Objectives: (things you should know and/or be able to do)

Laplace Domain Solutions:

- Be able to derive the particular solutions (in Laplace domain) for a well produced at a constant flowrate in a homogeneous reservoir for the following initial condition—subject to the following inner and outer boundary conditions:
  
  **Initial Condition** (Uniform Pressure Distribution)
  
  \[ p_D(r_D, t_D \leq 0) = 0 \]

  **Inner Boundary Condition** (Constant Flowrate at the Well)
  
  \[ r_D \frac{\partial p_D}{\partial r_D} \bigg|_{r_D = 1} = -1 \]

  **Outer Boundary Conditions**
  
  a. "Infinite-Acting" Reservoir
  
  \[ p_D(r_D \rightarrow \infty, t_D) = 0 \] (No reservoir boundary)

  b. "Prescribed Flux" at the Boundary
  
  \[ r_D \frac{\partial p_D}{\partial r_D} \bigg|_{r_D = r_{eD}} = q_{Dext}(t_D) \] (Specified flux across the reservoir boundary)

  c. Constant Pressure Boundary
  
  \[ p_D(r_{eD}, t_D) = 0 \] (Constant pressure at the reservoir boundary)
Objectives: (things you should know and/or be able to do)

- **Particular Solutions in the Laplace Domain**
  - "Infinite-acting" reservoir behavior: "cylindrical source" solution
    \[
    \bar{p}_D(r_D,u) = \frac{1}{u} \frac{K_0(\sqrt{u}r_D)}{\sqrt{u} K_1(\sqrt{u})}
    \]
  - "Infinite-acting" reservoir behavior: "line source" solution
    \[
    \bar{p}_D(r_D,u) = \frac{1}{u} K_0(\sqrt{u}r_D) \quad \text{(where } \sqrt{u} K_1(\sqrt{u}) \to 1; \text{ for } \sqrt{u} \to 0)\]
  - "Infinite-acting" reservoir behavior: "log approximation" solution
    \[
    \bar{p}_D(r_D,u) \approx \frac{1}{u} K_0(\sqrt{u}r_D) = \frac{1}{2u} \ln \left[ \frac{4}{e^{2\gamma} r_D^2} \right] \quad (\gamma = 0.577216 \ldots \text{ Euler's Constant})
    \]

- **Particular Solutions in the Laplace Domain**: (Continued)
  - Bounded circular res. — "no-flow" at the outer boundary \((i.e., \ q_{D\text{ext}}(t_D)=0)\)
    \[
    \bar{p}_D(r_D,u) = \frac{1}{u} \frac{K_0(\sqrt{u}r_D) I_1(\sqrt{u}r_D) + K_1(\sqrt{u}r_D) I_0(\sqrt{u}r_D)}{\sqrt{u} K_1(\sqrt{u}) I_1(\sqrt{u}r_D) - \sqrt{u} I_1(\sqrt{u}) K_1(\sqrt{u}r_D)}
    \]
  - Bounded circular reservoir — "constant pressure" at the outer boundary
    \[
    \bar{p}_D (r_D,u) = \frac{1}{u} \frac{K_0(\sqrt{u}r_D) I_0(\sqrt{u}r_D) - K_0(\sqrt{u}r_D) I_0(\sqrt{u}r_D)}{\sqrt{u} K_1(\sqrt{u}) I_0(\sqrt{u}r_D) + \sqrt{u} I_1(\sqrt{u}) K_0(\sqrt{u}r_D)}
    \]
Solution of the Radial Flow Diffusivity Equation in Terms of the Laplace Transform

from Department of Petroleum Engineering Course Notes (1994)
Solution for Radial Flow in a Homogeneous Reservoir: Infinite-Radius, No-Flow, and Constant Pressure Outer Boundaries - Laplace Transform Approach

