Objectives: (things you should know and/or be able to do)

Introduction/Fundamentals:

- Be able to state the definition of the Laplace transformation and its inverse.

**Definition of the Laplace Transform:**

\[
\tilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt \quad \text{or} \quad \frac{1}{s} \int_0^\infty e^{-sx} \tilde{f}(x) \, dx \quad \text{(using } x=st)\
\]

**Definition of the Inverse Laplace Transform:** (Mellin Inversion Integral)

\[
f(t) = \mathcal{L}^{-1}[\tilde{f}(s)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \tilde{f}(s) \, ds
\]

Self-Study

Introduction to Laplace Transforms
(Properties and Theorems)
Objectives: (things you should know and/or be able to do)

Properties of the Laplace Transform:

- Be familiar with the "unit step" function shown below.

![unit step function diagram](image)

The unit step function is given by:

\[ u(t-a) = \begin{cases} 
0 & t < a \\ 
1 & t \geq a 
\end{cases} \]

And its Laplace transform is:

\[ \tilde{f}(u) = \frac{1}{s} e^{-as} \]

- Be able to develop and apply the Laplace transform formulas for the discrete data functions shown below.
  - Step Data Function:
    \[ \tilde{f}(s) = \frac{1}{s} \sum_{i=1}^{n} (f_i-f_{i-1}) e^{-s(t_{i-1})} \quad (where \ t_0=0 \ and \ f_0=0) \]
  - Piecewise Linear Data Function: Roumboutsos and Stewart Method
    \[ \tilde{f}(u) = \frac{1}{s^2} m_1(1-e^{-st_1}) + \frac{1}{s^2} \sum_{i=2}^{n-1} m_i(e^{-s(t_{i-1})} - e^{-s(t_i)}) + \frac{1}{s^2} m_n e^{-s(t_{n-1})}, \text{ where } m_i = \frac{f_i-f_{i-1}}{t_i-t_{i-1}}. \]
  - Piecewise Log-Linear Data Function: Blasingame Method
    \[ \tilde{f}(s) = \frac{\alpha_1}{s^\nu_1} \Gamma(v_1,s_{11}) + \frac{\alpha_2}{s^\nu_2} \Gamma(v_2,s_{21}) - \frac{\alpha_2}{s^\nu_2} \Gamma(v_2,s_{11}) \]
    \[ + \cdots + \frac{\alpha_{n-1}}{s^\nu_{n-1}} \Gamma(v_{n-1},s_{(n-1),1}) - \frac{\alpha_{n-1}}{s^\nu_{n-1}} \Gamma(v_{n-1},s_{(n-1),n-2}) + \frac{\alpha_n}{s^\nu_n} \Gamma(v_n) - \frac{\alpha_n}{s^\nu_n} \Gamma(v_n,s_{(n-1),n-1}) \]

The slope and intercept terms (\(\alpha's\) and \(\nu's\)) are shown graphically in the attached notes. Also, \(\Gamma(x)\) is the Gamma function and \(\gamma(a,x)\) is the first incomplete Gamma function.
Objectives: (things you should know and/or be able to do)

Application of the Laplace Transform to Solve Linear Ordinary Differential Equations:
- Be able to develop the Laplace transform of a given differential equation and its initial condition(s). This requires the Laplace transform of each time-derivative, then substitution into the differential form, the result is an algebraic expression in terms of $s$ and $\tilde{f}(s)$.
  - Laplace Transform of a Generic Time-Dependent Derivative:
    \[
    \mathcal{L}\left[ \frac{d^n}{dt^n} f(t) \right] = s^n \tilde{f}(s) - s^{n-1} f(t=0) - s^{n-2} f'(t=0) - \ldots - s f^{n-2}(t=0) - f^{n-1}(t=0)
    \]
    or in terms of arbitrary constants ($c_i$'s)
    \[
    \mathcal{L}\left[ \frac{d^n}{dt^n} f(t) \right] = s^n \tilde{f}(s) - s^{n-1} c_0 - s^{n-2} c_1 - \ldots - s c_{n-2} - c_{n-1}
    \]
    where
    \[
    c_0 = f(t=0), \quad c_1 = f'(t=0), \quad c_2 = f''(t=0) \ldots \quad c_{n-2} = f^{n-2}(t=0), \quad c_{n-1} = f^{n-1}(t=0)
    \]
- Be able to resolve the algebra resulting from the Laplace transform of a given differential equation and its initial condition(s) into a closed and hopefully, invertible form.
- Be able to invert the closed form Laplace transform solution of a given differential equation using the fundamental properties of Laplace transforms, Laplace transform tables, partial fractions, and prayer.

Self-Study

Application of the Laplace Transform:
Solution of Ordinary Differential Equations
Objectives: (things you should know and/or be able to do)

Numerical Laplace Transform and Inversion:

- Be able to use the Gauss-Laguerre formula for numerical Laplace transformation. The Laguerre quadrature weights, \( w_k \), and abscissas, \( x_k \), can be obtained from Abramowitz and Stegun.

\[
\tilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt \approx \frac{1}{s} \sum_{k=1}^{n} w_k f\left(\frac{x_k}{s}\right)
\]

- Be familiar with the development of the Gaver formula for numerical Laplace transformation, and note its similarity to the Widder inversion formula given in the Cost (AIAA Journal) paper.
Objectives: (things you should know and/or be able to do)

Numerical Laplace Transform and Inversion: (Continued)

- Be able to use the Gaver and Gaver-Stehfest numerical inversion algorithms for the inversion of Laplace transforms.
  - The Gaver formula for numerical Laplace transform inversion is
    \[ f_{\text{Gaver}}(n,t) = \frac{\ln(2)}{t} \frac{(2n)!}{(n-1)!} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} f \left[ \frac{\ln(2)}{t} (n+k) \right] \]
  - The Gaver-Stehfest formula for numerical Laplace transform inversion is
    \[ f_{\text{Gaver-Stehfest}}(n,t) = \frac{\ln(2)}{t} \sum_{i=1}^{n} V_i \tilde{f} \left[ \frac{\ln(2)}{t} i \right] \]
    and the Stehfest extrapolation coefficients are given
    \[ V_i = (-1)^{n-i+1} \sum_{k=\left[\frac{i+1}{2}\right]}^{\min\left[i, \frac{n}{2}\right]} \frac{n}{k_2} (2k)! \left[ \frac{n-k}{2} \right]!k!(k-1)!(i-k)!(2k-i)! \]
- Be familiar with and be able to derive the following methods for Laplace transform inversion:
  - The ter Haar polynomial methods (the so-called Methods 1 and 2),
  - The ter Haar/Laguerre quadrature formula, and
  - The Laguerre quadrature/Polynomial substitution formula.
Use of the Laplace Transform to Solve Differential Equations
Introduction to Laplace Transforms
(Properties and Theorems)

