Objectives: (things you should know and/or be able to do)

- Be able to identify all of the components required to develop a governing (partial differential) equation for the flow of fluids in porous media.
  - Mass continuity equation (mass balance on the system):
    - Continuity equation.
  - Equation of motion (force balance):
    - Darcy's law (relates velocity to pressure gradient).
  - Equation of state (relates fluid density with pressure, temperature, and composition):
    - Equation of state for a slightly compressible liquid.
    - Real gas law for compressible gases.
  - Constitutive equation (relates the shear rate and shear stress of the fluid):
    - Unnecessary, we assume Newtonian fluids (viscosity term in Darcy's Law).
  - Energy conservation equation (energy balance on the system):
    - Unnecessary, we assume isothermal flow conditions.
Objectives: (things you should know and/or be able to do)

- **Mass Continuity Equation**: Be able to describe in words and in terms of mathematical expressions, the mass continuity relation for flow through porous media.

  **Words**:
  
  \[
  \begin{bmatrix}
  \text{rate of mass flow into the system during the interval, } \Delta t \\
  \end{bmatrix} - \begin{bmatrix}
  \text{rate of mass flow out of the system during the interval, } \Delta t \\
  \end{bmatrix} = \begin{bmatrix}
  \text{rate of mass accumulation in the system during the interval, } \Delta t \\
  \end{bmatrix}
  \]

  **Mathematical Expression**:
  \[
  \nabla \cdot (\rho \vec{v}) = - \frac{\partial (\phi \rho)}{\partial t}
  \]

- **Equation of Motion**: Be able to recognize that Darcy's law is a force balance on the system (relates flow velocity and pressure drop).

  **Darcy's law**: \[
  \vec{v} = - \frac{k}{\mu} (\nabla p + \rho \vec{g})
  \]
  (vector form)

- **Equation of State**: Be able to state the definition of compressibility.

  Definition of isothermal compressibility for a fluid: \[
  c = \frac{1}{\rho} \frac{\partial \rho}{\partial p}
  \]
Objectives: (things you should know and/or be able to do)

- **Diffusivity Equations for Liquid Flow ("pressure" form):** Be able to develop the "diffusivity" equations for the flow of a slightly compressible liquid in porous media.
  
  **"Gradient-Squared" Case:** General form for a slightly compressible liquid.
  
  \[ c (\nabla p)^2 + \nabla^2 p = \frac{\phi \mu c_t}{k} \frac{dp}{dt} \]

  **"Small and Constant Compressibility" Case:** Base relation for all developments in reservoir engineering and well testing.
  
  \[ \nabla^2 p = \frac{\phi \mu c_t}{k} \frac{dp}{dt} \]
Objectives: (things you should know and/or be able to do)

- **Diffusivity Equations for Liquid Flow ("pseudopressure/pseudotime" forms):** Be able to derive the pseudopressure/pseudotime forms of the diffusivity equation for cases where fluid density and viscosity are functions of pressure.

  **"Pseudopressure-Time" Form**

  \[
  \nabla^2 p_p = \frac{\phi \mu c_t}{k} \frac{dp_p}{dt}
  \]

  where the "pseudopressure" function, \( p_p \), is given by:

  \[\begin{align*}
  p_p &= \left[\frac{\mu B}{k}\right]_n \int_{P_{base}}^{p} \frac{k}{\mu B} \, dp \\
  &= (\mu B)_n \int_{P_{base}}^{p} \frac{1}{\mu B} \, dp
  \end{align*}\]

  and the "pseudotime" function, \( t_a \), is given by:

  \[t_a = (\mu c_t)_n \int_{0}^{t} \frac{1}{\mu(p)c_t(p)} \, dt\]

  **"Pseudopressure-Pseudotime" Form**

  \[
  \nabla^2 p_p = \frac{\phi}{k} (\mu c_t)_n \frac{dp_p}{dt_a}
  \]
Derivation of the Diffusivity Equation for Single-Phase Liquid Flow

from Department of Petroleum Engineering Course Notes (1997)
Diffusivity Equation for Single Phase Liquid Flow in any Flow Geometry

**General Introduction**

Our goal is to develop a mathematical expression (i.e., a partial differential equation) which describes the flow of a single fluid in a porous media with respect to time and distance (i.e., the so-called "diffusivity equation"). This relation should make no assumptions about rates or pressures which may be imposed on the system. We should begin by developing results for a general flow geometry, then adapt these results later to a specific flow geometry (e.g., radial, linear, or spherical flow)—as the need arises.

