla \cdot \mathbf{u}) \right) + k_2 (\nabla \cdot \mathbf{u}) \mathcal{F} \left( \mathbf{u}_0 \right) - k_0 (\nabla \cdot \mathbf{u}_0) + \left( \frac{\mu}{4} - \frac{\mu_0}{8} \right) \frac{1}{4} \mathcal{F} \left( k_0 (\nabla \cdot \mathbf{u}) \mathcal{F} \left( \mathbf{u}_0 \right) - \frac{\mu_0}{8} \mathcal{F} \left( k_0 (\nabla \cdot \mathbf{u}) \mathcal{F} \left( \mathbf{u}_0 \right) \right) \right) (40)
\]

We will take the inverse laplace transform of Eqs. 39 and 40 using the following tables

<table>
<thead>
<tr>
<th>( \mathcal{F} \left( \frac{1}{\mu} \right) )</th>
<th>( \mathcal{F} \left( \frac{1}{2} \right) )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_0 (\nabla \cdot \mathbf{u}) )</td>
<td>( E_1 \left( \frac{a^2}{4t} \right) )</td>
<td>Carlaw and Jaeger: Conduction of Heat in Solids, Table V, Eq. 26, p. 495.</td>
</tr>
<tr>
<td>( k_0 (\nabla \cdot \mathbf{u}) )</td>
<td>( \frac{1}{2t} ) ( \exp \left( -\frac{a^2}{4t} \right) )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.120, p. 1028.</td>
</tr>
<tr>
<td>( \frac{1}{\mu} ) ( k_2 (\nabla \cdot \mathbf{u}) )</td>
<td>( \frac{2t}{a^2} \exp \left( -\frac{a^2}{4t} \right) )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 304.</td>
</tr>
<tr>
<td>( \frac{1}{\mu} ) ( k_2 (\nabla \cdot \mathbf{u}) )</td>
<td>( \frac{2t}{a^2} ) ( \Gamma \left( \frac{3}{2}, \frac{a^2}{4t} \right) )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.13, p. 306.</td>
</tr>
<tr>
<td>( \frac{2t}{a^2} ) ( \left( \frac{1}{2t} + \frac{z}{a^2} \right) \exp \left( -\frac{a^2}{4t} \right) )</td>
<td></td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.14, p. 306.</td>
</tr>
</tbody>
</table>
Inverting Eq. 39 term-by-term using the previous table gives

\[
P_D(r, t) = \frac{1}{Z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{Z} \left( \frac{r_{D0}^2}{4t_0} \right) + \frac{Z}{r_{D0}^2} \exp \left( -\frac{r_{D0}^2}{4t_0} \right)
\]

\[
+ \left( \frac{r_0^2 - r_{D0}^2}{4} \right) \left( \frac{1}{Zt_0} + \frac{2}{Zr_{D0}^2} \right) \exp \left( -\frac{r_{D0}^2}{4t_0} \right)
\]

\[
- \left( \frac{r_0^2 - r_{D0}^2}{4} \right) \frac{1}{t_0} \exp \left( -\frac{r_{D0}^2}{4t_0} \right)
\]

Collecting

\[
P_D(r, t) = \frac{1}{Z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{Z} \left( \frac{r_{D0}^2}{4t_0} \right)
\]

\[
+ \left[ \frac{z}{r_{D0}^2} \left( \frac{1}{Zt_0} + \frac{2}{Zr_{D0}^2} \right) - c \left( \frac{1}{Zt_0} \right) \right] \exp \left(-\frac{r_{D0}^2}{4t_0} \right), \quad \text{where} \quad c = \left[ (r_0^2/4) - (r_{D0}^2/8) \right]
\]

cancelling the c/2t_0 terms

\[
P_D(r, t) = \frac{1}{Z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{Z} \left( \frac{r_{D0}^2}{4t_0} \right) + \frac{Z}{r_{D0}^2} \exp \left(-\frac{r_{D0}^2}{4t_0} \right)
\]