(from Petroleum Engineering 620 Course Notes -- 1992)
Introduction to Laplace Transforms

Definition of the Laplace transform

\[ \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \quad (t>0) \quad (1) \]

The Mellin inversion formula is given by

\[ f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds \quad (t>0) \quad (2) \]

Some important observations:

1. The laplace transform, \( \mathcal{L}\{f(t)\} \), defined by Eq. 1 shows that the range of the integral is from 0 to \( \infty \) — which means that the integral (i.e., the laplace transform) is only a function of the laplace transform parameter, \( s \), and is defined for all \( t \), \( 0 < t < \infty \) (i.e., all \( t \)). Note also that the limits of the integrand are the range for \( f(t) \) (i.e., \( 0 < t < \infty \)).

2. Similarly, the inverse laplace transform, as defined by Eq. 2, illustrates that the laplace transform parameter, \( s \), is a complex conjugate variable (i.e., \( s = x + iy \)). This proves that the inverse laplace transform is a complex conjugate operation.

The traditional approach is to attempt to resolve \( \mathcal{L}\{f(t)\} \) into an explicitly invertible form, which is generally obtained by consultation of reference tables. An alternative approach is to simply invert \( \mathcal{L}\{f(t)\} \) numerically, using an approach such as the Gaver-Stehfest algorithm.

Although computationally efficient in general, most popular laplace transform inversion algorithms produce accurate results only for certain classes of functions and/or over certain time intervals.

The Gaver-Stehfest algorithm is given as

\[ f(t) = \sum_{i=1}^{n} \frac{\Gamma(n)}{\Gamma(i)\Gamma(n-i)} \frac{t^i}{(2i)!} \int_{-\infty}^{+\infty} F(\alpha) d\alpha \]

where,

\[ a = \frac{\ln(z)}{t} \quad (4) \]

and

\[ v_{2} = (-1)^{n/2} \sum_{k=0}^{n/2} \frac{\Gamma(n-k)!}{\Gamma(k-k)!\Gamma(k-k)!\Gamma(k-k)!} \]


The Gaver-Stehfest algorithm typically gives good results for a very wide range of problems—however, in conventional programming languages (Fortran, C, Pascal, etc.) the Gaver-Stehfest algorithm suffers significantly from numerical roundoff (truncation errors). Recent advances in "precisionless" arithmetic (such as the package Mathematica (by Wolfram Research)) provide essentially exact results from the Gaver-Stehfest algorithm.

Typical application of the Gaver-Stehfest algorithm...
Involves $n$-values from 4 to 20, where the $n$-values must be tested on individual $F(s)$ functions.

**Linearity of the Laplace Transform**

Turning our efforts again to analytical considerations and the properties of the Laplace transform, we first note that the Laplace transform is a linear operator. Specifically, we note that

\[ \mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\} \]  

\[ = c_1 F_1(s) + c_2 F_2(s) \]  

Similarly, the inverse Laplace transform is given by

\[ \mathcal{L}^{-1}\left\{c_1 F_1(s) + c_2 F_2(s)\right\} = c_1 \mathcal{L}^{-1}\left\{F_1(s)\right\} + c_2 \mathcal{L}^{-1}\left\{F_2(s)\right\} \]  

\[ = c_1 f_1(t) + c_2 f_2(t) \]  

**Laplace Transform of a Product of Two Functions**

The Laplace transform of a product of two functions is a non-trivial case that yields the following integral in the complex plane:

\[ \mathcal{L}\{f(t)g(t)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(m)g(s-m)\,dm \]  

The $f(t)g(t)$ formulation arises in the solution of non-linear partial differential equations, but the right-hand-side (RHS) of Eq. 8 is not a useful form for the solution of such problems.

\[ \text{Laplace Transform of a Constant} \]

Recalling the definition of the Laplace transform, Eq. 1, we have

\[ \mathcal{F}(s) = \int_0^\infty f(t) e^{-st} dt \]  

\[ \text{Assuming } f(t) = c \text{ (a constant), we obtain} \]

\[ \mathcal{F}(s) = \int_0^\infty c e^{-st} dt \]

\[ = c \int_0^\infty e^{-st} dt \]

\[ = c \left[ -\frac{1}{s} e^{-st} \right]_0^\infty \]

\[ = -\frac{c}{s} \left[ e^{-s(\infty)} - e^{-s(0)} \right] \]

\[ = -\frac{c}{s} (e^{-\infty} - e^{-0}) \]

\[ = -\frac{c}{s} (0 - 1) \]

or

\[ \mathcal{F}(s) = \frac{c}{s} \quad \text{for } f(t) = c \]  

\[ \text{Laplace Transform of Time, } t \]

Assuming $f(t) = t$, we have

\[ \mathcal{F}(s) = \int_0^\infty t e^{-st} dt \]  

The $te^{-st}$ term cannot be integrated directly and requires that we use "integration by parts." Recalling

\[ \int u dv = uv - \int v du \]

We will let

\[ u = t \]

\[ dv = e^{-st} \]

\[ du = dt \]

\[ v = -\frac{1}{s} e^{-st} \]
\[ \tilde{f}(s) = -\frac{t}{s} e^{-st} \left[ \int_0^\infty \frac{1}{s} e^{-st} dt \right] \]

\[ = -\left[ (\infty) e^{-s(\infty)} - (0) e^{-s(0)} \right] \]

\[ + \frac{1}{s} \int_0^\infty e^{-st} dt \]

