Objectives: (things you should know and/or be able to do)

Real Domain Solutions (via Inversion of the Laplace Domain Solutions):
- Be able to derive the following particular solutions in the real domain using the appropriate Laplace transform solutions for an unfractured well produced at a constant flowrate in a homogeneous reservoir for the following outer boundary conditions:
  - "Infinite-acting" reservoir behavior (line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} \ E_1\left(\frac{r_D^2}{4t_D}\right) \]
  - "Infinite-acting" reservoir behavior (the so-called "log approximation," also a line source solution)
    \[ p_D(t_D, r_D) = \frac{1}{2} \ \ln\left(\frac{4}{r_D} \ \frac{t_D}{r_D^2}\right) \]
  - Bounded circular reservoir — "no-flow" at the outer boundary
    \[ p_D(t_D, r_D, r_e) = \frac{1}{2} \ E_1\left(\frac{r_D^2}{4t_D}\right) - \frac{1}{2} \ E_1\left(\frac{r_e^2}{4t_D}\right) + \frac{2t_D}{r_e^2} \ \exp\left(-\frac{r_e^2}{4t_D}\right) + \frac{r_D^2}{2r_e^2} - \frac{1}{4} \ \exp\left(-\frac{r_e^2}{4t_D}\right) \]
    and its "well testing" derivative function, \( p_D' = d/dt_D[p_D(r_D, t_D)] \) is given by
    \[ p_D'(t_D, r_D, r_e) = \frac{1}{2} \ \exp\left(-\frac{r_D^2}{4t_D}\right) + \frac{2t_D}{r_e^2} \ \exp\left(-\frac{r_e^2}{4t_D}\right) + \frac{1}{2t_D} \left[\frac{r_D^2}{4} - \frac{r_e^2}{8}\right] \ \exp\left(-\frac{r_e^2}{4t_D}\right) \]
Solutions of the Radial Flow Diffusivity Equation in the Real Domain

from Department of Petroleum Engineering Course Notes (1994)
Solutions for a Bounded Circular Reservoir: Infinite-Acting, No-Flow, and Constant Pressure Boundary Cases

The Laplace transform solutions under consideration are:

a. Infinite-Acting Reservoir Case:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{M \sqrt{v_a k_a}} \left( -\frac{1}{r^2} + \frac{1}{r} \right) \ln \left( \frac{r}{r_0} \right) \tag{1}
\]

b. No-Flow Boundary Case:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{M \sqrt{v_a k_a}} \left( I_1(r_0) + I_0(r_0) \right) \tag{4}
\]

where for \( \mu \to 0 \) Eq. 4 reduces to:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{M \sqrt{v_a k_a}} I_0(r_0) \tag{5}
\]

a. Constant Pressure Outer Boundary Case:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{M \sqrt{v_a k_a}} \left( I_1(r_0) - I_0(r_0) \right) \tag{6}
\]

where for \( \mu \to 0 \) Eq. 6 reduces to:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{M \sqrt{v_a k_a}} \ln \left( \frac{r}{r_0} \right) \tag{7}
\]

b. Line Source Solution:

Recalling Eq. 2 we have:

\[
\frac{\partial}{\partial \mu} \left( \frac{\phi_0}{r^2} \right) = \frac{1}{k_a} \left( \frac{\partial}{\partial r} \phi_0 \right) \tag{2}
\]

Multiplying through Eq. 2 by the Laplace transform parameter, \( \mu \), gives:

\[
\frac{\partial}{\partial \mu} \left( \frac{\phi_0}{r^2} \right) = \frac{1}{k_a} \frac{\partial}{\partial \mu} \phi_0 \tag{3}
\]

Recalling the time derivative theorem for Laplace transforms we have:

\[
\mathcal{L} \left\{ \frac{\partial}{\partial t} \phi(t) \right\} = \frac{d}{d \mu} \left\{ \mathcal{L} \left( \phi(t) \right) \right\} \tag{9}
\]

assuming that \( \phi(t=0) \), which is true by our initial condition, we can similarly write:

\[
\frac{d}{d \mu} \left[ \mathcal{L} \left( \phi(t) \right) \right] = \mathcal{L} \left\{ \frac{\partial}{\partial t} \phi(t) \right\} \tag{10}
\]

or in terms of our problem we have:

\[
\frac{d}{d \mu} \left[ \hat{\phi}_0(r, \mu) \right] = \mathcal{L} \left\{ \frac{\partial}{\partial r} \phi_0 \right\} \tag{11}
\]