The fundamental partial differential equation (the diffusion equation) is given in dimensionless form by:

\[
\frac{\partial \phi}{\partial \tau} + \frac{1}{\sqrt{\pi \beta}} \frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial \rho} \tag{1}
\]

or

\[
\frac{1}{\beta} \frac{\partial}{\partial \rho} \left[ \phi \frac{\partial \phi}{\partial \rho} \right] = \frac{\partial \phi}{\partial \rho} \tag{2}
\]

where

\[
\phi = \frac{r}{k} \quad \beta = \frac{r}{k} \quad \rho = \frac{r}{k} \quad \frac{\partial \phi}{\partial \rho} = \frac{t}{\mu} \tag{4}
\]

and \( \frac{\partial \phi}{\partial \rho} \) are given by

<table>
<thead>
<tr>
<th>Darcy Units</th>
<th>Field Units</th>
<th>ST Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{t}{\mu} )</td>
<td>2.35710^-8</td>
<td>5.17810^-8</td>
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<td>2.35710^-8</td>
<td>5.17810^-8</td>
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</tbody>
</table>

The initial condition is given as

\[
\phi(\rho, \tau = 0) = 0 \quad \text{(uniform pressure distribution)} \tag{5}
\]

The constant rate inner boundary condition is

\[
\frac{\partial \phi}{\partial \rho} \bigg|_{\rho = \rho_0} = 1 \quad \text{(constant flow rate at the well)} \tag{6}
\]

The outer boundary conditions are given by:

a. "Infinite-acting" outer boundary condition

\[
\phi(\rho = \infty, \tau) = 0 \tag{7}
\]

b. "No-flow" outer boundary condition

\[
\frac{\partial \phi}{\partial \rho} \bigg|_{\rho = \rho_0} = 0 \tag{8}
\]

c. "Constant pressure" outer boundary condition

\[
\phi(\rho_0, \tau) = \phi_{\text{ext}}(\tau) = 0 \quad \text{(constant at initial pressure)} \tag{9}
\]

d. "Specified flux" outer boundary condition

\[
\frac{\partial \phi}{\partial \rho} \bigg|_{\rho = \rho_0} = \phi_{\text{ext}}(\tau) \tag{10}
\]

Laplace Transform Formulation:

\[
\phi_\beta(\rho, \beta) = \mathcal{L}\{\phi(\rho, \tau)\}, \mu = \text{laplace transform parameter}
\]

Taking the laplace transform of eq. 2 gives

\[
\frac{1}{\beta} \frac{d}{d\beta} \left[ \phi_{\beta} \right] = \mu \phi_{\beta} \tag{12}
\]

We recognize from Eq. 6 that \( \phi(\rho, \tau = 0) = 0 \), combining Eqs. 6 and 12 we obtain

\[
\frac{1}{\beta} \frac{d}{d\beta} \left[ \phi_{\beta} \right] = \mu \phi_{\beta} \tag{13}
\]

Taking the Laplace transform of the inner boundary condition gives

\[
\left[ \frac{\partial \phi_{\beta}}{\partial \beta} \right]_{\beta = 0} = -1 \tag{14}
\]

Taking the laplace transform of the outer boundary conditions:

a. Laplace transform of the "infinite-acting" outer boundary condition

\[
\phi_{\beta}(\beta = \infty, \mu) = 0 \tag{15}
\]

b. Laplace transform of the "no-flow" outer boundary condition

\[
\left[ \frac{\partial \phi_{\beta}}{\partial \beta} \right]_{\beta = \rho_0} = 0 \tag{16}
\]

c. Laplace transform of the "constant pressure" outer boundary condition

\[
\lim_{\beta \to \infty} \phi_{\beta}(\rho_0, \mu) = \phi_{\text{ext}}(\tau) = 0 \quad \text{(constant at initial pressure)} \tag{17}
\]

d. Laplace transform of the "specified flux" outer boundary condition

\[
\left[ \frac{\partial \phi_{\beta}}{\partial \beta} \right]_{\beta = \rho_0} = \phi_{\text{ext}}(\tau) \tag{18}
\]