Consolidating, and completing the last integral,

\[ \tilde{f}(s) = \frac{1}{s} \left[ \frac{1}{s} e^{-st} \right]_0^\infty \]

\[ = -\frac{1}{s^2} \left[ e^{-s(\infty)} - e^{-s(0)} \right] \]

or finally

\[ \tilde{f}(s) = \frac{1}{s^2} \quad \text{for} \quad f(t) = t \quad (12) \]

**Laplace Transform of Time Squared, \( t^2 \)**

Assuming \( f(t) = t^2 \), we obtain

\[ \tilde{f}(s) = \int_0^\infty t^2 e^{-st} dt \quad (13) \]

The \( t^2 e^{-st} \) term cannot be integrated directly, which requires that we again use "integration by parts." The integration by parts formula is given by

\[ \int u dv = uv - \int v du \]

We let \( u = t^2 \) \quad \( dv = e^{-st} dt \)

\[ du = 2tdt \quad v = -\frac{1}{s} e^{-st} \]

\[ \tilde{f}(s) = -\frac{t^2}{s} e^{-st} \left[ \int_0^\infty -\frac{1}{s} e^{-st} dt \right] (st) dt \]

which reduces to

\[ \tilde{f}(s) = \frac{2}{s} \int_0^\infty t e^{-st} dt \quad (14) \]

Substituting the result from Eq. 11 (i.e., Eq. 12) into Eq. 14, we obtain

\[ \tilde{f}(s) = \frac{2}{s^3} \quad \text{for} \quad f(t) = t^2 \quad (15) \]

**Laplace Transform of Time to an Integer Exponent, \( t^n \)**

Assuming \( f(t) = t^n \), we obtain

\[ \tilde{f}(s) = \int_0^\infty t^n e^{-st} dt \quad (16) \]

As before, the \( t^n e^{-st} \) term cannot be integrated directly, and we must again use "integration by parts." By induction from previous results, we obtain

\[ \tilde{f}(s) = \frac{n!}{s^{n+1}} \quad \text{for} \quad f(t) = t^n \quad (17) \]
substituting Eq. 22 into Eq. 19
\[ \tilde{f}(s) = \frac{1}{s^{p+1}} \Gamma(p+1) \text{ for } f(t) = t^p \quad (p > -1) \] (23)

\[ \text{laplace Transform of the Time Derivative, } f''(t) \]

The laplace Transform of a derivative with respect to time is one of the most useful characteristics of the laplace Transform. In particular, the laplace transform of a time derivative yields an algebraic relation in terms of \( \tilde{f}(s) \).

\[ L\{f'(t)\} = \int_0^\infty f(t) e^{-st} dt \] (24)

The integration-by-parts formula is given by
\[ \int uv = \int u \, dv \]

Letting
\[ u = e^{-st} \quad \text{du} = -se^{-st} \, dt \quad \text{v} = f(t) \]

\[ L\{f'(t)\} = e^{-st} f(t) \bigg|_0^\infty - \int_0^\infty f(t) \left[-se^{-st}\right] \, dt \]

or
\[ L\{f'(t)\} = s\int_0^\infty f(t) e^{-st} \, dt + e^{-st} f(t) \bigg|_0^\infty \]

or
\[ L\{f'(t)\} = s\tilde{f}(s) + e^{-st} f(t=0) \]

Finally
\[ L\{f'(t)\} = s\tilde{f}(s) - f(t=0) \] (25)
For the general case of the $n$-th derivative with respect to time, we have:

$$L\{f^{(n)}(t)\} = s^n \tilde{f}(s) - s^{n-1} f(o) - s^{n-2} f'(o) - \ldots - f^{(n-1)}(o)$$

We note that Eq. 26 is an algebraic function in $s$ and $\tilde{f}(s)$, and includes the "constants" $f(o)$, $f'(o)$, and so on. Eq. 26 forms the basis for using the Laplace transform to solve linear ordinary differential equations.

Laplace Transform of Partial (Non-Time) Derivatives

The Laplace transform of a partial derivative (i.e., a non-time derivative) is a necessary component of the solution of partial differential equations using Laplace transforms. It is somewhat obvious that the Laplace transform of a non-time dependent derivative (or, for that matter, a non-time dependent integral) will simply result in the derivative (or integral) of the Laplace transform with respect to the (non-time) variable in question.

To illustrate the proof of this concept, we take the Laplace transform of the partial derivative of $f(t)$, with respect to $x$.

$$L\left\{ \frac{\partial f(t)}{\partial x} \right\} = \int_0^\infty \frac{\partial f(t)}{\partial x} e^{-st} dt$$

(27)

Since the integral is with respect to time, and the (partial) derivative is with respect to a variable other than time (i.e., x), we can "move" the derivative outside the integral.

This gives:

$$\int_0^\infty \left[ \frac{\partial f(t)}{\partial x} \right] e^{-st} dt = \frac{d}{dx} \left[ \int_0^\infty f(t) e^{-st} dt \right]$$

$$= \frac{d}{dx} \tilde{f}(s) = \frac{d}{dx} \tilde{f}(s)$$

This result reduces to an ordinary derivative in $\tilde{f}(s)$ because the time variable has been eliminated by the application of the Laplace transform. The final form is:

$$L\left\{ \frac{\partial f(t)}{\partial x} \right\} = \frac{d}{dx} \tilde{f}(s)$$

(28)

Extending Eq. 28 to the general case of an $n$-th derivative, we have

$$L\left\{ \frac{\partial^n f(t)}{\partial x^n} \right\} = \frac{d^n}{dx^n} \tilde{f}(s)$$

(29)

As alluded to earlier in this development, the net effect of applying the Laplace transform to a partial derivative of a function $f(x,t)$ is to "convert" this to an ordinary derivative (in x) of the Laplace transformed function, $\tilde{f}(x,s)$. In other words, $\tilde{f}(s)$ is not a function of time, so the partial derivatives of $\tilde{f}(s)$ can be written as ordinary derivatives.
Theorems for the Application of the Laplace Transform to Common Problems in Engineering

Translation Theorem

\[ L\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt \]

