Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 5 — Dual Porosity Reservoirs:
Warren and Root Approach—Pseudosteady-State Interporosity Flow Behavior

Half our standards come from our first masters,
the other half from our first loves.

— Santayana (1906)

**Topic:** Dual Porosity Reservoirs: Warren and Root Approach—Pseudosteady-State Interporosity Flow Behavior

**Objectives:** (things you should know and/or be able to do)

- Be familiar with the "fracture" and "matrix" models developed by Warren and Root and also be familiar with the Laplace and real domain results given by Warren and Root for pseudosteady-state interporosity flow. These relations are:
  - **Laplace domain results:**
    - Warren and Root "interporosity flow function:"
      \[
      f(u) = \frac{\lambda + \omega(1-\omega)u}{\lambda + (1-\omega)u}
      \]
    - Cylindrical source formulation: W&R Eq. 14
      \[
      \bar{p}_D(u, r_D, \omega, \lambda, s) = \frac{1}{u} \frac{K_0(\sqrt{uf(u)} r_D)}{K_1(\sqrt{uf(u)})} + \frac{s}{u}
      \]
    - Line source formulation:
      \[
      \bar{p}_D(u, r_D, \omega, \lambda, s) = \frac{1}{u} K_0(\sqrt{uf(u)} r_D) + \frac{s}{u}
      \]
    - "Log approximation" formulation:
      \[
      \bar{p}_D(u, r_D, \omega, \lambda, s) = \frac{1}{2u} \ln \left[ \frac{4}{e^{2\gamma} r_D^2 uf(u)} \right] + \frac{s}{u}
      \]
  - **Real domain results:**
    - Line source solution: (W&R Eq. 15—including the skin factor, s)
      \[
      p_D(t_D, r_D, \omega, \lambda, s) = \frac{1}{2} \ln \left[ \frac{4}{e^{2\gamma} r_D^2} \right] - \frac{1}{2} E_i \left[ \frac{\lambda}{\omega(1-\omega)} t_D \right] + \frac{1}{2} E_i \left[ \frac{\lambda}{(1-\omega)} t_D \right] + s
      \]
    - Well testing derivative of the time domain solution:
      \[
      p_D'(t_D, r_D, \omega, \lambda) = \frac{1}{2} + \frac{1}{2} \exp \left[ \frac{-\lambda}{\omega(1-\omega)} t_D \right] - \frac{1}{2} \exp \left[ \frac{-\lambda}{(1-\omega)} t_D \right]
      \]
- You should be able to distinguish between the pseudosteady-state and transient interporosity flow models (the Warren and Root, and de Swaan/Najurieta models, respectively). (You should also be aware of the various interporosity flow models proposed by Moench.)
- Be familiar with and be able to define the following parameters for dual porosity reservoir systems:
  - **Interporosity Flow Parameter:**
    \[
    \lambda = \alpha \frac{k_m}{k_w}
    \] (where \(\alpha\) is the matrix block shape parameter)
  - **Storativity Ratio:**
    \[
    \omega = \frac{(\phi Vc)_f}{((\phi Vc)_f + (\phi Vc)_m)}
    \]
Lecture Outline:

- Concepts of the Warren and Root dual porosity model
  - Comparison of reservoir and mathematical models.
  - Governing partial differential equations.
  - Use of a pseudosteady-state relation for matrix flow.

- Development of the Laplace space solution (W&R Eq. 14)
  - Fundamental relations (partial differential equations).
  - Interface condition (pseudosteady-state relation for matrix flow).
  - Definition of dimensionless variables.
  - "Interporosity Flow Function," \( f(u) \).
  - Development of general solution in the Laplace domain.
  - Development of the particular solution in the Laplace domain for a well produced at a constant rate in an infinite-acting dual porosity reservoir.

- Development of the real space solution (W&R Eq. 15)
  - Use of "log approximation" in Laplace domain for W&R Eq. 14.
  - Expansion and separation, isolate invertible forms—invert to yield W&R Eq. 15.
  - Discuss applications of the Schapery inversion approach (i.e., use of \( u=1/(e^t) \)).

- Discussion of the various patterns of flow behavior for an unfractured well in an infinite-acting dual porosity reservoir system.
  - \( p_{wD} \) and \( p_{wD} \) vs. \( t_D \) (various \( \lambda \) and \( \omega \) values — Warren and Root\(^1\) Solution)
  - \( p_{wD} \) vs. \( t_D \lambda/4 \) (Stewart and Ascharsobbi\(^2\) type curve)
  - \( p_{wD} \) vs. \( t_D \lambda/(1-\omega) \) (Onur, et al\(^5\) type curve)

References:

Solutions:


Data Analysis Methods:


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Reading Assignment:
- Review attached notes.
- Derivation of the Warren and Root approach: Pseudosteady-state interporosity flow

Exercises: For your own practice/skills building—do NOT turn in!