\[
+ \frac{2}{r_{D0}^2} \left( \frac{r_0^2 - r_{D0}^2}{4} \right) \exp \left(-\frac{r_{D0}^2}{4t_0} \right)
\]

which yields the following reduction

\[
P_D(r, t) = \frac{1}{Z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{Z} \left( \frac{r_{D0}^2}{4t_0} \right) + t_0 \exp \left(-\frac{r_{D0}^2}{4t_0} \right) + \frac{Z}{r_{D0}^2} \left( \frac{r_0^2}{4} - \frac{1}{4t_0} \right) \exp \left(-\frac{r_{D0}^2}{4t_0} \right) \quad (41)
\]

Segmenting the solution into particular flow regimes

\[
P_D(r, t) = \frac{1}{Z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2}{r_{D0}^2} \exp \left(-\frac{r_{D0}^2}{4t_0} \right)
\]

\[
\text{--- Infinite-Acting-Reservoir Term (Reservoir Size)}
\]

\[
\text{--- Material Balance Term (Reservoir Size)}
\]

\[
- \frac{1}{Z} \left( \frac{r_{D0}^2}{4t_0} \right) + \frac{Z}{r_{D0}^2} \left( \frac{r_0^2}{4} - \frac{1}{4t_0} \right) \exp \left(-\frac{r_{D0}^2}{4t_0} \right)
\]

\[
\text{--- Reservoir Shape Effects Terms}
\]
Due to conflicting results obtained by inverting Eq. 40 term-by-term, we will proceed by differentiating Eq. 42.

Note that
\[
\frac{d}{dt_D} \frac{d E_1(x)}{dx} = \frac{d}{dt_D} \frac{d}{dx} \left[ \frac{d x}{d t_D} \left[ - \exp(-x) \right] \right] = \frac{d}{dt_D} \left[ \frac{d x}{d t_D} \left[ \frac{-\exp(-x)}{x} \right] \right] \tag{43}
\]
and
\[
\frac{d}{dt_D} \frac{d \exp(x)}{dx} = \frac{d}{dt_D} \frac{d}{dx} \left[ \frac{d x}{d t_D} \left[ - \exp(-x) \right] \right] = \frac{d}{dt_D} \left[ \frac{d x}{d t_D} \left[ - \exp(-x) \right] \right] \tag{44}
\]

Differentiating Eq. 42 term-by-term
\[
\frac{d}{dt_D} \left[ \frac{1}{2} \frac{d x}{d t_D} \left( \frac{r_D^2}{4 t_D} \right) \right] = \frac{1}{2} \frac{d}{dt_D} \left( \frac{r_D^2}{4 t_D} \right) \left( -\frac{4 t_D}{r_D^2} \right) \exp \left( \frac{-r_D^2}{4 t_D} \right) \nonumber
\]
\[
= \frac{1}{2} \left( \frac{-r_D^2}{4 t_D^2} \right) \left( -\frac{4 t_D}{r_D^2} \right) \exp \left( \frac{-r_D^2}{4 t_D} \right) \nonumber
\]
\[
= \frac{1}{2} \left( \frac{-r_D^2}{4 t_D^2} \right) \exp \left( \frac{-r_D^2}{4 t_D} \right) \left( \frac{-r_D^2}{4 t_D} \right) \tag{45}
\]
or
\[
\frac{d}{dt_D} \left[ \frac{1}{2} \frac{d x}{d t_D} \left( \frac{r_D^2}{4 t_D} \right) \right] = \frac{1}{2t_D} \exp \left( \frac{-r_D^2}{4 t_D} \right) \tag{45}
\]

