Objectives: (things you should know and/or be able to do)

Infinite-Acting Reservoir Case:

- Be able to develop the real domain (time) solution (i.e., the Exponential Integral solution) for the constant rate inner boundary condition and the infinite-acting reservoir outer boundary condition—using both the Laplace transform as well as the Boltzmann transform approaches. Also be able to develop the "log-approximation" for this solution.

  ■ Boltzmann Transform of the Diffusivity Equation:

  \[
  \frac{d^2 p_D}{d \varepsilon_D^2} + \left[ 1 + \frac{1}{\varepsilon_D} \right] \frac{dp_D}{d \varepsilon_D} = 0 \quad \text{(infinite-acting reservoir case only)}
  \]

- Real domain (time) solutions: (Continued)

  ■ "Exponential Integral" Solution for the Diffusivity Equation:

  \[
  p_D(t_D, r_D) = \frac{1}{2} \ E_1 \left[ \frac{r_D^2}{4 t_D} \right]
  \]

  ■ "Log Approximation" Solution for the Diffusivity Equation:

  \[
  p_D(t_D, r_D) = \frac{1}{2} \ \ln \left[ \frac{4}{e^\gamma} \frac{t_D}{r_D^2} \right]
  \]
Solution of the Radial Flow Diffusivity Equation
via the Boltzmann Transform

from Department of Petroleum Engineering Course Notes (1994)
Radial Flow Solution for an Infinite-Aging Homogeneous Reservoir:

Boltzmann Transform Approach

This method has been demonstrated by a variety of authors—the approach we choose was presented by J.L. Johnston in the 2nd edition of the Lee-Well testing text.

The basic partial differential equation is given in dimensionless form as

\[
\frac{\partial \phi}{\partial \xi} + \frac{1}{2} \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \eta}
\]

where \( \xi = \frac{x}{r} \) and \( \eta = \frac{h}{k} \) \((r_0, h_0)\) (1)

\( t = \frac{t_0}{k \varepsilon} \) (2)

where \( t_0 \) and \( h_0 \) are given by

\[ \begin{align*}
\text{Darcy Units} & \quad \text{Field Units} & \quad \text{SI Units} \\
\varepsilon & = 1 & \quad \varepsilon = \text{2.687 x 10}^{-3} & \quad \varepsilon = 2.687 \times 10^{-5} \\
\frac{h_0}{r_0} & = 2 & \quad \frac{h_0}{r_0} = 0.811 \times 10^{-3} & \quad \frac{h_0}{r_0} = 5.256 \times 10^{-4}
\end{align*} \]

The "initial" condition is given as

\( \frac{\partial \phi}{\partial \eta} \mid_{\eta=0} = 0 \) (uniform pressure distribution) (3)

The constant rate inner boundary condition is

\[ \left[ \frac{\partial \phi}{\partial \xi} \right] \mid_{\xi=0} = -1 \] (constant flow rate at the well) (4)

The "infinite-acting" outer boundary condition is given by

\( \phi(\xi = 700, \eta) = 0 \) (5)

Rewriting eq. 1 we have

\[ \frac{1}{r_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) = \frac{\partial \phi}{\partial \eta} \] (6)

The Boltzmann transform variable, \( s_0 \), is defined as

\[ s_0 = \frac{1}{s_0} \] (7)

Where for our problem we have

\( a = \frac{1}{4}, b = 2, c = -1 \)

which yields

\( s_0 = \frac{1}{4} \) (8)

Expanding the \( \frac{1}{s_0^2} \) term we have

\[ \frac{1}{r_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{1}{r_0^2} \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \eta} \] (11)

Applying the chain rule

\[ \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \xi} \frac{\partial \xi}{\partial \eta} \]

which combined with eq. 11 gives

\[ \frac{\partial \phi}{\partial \xi} \frac{\partial \xi}{\partial \eta} + \frac{1}{r_0^2} \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \eta} \]