Combining Eqs. 10 and 13:

\[
\frac{d}{d \mu} \left[ \hat{\phi}_0(r, \mu) \right] = \mathcal{L} \left\{ \frac{\partial}{\partial r} \phi_0 \right\} \tag{14}
\]

Inversion of Eqs. 2 and 10 is accomplished by the use of Laplace transform tables, where the results of inversion are given below:

\[
\begin{align*}
\frac{1}{\sqrt{\pi \mu}} & \quad \frac{1}{\sqrt{\pi \mu}} \exp \left( -\frac{a^2}{4 \mu} \right) \tag{12} \\
& \quad \frac{1}{\sqrt{\pi \mu}} \exp \left( -\frac{a^2}{4 \mu} \right) \tag{12}
\end{align*}
\]

References:


Making the appropriate substitutions:

\[
\hat{\phi}_0(r, \mu) = \frac{1}{\sqrt{\pi \mu}} \exp \left( -\frac{a^2}{4 \mu} \right) \tag{15}
\]

\[
\frac{d}{d \mu} \left[ \hat{\phi}_0(r, \mu) \right] = \frac{1}{\sqrt{\pi \mu}} \exp \left( -\frac{a^2}{4 \mu} \right) \tag{16}
\]
Defining the so-called "well testing derivative" we have

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{2z} \frac{d}{dt} \left[ \ln \left( \frac{4}{e^{2\pi} r_0^2} \right) \right] \]  

(1a)

Substituting Eq. 14 into Eq. 17 we have

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{2z} \exp \left( -\frac{r_0^2}{4zd} \right) \]  

(18)

Isolating the \( t_0 \) term in Eq. 20 we have

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{z} \ln(t_0) + \frac{1}{z} \ln \left( \frac{4}{e^{2\pi} r_0^2} \right) \]  

(21)

Substituting Eq. 21 into Eq. 17 to determine the well testing derivative we have

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = t_0 \left[ \frac{d}{dt} \left( \frac{1}{z} \ln(t_0) \right) \right] + t_0 \left[ \frac{d}{dt} \left( \frac{1}{z} \ln \left( \frac{4}{e^{2\pi} r_0^2} \right) \right) \right] \]

which reduces to

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{z} \]  

(22)

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{z} \left[ \frac{1}{z} \right] = \frac{1}{z} \]

(23)

Solution for a No-Flow Outer Boundary:

It is not possible to invert the complete solution (Eq. 4) for this case so we will attempt an approximate solution of the line source form (Eq. 5). Recalling Eq. 5 we have

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{z} \left( \frac{1}{z} \right) \ln \left( \frac{4}{e^{2\pi} r_0^2} \right) \]

(5)

We immediately recognize that the first term in Eq. 5 is the solution for an infinite-active-reservoir, and given the linearity of the inverse Laplace transform, we can invert Eq. 5 to yield

\[ \frac{\partial \theta(r,t)}{\partial \ln(t)} = \frac{1}{z} \ln \left( \frac{4}{e^{2\pi} r_0^2} \right) \]

(24)

So what is our strategy to invert the second term in Eq. 23?

First we will use recursion relations to express the \( k_n(z) \) Bessel functions then consider a two-term expansion of the resulting \( \frac{I_n(z)}{I_0(z)} \) ratio. Recall that as \( z \to 0 \) that \( I_n(z) \to 1 \) and \( I_0(z) \to 0 \), which permits polynomial expansions.

From Abramowitz and Stegun, Handbook of Mathematical Functions, (Eq. 9.6.15, p. 372), we have

\[ I_0(z)k_0(z) + I_1(z)k_1(z) = \frac{1}{z} \]

using \( n=0 \)

\[ I_0(z)k_0(z) + I_1(z)k_1(z) = \frac{1}{z} \]

(25)

(Solutions of the Radial Flow Diffusivity Equation in the Real Domain)
Solutions of the Radial Flow Diffusivity Equation in the Real Domain

Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 2 — Solution of the Radial Flow Diffusivity Equation (Real Domain Solutions)

Using \( n = 1 \) we have

\[
I_n(z) = I_{n-1}(z) + z I_{n-1}(z) = \frac{1}{z}
\]

(26)

Equating Eqs. 25 and 26 we have

\[
I_{n-1}(z) k_0(z) + I_{n-1}(z) k_0(z) = I_{n-1}(z) k_0(z) + I_{n-1}(z) k_0(z)
\]

or solving for \( k_0(z) \)

\[
k_0(z) = \frac{I_{n-1}(z)}{I_{n-1}(z)} = \frac{k_0(z) - k_0(z)}{z}
\]