Similarly for \( \frac{d}{dt_D} \left[ \frac{1}{2} \frac{d x}{d t_D} \left( \frac{r_D^2}{4 t_D} \right) \right] \) we have
\[
\frac{d}{dt_D} \left[ \frac{1}{2} \frac{d x}{d t_D} \left( \frac{r_D^2}{4 t_D} \right) \right] = \frac{1}{2t_D} \exp \left( \frac{-r_D^2}{4 t_D} \right) \tag{46}
\]

Next we have
\[
\frac{d}{dt_D} \left[ \frac{z}{r_D^2} \frac{t_D}{t_D} \exp \left( \frac{-r_D^2}{4 t_D} \right) \right] = \frac{z}{r_D^2} \left[ \exp \left( \frac{-r_D^2}{4 t_D} \right) \frac{dt_D}{dt_D} + t_D \frac{d}{dt_D} \left[ \exp \left( \frac{-r_D^2}{4 t_D} \right) \right] \right] \nonumber
\]
\[
= \frac{z}{r_D^2} \left[ 1 + t_D \frac{d}{dt_D} \left( \frac{-r_D^2}{4 t_D} \right) \right] \exp \left( \frac{-r_D^2}{4 t_D} \right) \nonumber
\]
\[
= \frac{z}{r_D^2} \left[ 1 + \frac{r_D^2}{4 t_D} \right] \exp \left( \frac{-r_D^2}{4 t_D} \right) = \left[ \frac{z}{r_D^2} + \frac{1}{4 t_D} \right] \exp \left( \frac{-r_D^2}{4 t_D} \right) \tag{47}
\]

Similarly
\[
\frac{d}{dt_D} \left[ \left( \frac{r_D^2}{4 t_D} - \frac{1}{4} \right) \frac{r_D^2}{4 t_D} \exp \left( \frac{-r_D^2}{4 t_D} \right) \right] = \left( \frac{r_D^2}{4 t_D} - \frac{1}{4} \right) \frac{r_D^2}{4 t_D} \exp \left( \frac{-r_D^2}{4 t_D} \right) \nonumber
\]
\[
= \frac{1}{2t_D^2} \left[ \frac{r_D^2}{4} - \frac{r_D^2}{8} \right] \exp \left( \frac{-r_D^2}{4 t_D} \right) \tag{48}
\]
Collecting the derivative terms we have
\[ \frac{d}{dt} \left[ P_B(r_0,t_0) \right] = \frac{1}{2t_0} \exp \left( -\frac{r_0^2}{4t_0} \right) - \frac{1}{2t_0} \exp \left( -\frac{r_0^2}{4t_0} \right) \]
\[ + \left[ \frac{2}{r_0^2} + \frac{1}{z t_0} \right] \exp \left( -\frac{r_0^2}{4t_0} \right) + \frac{1}{z t_0} \left[ \frac{r_0^2}{4} + \frac{r_0^2 - r_0^2}{8} \right] \exp \left( -\frac{r_0^2}{4t_0} \right) \]

Collecting further
\[ \frac{d}{dt} \left[ P_B(r_0,t_0) \right] = \frac{1}{2t_0} \exp \left( -\frac{r_0^2}{4t_0} \right) + \left[ \frac{1}{2z t_0} \frac{r_0^2}{r_0^2} + \frac{1}{2z t_0} \frac{r_0^2}{r_0^2} \right] \exp \left( -\frac{r_0^2}{4t_0} \right) \]

Multiplying through by \( t_0 \) we have
\[ P_B'(r_0,t_0) = t_0 \frac{d}{dt} \left[ P_B(r_0,t_0) \right] \]

or
\[ P_B'(r_0,t_0) = \frac{1}{2z t_0} \left( \frac{r_0^2}{4} + \frac{r_0^2 - r_0^2}{8} \right) \exp \left( -\frac{r_0^2}{4t_0} \right) \]

\[ \text{Infinite-Acting Reservoir Term} \]

\[ \text{Material Balance Term (Reservoir Size)} \]

\[ \text{Reservoir Shape Effects Term} \]