**Derivation of the Single-Phase Diffusivity Equation (Density Formulation)**

In order to develop a partial differential equation for the flow of fluids in porous media the following physical concepts must be considered:

1. Mass continuity equation (mass balance on the system):
   - Continuity equation.
2. Equation of motion (force balance):
   - Darcy's law (relates velocity to pressure gradient).
3. Equation of state (relates fluid density with pressure, temperature, and composition):
   - Equation of state for a slightly compressible liquid.
   - Real gas law for compressible gases.
4. Constitutive equation (relates the shear rate and shear stress of the fluid):
   - Unnecessary, we assume Newtonian fluids (viscosity term in Darcy's Law).
5. Energy conservation equation (energy balance on the system):
   - Unnecessary, we assume isothermal flow conditions.

The mass continuity equation is described in words as

\[
\left[ \text{rate of mass flow into the system} \right] - \left[ \text{rate of mass flow out of the system} \right] = \left[ \text{rate of mass accumulation in the system during the interval, } \Delta t \right] \quad \ldots (1)
\]

The mathematical form of the mass continuity equation is

\[
\nabla \cdot (\rho \mathbf{v}) = -\frac{\partial (\rho \phi)}{\partial t} \quad \ldots (2)
\]

where

\[
\mathbf{v} = \text{fluid velocity vector} \\
\rho = \text{fluid density} \\
\phi = \text{porosity, fraction of bulk volume} \\
t = \text{time}
\]

The equation of motion used for fluid flow in porous media is known as Darcy's law. Although Darcy's law was empirically derived by performing experiments of water flow through sand packs, and while this concept was only partially verified in an analytical sense (using Navier-Stokes theory), we consider Darcy's law accurate for the laminar flow of fluids in porous media. Darcy's law is given (in vector form) as

\[
\mathbf{v} = \frac{k}{\mu} (\nabla P + \rho \mathbf{g}) \quad \ldots (3)
\]

where

\[
P = \text{pressure gradient} \\
k = \text{effective permeability} \\
\mu = \text{fluid viscosity} \\
\mathbf{g} = \text{gravity vector}
\]

Combining Eqs. 2 and 3 gives us

\[
\nabla \cdot \left[ \frac{k}{\mu} (\nabla P + \rho \mathbf{g}) \right] = \frac{\partial (\rho \phi)}{\partial t} \quad \ldots (4)
\]

If we assume horizontal flow, the gravity term can be dropped, which reduces Eq. 4 to

\[
\nabla \cdot \left[ \frac{k}{\mu} \nabla P \right] = \frac{\partial (\rho \phi)}{\partial t} \quad \ldots (5)
\]

The general \( \nabla \) operator is defined as

\[
\nabla \cdot a = \frac{\partial a}{\partial y} \quad \ldots (6)
\]

where pressure or pseudopressure will be used as the "y" variable. The \( \nabla \cdot a \) terms are given by:

**Linear Flow:**

\[
\nabla \cdot a = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial y} + \frac{\partial a}{\partial r}
\]

**Radial Flow:**

\[
\nabla \cdot a = \frac{1}{r \partial r} \left( \frac{\partial}{\partial r} \right) \left( r \frac{\partial a}{\partial r} \right) + \frac{1}{r^2 \partial \theta} \left( \frac{\partial a}{\partial \theta} \right)
\]

For the developments in this section, we will leave the results in the \( \nabla \) notation (indicating a general flow geometry). In later developments the appropriate operator(s) for a particular flow geometry (e.g., radial flow) will be substituted.

Eq. 5 is the fundamental form of the diffusivity equation in terms of fluid density and it represents the starting point for our developments in terms of pressure, pressure-squared, and pseudopressure. At this point, the compressibility equation-of-state has not been incorporated into the diffusivity equation because this substitution is not yet necessary. In fact, the diffusivity equation can be worked entirely in terms of density—however, this form would not be practical for field applications.

We note that Eq. 5 includes the constitutive equation implicitly via the viscosity term in Darcy's law (Eq. 3). Virtually all well testing conditions are considered isothermal, hence the conservation of energy equation can be neglected.