Derivation of Solutions for Naturally Fractured Reservoirs:
- Derive the $p_D$ and $p_D'$ results using the Najurieta-ter Haar substitution ($u=1/(e_{nD})$) in the Laplace domain form of the "log approximation" function (given below):
  - Najurieta-ter Haar substitution ($u=1/(e_{nD})$):
    \[
    p_D(t_D) = u\bar{p}_D(u)|_{u=1/(e_{nD})}
    \]
  - Laplace domain form of the "log approximation" function:
    \[
    \bar{p}_D(u,r_D,\omega,\lambda,s) = \frac{1}{2u} \ln \left[ \frac{4}{e^{2\gamma} r_D^2 u \bar{f}(u)} \right] + \frac{s}{u}
    \]
- The $p_D$ solution proposed by Aguilera for the case of pseudosteady-state interporosity flow is given as
  \[
  p_D(t_D,\omega,\lambda,s) = \frac{1}{2} E_I \left[ \frac{1}{4t_D} \eta_D \right] + s
  \]
  where the $\eta_D$ function for this case is given by:
  \[
  \eta_D = \omega + (1-\omega) \left[ 1 - \exp \left[ \frac{-t_D}{\tau_D} \right] \right]
  \]
  Where $\tau_D$ is a constant that is not exactly well-defined—but it is suggested that $\tau_D$ is of the form:
  \[
  \tau_D = \frac{\xi}{\lambda}, \text{ where } \xi \text{ is estimated to vary from 1 to 4 (but probably varies more...)}
  \]
You are to derive the $p_D'$ result using Aguilera's solution for $p_D$.

References:
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**Exercises:** For your own practice/skills building—do **NOT** turn in!

**Computation of Well Performance in an Infinite-Acting, Dual Porosity Reservoir:**
- You are to plot the following $p_D$ and $p_D'$ solutions as noted:

**Base Case:**
- Plot the $p_D$ and $p_D'$ functions obtained using the Gaver-Stehfest algorithm. The appropriate Laplace domain result is taken from the Warren and Root paper (Eq. 14). This relation is:

$$\bar{p}_D(u, r_D, \omega, \lambda, s) = \frac{1}{u} \frac{K_0(\sqrt{\mu u(s)}r_D)}{\sqrt{\mu u(s)} K_1(\sqrt{\mu u(s)})} + \frac{s}{u}$$

**Reference:**

"Log Approximation" Results:
- Plot the $p_D$ and $p_D'$ results derived using the "log approximation" function. These are:

$$p_D(t_D, r_D, \omega, \lambda, s) = \frac{1}{2} \ln \left[ \frac{4}{e^\gamma r_D} \right] - \frac{1}{2} E_1 \left[ \frac{-\lambda}{\omega(1-\omega)} t_D \right] + \frac{1}{2} E_1 \left[ \frac{-\lambda}{(1-\omega)} t_D \right] + s$$

$$p_D'(t_D, r_D, \omega, \lambda) = \frac{1}{2} \left[ 1 + \frac{1}{2} \exp \left[ \frac{-\lambda}{\omega(1-\omega)} t_D \right] - \frac{1}{2} \exp \left[ \frac{-\lambda}{(1-\omega)} t_D \right] \right]$$

These results are derived in the attached notes.

"Najurieta-ter Haar" Substitution Results:
- Plot the $p_D$ and $p_D'$ results obtained by using the Najurieta-ter Haar substitution in the "log approximation" form of the Laplace domain solution—use your solution from the Najurieta-ter Haar substitution problem.

**Aguilera Solution Approach:**
- Plot the $p_D$ and $p_D'$ results obtained from Aguilera's approximate solution for the case of pseudosteady-state interporosity flow—use your $p_D'$ solution from the Aguilera problem. You should vary the $c$-value in the $\tau_D = c/\lambda$ relation to obtain the "best" results—be sure to discuss this case in depth.

**Plotting Requirements:**
All plots are to be made using the following criteria:

- $\lambda = 5 \times 10^{-9}, 5 \times 10^{-6}, 5 \times 10^{-3}, 10^{-3}, 10^{-2}, 10^{-1}$, and $s = 0$ for $10^0 \leq t_D \leq 10^{10}$.
- Using each of the $p_D$ solutions, you are to prepare a semilog plot analogous to Fig. 5 given in the Warren and Root paper. Graph all solutions on the same plot.
- Using each of the $p_D$ and $p_D'$ solutions, you are to prepare a log-log plot of $p_D$ and $p_D'$ Graph each of the 4 solutions (i.e., for $p_D$ or $p_D'$) on the same plot, but graph the $p_D$ and $p_D'$ functions on separate plots.
Exercises: For your own practice/skills building—do NOT turn in!

Paper Reviews:

- You are to provide a critical and detailed review (at least 1 page) for the following paper(s):

For each paper you are to address the following questions: (Type or write neatly)

- **Problem:**
  - What is/are the problem(s) solved?
  - What are the underlying physical principles used in the solution(s)?

- **Assumptions and Limitations:**
  - What are the assumptions and limitations of the solutions/results?
  - How serious are these assumptions and limitations?