Solution for Constant Pressure Outer Boundary:

Similar to the no-flow outer boundary case, we cannot directly invert Eq. 6, so we will attempt an approximate solution of the line source form (Eq. 7). Recalling Eq. 7 we have
\[ P_B(r_0,\eta) = \frac{1}{4} \frac{k_0(\eta r_0)}{M} - \frac{1}{4} \frac{k_0(\eta r_0)}{M} I_0(\eta r_0) \] (line source) (7)

Recalling the polynomial approximation for \( I_0(\eta) \) (Eq. 31) we have
\[ I_0(\eta) = 1 + \eta^2 + \frac{\eta^4}{64} + \frac{\eta^6}{2304} + \ldots \] (81)

Using a two-term approximation for the \( I_0(\eta r_0) / I_0(\eta r_0) \)
\[ \frac{I_0(\eta r_0)}{I_0(\eta r_0)} = 1 + \frac{\eta r_0^2}{4 \eta r_0^2} \]

Using a two-term binomial series for \( (1 + \frac{\eta r_0^2}{14})^{-1} \) we have
\[ \frac{I_0(\eta r_0)}{I_0(\eta r_0)} = \left( 1 + \frac{\eta r_0^2}{4 \eta r_0^2} \right) \left( 1 - \frac{\eta r_0^2}{4 \eta r_0^2} \right) \]

or
\[ \frac{I_0(\eta r_0)}{I_0(\eta r_0)} = \frac{1 + \eta r_0^2 - \eta r_0^2}{4} - \frac{1}{16} \] (50)
Substituting Eq. 50 into Eq. 47 gives us

$$
 \frac{\bar{Q}_D(r_0, \mu)}{\bar{Q}_D(r_0, \mu)} = \frac{1}{\mu} k_0(\sqrt{r_0}) - \frac{1}{\mu} k_0(\sqrt{r_0}) (1 + \frac{\mu r_0^2 - \mu r_0^2}{4} - \mu r_0^2 \frac{r_0^2}{16})
$$

or

$$
 \frac{\bar{Q}_D(r_0, \mu)}{\bar{Q}_D(r_0, \mu)} = \frac{1}{\mu} k_0(\sqrt{r_0}) - \frac{1}{\mu} k_0(\sqrt{r_0}) + \frac{1}{16} (r_0^2 - r_0^2) k_0(\sqrt{r_0}) - \mu r_0^2 \frac{r_0^2}{16} k_0(\sqrt{r_0})
$$

for simplicity, we will ignore the term in Eq. 51, this gives

$$
 \frac{\bar{Q}_D(r_0, \mu)}{\bar{Q}_D(r_0, \mu)} = \frac{1}{\mu} k_0(\sqrt{r_0}) - \frac{1}{\mu} k_0(\sqrt{r_0}) + \frac{1}{16} (r_0^2 - r_0^2) k_0(\sqrt{r_0})
$$

From our previous efforts we recall that

$$
 \frac{1}{2} k_0(\sqrt{a}) = \frac{1}{2} E_1 \left( \frac{a^2}{4t} \right)
$$

Inverting Eq. 52 term-by-term we have

$$
 \frac{1}{2} k_0(\sqrt{a}) = \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) + \frac{1}{8} \exp \left( -\frac{r_0^2}{4t_0} \right)
$$

Reservoir Shape Effects Terms

Differentiating Eq. 53 term-by-term, we simply recall Eqs. 45 and 46

$$
 \frac{d}{dt_0} \left[ \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) \right] = \frac{1}{2} \exp \left( -\frac{r_0^2}{4t_0} \right)
$$

$$
 \frac{d}{dt_0} \left[ \frac{1}{2} E_1 \left( \frac{r_0^2}{4t_0} \right) \right] = \frac{1}{2} \exp \left( -\frac{r_0^2}{4t_0} \right)
$$