(27)

Recalling the first recursion relation in Eq. 9.6.21, p. 376, Abramowitz and Stegun, Handbook of Mathematical Functions, in terms of \( I_n(z) \)

\[
I_n(z) - I_{n-1}(z) = \frac{2n}{z} I_{n-1}(z)
\]

for \( n = 1 \) we have

\[
I_1(z) - I_0(z) = \frac{2}{z} I_0(z)
\]

rearranging

\[
I_1(z) = \frac{z}{2} I_0(z)
\]

(28)

Substituting Eq. 28 into Eq. 27 we have

\[
k_0(z) = \frac{z}{2} [k_0(z) - k_0(z)]
\]

(29)

Establishing the \( I_0(z) / I_1(z) \) ratio using Eqs. 33 and 34 we have

\[
\begin{align*}
I_0(z) &= \frac{z}{(1 + a^2 z)} \\
I_1(z) &= \frac{a}{1 + a^2 z}
\end{align*}
\]

(30)

assuming \( a^2 z < 1 \) we can express \((1 + a^2 z)^{-1}\) as a binomial series

\[
(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots
\]

(1x1.6.1)

using a two term expansion of \((1 + a^2 z)^{-1}\) we have

\[
(1 + a^2 z)^{-1} = 1 - a^2 z
\]

(31)

Substituting Eq. 31 into Eq. 30 we have

\[
I_0(z) = \frac{z}{(1 + a^2 z)(1 - a^2 z)}
\]

(32)

Expanding

\[
I_0(z) = \frac{z}{a} \left( 1 + \frac{b^2}{4} - \frac{a^2}{4} - \frac{a^2 b^2}{32} \right)
\]

(33)

neglecting the \( a^2 b^2 / 32 \) term we have

\[
I_0(z) = \frac{z}{a} \left( 1 + \frac{b^2}{4} \right)
\]

(34)

Recalling that \( b = \sqrt{\mu} \) and \( a = \sqrt{\mu} \) and substituting Eq. 34 into Eq. 30

\[
p_r(t, r_p) = \frac{q_{in}(t) f(t)}{4\pi k_0} \left[ \frac{1}{2} k_0 f(t) - k_0 f(t) \right] \left( 1 - \frac{a^2 z^2 + b^2 z^2}{4} \right)
\]

(35)

Cancelling the \( a^2 z^2 \) and \( b^2 z^2 \) terms and using \( a = \sqrt{\mu} \) and \( b = \sqrt{\mu} \)

\[
p_r(t, r_p) = \frac{q_{in}(t) f(t)}{4\pi k_0} \left[ \frac{1}{2} k_0 f(t) - k_0 f(t) \right] \left( 1 - \frac{a^2 z^2 + b^2 z^2}{4} \right)
\]

Continuing the expansion

\[
p_r(t, r_p) = \frac{q_{in}(t) f(t)}{4\pi k_0} \left[ \frac{1}{2} k_0 f(t) - k_0 f(t) \right] \left( 1 - \frac{a^2 z^2 + b^2 z^2}{4} \right)
\]

(36)

For reference, we note the Laplace transform of Eq. 38

\[
p_r(s, r_p) = \frac{1}{s} \left[ k_0 f(s) - k_0 f(s) \right] \left( 1 - \frac{a^2 z^2 + b^2 z^2}{4} \right)
\]

(37)

(Solutions of the Radial Flow Diffusivity Equation in the Real Domain)
Multiplying through Eq. 29 by the laplace transform parameter \( s \) gives

\[
\mathcal{L}\left[ \frac{\partial^2 C}{\partial r^2} \right] = k_0 \mathcal{L}\left[ \frac{\partial C}{\partial r} \right] + k_1 \mathcal{L}\left[ \frac{\partial C}{\partial r} \right] - k_0 \mathcal{L}\left[ C \right] + \left( \frac{r_0^2 - r_0^2}{4} \right) \mathcal{L}\left[ C \right] - \left( \frac{r_0^2 - r_0^2}{8} \right) \mathcal{L}\left[ C \right]
\]

We will take the inverse laplace transform of Eqs. 39 and 40 using the following tables:

<table>
<thead>
<tr>
<th>( s )</th>
<th>( f(s) )</th>
<th>Reference(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{s} )</td>
<td>( \frac{1}{s} )</td>
<td>Abramowitz and Stegun: Handbook of Mathematical Functions, Table 9.3, Eq. 21.3.121, p. 1023.</td>
</tr>
<tr>
<td>( \frac{1}{s} )</td>
<td>( \frac{2t}{s} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 506.</td>
</tr>
<tr>
<td>( \frac{1}{s} )</td>
<td>( \frac{z}{s} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 506.</td>
</tr>
<tr>
<td>( \frac{z}{s} )</td>
<td>( \frac{\Gamma(z, a^2)}{a^2} )</td>
<td>Roberts and Kaufman: Table of Laplace Transforms, Section 2, Eq. 13.2.1, p. 506.</td>
</tr>
</tbody>
</table>

Inverting Eq. 39 term-by-term using the previous table gives

\[
q_0(r_0, t_0) = \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2t_0}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

\[
+ \left( \frac{r_0^2 - r_0^2}{4} + \frac{z}{2t_0} \right) \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

\[
- \left( \frac{r_0^2 - r_0^2}{8} \right) \frac{1}{2t_0} \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

Collecting

\[
q_0(r_0, t_0) = \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2t_0}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

\[
+ \left[ \frac{z}{2t_0} + c \left( \frac{z}{2t_0} \right) - c \left( \frac{1}{2t_0} \right) \right] \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

where

\[
c = \left( r_0^2 \right) - \left( r_0^2 \right)
\]

Cancelling the \( c \) terms

\[
q_0(r_0, t_0) = \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2t_0}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

\[
+ \frac{z}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

which yields the following reduction

\[
q_0(r_0, t_0) = \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) - \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2t_0}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right) + \left( \frac{r_0^2 - r_0^2}{4} \right) \exp\left( \frac{-r_0^2}{4t_0} \right)
\]

Segmenting the solution into particular flow regimes

\[
q_0(r_0, t_0) = \frac{1}{z} \left( \frac{r_0^2}{4t_0} \right) + \frac{2t_0}{r_0^2} \exp\left( \frac{-r_0^2}{4t_0} \right) / \left( \text{Material Balance Term (Reservoir Size)} \right)
\]

\[
\left( \text{Infinite-Acting-Reservoir Term (Reservoir Size)} \right)
\]

\[
\left( \text{Reservoir Shape Effects Terms} \right)
\]

(Solutions of the Radial Flow Diffusivity Equation in the Real Domain)
Due to conflicting results obtained by inverting Eq. 40 term-by-term, we will proceed by differentiating Eq. 42.

Note that
\[
\frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(48)

and
\[
\frac{d}{d\zeta} \exp(x) = \frac{dx}{d\zeta} \frac{d\exp(x)}{dx} = \frac{dx}{d\zeta} \left[ -\exp(x) \right]
\]
(49)

Differentiating Eq. 42 term-by-term,
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(50)

or
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(51)

Similarly, for \( d/d\zeta \left[ \frac{dE_i(x)}{dx} \right] \) we have
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(52)

Next we have
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(53)

Collecting the derivative terms we have
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(54)

Collecting further
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(55)

Multiplying through by \( \zeta \), we have

or
\[
\frac{d}{d\zeta} \left[ \frac{dE_i(x)}{dx} \right] = \frac{d}{d\zeta} \left( \frac{dE_i(x)}{dx} \right) = \frac{dx}{d\zeta} \left[ -\exp(-x) \right]
\]
(56)

Solution for Constant Pressure Outer Boundary:

Similar to the no-flow outer boundary case, we cannot directly invert Eq. 36, so we will attempt an approximate solution of the line source term (Eq. 11). Recalling Eq. 7 we have
\[
\bar{f}_0(r_0, \mu) = \frac{1}{\mu} \ln(\mu r_0) - \frac{1}{\mu} \ln(\mu r_0) I_0(\mu r_0) \text{ (line source)}
\]
(57)

Recalling the polynomial approximation for \( I_0(x) \) (Eq. 31), we have
\[
I_0(x) = 1 + \frac{2x}{4} + 4x^2 + \frac{16x^4}{8}
\]
(58)

using a two-term approximation for the \( I_0(\mu r_0) \), we have
\[
I_0(\mu r_0) = 1 + \mu r_0^2 + \frac{16}{4}
\]
(59)

using a two-term binomial series for \( (1 + \mu r_0^2) \), we have
\[
I_0(\mu r_0) = 1 + \mu r_0^2 - \frac{1}{16}
\]
(60)
Collecting the derivatives we have
\[
\frac{d}{dt_0} \left( \frac{\rho_0(0, t_0)}{4 t_0} \right) = \frac{1}{2 t_0} \exp \left( -\frac{r_0^2}{2 t_0} \right) - \frac{1}{2 t_0} \exp \left( -\frac{r_0^2}{4 t_0} \right) + \frac{r_0^2 - r_0^2}{2 t_0} \exp \left( -\frac{r_0^2}{4 t_0} \right)
\]