- **Practical Applications:**
  - What are the practical applications of the solutions/results?
  - If there are no obvious "practical" applications, then how could the solutions/results be used in practice?

- **Discussion:**
  - Discuss the author(s)'s view of the solutions/results.
  - Discuss your own view of the solutions/results.

- **Recommendations/Extensions:**
  - How could the solutions/results be extended or improved?
  - Are there applications other than those given by the author(s) where the solution(s) or the concepts used in the solution(s) could be applied?
Example Plots:

**Dual Porosity Reservoirs:** Semilog Plot (Fig. 5 — Warren and Root paper)

- **Legend:**
  - \( \alpha = 1 \times 10^{-3} \)
  - \( \alpha = 1 \times 10^{-4} \)
  - \( \alpha = 1 \times 10^{-5} \)

- **Legend:** Approximations
  - \( \alpha = 1 \times 10^{-3}, \lambda = 5 \times 10^{-6} \)
  - Aguilera Case
  - Najarieta-tar Haar Case

- **Both numerical inversion and semi-analytical solutions are plotted.**

**Dual Porosity Reservoirs:** Log-Log Plot—\( p_D \) cases (Warren and Root Model)

- **Model Legend:**
  - Solution for an Unfractured Well Produced at a Constant Flowrate in a Naturally-Fractured Reservoir, Pseudosteady-State Interporosity Flow (\( \alpha = 0, C_P = 0 \)).

- **Legend:** Approximations
  - \( \alpha = 1 \times 10^{-3}, \lambda = 5 \times 10^{-6} \)
  - Aguilera Case
  - Najarieta-tar Haar Case

- **Both numerical inversion and semi-analytical solutions are plotted.**
Dual Porosity Reservoirs: Log-Log Plot—$p_D'$ cases (Warren and Root Model)
"Stewart and Ascharsobbi" Type Curve: $p_{wd}$ vs. $t_{D\lambda/4}$ — Various $\lambda$ and $\omega$ Values

"Onur, Satman, and Reynolds" Type Curve: $p_{wd}$ vs. $t_{D\lambda/(1-\omega)}$ — Various $\lambda$ and $\omega$ Values
"Stewart and Aschrobbi" Type Curve: $p_{WD}$ vs. $t_D \lambda / 4$ — Various $\lambda$ and $\omega$ Values

"Onur, Satman, and Reynolds" Type Curve: $p_{WD}'$ vs. $t_D \lambda / (1-\omega)$ — Various $\lambda$ and $\omega$ Values
Derivation of the Warren and Root Approach: Pseudosteady-State Interporosity Flow

(from Petroleum Engineering 620 Course Notes — 1994)
Performance of Dual Porosity/Natually Fractured Reservoirs--Pseudosteady-State Interporosity Flow (Warren and Root Approach)

The analytical modelling of reservoir performance for naturally fractured/dual porosity reservoirs was initiated by Warren and Root, "The Behavior of Naturally Fractured Reservoirs," p. 245-55, SPE, 1963. This work gives us rigorous solutions in the Laplace domain for performance in an infinite-acting reservoir, as well as approximations in the real domain.

The true and idealized physical models for our "naturally fractured" reservoir is shown below.

![Diagram showing actual and model reservoirs with labels: VUGS, MATRIX, FRACTURE, MATRIX, FRACTURES]

**FIG. 1 — IDEALIZATION OF THE HETEROGENEOUS POROUS MEDIUM.**

Considering the flow of a "slightly compressible" liquid we can write the diffusivity equations for the "fracture" and "matrix" systems as

\[
\frac{k_f}{\mu} \frac{\partial p_f}{\partial t} = \frac{\partial}{\partial t} \left( \phi c_f \frac{dp_f}{dt} \right) \quad \text{Fracture Relation} \quad (1)
\]

\[
\frac{k_m}{\mu} \frac{\partial p_m}{\partial t} = \frac{\partial}{\partial t} \left( \phi c_m \frac{dp_m}{dt} \right) + q \quad \text{Matrix Relation} \quad (2)
\]

Solving Eqs. 1 and 2 for the flow rate, \( q \), and equating gives

\[
\frac{k_f}{\mu} \frac{\partial p_f}{\partial t} + \frac{k_m}{\mu} \frac{\partial p_m}{\partial t} = \frac{\partial}{\partial t} \left( \phi c_f \frac{dp_f}{dt} \right) + \frac{\partial}{\partial t} \left( \phi c_m \frac{dp_m}{dt} \right) \quad (3)
\]

where Eq. 3 is identical to Eq. 6 in the Warren and Root paper.
warren and Root suggested that the behavior of the matrix blocks could be considered to be at pseudosteady-state flow conditions. In this case the flowrate out of the matrix is

\[ q = -[\Phi t]_m \frac{\partial p_m}{\partial t} \quad (4) \]

In addition, warren and Root suggest an "interface" condition to describe the pressure drop across the fracture face. This relation is given as

\[ q = \alpha \frac{k_m}{m} (p_m - p_f) \quad (5) \]

where \( \alpha \) is a geometric factor given as \( \alpha = 4n(n+1)l^2 \), where \( n \) is the number of normal sets of fractures and \( l \) is a characteristic length. The \( \alpha \) parameter is not especially important because we will "lump" \( \alpha \) and other parameters into a physical constant to be obtained from data analysis.