Differentiating the last term in Eq. 53 we have

$$
 \frac{d}{dt_0} \left[ \frac{(r_0^2 - r_0^2)}{8} \exp \left( -\frac{r_0^2}{4t_0} \right) \right] = \frac{(r_0^2 - r_0^2)}{8} \frac{d}{dt_0} \left[ \frac{1}{t_0} \exp \left( -\frac{r_0^2}{4t_0} \right) \right]
$$

$$
 = \left( \frac{r_0^2 - r_0^2}{8} \right) \left[ \exp \left( -\frac{r_0^2}{4t_0} \right) \frac{d}{dt_0} \left( \frac{1}{t_0} \right) + \frac{1}{t_0} \frac{d}{dt_0} \left[ \exp \left( -\frac{r_0^2}{4t_0} \right) \right] \right]
$$

$$
 = \left( \frac{r_0^2 - r_0^2}{8} \right) \left[ \frac{r_0^2}{4t_0} - 1 \right] \exp \left( -\frac{r_0^2}{4t_0} \right)
$$
Collecting the derivatives we have
\[
\frac{d}{dt_0} \left[ \rho_0(\rho_0, t_0) \right] = \frac{1}{2t_0} \exp \left( \frac{-\rho_0^2}{4t_0} \right) - \frac{1}{2t_0} \exp \left( \frac{-\rho_0^2}{4t_0} \right) 
+ \left( \frac{\rho_0^2 - \rho_0^2}{8t_0^2} \right) \left[ \frac{\rho_0^2}{4t_0} - 1 \right] \exp \left( \frac{-\rho_0^2}{4t_0} \right)
\]

Multiplying through by \( t_0 \) yields the well testing derivative
\[
\rho_0'(\rho_0, t_0) = \frac{1}{2} \exp \left( \frac{-\rho_0^2}{4t_0} \right) - \frac{1}{2t_0} \exp \left( \frac{-\rho_0^2}{4t_0} \right) + \left( \frac{\rho_0^2 - \rho_0^2}{8t_0^2} \right) \left[ \frac{\rho_0^2}{4t_0} - 1 \right] \exp \left( \frac{-\rho_0^2}{4t_0} \right) (55)
\]

---

**Summary of Results:**

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
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<tr>
<td>a. Infinite-acting reservoir</td>
<td>( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) )</td>
</tr>
<tr>
<td>Laplace Domain/Cylindrical Source Solution</td>
<td>( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) )</td>
</tr>
<tr>
<td>Laplace Domain/Line Source Solution</td>
<td>( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) )</td>
</tr>
<tr>
<td>Laplace Domain/&quot;log&quot; approximation</td>
<td>( \rho_0(\rho_0, \mu) = \frac{1}{2m} \ln \left( \frac{\sqrt{\mu} \rho_0}{\pi} \right) )</td>
</tr>
<tr>
<td>Real Domain/Cylindrical Source Solution</td>
<td>( \rho_0(\rho_0, t_0) = \mathcal{L}^{-1} \left{ \frac{1}{m \sqrt{\mu} k_1(\mu \rho_0)} \right} = \text{not invertable} )</td>
</tr>
<tr>
<td>Real Domain/Line Source Solution</td>
<td>( \rho_0(\rho_0, t_0) = \frac{1}{2} \exp \left( \frac{-\rho_0^2}{4t_0} \right) )</td>
</tr>
<tr>
<td>Real Domain/Derivative of the Line Source Solution</td>
<td>( \rho_0'(\rho_0, t_0) = \frac{1}{2} \exp \left( \frac{-\rho_0^2}{4t_0} \right) )</td>
</tr>
<tr>
<td>Real Domain/&quot;log&quot; approximation</td>
<td>( \rho_0(\rho_0, t_0) = \frac{1}{2} \ln \left( \frac{\sqrt{\mu} \rho_0}{\pi} \right) )</td>
</tr>
<tr>
<td>Real Domain/Derivative of the &quot;log&quot; approximation</td>
<td>( \rho_0'(\rho_0, t_0) = \frac{1}{2} )</td>
</tr>
</tbody>
</table>