Multiplying through by \( t_0 \) yields the well testing derivative
\[
\rho'_0(0, t_0) = \frac{1}{2} \exp \left( -\frac{r_0^2}{2 t_0} \right) - \frac{1}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right) + \frac{r_0^2 - r_0^2}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right)
\]

Summaries of Results:

<table>
<thead>
<tr>
<th>Case</th>
<th>Solution</th>
</tr>
</thead>
</table>
| a. infinite acting reservoir       | L.F.D.: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | L.F.L.: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | L.F. "lag" approximation: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
| Real Domain / Cylindrical Source   | \[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | Real Domain / Line Source: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | Real Domain / "lag" approximation: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
| Real Domain / Derivative of the    | \[ \rho'_0(r, t) = \frac{1}{2} \exp \left( -\frac{r_0^2}{2 t_0} \right) - \frac{1}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right) + \frac{r_0^2 - r_0^2}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right) \] |
|                                    | Real Domain / Derivative of the "lag" approximation: 
\[ \rho'_0(r, t) = \frac{1}{2} \exp \left( -\frac{r_0^2}{2 t_0} \right) - \frac{1}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right) + \frac{r_0^2 - r_0^2}{2} \exp \left( -\frac{r_0^2}{4 t_0} \right) \] |
| b. bounded circular reservoir      | L.F. Cyl Source: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | L.F. Line Source: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |
|                                    | L.F. "lag" approximation: 
\[ \bar{\rho}_0(r, t) = \frac{1}{4t_0} \left( \frac{kr_0^2}{r^2} \right) \] |

(Solutions of the Radial Flow Diffusivity Equation in the Real Domain)
Case 1: Bounded circular reservoir (no-flow boundary - continued)

Real Domain / Line Source Solution

\[ p(r, t) = \frac{1}{2} \left( \frac{r}{r_0} \right)^2 \ln \left( \frac{r}{r_0} \right) - \frac{1}{2} \left( \frac{r}{r_0} \right)^2 \ln \left( \frac{r}{r_0} \right) + \frac{z}{2} \int_0^t \frac{r^2}{r_0^2} \exp \left( -\frac{r^2}{2r_0^2} \right) \, dt \]

Real Domain / Derivative of Line Source Solution

\[ \frac{\partial p}{\partial r}(r_0, t) = \frac{1}{2} \exp \left( -\frac{r_0^2}{2} \right) + \frac{z}{2} \int_0^t \frac{r_0^2 - r^2}{r_0^2} \exp \left( -\frac{r_0^2}{4r_0^2} \right) \, dt \]

Case 2: Bounded circular reservoir (constant pressure boundary)

Laplace Domain / Line Source

\[ \phi(r, \mu) = \frac{1}{\mu} \frac{k_v}{k_h} \frac{I_0(\mu r_0)}{I_0(\mu r_0)} - \frac{k_v}{k_h} \frac{I_0(\mu r_0)}{I_0(\mu r_0)} + \frac{1}{\mu} \frac{k_v}{k_h} \frac{I_0(\mu r_0)}{I_0(\mu r_0)} \]

Laplace Domain / Line Source

\[ \phi(r_0, \mu) = \frac{1}{\mu} \frac{k_v}{k_h} \frac{I_0(\mu r_0)}{I_0(\mu r_0)} - \frac{k_v}{k_h} \frac{I_0(\mu r_0)}{I_0(\mu r_0)} \]

Real Domain / Line Source Solution

\[ \rho_c(r, t) = \frac{1}{2} \left( \frac{r}{r_0} \right)^2 \ln \left( \frac{r}{r_0} \right) - \frac{1}{2} \left( \frac{r}{r_0} \right)^2 \ln \left( \frac{r}{r_0} \right) + \frac{z}{2} \int_0^t \frac{r_0^2}{r_0^2} \exp \left( -\frac{r_0^2}{2r_0^2} \right) \, dt \]

Real Domain / Derivative of Line Source Solution

\[ \frac{\partial \rho_c}{\partial r}(r_0, t) = \frac{1}{2} \exp \left( -\frac{r_0^2}{2} \right) + \frac{z}{2} \int_0^t \frac{r_0^2 - r^2}{r_0^2} \exp \left( -\frac{r_0^2}{4r_0^2} \right) \, dt \]
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir—Various $r_D$:
Dimensionless Pressure and Derivative—Radial Flow Case (SPE 25479)