Equate Eqs. 4 and 5, and solving for the matrix pressure derivative, \( \frac{\partial p_m}{\partial t} \), we have

\[ \frac{\partial p_m}{\partial t} = \frac{\alpha}{m \Phi t} (p_f - p_m) \quad \text{Interface Condition} \quad (6) \]

Substituting Eq. 4 into Eq. 1 gives us a fracture pressure relation that includes the matrix pressure change with respect to time. This result is

\[ \frac{\partial^2 p_f}{\partial t^2} = \frac{m}{k_f} \left[ \frac{\Phi t}{\Phi t_f} \frac{\partial p_f}{\partial t} + \frac{[\Phi t]_m}{\Phi t} \frac{\partial p_m}{\partial t} \right] \quad \text{Fracture Pressure} \quad (7) \]

Using the following definitions for dimensionless time and dimensionless pressure

\[ t_d = \frac{k_f}{m/[\Phi t]_f + [\Phi t]_m) \Phi t} t \quad \text{(darcy units)} \quad (8) \]

\[ p_d = \frac{2\pi k_f h (p_f - p)}{q_b m} \quad \text{(darcy units)} \quad (9) \]
Solving Eq. 8 for time, $t$, we have

$$t = \frac{\mu}{k_f} \left[ \left( \frac{\phi c^f_f + \phi c^f_m}{\phi c^f_m} \right) r_w^2 \right]$$

(10)

Solving Eq. 9 for the matrix pressure, $p_m$, gives

$$p_m = p_i - \frac{\frac{\mu}{k_f}}{\frac{2\pi}{k_f h}} q_{\text{Em}}$$

(11)

Solving Eq. 9 for the fracture pressure, $p_f$, we obtain

$$p_f = p_i - \frac{\frac{2\pi}{k_f h}}{\frac{2\pi}{k_f h}} q_{\text{Ef}}$$

(12)

Substituting Eqs. 10-12 into Eq. 6 gives

$$\frac{d}{dt} \left( \frac{Q_{\text{Em}}}{k_f} \right) = \frac{\alpha \lambda m}{k_f} \left[ \frac{1}{\mu} \left( \frac{p_i - \frac{2\pi}{k_f h} q_{\text{Em}}}{p_f} \right) - \frac{1}{2\pi} \right]$$

Reducing

$$\frac{1}{\frac{2\pi}{k_f h}} q_{\text{Em}} = \frac{\alpha \lambda m}{k_f} \left[ \frac{1}{\mu} \left( \frac{p_i - \frac{2\pi}{k_f h} q_{\text{Em}}}{p_f} \right) - \frac{1}{2\pi} \right]$$

cancelling and isolating the $\frac{Q_{\text{Em}}}{k_f}$ term gives

$$\frac{d}{dt} \left( \frac{Q_{\text{Em}}}{k_f} \right) = \frac{\alpha \lambda m}{k_f} \left( \phi c^f_f + \phi c^f_m \right) \left( p_i - p_m \right)$$

(13)

Defining the following "lumped" parameters we have

$$\lambda = \frac{\alpha \lambda m}{k_f} r_w^2$$

(Interporosity Flow Coefficient)

(14)

and

$$\omega = \frac{\phi c^f_f}{\phi c^f_f + \phi c^f_m}$$

(15)

or rewriting

$$1 - \omega = \frac{\phi c^f_m}{\phi c^f_f + \phi c^f_m}$$

(16)

Substituting Eqs. 14 and 16 into Eq. 13 gives

$$\frac{d}{dt} \left( \frac{Q_{\text{Em}}}{k_f} \right) = \lambda \left( p_f - p_m \right)$$

(Dimensionless Interface Condition)

(17)
Substituting Eqs. 10-12 into Eq. 6 gives
\[ \nabla^2 \left( \Psi - \frac{1}{2\pi} \frac{q_{bf}}{k_f} \right) \]
\[ = \frac{1}{k_f} \left[ \frac{1}{[m \chi_f]_{bf} + \chi_f} \right] \left( \frac{q_{bf} - \frac{1}{2\pi} \frac{q_{bf}}{k_f} P_m}{k_f} \right) \]
\[ \frac{\partial}{\partial t} \left( \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \right) \]
\[ + \chi_f \frac{\partial}{\partial t} \left( \frac{q_{bf} - \frac{1}{2\pi} \frac{q_{bf}}{k_f} P_m}{k_f} \right) \]

Reducing
\[ - \frac{1}{2\pi} \frac{q_{bf}}{k_f} \nabla^2 \Psi_{bf} = \frac{1}{2\pi} \frac{q_{bf}}{k_f} \left( \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \right) \]
\[ \frac{\partial}{\partial t} \left( \frac{1}{k_f} \right) \]