(30)

letting \( u = s-a \),

\[ L\{e^{at}f(t)\} = \int_0^\infty f(t)e^{-(s-a)t}dt \]

which is simply the definition of the Laplace transform. Therefore

\[ L\{e^{at}f(t)\} = \mathcal{F}(s-a) \]

(31)

Scale Change

\[ L\{f(at)\} = \int_0^\infty f(at)e^{-st}dt \]

(32)

defining the following substitution

\[ u = at, \quad du = adt \]

we have

at \( t = 0, \quad u = 0 \)

at \( t = \infty, \quad u = \infty \)

where these substitutions yield

\[ L\{f(at)\} = \int_0^\infty f(u)e^{-\frac{1}{a}su}du \]

\[ = \frac{1}{a} \int_0^\infty f(u)e^{-\frac{1}{a}su}du \]

Recalling the definition of the Laplace transform

\[ L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \mathcal{F}(s) \]

Eq. 33 can be written as

\[ L\{f(at)\} = \frac{1}{a} \mathcal{F}(\frac{s}{a}) \]

(34)

Similarly

\[ L\{f\left(\frac{t}{c}\right)\} = c \mathcal{F}(cs) \]

(35)

Derivatives in the Laplace Domain

The definition of the Laplace transform is

\[ L\{f(t)\} = \mathcal{F}(s) = \int_0^\infty f(t)e^{-st}dt \]

(1)

Taking the derivative of Eq. 1 with respect to \( s \), we have:

\[ \frac{d}{ds}\mathcal{F}(s) = \mathcal{F}'(s) = \frac{d}{ds} \left[ \int_0^\infty f(t)e^{-st}dt \right] \]

(36)

Leibnitz’s rule for the differentiation of an integral is given by

\[ \frac{d}{dz} \left[ \int_a^b f(x,z)dx \right] = \int_a^b \frac{df(x,z)}{dz}dx + \int_a^b f(b,z)\frac{dz}{dz} + \int_a^b f(a,z)\frac{dz}{dz} \]

(37)
Applying Eq. 37 to Eq. 36, we have
\[ \frac{d}{ds} \left[ \int_0^\infty f(t) e^{-st} dt \right] = \int_0^\infty \frac{d}{ds} \left[ f(t) e^{-st} \right] dt \]
+ \left[ f(t \to 0) e^{-s(0+0)} \frac{d}{ds} \right] f(0) - f(t \to 0) e^{-s(0-0)} \frac{d}{ds} \left[ f(0) \right] \]

or, reducing
\[ \frac{d}{ds} \left[ \int_0^\infty f(t) e^{-st} dt \right] = \int_0^\infty \frac{d}{ds} \left[ f(t) e^{-st} \right] dt \] (38)

Isolating the integrand term,
\[ \frac{d}{ds} \left[ f(t) e^{-st} \right] = e^{-st} \frac{d}{ds} \left[ f(t) \right] + f(t) \frac{d}{ds} \left[ e^{-st} \right] \]

which reduces to
\[ \frac{d}{ds} \left[ f(t) e^{-st} \right] = -f(t) e^{-st} \] (39)

By inspection we note that the "n-th" derivative is given by:
\[ \frac{d^n}{ds^n} \left[ f(t) e^{-st} \right] = (-1)^n f(t) t^n e^{-st} \] (40)

Substitution of Eq. 39 into Eq. 38 gives us
\[ \frac{d}{ds} \left[ \int_0^\infty f(t) e^{-st} dt \right] = f(s) = -\int_0^\infty f(t) t e^{-st} dt \] (41)

letting \( q(t) = tf(t) \)
\[ f(s) = -\int_0^\infty q(t) e^{-st} dt = -\tilde{q}(s) \] (42)

or \( \tilde{f}(s) = -L \{ tf(t) \} \)

The general derivative is given by:
\[ \frac{d^n}{ds^n} \tilde{f}(s) = \tilde{f}^n(s) = \frac{d^n}{ds^n} \left[ \int_0^\infty f(t) e^{-st} dt \right] \] (44)

By induction from Eq. 38, we have:
\[ \frac{d^n}{ds^n} \tilde{f}(s) = \tilde{f}^n(s) = \frac{d^n}{ds^n} \left[ \int_0^\infty f(t) e^{-st} dt \right] \] (45)

And from Eq. 40 we have:
\[ \frac{d^n}{ds^n} \tilde{f}(s) = (-1)^n f(t) t^n e^{-st} \] (46)

Substituting Eq. 40 into Eq. 45 gives us
\[ \frac{d^n}{ds^n} \tilde{f}(s) = (-1)^n \int_0^\infty \tilde{f}(t) t^n e^{-st} dt \] (46)

or \[ \frac{d^n}{ds^n} \tilde{f}(s) = (-1)^n \sum_{i} \tilde{f}(i) t^i \] (47)

An interesting note is that Widder (Widder, D.V., The Laplace Transform, Princeton University Press, Princeton, N.J. (1946)) obtained an approximate inversion formula for the Laplace transform that is surprisingly similar in form to Eq. 47. This result is
\[ \frac{d^n}{ds^n} \tilde{f}(s) \approx (-1)^n \sum_{i} \tilde{f}(i) t^i \bigg|_{t=\frac{n}{s}} \] (48)
Solving Eq. 48 for \( f(t) \), we have

\[
 f(t) = \lim_{n \to \infty} \left[ (-1)^n \sum_{n=0}^{\infty} \frac{d^n}{ds^n} F(s) \right] s^n t^n
\]

Eqs. 48 and 49 are derived in full detail by Cost (Cost, T.L.; "Approximate Laplace Transform Inversion in Viscoelastic Stress Analysis," AIAA Journal, Dec. 1964, 267-266). As one may imagine, Widder formula (Eq. 49) is not particularly well suited for computations because \( F(s) \) is never "infinitely" differentiable.

Another interesting note is that Widder formula is quite similar to the result proposed by Gaver (Gaver, D.P., Jr.; "Observing Stochastic Processes and Approximate Transform Inversion," Operations Research, vol. 14, No. 3 (1966), 444-459). In simple terms, the Gaver formula uses finite differences to represent the derivatives in \( F(s) \).

