Objectives: (things you should know and/or be able to do)

Infinite-Acting Reservoir Case:

- Be able to develop the real domain (time) solution (i.e., the Exponential Integral solution) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition—using both the Laplace transform as well as the Boltzmann transform approaches. Also be able to develop the "log-approximation" for this solution.

Boltzmann Transform of the Diffusivity Equation:

\[ \frac{d^2 p_D}{d \varepsilon_D^2} + \left[ 1 + \frac{1}{\varepsilon_D} \right] \frac{dp_D}{d \varepsilon_D} = 0 \quad \text{(infinite-acting reservoir case only)} \]

- Real domain (time) solutions: (Continued)

  "Exponential Integral" Solution for the Diffusivity Equation:

  \[ p_D(t_D, r_D) = \frac{1}{2} E_1 \left[ \frac{r_D^2}{4t_D} \right] \]

  "Log Approximation" Solution for the Diffusivity Equation:

  \[ p_D(t_D, r_D) = \frac{1}{2} \ln \left[ \frac{4}{e \gamma} \frac{t_D}{r_D^2} \right] \]
Solution of the Radial Flow Diffusivity Equation via the Boltzmann Transform

from Department of Petroleum Engineering Course Notes (1994)
Radial Flow Solution for an Infinite-Acting Homogeneous Reservoir

Boltzmann Transform Approach

This method has been demonstrated by a variety of authors -- the approach we choose was presented by J.L. Johnston in the 2nd edition of the Lee Well Test text.

The basic partial differential equation is given in dimensionless form as

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial \theta}$$

where $\frac{\partial \phi}{\partial \theta} = 0$.

This condition is given as

$$\phi |_{r=\infty} = 0$$

The constant rate inner boundary condition is

$$\frac{\partial \phi}{\partial r} |_{r=r_0} = 0$$

and the "infinite-acting" outer boundary condition is given by

$$\frac{\partial \phi}{\partial r} |_{r=\infty} = 0$$

Rewriting Eq. 1 we have

$$\frac{1}{r^2} \left[ r^2 \frac{\partial \phi}{\partial r} \right] = \frac{\partial \phi}{\partial \theta}$$

The Boltzmann transform variable, $\xi$, is defined as

$$\xi = \frac{r^2}{\zeta_0^2}$$

where for our problem we have

$$\xi = \frac{r^2}{\zeta_0^2}$$

which yields

$$\frac{\partial \xi}{\partial r} = \frac{r}{\xi} \frac{\partial \xi}{\partial \xi}$$

Expanding the $\frac{\partial \phi}{\partial \theta}$ term we have

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial \xi}$$

Applying the chain rule,

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \theta}$$

which combined with Eq. 11 gives

$$\frac{\partial \phi}{\partial \xi} - \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} = 0$$

Expanding

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \phi}{\partial \xi}$$

Isolating terms

$$\frac{\partial \phi}{\partial \xi} = 0$$

Dividing through by $(\frac{\partial \phi}{\partial \xi})^2$ gives

$$\frac{\partial \phi}{\partial \xi} + \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

Reducing the $\frac{\partial \phi}{\partial \xi}$ term we have

$$\frac{\partial \phi}{\partial \xi} + \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

Which can be further reduced to

$$\frac{\partial \phi}{\partial \xi} + \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

Completing the factorization of the $\frac{\partial \phi}{\partial \xi}$ gives

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

Using Eq. 10 we take the following derivatives

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{(\frac{\partial \phi}{\partial \xi})^2} \left[ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \xi} \right] = 0$$

(Solution of the Radial Flow Diffusivity Equation via the Boltzmann Transform)
Substituting Eqs. 15-18 into Eq. 12 gives

\[ \frac{\partial^2 \psi}{\partial \xi^2} + \left[ \frac{1}{1 + \frac{r_0}{r}} - \frac{1}{r} \right] \frac{\partial \psi}{\partial r} = 0 \]

which can be rearranged to yield

\[ \frac{d^2 \psi}{d \xi^2} = -\frac{1}{\xi} \]

Making the following variable of substitution

\[ \eta = \frac{d \psi}{d \xi} \]

Substituting Eq. 20 into Eq. 16, and noting the use of ordinary derivatives

\[ \frac{d \eta}{d \xi} + \left[ 1 + \frac{1}{\xi} \right] \eta = 0 \]

where \( \xi > 0 \), \( \psi \to 0 \) as \( \xi \to \infty \), which gives

\[ \psi(\xi = 0) = 0 \]

We must now establish the initial and boundary conditions in terms of the Boltzmann transform variable, \( \xi \). Recalling the initial condition, Eq. 4, we have

\[ \psi_0 (r_0, t_0 = 0) = 0 \]

or simply

\[ \psi_0 (t_0 = 0) = 0 \]

where \( t_0 > 0 \), \( \psi_0 \to 0 \) as \( t_0 \to \infty \), which yields

\[ \psi_0 (t_0 = 0) = 0 \]

Recalling the outer boundary condition, Eq. 7, we have

\[ \psi_0 (r_0 = \infty, t_0) = 0 \]

or as \( r_0 \to \infty \), \( \psi_0 \to 0 \), which yields

\[ \psi_0 (r_0 = \infty) = 0 \]

Which reduces to

\[ \psi_0 \]

where \( \alpha = \exp[\xi] \), i.e., the constant of integration. Recalling Eq. 20 and combining gives

\[ \frac{d \psi_0}{d \xi} = \alpha \exp[-\xi] \]

or

\[ \alpha = \frac{1}{\xi} \]

Substitution of Eq. 25 into Eq. 19 gives

\[ \alpha \lim_{\xi \to 0} \left[ \exp[-\xi] \right] = \frac{1}{2} \]

or

\[ \alpha = \frac{1}{2} \]

Substitution of Eq. 26 into Eq. 22 gives

\[ \frac{d \psi_0}{d \xi} = \frac{1}{2} \exp[-\xi] \]

Resulting in

\[ \psi_0 (r_0 = \infty) = 0 \]

or

\[ \psi_0 (r_0 = \infty) = 0 \]
Separating and integrating Eq. 25 gives

$$
\int_{r_0}^{R} \frac{dP}{dP} = -\frac{1}{Z} \int_{0}^{r_0} \frac{1}{\phi_0} e^{-t_0} \, d\phi
$$

where we note that $P = 0$ at $t = \infty$ is the initial/outer boundary condition. Completing the integration and reversing the limits we have

$$
P_0 = \frac{1}{Z} \int_{0}^{\infty} \frac{1}{\phi_0} e^{-t_0} \, d\phi
$$

We note that the integral in Eq. 26 is the exponential integral, $E_i(x)$, which is given by

$$
E_i(x) = \int_{x}^{\infty} \frac{1}{y} e^{-y} \, dy
$$

Combining Eqs. 26 and 27 gives our final result

$$
P_0(\phi_0, t_0) = \frac{1}{Z} E_i(\frac{\phi_0^2}{4t_0})
$$