b. Bounded circular reservoir/no-flow outer boundary

| Laplace Domain/Cyl Source | \( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) I_1(\sqrt{\mu} \rho_0) \left\{ \frac{k_1(\sqrt{\mu} \rho_0)}{\mu \sqrt{\mu} k_1(\mu \rho_0)} \right\} \) |
| Laplace Domain/Line Source | \( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) + \frac{1}{m} k_1(\sqrt{\mu} \rho_0) I_0(\sqrt{\mu} \rho_0) \) |
| Laplace Domain/Line Source | \( \rho_0(\rho_0, \mu) = \frac{1}{m} k_0(\sqrt{\mu} \rho_0) + \frac{1}{m} k_1(\sqrt{\mu} \rho_0) I_0(\sqrt{\mu} \rho_0) \) |
Case | Solution
--- | ---
b. bounded circular reservoir / no-flow boundary - continued
Real Domain / Line Source Soln. \( p(r, t) = \frac{1}{2} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{2} \left( \frac{r_e^2}{4t_0} \right) + \frac{z}{2} \frac{t_0}{t} \exp \left( \frac{-r_e^2}{4t_0} \right) \)
+ \( \frac{1}{2} \left( \frac{r_0^2 - r_e^2}{4t_0} \right) \exp \left( \frac{-r_e^2}{4t_0} \right) \)
Real Domain / Derivative of Line Source Soln. \( p^t(r, t) = \frac{1}{2} \frac{t_0}{t} \exp \left( \frac{-r_e^2}{4t_0} \right) + \frac{z}{2} \frac{t_0}{t} \exp \left( \frac{-r_e^2}{4t_0} \right) \)
+ \( \frac{1}{2} \frac{t_0}{t} \left( \frac{r_0^2 - r_e^2}{4} \right) \exp \left( \frac{-r_e^2}{4t_0} \right) \)
c. bounded circular reservoir / constant pressure boundary
Laplace Domain / Cyl. Source \( \Phi_d(r, \theta) = \frac{1}{\sqrt{4 \pi}} \frac{k_d}{k_e} \left( \frac{r_0^2}{4l_0} \right) - \frac{1}{\sqrt{4 \pi}} \frac{k_e}{k_d} \left( \frac{r_e^2}{4l_e} \right) - \frac{1}{\sqrt{4 \pi}} \frac{k_d}{k_e} \left( \frac{r_0^2}{4l_0} \right) \)
Laplace Domain / Line Source \( \Phi_d(r, \theta) = \frac{1}{\sqrt{4 \pi}} \frac{k_0}{k_e} \left( \frac{r_0^2}{4l_0} \right) - \frac{1}{\sqrt{4 \pi}} \frac{k_e}{k_0} \left( \frac{r_e^2}{4l_e} \right) \)
Real Domain / Line Source Soln. \( p_d(r, t) = \frac{1}{2} \frac{t_0}{t} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{2} \frac{t_0}{t} \left( \frac{r_e^2}{4t_0} \right) \)
+ \( \frac{1}{2} \frac{t_0}{t} \left( \frac{r_0^2 - r_e^2}{4t_0} \right) \exp \left( \frac{-r_e^2}{4t_0} \right) \)
Real Domain / Derivative of Line Source Soln. \( p^t_d(r, t) = \frac{1}{2} \frac{t_0}{t} \exp \left( \frac{-r_e^2}{4t_0} \right) + \frac{1}{2} \frac{t_0}{t} \exp \left( \frac{-r_e^2}{4t_0} \right) \)
+ \( \frac{1}{2} \frac{t_0}{t} \left( \frac{r_0^2 - r_e^2}{4t_0} \right) \exp \left( \frac{-r_e^2}{4t_0} \right) \)