**Laplace Transform of an Integral taken with Respect to Time**

Our objective is to obtain:

\[
 L\{ \int_0^t f(t) \, dt \} = \int_0^\infty \left[ \int_0^t f(t) \, dt \right] \, ds
\]

The best approach is to simply define the integral as a function \( q(t) \):

\[
 q(t) = \int_0^t f(t) \, dt
\]

and

\[
 q'(t) = \frac{d}{dt} [q(t)] = f(t)
\]

Taking the Laplace transform of \( q(t) \) we have

\[
 L\{ q'(t) \} = s \tilde{q}(s) - q(0) \quad \text{from Eq. 25}
\]

Assuming that \( q(0) = 0 \) (i.e., that \( \int_0^\infty f(t) \, dt = 0 \)), we have

\[
 \tilde{q}(s) = \frac{1}{s} L\{ q'(t) \} \quad \text{(50)}
\]

Taking the inverse Laplace transform of Eq. 50

\[
 q(t) = L^{-1} \left( \frac{1}{s} L\{ q'(t) \} \right)
\]

Recalling our previous definitions

\[
 q(t) = \int_0^t f(t) \, dt \quad \text{and} \quad q'(t) = f(t)
\]

Substituting these relations into Eq. 51 gives

\[
 \int_0^t f(t) \, dt = L^{-1} \left( \frac{1}{s} L\{ f(t) \} \right)
\]

Or

\[
 \int_0^t f(t) \, dt = L^{-1} \left( \frac{1}{s} \tilde{f}(s) \right)
\]

Alternatively

\[
 L\{ \int_0^t f(t) \, dt \} = \frac{1}{s} \tilde{f}(s)
\]

**Laplace Transform of an Integral taken with Respect to a Variable Other than Time**

Our objective is to obtain

\[
 L\{ \int_a^b f(x, t) \, dx \} = \int_0^\infty \left[ \int_a^b f(x, t) \, dx \right] e^{-st} \, dt
\]
Focusing on the right-hand-side (RHS) integral
\[
\int_0^\infty \left[ \int_a^b f(x,t) dx \right] e^{-st} dt = \int_a^b \left[ \int_0^\infty f(x,t) e^{-st} dt \right] dx
\]

\[= \int_a^b \mathcal{L} \{ f(x,t) \} \, dx \]
\[= \int_a^b \bar{f}(x,s) \, dx \]

Finally:
\[
\mathcal{L} \{ \int_a^b f(x,t) dx \} = \int_a^b \mathcal{L} \{ f(x,t) \} \, dx \quad (55)
\]
or
\[
\mathcal{L} \{ \int_a^b f(x,t) dx \} = \int_a^b \bar{f}(x,s) \, dx \quad (56)
\]

**Convolution Theorem**

Our goal is to combine (or "convolve") two separate functions in the Laplace domain. While not obvious at the outset, this effort will result in a multiplication of the Laplace transform of these functions—which yields a "convolution" integral in the real domain.

Without proof, the convolution integral is given by:
\[
\int_0^t f(u) \varphi(t-u) \, du = \mathcal{L}^{-1} \{ \mathcal{L} \{ f(s) \} \mathcal{L} \{ \varphi(s) \} \} \quad (57)
\]
The convolution integral has particular applications in signal processing and other such cases where multiple inputs are combined to yield an output.
Laplace Transform of Data Functions
Properties of the Laplace Transform: Piecewise Step Data Function

The Laplace transform of a piecewise data function is given by

\[ \tilde{f}(s) = \sum_{i=1}^{n} f_i(t_i) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f_i(t) e^{-st} \, dt \]

or

\[ \tilde{f}(s) = \frac{1}{s} \left[ e^{-st_0} - e^{-st} \right] \]

Substituting Eq. 4 into Eq. 1 gives us

\[ \tilde{f}(s) = \frac{1}{s} \left[ e^{-st_0} - e^{-st} \right] + \frac{f_0}{s} \left[ e^{-st_0} - e^{-st} \right] + \frac{f_1}{s} \left[ e^{-st_0} - e^{-st} \right] + \cdots + \frac{f_n}{s} \left[ e^{-st_0} - e^{-st} \right] \]

or

\[ \tilde{f}(s) = \frac{1}{s} e^{-st_0} + \frac{1}{s} \left( f_0 - f_1 \right) e^{-st_1} + \frac{1}{s} \left( f_1 - f_2 \right) e^{-st_2} + \cdots + \frac{1}{s} \left( f_n - f_{n-1} \right) e^{-st_{n-1}} + \frac{f_n}{s} e^{-st_n} \]

Recalling that \( t_0 = 0 \) and \( t_n = \infty \), we have

\[ \tilde{f}(s) = \frac{1}{s} + \frac{1}{s} (f_0 - f_1) e^{-st_1} + \frac{1}{s} (f_1 - f_2) e^{-st_2} + \cdots + \frac{1}{s} (f_{n-1} - f_n) e^{-st_{n-1}} \]

or in summation form we obtain

\[ \tilde{f}(s) = \frac{1}{s} \sum_{i=1}^{n} (f_i - f_{i-1}) e^{-st_{i-1}} \text{ where } t_0 = 0, t_0 = 0 \]

Example

\[ f(t) \]

\[ f(t) = \frac{1}{s} \sum_{i=1}^{n} (f_i - f_{i-1}) e^{-st_{i-1}} \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \tilde{f}(s) = \frac{1}{s} \left( e^{-s0} - e^{-s1} \right) + \frac{1}{s} \left( e^{-s1} - e^{-s2} \right) + \cdots + \frac{1}{s} \left( e^{-s(n-1)} - e^{-s(n-2)} \right) \]

\[ \tilde{f}(s) = \frac{1}{s} \left( e^{-s0} - e^{-s1} \right) + \frac{1}{s} \left( e^{-s1} - e^{-s2} \right) + \cdots + \frac{1}{s} \left( e^{-s(n-1)} - e^{-s(n-2)} \right) \]
**Properties of the Laplace Transform: Piecewise Linear Data Function**

The objective is to develop a Laplace transform expression for a piecewise linear data function. This piecewise function is given graphically as shown below.