Expanding

\[ \frac{1}{r_0^2} \left[ \frac{1}{s_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \right] = \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} + \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \eta} \]

Isolating terms

\[ \frac{1}{s_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial \phi}{\partial \eta} \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \]

Dividing through by \( (s_0^2/r_0^2)^2 \) gives

\[ \frac{1}{s_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial \phi}{\partial \eta} \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \]

Reducing the \( x(1/k_0^2) \) term we have

\[ \frac{1}{s_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \]

Which can be further reduced to

\[ \frac{1}{s_0^2} \frac{\partial \phi}{\partial \xi} + \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \]

Completing the factorization of \( (s_0^2/r_0^2)^2 \) gives

\[ \frac{1}{s_0^2} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{1}{s_0^2} \frac{\partial \phi}{\partial \eta} \]

Using eq. 10 we take the following derivatives

\[ \frac{\partial \phi}{\partial \xi} = \frac{1}{4} \left( \frac{1}{s_0^2} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]

\[ \frac{\partial \phi}{\partial \eta} = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]

\[ \frac{\partial \phi}{\partial \xi} \left( \frac{\partial \phi}{\partial \eta} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]

\[ \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \xi} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]

\[ \frac{\partial \phi}{\partial \xi} \left( \frac{\partial \phi}{\partial \eta} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]

\[ \frac{\partial \phi}{\partial \eta} \left( \frac{\partial \phi}{\partial \xi} \right) = \frac{1}{4} \left( \frac{1}{s_0^2} \right) \]
Substituting Eqs. 18-19 into Eq. 12 gives
\[
\frac{\partial^2 \theta}{\partial z^2} + \left[ \frac{1}{(2\pi r_0)^2} \frac{2}{r^2} + \frac{1}{r} \right] \frac{\partial \theta}{\partial r} - \frac{1}{(2\pi r_0)^2} \frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial z} = 0
\]
Reducing
\[
\frac{\partial^2 \theta}{\partial z^2} + \left[ \frac{1}{2} \frac{1}{\epsilon_0} + \frac{1}{2\epsilon_0} \right] \frac{\partial \theta}{\partial z} = 0
\]
Or finally we have
\[
\frac{\partial^2 \theta}{\partial z^2} + \frac{1}{\epsilon_0} \frac{1}{\epsilon_0} \frac{\partial \theta}{\partial z} = 0
\]
Where Eq. 16 is our “Boltzmann” transformed differential equation.

We must now establish the initial and boundary conditions in terms of the Boltzmann transform variable \( \epsilon_0 \). Recalling the initial condition, Eq. 5, we have
\[
\theta(r_0, z_0 \leq 0) = 0
\]
where for \( r_0 \to 0; \epsilon_0 \to \infty \), which gives
\[
\theta(r_0, \epsilon_0) = 0
\]
Recalling the outer boundary condition, Eq. 7, we have
\[
\theta(r_0, z_\infty) = 0
\]
or at \( r_0 \to 0; \epsilon_0 \to \infty \), which yields
\[
\theta(r_0, \epsilon_0) = 0
\]
Where Eqs. 17 and 18 are the same, which illustrates that the Boltzmann transform “collapses” 2 conditions into 1. Combining this observation with the inner boundary condition, we have 2 “boundary” conditions. Coupling this observation with the fact that Eq.16 is only a function of the Boltzmann variable \( \epsilon_0 \) we can solve Eq. 14 uniquely. Note that the “collapsing” of the initial and outer boundary conditions must occur in the Boltzmann transform is technically invalid.