Cancelling and isolating the \( \nabla^2 \Psi_{bf} \) term
\[ \nabla^2 \Psi_{bf} = \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial P_m}{\partial t} \]
\[ + \chi_f \frac{\partial}{\partial t} \left( \frac{q_{bf} - \frac{1}{2\pi} \frac{q_{bf}}{k_f} P_m}{k_f} \right) \]

Substituting Eqs. 15 and 16 gives
\[ \nabla^2 \Psi_{bf} = \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial P_m}{\partial t} \]
\[ + \chi_f \frac{\partial}{\partial t} \left( \frac{q_{bf} - \frac{1}{2\pi} \frac{q_{bf}}{k_f} P_m}{k_f} \right) \]
\[ \text{Recalling the } \nabla^2 \text{ operator we have} \]
\[ \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial P_m}{\partial t} \]
\[ + \chi_f \frac{\partial}{\partial t} \left( \frac{q_{bf} - \frac{1}{2\pi} \frac{q_{bf}}{k_f} P_m}{k_f} \right) \]
\[ \text{Defining a "dimensionless radius" we have} \]
\[ r = \frac{r}{r_m} \]
\[ \text{or} \]
\[ r = r_m \]
\[ \text{Substituting Eq. 21 into the } \nabla^2 \Psi_{bf} \text{ operator we obtain} \]
\[ \nabla^2 \Psi_{bf} = \frac{1}{(k_f)} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial P_m}{\partial t} \]
\[ \text{Expanding} \]
\[ \nabla^2 \Psi_{bf} = \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial q_{bf}}{\partial t} \]
\[ \text{Substituting Eq. 22 into Eq. 16 and cancelling the } r_m^2 \text{ gives} \]
\[ \frac{1}{k_f} \left( \frac{m \chi_f + \chi_m}{k_f} \right) \frac{\partial q_{bf}}{\partial t} = \omega \frac{\partial q_{bf}}{\partial t} + (1-\omega) \frac{\partial P_m}{\partial t} \]
Summarizing our major results so far

\[ \frac{L^{Dm}}{D^0} = \lambda \frac{(\rho_f - \rho_m)}{(1-w)} \]  
\text{(Dimensionless Interface Condition)} \tag{17}

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_D}{dr_D} \right] = w \frac{d\rho_f}{dr_D} + (1-w) \frac{d\rho_m}{dr_D} \]  
\text{(Dimensionless Fracture Pressure)} \tag{23}

Taking the Laplace transform of Eq. 17 we have

\[ \mathcal{L}\left( \left[ \rho_f^{(w)} - \rho_m^{(w)} \right] \right) = \lambda \left[ \mathcal{L}\left( \rho_f^{(w)} - \rho_m^{(w)} \right) \right] \]

or

\[ \left[ \mathcal{L}\left( (1-w) \right) + \lambda \right] \mathcal{L}\left( \rho_f^{(w)} \right) = \lambda \mathcal{L}\left( \rho_f^{(w)} \right) \]

and finally

\[ \mathcal{L}\left( \rho_f^{(w)} \right) = \frac{\lambda}{\lambda + (1-w)w} \mathcal{L}\left( \rho_f^{(w)} \right) \tag{24} \]

And taking the Laplace transform of Eq. 23 gives

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_D}{dr_D} \right] = w \left[ \mathcal{L}\left( \rho_f^{(w)} - \rho_m^{(w)} \right) + (1-w) \mathcal{L}\left( \rho_m^{(w)} - \rho_m^{(w)} \right) \right] \]


or

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_f}{dr_D} \right] = w \mathcal{L}\left( \rho_f^{(w)} \right) + (1-w) \mathcal{L}\left( \rho_m^{(w)} \right) \tag{25} \]

Substituting Eq. 24 into Eq. 25 gives

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_f}{dr_D} \right] = w \frac{\mathcal{L}\left( \rho_f^{(w)} \right) + (1-w) \frac{\lambda}{\lambda + (1-w)w}}{\frac{\lambda + (1-w)w}{\lambda + (1-w)w}} \mathcal{L}\left( \rho_f^{(w)} \right) \]

Collecting

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_f}{dr_D} \right] = \frac{w \left[ \lambda + (1-w)w \right] + (1-w) \lambda \mathcal{L}\left( \rho_f^{(w)} \right)}{\lambda + (1-w)w} \]

or finally

\[ \frac{1}{R_D} \frac{d}{dr_D} \left[ R_D \frac{d\rho_f}{dr_D} \right] = \frac{\lambda + (1-w)w}{\lambda + (1-w)w} \mathcal{L}\left( \rho_f^{(w)} \right) \tag{26} \]