The piecewise functionals are given by

\[
  f(t) = \begin{cases} 
    f_1(t) = m_1 t + b_1 = m_1 (t-t_0) + f_0 & 0 \leq t \leq t_1 \\
    f_2(t) = m_2 t + b_2 = m_2 (t-t_1) + f_1 & t_1 \leq t \leq t_2 \\
    \vdots \\
    f_n(t) = m_n t + b_n = m_n (t-t_{n-1}) + f_{n-1} & t_{n-1} \leq t \leq \infty 
  \end{cases}
\]

Where \( f(t) \) is given by

\[
  f(t) = m_i (t-t_{i-1}) + f_{i-1} 
\]

And the Laplace transform, \( F(s) \), is given by

\[
  F(s) = \mathcal{L}\{f(t)\} = \int_0^{t_1} f(t)e^{-st}dt + \int_{t_1}^{t_2} f(t)e^{-st}dt + \ldots + \int_{t_{n-1}}^{\infty} f(t)e^{-st}dt 
\]

or

\[
  F(s) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(t)e^{-st}dt 
\]

Substituting Eq. 2 into Eq. 4

\[
  \bar{f}(s) = \mathcal{L}\{f(t)\} = \int_0^{t_1} f(t)e^{-st}dt + \int_{t_1}^{t_2} f(t)e^{-st}dt + \ldots + \int_{t_{n-1}}^{\infty} f(t)e^{-st}dt 
\]

or

\[
  \bar{f}(s) = \sum_{i=1}^{n} \bar{f}_i(s). 
\]

Where

\[
  \bar{f}_i(s) = \int_{t_{i-1}}^{t_i} f(t)e^{-st}dt 
\]

Completing the integration in Eq. 7, we have

\[
  \bar{f}_i(s) = f_i(t_i) - \frac{m_i}{s} \int_{t_{i-1}}^{t_i} e^{-st}dt + m_i \int_{t_{i-1}}^{t_i} e^{-st}dt - m_i t_i \int_{t_{i-1}}^{t_i} e^{-st}dt 
\]

or

\[
  \bar{f}_i(s) = m_i t_i \int_{t_{i-1}}^{t_i} e^{-st}dt 
\]

Isolating the last term in Eq. 8, we have

\[
  \bar{f}_i(s) = \int_{t_{i-1}}^{t_i} e^{-st}dt 
\]

Using integration-by-parts

\[
  \int uv = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\nu = \int u d\n
\]

The final form is given by

\[
  \int_{t_{i-1}}^{t_i} e^{-st}dt = -\frac{e^{-st}}{s} \bigg|_{t_{i-1}}^{t_i} 
\]
Properties of the Laplace Transform: Piecewise Linear Data Function

Substituting Eq. 10 into Eq. 8 we obtain

\[ \tilde{f}_i(s) = \frac{1}{s^2} \left[ e^{-st_i} - e^{-st_{i-1}} \right] \]

Completing the integration

\[ \tilde{f}_i(s) = \frac{1}{s^2} \left[ e^{-st_i} - e^{-st_{i-1}} - \frac{1}{s} m_i t_i e^{-st_i} + \frac{1}{s} m_i t_{i-1} e^{-st_{i-1}} \right] \]

Rewriting Eq. 12

\[ \tilde{f}(s) = \frac{1}{s^2} \sum_{i=1}^{n} \left( f_i e^{-st_i} - f_i e^{-st_{i-1}} \right) - \frac{1}{s} \sum_{i=1}^{n} m_i \left[ e^{-st_i} - e^{-st_{i-1}} \right] \] (13)

Isolating the first summation we have

\[ \frac{1}{s^2} \sum_{i=1}^{n} \left( f_i e^{-st_i} - f_i e^{-st_{i-1}} \right) = f_0 e^{-st_0} - f_0 e^{-st_1} + f_1 e^{-st_1} - f_1 e^{-st_2} \]

by induction, we are left with

\[ \frac{1}{s^2} \sum_{i=1}^{n} \left( f_i e^{-st_i} - f_i e^{-st_{i-1}} \right) = f_0 e^{-st_0} - f_n e^{-st_n} \] (14)

But we recall that \( t_0 = 0, t_n = \infty \) and \( f_0 = 0, f_n = 0 \). This reduces Eq. 14 to

\[ \frac{1}{s^2} \sum_{i=1}^{n} \left( f_i e^{-st_i} - f_i e^{-st_{i-1}} \right) = 0 e^{-st_0} - f_n e^{-st_n} = 0 \]

where this reduces Eq. 13 to

\[ \tilde{f}(s) = \frac{1}{s^2} \sum_{i=1}^{n} m_i \left[ e^{-st_i} - e^{-st_{i-1}} \right] \]

Recall that

\[ m_i = \frac{f_i - f_{i-1}}{t_i - t_{i-1}} \]

which reduces our result to

\[ \tilde{f}(s) = \frac{1}{s^2} \sum_{i=1}^{n} m_i \left[ e^{-st_i} - e^{-st_{i-1}} \right] \]

Substituting Eq. 11 into Eq. 6 gives

\[ \tilde{f}(s) = \frac{1}{s^2} \tilde{f}(s) = \frac{1}{s^2} \left( \sum_{i=1}^{n} \left( f_i e^{-st_i} - f_i e^{-st_{i-1}} \right) - \frac{1}{s} \sum_{i=1}^{n} m_i \left[ e^{-st_i} - e^{-st_{i-1}} \right] \right) \] (12)

Again recalling that \( t_0 = 0 \) and \( t_n = \infty \), we reduce Eq. 15 to

\[ \tilde{f}(s) = \frac{1}{s^2} \sum_{i=1}^{n} m_i \left[ e^{-st_i} - e^{-st_{i-1}} \right] \] (15)
Properties of the Laplace Transform: Piecewise Linear Data Function

\[ f(t) = \frac{1}{s^2} m_1 \left[ e^{-s(0)} - e^{-st_1} \right] + \frac{1}{s^2} \sum_{i=2}^{n-1} m_i \left[ e^{-st_{i-1}} - e^{-st_i} \right] + \frac{1}{s^2} m_n \left[ e^{-st_{n-1}} - e^{-s(\infty)} \right] \]

or

\[ f(s) = \frac{1}{s^2} m_1 \left[ 1 - e^{-st_1} \right] + \frac{1}{s^2} \sum_{i=2}^{n-1} m_i \left[ e^{-st_{i-1}} - e^{-st_i} \right] + \frac{1}{s^2} m_n e^{-st_{n-1}} \]
Properties of the Laplace Transform: Piecewise Power-Law Data Function