Assuming that the permeability, \( k \), and the fluid viscosity, \( \mu \), are constant (i.e., the usual "liquid" flow assumptions), Eq. 5 reduces to

\[
\nabla \cdot \left[ \rho \mathbf{v} \right] = \frac{\mu \partial (\rho \phi)}{k} \quad \ldots (6)
\]

**Derivation of the Single-Phase Diffusivity Equation (Slightly Compressible Liquid Case)**

If we expand the terms on the left-hand-side of Eq. 6 using the product rule, and we expand the term on the right-hand-side using the chain rule, then we obtain

\[
\nabla \cdot \rho \mathbf{v} = \frac{\mu \partial (\rho \phi)}{k} \quad \ldots (7)
\]
Using the chain rule for the grad terms ($\nabla$), the first term on the left-hand-side of Eq. 7 can be written as
$$\nabla \cdot \nabla p = \frac{\partial^2}{\partial p^2} p \nabla^2 p = \frac{\partial^2}{\partial p^2} (\nabla p)^2$$

The second term on the left-hand-side of Eq. 7 can also be reduced using the definition of the $\nabla$ operator. This gives
$$\rho \nabla \cdot \nabla p = \rho \nabla^2 p$$

Substituting Eqs. 8 and 9 into Eq. 7, we obtain
$$\frac{\partial}{\partial p} (\nabla p)^2 + \rho \nabla^2 p = \frac{\mu}{k} \frac{\partial \phi}{\partial \phi}$$

By applying the product rule to the $\frac{\partial \phi}{\partial \phi}$ term in Eq. 10 we have
$$\frac{\partial \phi}{\partial \phi} = \phi \frac{\partial}{\partial \phi} + \rho \frac{\partial}{\partial \phi}$$

Factoring out the $\phi$ terms gives us
$$\frac{\partial \phi}{\partial \phi} = \phi \left[ \frac{1}{\rho} \frac{\partial \phi}{\partial \phi} + \frac{1}{\phi} \frac{\partial \phi}{\partial \phi} \right]$$

The definition of fluid compressibility is given as
$$c = \frac{1}{\rho} \frac{\partial \rho}{\partial p}$$

and, the definition of pore-volume compressibility is given as
$$c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial p}$$

Combining the definition of compressibility with the expansion of $\frac{\partial \phi}{\partial \phi}$ we obtain
$$\frac{\partial \phi}{\partial \phi} = \phi (c + c_f)$$

The "total" compressibility is defined as $c_t = c + c_f$, and the last relation can be written as
$$\frac{\partial \phi}{\partial \phi} = \phi c_t$$

Substituting Eq. 13 into Eq. 10, we obtain
$$\frac{\partial}{\partial p} (\nabla p)^2 + \rho \nabla^2 p = \frac{\phi c_t}{k} \frac{\partial}{\partial \phi}$$

Dividing through by the fluid density gives
$$\frac{1}{\rho} \frac{\partial}{\partial p} (\nabla p)^2 + \frac{\partial^2}{\partial p^2} \frac{\rho}{\phi c_t} \frac{\partial}{\partial \phi}$$

Using the definition of fluid compressibility (Eq. 11) we obtain our "final" form
$$c (\nabla p)^2 + \nabla^2 p = \frac{\phi c_t}{k} \frac{\partial}{\partial \phi}$$

The $c (\nabla p)^2$ term in Eq. 14 is clearly nonlinear, as this term squares the gradient of pressure (which is nonlinear in and of itself) and multiplies this group by the compressibility (which is generally a weak function of pressure for liquids above the bubblepoint pressure). Solving Eq. 14 by analytical means is very difficult, and the solution given by Finjord and Aadnoy (which is a perturbation solution) is not well suited for the practical analysis of well test and/or production data.

However, if we assume a small and constant compressibility (as required by the equation of state for a "slightly" compressible liquid), then $c (\nabla p)^2$ can be neglected.

For reference, the equation of state for a "slightly" compressible liquid is given as
$$p = \rho_0 \exp \left[ c (p - p_0) \right]$$

Therefore, if the $c (\nabla p)^2$ term in Eq. 14 is neglected, Eq. 14 relation becomes
$$\nabla^2 p = \frac{\phi c_t}{k} \frac{\partial}{\partial \phi}$$

Eq. 16 is familiar as a typical diffusion-type, partial differential equation and it can be solved analytically for a variety of flow geometries and boundary conditions (all of which assume that the diffusivity term, $(\phi c_t)/k$, is constant).

Derivation of the Single Phase Diffusivity Equation (Pseudopressure and Pseudotime Case)

The developments shown in the previous section focused on the derivation of the diffusivity equation for single-phase liquid flow where the liquid was considered to be of small and constant compressibility. Generally, these conditions are only satisfied for black oil fluids above the bubblepoint pressure. For black oil fluids below the bubble point pressure (and for natural gases) Eq. 16 will be an approximation at best, and a complete failure at worst.