Using shorthand notation we define

\[ f(w) = \frac{\left[ \lambda + w(1-w) \right]}{\lambda + (1-w)w} \]  
\text{(27)}
Substituting Eq. 27 into Eq. 26 we have
\[
\frac{1}{r_0} \frac{d}{dr_0} \left[ r_0 \frac{d \Phi_f(u)}{dr_0} \right] = u f(u) \Phi_f(u)
\]  
(28)

Multiplying through Eq. 28 by \( r_0^2 \) gives
\[
r_0^2 \frac{d}{dr_0} \left[ r_0 \frac{d \Phi_f(u)}{dr_0} \right] = u f(u) r_0^2 \Phi_f(u)
\]  
(29)

Defining a variable of substitution, \( z \), as follows
\[
z = \sqrt{u f(u)} \ \frac{r_0}{r_0} = \sqrt{a} \ r_0
\]  
(30)
or
\[
r_0 = \frac{1}{\sqrt{a}} z
\]  
(31)

where
\[
\sqrt{a} = \sqrt{u f(u)}
\]  
(32)

Applying the chain rule on the \( r_0 d()/dr_0 \) terms in Eq. 29 gives
\[
r_0 z^2 \frac{d}{dz} \left[ r_0 \frac{z^2}{r_0} \frac{d \Phi_f(z)}{dr_0} \frac{dz}{r_0} \right] = z^2 \Phi_f(z)
\]  
(33)

where
\[
z^2 \frac{dz}{dr_0} = \sqrt{a} \frac{d \Phi_f(u)}{dr_0}
\]  
(34)

Substituting Eqs. 31 and 34 into Eq. 33 we obtain
\[
z^2 \frac{d}{dz} \left[ \sqrt{a} \frac{z^2}{r_0} \frac{d \Phi_f(z)}{dz} \right] = z^2 \Phi_f(z)
\]  
(35)

Cancelling the "a" terms we have
\[
z \frac{d}{dz} \left[ z \frac{d \Phi_f(z)}{dz} \right] = z^2 \Phi_f(z)
\]  
(36)

Expanding the left-hand-side of Eq. 35 gives
\[
z^2 \frac{d^2 \Phi_f(z)}{dz^2} + z \frac{d \Phi_f(z)}{dz} = z^2 \Phi_f(z)
\]  
(37)

From Abramowitz and Stegun, "Handbook of Mathematical Functions," (p. 374, Eq. 9.1.1), the modified Bessel differential equation is given by
\[
z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + \left( z^2 + \nu^2 \right) w = 0
\]  
(37)

The general solution of Eq. 37 is given by
\[
w = A I_{\nu}(z) + B K_{\nu}(z)
\]  
(38)
where the functions $I_v(z)$ and $K_v(z)$ are the modified Bessel functions of the first and second kinds, respectively. This implies that the general solution of Eq. 37 is
\begin{equation}
\tilde{D}_f(z) = A I_0(z) + B K_0(z)
\end{equation}
Substituting Eq. 38 into Eq. 39 we have
\begin{equation}
\tilde{D}_f(r_0,\nu) = A I_0(\sqrt{\nu} r_0) + B K_0(\sqrt{\nu} r_0)
\end{equation}
where
\begin{equation}
\sqrt{\nu} = \sqrt{\mu t / \omega}
\end{equation}
Substituting Eq. 32 into Eq. 40 gives
\begin{equation}
\tilde{D}_f(r_0,\nu) = A I_0(\sqrt{\nu} r_0) + B K_0(\sqrt{\nu} r_0)
\end{equation}
Comparing to the homogeneous case we have
\begin{equation}
\tilde{D}_h(r_0,\nu) = A I_0(\sqrt{\nu} r_0) + B K_0(\sqrt{\nu} r_0)
\end{equation}
The governing identity that relates the homogeneous reservoir solution in the real domain, $\tilde{D}_h(r_0, t)$, with the Laplace transform solution for the naturally fractured/dual porosity reservoir case, $\tilde{D}_f(r_0, \nu)$, is given by
\begin{equation}
\tilde{D}_f(r_0, \nu) = f(t) \int_0^\infty \tilde{D}_h(r_0, \tau) \exp[-\nu \mu t / \omega] d\tau
\end{equation}
where Eq. 43 was proposed by Thompson, Manrique, and Jelmer, "Efficient Algorithms for Computing the Bounded Reservoir Horizontal Well Pressure Response," paper SPE 21827, Denver 1991.

As with previous efforts, we require the $\frac{d}{dr_0} [\tilde{D}_f(r_0, \nu)]$ term. Using our "a" notation, the derivative of Eq. 40 is
\begin{equation}
\frac{d}{dr_0} [\tilde{D}_f(r_0, \nu)] = A \sqrt{\nu} I_1(\sqrt{\nu} r_0) - B \sqrt{\nu} K_1(\sqrt{\nu} r_0) \quad [\sqrt{\nu} = \sqrt{\nu t / \omega}]
\end{equation}
Summarizing the relevant results (where $\sqrt{\nu} = \sqrt{\nu t / \omega}$)
\begin{equation}
\tilde{D}_f(r_0, \nu) = A I_0(\sqrt{\nu} r_0) + B K_0(\sqrt{\nu} r_0)
\end{equation}
and
\begin{equation}
\frac{d}{dr_0} [\tilde{D}_f(r_0, \nu)] = A \sqrt{\nu} r_0 I_1(\sqrt{\nu} r_0) - B \sqrt{\nu} r_0 K_1(\sqrt{\nu} r_0)
\end{equation}
where Eqs. 40 and 44 are identical to the homogeneous reservoir case where $\sqrt{\nu t} = 1$. This result suggests that substituting $\sqrt{\nu t / \omega}$ for $\sqrt{\nu}$ in the homogeneous solution will yield the
dual porosity- naturally fractured reservoir solution. We note that Eq. 43 also verifies this concept.