Recalling the constant rate inner boundary condition, Eq. 6,
\[
\left[ \frac{\partial \theta}{\partial z} \right]_{z_0 = 0} = \frac{-1}{2}\epsilon_0 \quad \text{or} \quad \left[ \frac{\partial \theta}{\partial z} \right]_{z_0 = 0} = -1 \quad \text{(line source condition)} (6)
\]
or
\[
\left[ \frac{\partial \theta}{\partial z} \right]_{z_0 = 0} = -\frac{1}{2} \frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial z} = -1
\]
which can be rearranged to yield
\[
\frac{\epsilon_0 \frac{\partial \theta}{\partial z}}{\theta_0 - \epsilon_0} = -\frac{1}{2}
\]
Making the following variable of substitution
\[
\psi = \frac{\epsilon_0}{\theta_0}
\]
Substituting Eq. 20 into Eq. 16, and noting the use of ordinary derivatives
\[
\frac{d\psi}{d\epsilon_0} + \left[ \frac{1}{\epsilon_0} + \frac{1}{\epsilon_0} \right] \psi = 0
\]
\[
\frac{1}{\psi} \frac{d\psi}{d\epsilon_0} = -\left[ \frac{1}{\epsilon_0} + \frac{1}{\epsilon_0} \right] \frac{d\epsilon_0}{d\epsilon_0} = -\frac{d\epsilon_0}{\epsilon_0} - \frac{d\epsilon_0}{\epsilon_0}
\]
Integrating
\[
\ln(\psi) = -\epsilon_0 - \ln\epsilon_0 \quad \beta \quad \text{constant of integration}
\]
Exponentiating
\[
\psi = \exp\left[-\epsilon_0 - \ln\epsilon_0 + \beta\right]
\]
or
\[
\psi = \exp\left[-\epsilon_0\right] \exp\left[-\ln\epsilon_0\right] \exp[\beta]
\]
which reduces to
\[
\psi = \tilde{\epsilon_0} \exp\left[-\tilde{\epsilon_0}\right]
\]
where \( \tilde{\epsilon_0} = \exp[\beta] \), i.e., the constant of integration. Recalling Eq. 20 and combining gives
\[
\frac{d\epsilon_0}{d\epsilon_0} = \tilde{\epsilon_0} \exp\left[-\tilde{\epsilon_0}\right]
\]
Multiplying through by \( \epsilon_0 \) gives
\[
\epsilon_0 \frac{d\epsilon_0}{d\epsilon_0} = \tilde{\epsilon_0} \exp\left[-\tilde{\epsilon_0}\right]
\]
Substitution of Eq. 22 into Eq. 19 gives
\[
\lim_{\epsilon_0 \to \infty} \left[ \exp\left[-\tilde{\epsilon_0}\right] \right] = -\frac{1}{2}
\]
or
\[
\tilde{\epsilon_0} = -\frac{1}{2}
\]
Substitution of Eq. 24 into Eq. 22 gives
\[
\frac{d\epsilon_0}{d\epsilon_0} = -\frac{1}{2} \exp\left[-\tilde{\epsilon_0}\right]
\]
Which can be rearranged to yield
\[
\frac{\epsilon_0 \frac{\partial \theta}{\partial z}}{\theta_0 - \epsilon_0} = -\frac{1}{2}
\]
Separating and integrating Eq. 25 gives

$$\int_{R=0}^{R_0} \frac{dR}{R} = -\frac{1}{2} \int_{t_0}^{t} \frac{1}{x_0} \frac{1}{R_0^2} e^{-\frac{R^2}{4R_0^2}} \, dt_0$$

where we note that $R_0 = 0$ at $t = t_0$ is the initial outer boundary condition. Completing the integration and reversing the limits we have

$$R_0 = \frac{1}{2} \int_{t_0}^{t} \frac{1}{x_0} \frac{1}{R_0^2} e^{-\frac{y^2}{4R_0^2}} \, dy$$

(26)

We note that the integral in Eq. 26 is the exponential integral, $E_i(y)$, which is given by

$$E_i(y) = \int_{y}^{\infty} \frac{1}{x} e^{-\frac{x^2}{4}} \, dx$$

(27)

Combining Eqs. 26 and 27 gives our final result

$$R_0 (R_0, t_0) = \frac{1}{2} \left( \frac{R_0^2}{4t_0} \right)$$

(28)