Therefore, we must develop a form of the diffusivity equation where a general behavior for $\rho$, $\mu$, and $k$ can be assumed. The, $(\phi k)/\mu$, term in Eq. 5 will be treated as general, with no assumptions on its behavior with respect to pressure. Recalling Eq. 5, we have
$$\nabla \cdot \left[ \frac{\phi k}{\mu} \nabla p \right] = \frac{\partial \phi}{\partial \phi}$$

For a black oil fluid, the oil density is given by
$$\rho_o = \frac{1}{B_o} \left( \rho_{o,sc} + 0.01357 R_5 \rho_g \right)$$

where
$$\rho_o = \text{Oil density at reservoir conditions, lbm/ft}^3$$
$$\rho_{o,sc} = \text{Oil density at stock tank (or standard) conditions, lbm/ft}^3$$
$$B_o = \text{Oil formation volume factor, RB/STB}$$
$$R_5 = \text{Gas-oil ratio, scf/STB (constant for p>p_b)}$$
$$\rho_g = \text{Stock tank gas gravity (air=1.0)}$$

For gases, the density is given by the real gas law as
$$\rho_g = \frac{p M}{z R T}$$

where
$$\rho_g = \text{Gas density at reservoir conditions, lbm/ft}^3$$
$$p = \text{Reservoir pressure, psia}$$
$$M = \text{Molecular weight of the gas, lbm/lb-mole}$$
$$z = \text{z-factor at reservoir pressure}$$
$$R = \text{Universal gas constant (10.732 [psia-ft]/[lb-mole-oR])}$$
$$T = \text{Reservoir temperature, oR}$$

and the gas formation volume factor is given by
Combining these developments and using a generic form given by Eqs. 21 and 22, we obtain the following general relation as our starting point:

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right]$$

Expanding the right-hand-side of Eq. 27 using the chain rule gives us:

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \frac{\partial}{\partial \phi} \left[ \frac{\phi}{B} \right] \frac{\partial \phi}{\partial t}$$

Using the product rule we have:

$$\frac{\partial}{\partial \phi} \left[ \frac{\phi}{B} \right] = \frac{1}{B} \frac{\partial \phi}{\partial \phi} - \frac{\phi}{B} \frac{\partial B}{\partial \phi} \frac{\partial \phi}{\partial t}$$

which after some rearranging into a more familiar form gives us:

$$\frac{\partial}{\partial \phi} \left[ \frac{\phi}{B} \right] = \frac{1}{B} \frac{\partial \phi}{\partial \phi} - \frac{1}{B} \frac{\partial B}{\partial \phi} \frac{\partial \phi}{\partial t}$$

Recalling the definition of fluid compressibility, we have:

$$\frac{1}{B} \frac{\partial B}{\partial \phi} = \frac{1}{c} \frac{\partial \phi}{\partial \phi}$$

Recalling the definition of pore volume compressibility, we have:

$$c_T = \frac{1}{B} \frac{\partial \phi}{\partial \phi}$$

Substituting Eqs. 12 and 29 into Eq. 28, and then using the definition, $c_T = c + c_f$, in this result, gives us the following:

$$\frac{\partial}{\partial t} \left[ \frac{\phi}{B} \right] = \frac{c_T}{B} \frac{\partial \phi}{\partial t}$$

Substituting Eq. 30 back into Eq. 27 we obtain:

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \frac{\phi c_T}{B} \frac{\partial \phi}{\partial t}$$

Combining Eqs. 24 and 25 and solving for $\frac{k}{\mu B} \nabla p$ we have:

$$\frac{k}{\mu B} \nabla p = \left[ \frac{k}{\mu B} \right]_{p_{base}} \nabla p_p$$

Also, if we combine Eqs. 25 and 26 and solve for $\frac{\phi}{\partial t}$ we obtain:

$$\frac{\phi}{\partial t} = \left[ \frac{k}{\mu B} \right]_{p_{base}} \frac{\phi}{\partial t}$$

Substituting Eqs. 32 and 33 into Eq. 31 yields:

$$\nabla \cdot \left[ \frac{k}{\mu B} \nabla p \right] = \frac{\phi c_T}{B} \frac{k}{\mu B} \frac{\phi}{\partial t}$$