Solution for an Infinite-Acting Reservoir:
The solution for a vertical well producing at a constant flowrate in an infinite-acting homogeneous reservoir was derived previously. The "like source" solution is given as

\[ \Phi_{th}(r_0, \mu) = \frac{1}{\mu} k_0 \sqrt{\mu r_0} \mu \]  \hspace{1cm} (45)

replacing \( \sqrt{\mu} \) by \( \sqrt{\mu f(u)} \) gives us the dual porosity- naturally fractured reservoir solution. This result is

\[ \Phi_{df}(r_0, \mu) = \frac{1}{\mu} k_0 \sqrt{\mu f(u) r_0} \mu \]  \hspace{1cm} (46)

where

\[ f(u) = \frac{\lambda + \omega (1-u) \mu}{\lambda + (1-u) \mu} \]  \hspace{1cm} (47)

Obviously, Eq. 46 cannot be analytically inverted due to the algebraic form of the function, \( f(u) \). However, following the work of Warren and Root, we will use a logarithmic approximation of \( k_0(x) \). Our previous developments show that

\[ k_0(x) \approx \frac{1}{2} \ln \left( \frac{4}{e^{2x} x^2} \right) \quad \text{as } x \to 0 \]  \hspace{1cm} (47)

Substituting Eq. 47 into Eq. 46 gives

\[ \Phi_{df}(r_0, \mu) = \frac{1}{2\mu} \ln \left( \frac{4}{e^{2\gamma} r_0^2 \mu f(u)} \right) \]  \hspace{1cm} (48)

Expanding the logarithmic term gives

\[ \Phi_{df}(r_0, \mu) = \frac{1}{2\mu} \ln \left( \frac{4}{1} \right) - \frac{1}{2\mu} \ln(u) - \frac{1}{2\mu} \ln[f(u)] \]  \hspace{1cm} (49)

Isolating the \( f(u) \) term

\[ \ln[f(u)] = \ln \left[ \frac{\lambda + \omega (1-u) \mu}{\lambda + (1-u) \mu} \right] \]

Dividing thru the logarithmic argument by \( \lambda \) gives

\[ \ln[f(u)] = \ln \left[ \frac{1 + \omega (1-u) \mu}{1 + (1-u) \mu} \right] \]
Expanding the logarithm we obtain
\[ \ln [f(w)] = \ln[1 + \frac{\omega(\omega-w)}{\lambda}] - \ln[1 + \frac{(\omega-w)}{\lambda}] \] (50)

Substituting Eq. 50 into Eq. 49 gives
\[ \tilde{f}_f(t, \omega) = \frac{1}{z} \left[ \frac{1}{M} \ln \left( \frac{4}{\omega^2 \omega^2} \right) - \ln(\omega) + \frac{1}{M} \ln \left( \frac{1 + \omega(\omega-w)}{\lambda} \right) + \frac{1}{M} \ln \left( \frac{1 + (\omega-w)}{\lambda} \right) \right] \] (51)

The analytical inversion of Eq. 51 is possible using the table of transforms given below

<table>
<thead>
<tr>
<th>( \hat{f}(\omega) )</th>
<th>( f(t) )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\mu} \ln \left( \frac{4}{\omega^2 \omega^2} \right) ) or ( \frac{1}{\mu} ) constant</td>
<td>( \frac{1}{\omega^2} \ln(\omega) ) or ( \ln(\omega^2) ) constant</td>
<td>(trivial) Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.1, p. 1021.</td>
</tr>
<tr>
<td>( \frac{1}{\mu} \ln(\omega) ) or ( \ln(\omega^2) )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.98, p. 1027. and Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 4.1, p. 248.</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{\mu} \ln \left( \frac{1 + \omega^2}{\alpha} \right) ) or ( \frac{1}{\mu} ) constant</td>
<td>-Ei(-( \alpha t )) or Ei(( \alpha t ))</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 29.3, Eq. 29.3.103, p. 1027 and Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 4.1.11, p. 260.</td>
</tr>
</tbody>
</table>