From Gradshteyn & Ryzhik, we have:
\[ \int_0^\infty t^{\nu-1} e^{-st} dt = s^{-\nu} \Gamma(\nu) \quad (s) \]
\[ \int_0^\infty t^{\nu-1} e^{-st} dt = s^{-\nu} \Gamma(\nu, st) \quad (6) \]
\[ \int_s^{\infty} t^{\nu-1} e^{-st} dt = s^{-\nu} \gamma(\nu, st) \quad (7) \]

Substituting Eqs. 5-7 into Eq. 4 we obtain:
\[ \int_{t_i}^{t_{i+1}} \frac{e^{-st}}{s} dt = \frac{1}{\nu_i} \left[ \Gamma(\nu_i) - \Gamma(\nu_i, st_i) - \gamma(\nu_i, st_i) \right] \quad (8) \]

Substituting Eq. 8 into Eq. 3 gives:
\[ \tilde{f}_i(s) = \frac{x_i}{\nu_i} \left[ \Gamma(\nu_i) - \Gamma(\nu_i, st_i) - \gamma(\nu_i, st_i) \right] \quad (9) \]

The functions given in Eq. 9 are:
\[ \Gamma(\nu, z) = \int_0^\infty t^{\nu-1} e^{-t} dt \quad \text{the Gamma function} \quad (10) \]
\[ \gamma(\nu, z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{\nu+n}}{n! (\nu+n)} \quad \text{is the first Incomplete Gamma function} \quad (11) \]
\[ \Gamma(\nu, z) = \Gamma(\nu) - \gamma(\nu, z) \quad \text{is the second Incomplete Gamma function} \quad (12) \]

Substituting Eq. 12 into Eq. 9 gives:
\[ \tilde{f}_i(s) = \frac{x_i}{\nu_i} \left[ \gamma(\nu_i, st_i) - \gamma(\nu_i, st_{i-1}) \right] \quad (13) \]

Presenting the first and last few terms using Eq. 13:
\[ i=1 \quad \tilde{f}_1(s) = \frac{x_1}{\nu_1} \left[ \gamma(\nu_1, st_1) - \gamma(\nu_1, st_0) \right] \quad (14) \]
\[ i=2 \quad \tilde{f}_2(s) = \frac{x_2}{\nu_2} \left[ \gamma(\nu_2, st_2) - \gamma(\nu_2, st_1) \right] \quad (15) \]
Properties of the Laplace Transform: Piecewise Power-Law Data Function

For the last partition, \( t_n = \infty \). Considering the first incomplete Gamma function we recall Eq. 12, this gives

\[
\Gamma(v, z) = \Gamma(v) - \psi(v, z) \tag{12}
\]

Considering \( z = \infty \),

\[
\Gamma(v, \infty) = \Gamma(v) - \psi(v, \infty)
\]

The definition of the second incomplete Gamma function is given as

\[
\Gamma(v, z) = \int_z^\infty e^{-t} t^{v-1} dt \tag{18}
\]

Using \( z = \infty \) in Eq. 13 we have

\[
\Gamma(v, \infty) = 0 \quad \left( ; \psi(v, \infty) = \Gamma(v) \right) \tag{19}
\]

Assuming \( t_n = \infty \), and substituting Eq. 19 into Eq. 17 we obtain

\[
i = n, t_i = \infty \quad F_i(z) = \frac{x_n}{s^{n}} \frac{\Gamma(v_n) - \gamma(v_n, s t_{n-1})}{s^{n}} \tag{20}
\]

Substituting Eqs. 14, 15, 16, and 20 into Eq. 1, we have

\[
F(s) = \frac{x_1}{s^0} \frac{\gamma(v_1, s t_1)}{s^{0}} + \frac{x_2}{s^{0}} \frac{\gamma(v_2, s t_2) - \gamma(v_1, s t_1)}{s^{0}} \tag{21}
\]

\[
+ \frac{x_3}{s^{0}} \frac{\gamma(v_3, s t_3) - \gamma(v_2, s t_2)}{s^{0}} + \ldots + \frac{x_{n-1}}{s^{0}} \frac{\gamma(v_{n-1}, s t_{n-1}) - \gamma(v_{n-2}, s t_{n-2})}{s^{0}} \tag{21}
\]

\[
+ \frac{x_n}{s^{0}} \frac{\gamma(v_n, s t_n) - \gamma(v_{n-1}, s t_{n-1})}{s^{0}} \tag{21}
\]

A somewhat misleading issue is the computational efficiency of Eq. 21. Eq. 21 is generally superior to the method of Kombosco and Stewart in terms of accuracy—but at a significant cost in computation time—on the order of a factor of 10 or more than the Kombosco and Stewart approach.
Numerical Laplace Transform and Inversion
Numerical Laplace Transform and Inversion

- Gauss-Laguerre formula for numerical Laplace transformation:
  First we start with the definition of the Laplace transform, and we then use the transform variable $x=st$, which gives

  $$\tilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt \quad \text{or} \quad \frac{1}{s} \int_0^\infty e^{-x}f\left(\frac{x}{s}\right)\,dx \quad \text{(using } x=st)$$

  Recalling the Laguerre quadrature formula we have

  $$\int_0^\infty e^{-x}f(x)\,dx \approx \sum_{k=1}^n w_k f(x_k)$$

  where the Laguerre quadrature weights, $w_k$, and abscissas, $x_k$, can be obtained from Abramowitz and Stegun. Comparing these relations, we have

  $$\tilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\,dt \approx \frac{1}{s} \sum_{k=1}^n w_k f\left(\frac{x_k}{s}\right)$$