Canceling the $\frac{k}{\mu B} \nabla p$ terms we obtain:

$$\nabla \cdot \nabla p = \frac{\phi c_T}{k} \frac{\phi}{\partial t}$$

Recalling that $\nabla \cdot \nabla a = \nabla^2 a$, we obtain the following as our final result:

$$\nabla^2 p = \frac{\phi c_T}{k} \frac{\phi}{\partial t}$$
While the left-hand-side of Eq. 34 has no nonlinear terms, the $\mu$, $c_l$, and $k$ terms on the right-hand-side are not assumed constant and represent non-linearities due to the multiplication of these terms with $\partial p/\partial t$, as well as with each other. In order to develop analytical solutions using Eq. 34, we must either assume $(\mu c_l)/k$ to be constant, or we must develop a linearizing pseudotime that accounts for changes in $(\mu c_l)/k$ with respect to pressure.

For gas reservoirs, we typically assume that the effective permeability, $k$, is constant, and we also assume that $\mu c_l$ is constant during transient drawdown flow conditions.

Because we know that the $(\mu c_l)/k$ term is not really constant, we must then develop another linearizing function (in this case a pseudotime) in order to account for the variation in the $(\mu c_l)/k$ term. Agarwal\(^4\) proposed the following "intuitive" pseudotime function

$$t_a = (\mu c_l)_n \int_0^t \frac{1}{\mu (p)c_l(p)} \, dt \tag{35}$$

Lee and Holditch\(^5\) verified this function for the analysis of pressure buildup tests and gave analytic criteria for the application of Eq. 35 using wellbore pressures to evaluate $\mu$ and $c_l$ (but these results are only valid for the transient pressure buildup case, pseudotime is not used for the analysis or modelling of transient pressure drawdown behavior).

Frain and Wattenberger\(^6\) proposed a slightly different form of the Agarwal\(^4\) pseudotime function for boundary-dominated (pseudosteady-state) flow. This pseudotime is given by

$$t_a = (\mu c_l)_n \int_0^t \frac{1}{\mu (p)c_l(p)} \, dt \tag{36}$$

where the average reservoir pressure, $\overline{p}$, is used in place of the wellbore pressure, for the evaluation of $\mu c_l$. While Eq. 36 is rigorously correct for boundary-dominated flow, simulation studies suggests that Eq. 36 is also correct for modelling gas well performance during transient flow.

Our present objective is to combine the definition of the pseudotime with our general form of the diffusivity equation, Eq. 34. Taking the partial derivative of Eq. 35 with respect to time gives

$$\frac{\partial t_a}{\partial t} = (\mu c_l)_n \frac{\partial p}{\partial t} \tag{37}$$

Applying the chain rule to the $\frac{\partial p}{\partial t}$ term in Eq. 34, we have

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial t} \frac{\partial p}{\partial t} \tag{38}$$

Combining this result with Eq. 37 gives us

$$\mu c_l \frac{\partial p}{\partial t} = (\mu c_l)_n \frac{\partial p}{\partial t} \tag{39}$$

Substituting Eq. 34 into Eq. 38, we obtain

$$\nabla^2 p_n = \frac{\phi}{k} (\mu c_l)_n \frac{\partial p}{\partial t} \tag{40}$$

where the normalized pseudopressure function, $p_n$, is given by

$$p_n = \left[ \frac{\mu B}{k} \right]_n \int_{P_{base}}^P \frac{k}{\mu B} \, dp \tag{41}$$

and the normalized pseudotime function, $t_a$, is

$$t_a = (\mu c_l)_n \int_0^t \frac{1}{\mu (p)c_l(p)} \, dt \tag{42}$$

If we assume that the effective permeability, $k$, is constant (as is typically done), Eq. 23 reduces to

$$p_n = (\mu B)_n \int_{P_{base}}^P \frac{1}{\mu B} \, dp \tag{43}$$

Finally, we note that the definitions of pseudopressure and pseudotime are valid for the single-phase flow of both gases and liquids. The pseudopressure and pseudotime concepts can be extended to multiphase flow, but these results require knowledge of both the pressure and saturation histories, and, as such, these results are considered to be theoretical, with little (if any) practical applications possible without full-scale reservoir simulation.

Because the pseudopressure/pseudotime diffusivity equation is written in a general form (i.e., the "liquid equivalent" form), we can use the solutions derived for the "slightly compressible liquid" case to model the single-phase flow of both gases and compressible liquids—assuming that the appropriate pseudopressure and pseudotime functions are used for each case.
Variation Of Average Compressibility/
Mobility Ratio with Time

Figure 5: Variation of average compressibility/mobility ratio with time.