Using the table given above, the inverse Laplace transform of Eq. 51 is
\[ \tilde{f}_f(t, \omega) = \frac{1}{z} \left[ \ln \left( \frac{4}{\omega^2 \omega^2} \right) + \ln \left( \frac{\omega^2}{\alpha} \right) - E_i \left( \frac{\lambda}{\omega(\omega-w)} \right) + E_i \left( \frac{\lambda}{(\omega-w)} \right) \right] \] (52)
Collecting terms in Eq. 52 gives

$$P_d^f(r_0, t_0) = \frac{1}{2} \ln\left( \frac{4}{e^2} \frac{t_0}{r_p^2} \right) - \frac{1}{2} E_1\left( \frac{\lambda}{w(1-w)} \frac{t_0}{r_p^2} \right) + \frac{1}{2} E_1\left( \frac{\lambda}{(1-w)} t_0 \right)$$  \hspace{1cm} (53)

We would also like to use Eq. 53 to determine the "well testing" derivative. This derivative is given by:

$$P_d^f(r_0, t_0) = t_0 \frac{d}{dt} \frac{\partial}{\partial t} [P_d^f(r_0, t_0)]$$  \hspace{1cm} (54)

The general forms of the time derivatives are given below:

$$f(t) = \frac{d}{dt} [f(t)]$$

$$\ln(at)$$

or

$$\ln(a) + \ln(t)$$

$$E_1(t)$$

$$E_1(at)$$

$$- \frac{1}{t} \exp(-t)$$

$$\frac{d}{dt} \left[ E_1(\xi) \right] = \frac{d}{d\xi} \left[ E_1(\xi) \right]$$

if \( \xi = at \)

$$\frac{d}{dt} \left[ E_1(at) \right] = a \left( - \frac{1}{at} \exp(-at) \right)$$

or

$$\frac{d}{dt} \left[ E_1(at) \right] = - \frac{1}{t} \exp(-at)$$

Using these relations, the well testing derivative is given by:

$$P_d^f(r_0, t_0) = t_0 \left[ \frac{1}{t_0} + \frac{1}{t_0} \exp\left( -\frac{\lambda}{w(1-w)} t_0 \right) - \frac{1}{t_0} \exp\left( -\frac{\lambda}{(1-w)} t_0 \right) \right]$$

Expanding

$$P_d^f(r_0, t_0) = \frac{1}{z} + \frac{1}{z} \exp\left( -\frac{\lambda}{w(1-w)} t_0 \right) - \frac{1}{z} \exp\left( -\frac{\lambda}{(1-w)} t_0 \right)$$  \hspace{1cm} (55)
Summary of Results: Warren and Root Solution for a Well in an Infinite-Acting, Dual Porosity Reservoir

Warren and Root Interporosity Flow Function: \( f(w) \)
\[
f(w) = \frac{\lambda + w(1-w)\mu}{\lambda + (1-w)\mu}
\]  

Thompson, Manrique, and Jelkert Transform Relation:
\[
\tilde{P}_d(r_0,\mu) = f(w) \int_0^\infty \tilde{P}_h(r_0,\tau) \exp\left[-w\mu\tau\right] d\tau
\]  

Where Eq. 43 is valid for any case, regardless of geometry, well type (vertical, fractured, or horizontal wells), or flow regime (transient, post-transient, and boundary-dominated flow conditions).

Radial Flow Solutions - Infinite-Acting Reservoirs:
The line source solution is given by
\[
\tilde{P}_d(r_0,\mu) = \frac{1}{r_0} k_0\left(\ln(w) - \ln(w)\right)
\]  

And the "log approximation" is given by
\[
\tilde{P}_d(r_0,\mu) = \frac{1}{2w} \ln\left(\frac{4}{e^{2r_0^2/\mu}}\right)
\]

Combining Eqs. 27 and 48 we obtain
\[
\tilde{P}_d(r_0,\mu) = \frac{1}{z} \left[\frac{1}{\mu} \ln\left(\frac{4}{e^{2r_0^2/\mu}}\right) - \frac{1}{\mu} \ln\left(\frac{1}{\mu}\right) - \frac{1}{\mu} \ln\left(\frac{1+w(1-w)\mu}{\lambda}\right) + \frac{1}{\mu} \ln\left(\frac{1+(1-w)\mu}{\lambda}\right)\right]
\]  

Where the inverse Laplace transform of Eq. 51 is given by
\[
\tilde{P}_d(r_0,t_0) = \frac{1}{z} \ln\left(\frac{4}{e^{2r_0^2/\mu}}\right) - \frac{1}{z} \frac{\epsilon_1}{\epsilon_1} \left(\frac{\lambda}{\mu t_0}\right) + \frac{1}{z} \frac{\epsilon_1}{\epsilon_1} \left(\frac{\lambda}{t_0}\right)
\]  

And the "well testing" derivative is given by
\[
\tilde{P}_d'(r_0,t_0) = \frac{1}{z} \left[\ln\left(\frac{4}{e^{2r_0^2/\mu}}\right) - \ln\left(\frac{1}{\mu}\right) - \ln\left(\frac{1+w(1-w)\mu}{\lambda}\right) + \ln\left(\frac{1+(1-w)\mu}{\lambda}\right)\right]
\]