- Comparison of the Widder (derivative) and Gaver (difference) formulas for numerical Laplace transform inversion:
  The Widder formula is

  $$f_{\text{Widder}}(n,t) = \left[ (-1)^n \frac{s^{n+1}}{n!} \frac{d^n}{ds^n} \tilde{f}(s) \right]_{s=\frac{n}{t}}$$

  and the Gaver difference formula is

  $$f_{\text{Gaver}}(n,t) = \left[ \frac{(2n)!}{(n-1)!n!} (-1)^n s \Delta^n f\left(ns\right) \right]_{s=\frac{\ln(2)}{t}}$$

  Although neither formula is particularly well suited to numerical computations we do see the similarities of the Gaver concept and the Widder inversion formula. We also note that these approximations approach the correct solution for $n \to \infty$. 
Numerical Laplace Transform and Inversion

- Gaver formula for numerical Laplace transform inversion:

\[ f_{\text{Gaver}}(m,n,t) = a \frac{(n+m)!}{(m-1)!n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \tilde{f}(a[m+k]) \]

letting \( n=m \) and \( a=\ln(2)/t \), and using the definition of the binomial coefficient

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

we obtain

\[ f_{\text{Gaver}}(n,t) = \frac{\ln(2)}{t} \frac{(2n)!}{(n-1)!} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} \tilde{f} \left( \frac{\ln(2)}{t} (n+k) \right) \]

- The Gaver-Stehfest formula for numerical Laplace transform inversion:

\[ f_{\text{Gaver-Stehfest}}(n,t) = \frac{\ln(2)}{t} \sum_{i=1}^{n} V_i \tilde{f} \left( \frac{\ln(2)}{t} i \right) \]

And the Stehfest extrapolation coefficients are given

\[ V_i = (-1)_2^{n+i} \sum_{k=\left[\frac{i+1}{2}\right]}^{\left[\frac{n}{2}\right]} \frac{n}{k_2 (2k)!} \left[ \frac{n-k}{2} \right]!k!(k-1)!(i-k)!(2k-i)! \]
Numerical Laplace Transform and Inversion

Gaver method example: \( f(t) = \exp(-t); \tilde{f}(s) = 1/(1+s); \) \( f(1) = 0.367879 \) (exact)

Example: Gaver Laplace Transform Inversion Algorithm
\[ [f(s) = \exp(-t); \ exp(-1) = 0.367879... ] \]

Note Divergence of Gaver Series at \( n = 18 \)
\[ \exp(-1) = 0.367879... \] (exact)

Example: Gaver Extrapolation Formula (Polynomial in \( 1/n \))
\[ [f(s) = \exp(-t); \ exp(-1) = 0.367879... ] \]

Note Divergence of Gaver Series

5-term poly (at \( 1/n = 0 \)) = 0.367975 (Least Squares)
\[ \exp(-1) = 0.367879... \] (exact)
Numerical Laplace Transform and Inversion

- The ter Haar polynomial methods for Laplace transform inversion are given as
  **Method 1**: \( u^f(u) \) approach

  \[
  \frac{(k!)^{1/k}}{t} \int \frac{(k!)^{1/k}}{t} = a_0 + a_1 1! \frac{t}{(k!)^{1/k}} + a_2 2! \frac{t^2}{(k!)^{2/k}} + \ldots + a_n n! \frac{t^n}{(k!)^{n/k}}
  \]

  **Method 2**: \( \tilde{f}(u) \) approach

  \[
  \frac{1}{k!} \int \frac{1}{k!} = a_0 \left[ \frac{t^k}{k!} \right]^{k+1} + a_1 1! \left[ \frac{t^k}{k!} \right]^{k+2} + a_2 2! \left[ \frac{t^k}{k!} \right]^{k+3} + \ldots + a_n n! \left[ \frac{t^k}{k!} \right]^{k+n+1}
  \]

  Note that both Methods 1 and 2 are solved as a linear system for the \( a_i \) coefficients (given \( n \) values of \( t \) or \( k \), or a combination of both \( t \) and \( k \)). Also, for both Methods 1 and 2, \( f(t) \) is given as

  \[ f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n \]

- The ter Haar/Laguerre quadrature formula for numerical Laplace transform inversion is given by

  \[
  \tilde{f}(u) = c_1 \sum_{i=1}^{n} \frac{w_i x_i f(u)}{x_i} + c_2 \sum_{i=1}^{n} \frac{w_i x_i^2 f(u)}{x_i^2} + c_3 \sum_{i=1}^{n} \frac{w_i x_i^3 f(u)}{x_i^3} + c_4 \sum_{i=1}^{n} \frac{w_i x_i^2 f(u)}{x_i^2} + \ldots + c_m \sum_{i=1}^{n} \frac{w_i x_i^{m-2} f(u)}{x_i^{m-1}}
  \]

  (yields a linear system for the \( c_i \)'s)

  - Where the "correction function," \( c(t) \) is given by

    \[ c(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + \ldots + c_m t^{m-1} \]

  - And \( f(t) \) is defined as

    \[ f(t) \equiv \frac{c(t)}{t} \tilde{f}(1) \]
The Laguerre quadrature/Polynomial substitution formula for numerical Laplace transform inversion is given by

\[ \tilde{f}(u) = a_1 \sum_{i=1}^{n} \frac{w_i}{u} + a_2 \sum_{i=1}^{n} \frac{w_i}{u} g\left(\frac{x_i}{u}\right) + a_3 \sum_{i=1}^{n} \frac{w_i}{u} g\left(\frac{x_i}{u}\right)^2 + a_4 \sum_{i=1}^{n} \frac{w_i}{u} g\left(\frac{x_i}{u}\right)^3 + \ldots + a_m \sum_{i=1}^{n} \frac{w_i}{u} g\left(\frac{x_i}{u}\right)^{m-1} \]

(yields a linear system for the \(a_i\)'s)

- where \(f(t)\) is defined as
  \[ f(t) = a_1 + a_2 g(t) + a_3 g(t)^2 + \ldots + a_m g(t)^{m-1} \]

- and where \(g(t)\) is defined as a general function
  \[ g(t) = t, \exp(t), \ln(t), \text{etc.} \]