Multiplying through Eq. 18 by \( \rho_0 \), we have

\[
\frac{\rho_0}{\partial \beta} \left[ \phi_{\beta} \right] = \mu \phi_{\beta} \tag{19}
\]

Defining a variable of substitution, \( \tau = \sqrt{\rho_0 \beta} \)

or

\[
\phi_{\beta} \bigg|_{\beta = \rho_0} = \frac{\rho_0}{\sqrt{\beta}} \tag{20}
\]

Applying the chain rule on the \( d/d\beta \) terms in Eq. 19 we obtain

\[
\frac{\rho_0}{\partial \beta} \left[ \phi_{\beta} \right] = \mu \phi_{\beta} \tag{21}
\]

where

\[
\frac{\rho_0}{\partial \beta} \left[ \phi_{\beta} \right] = \sqrt{\rho_0} \tag{22}
\]

Substituting Eqs. 21 and 22 into Eq. 22 we have

\[
\frac{\rho_0}{\partial \beta} \left[ \phi_{\beta} \right] = \sqrt{\rho_0} \tag{23}
\]
Cancelling the \( \sqrt{r} \) terms on the left-hand-side we obtain
\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{dw}{dr} \right] = \frac{2}{r} S_0 \tag{24}
\]
Expanding the left-hand-side terms we have
\[
\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} = \frac{2}{r} S_0 \tag{25}
\]
From Abramowitz and Stegun, Handbook of Mathematical Functions, (p. 354, Eq. 9.6.1), the modified Bessel differential equation is given by
\[
\frac{d^2 w}{dx^2} + \frac{1}{x} \frac{dw}{dx} = \frac{2}{x} v^2 w \tag{26}
\]
The general solution of Eq. 26 is given by
\[
w = A I_0(x) + B K_0(x) \tag{27}
\]
where the functions \( I_0(x) \) and \( K_0(x) \) are the modified Bessel functions of the first and second kinds, respectively. By inspection, our general solution is
\[
b_0(r) = A I_0(\sqrt{r}) + B K_0(\sqrt{r}) \tag{28}
\]
or, substituting \( \sqrt{r} = \sqrt{r_0} \) (Eq. 20) into Eq. 28 we have
\[
b_0(r_0, w) = A I_0(\sqrt{r_0}) + B K_0(\sqrt{r_0}) \tag{29}
\]
In order to develop our particular solutions (i.e. to solve for the \( A \) and \( B \) parameters for each set of boundary conditions), we require the \( \frac{db_0}{dr_0} \) term. Using the chain rule we obtain
\[
\frac{db_0}{dr_0} = \frac{db_0}{dr} \frac{dr}{dr_0} \tag{30}
\]
Substituting Eq. 22 into Eq. 30
\[
\frac{db_0}{dr} = \sqrt{r} \frac{dw}{dr} \tag{31}
\]
and the \( \frac{db_0}{dr} \) term is given by
\[
\frac{db_0}{dr} = A I_1(\sqrt{r}) + B K_1(\sqrt{r}) \tag{32}
\]
From Abramowitz and Stegun, Handbook of Mathematical Functions, we have
\[
\frac{db_0}{dr} = I_1(\sqrt{r}) \tag{33}
\]
\[
\frac{db_0}{dr} = -K_1(\sqrt{r}) \tag{34}
\]
Substituting Eqs. 33 and 34 into Eq. 32 we have
\[
\frac{db_0}{dr} = A I_1(\sqrt{r}) - B K_1(\sqrt{r}) \tag{35}
\]
Combining Eqs. 30 and 35, and substituting \( \sqrt{r} = \sqrt{r_0} \) (Eq. 22) into Eq. 35 we obtain
\[
\frac{db_0}{dr} = A I_1(\sqrt{r_0}) - B K_1(\sqrt{r_0}) \tag{36}
\]
Multiplying through by \( r_0 \) gives
\[
[r_0 \frac{db_0}{dr}] = A I_1(\sqrt{r_0}) r_0 - B K_1(\sqrt{r_0}) r_0 \tag{37}
\]
Summarizing our efforts so far:

**General Solution in Laplace Domain**
\[
b_0(\nu, \omega) = A I_0(\sqrt{\nu r_0}) + B K_0(\sqrt{\nu r_0}) \tag{38}
\]
**Radial Derivative of the General Solution in Laplace Domain**
\[
[\nu \frac{db_0}{d\nu}] = A I_1(\sqrt{\nu r_0}) - B K_1(\sqrt{\nu r_0}) \tag{39}
\]
**Laplace transform of boundary conditions**

- **Inner boundary condition**
  \[
  \left[ \frac{db_0}{dr} \right]_{r=r_0} = \frac{0}{1} \quad \text{(constant rate at well)} \tag{40}
  \]
- **Outer boundary conditions**
  a. **Infinite-acting** reservoir
  \[
  b_0(r_0, \omega) = 0 \tag{41}
  \]
  b. **No-flow** outer boundary condition
  \[
  \left[ \frac{db_0}{dr} \right]_{r=r_0} = 0 \tag{42}
  \]
  c. **Constant pressure** outer boundary condition
  \[
  \left[ \frac{db_0}{dr} \right]_{r=r_0} = -0 \tag{43}
  \]
  d. **Prescribed flux** outer boundary condition
  \[
  \left[ \frac{db_0}{dr} \right]_{r=r_0} = \frac{\omega}{2} \tag{44}
  \]

Our goal is to use the boundary conditions to determine the \( A \) and \( B \) parameters. Our first step is to use the constant rate inner boundary condition (Eq. 41) as a starting point then combine this condition with each outer boundary.
condition in order to determine $A$ and $B$ for each case.

Starting with the inner boundary condition (Eq. 14) and the derivative of the general solution (Eq. 37) we have

$$A\sqrt{\alpha} I_0(\alpha r) - B\sqrt{\alpha} I_1(\alpha r) = \frac{-1}{\mu}$$

or

$$A\sqrt{\alpha} I_0(\alpha r) - B\sqrt{\alpha} I_1(\alpha r) = \frac{-1}{\mu}$$

(Eq. 38)

Outer Boundary Case 1: Infinite-acting reservoir

Combining Eqs. 29 and 15 we have

$$\lim_{r \to \infty} \left[ A I_0(\alpha r) + B I_1(\alpha r) \right] = 0$$

(Eq. 39)

Given that we are taking the limit as $r \to \infty$, we must establish the behavior of $I_0(x)$ and $I_1(x)$. Considering the behavior of $I_0(x)$ and $I_1(x)$ we have

$$I_0(x) \to 0$$

$$I_1(x) \to 0$$

and

$$\lim_{x \to 0} I_0(x) = 0$$

$$\lim_{x \to 0} I_1(x) = 0$$

Since $I_0(x) \to 0$ and $I_1(x) \to 0$, then $A(0) + B(0) = 0$; therefore $A(0) = 0$ in order for the solution to be bounded. Setting $A = 0$ we solve Eq. 38 for $B$, which gives

$$B = \frac{1}{\mu}$$

(Eq. 40)

and of course

$$A = 0$$

Substituting Eqs. 40 and 41 into the general solution (Eq. 29) we obtain the particular solution for the infinite-acting reservoir case. This result is

$$I_0(r, \mu) = \frac{1}{\mu} \frac{k_0(\mu r^2)}{\sqrt{\pi k(\mu)}}$$

(Eq. 42)

Eq. 42 is called the cylindrical source solution.

Unfortunately, Eq. 42 is not readily invertible; therefore we will attempt to reduce Eq. 42 into a more usable form. From Abramowitz and Stegun, Handbook of Mathematical Functions, p. 375, Eq. 9.6.19 (for $n=1$) we have

$$k_1(x) = \frac{1}{x}$$

(Eq. 43)

or, multiplying through by $x$ we have

$$x k_1(x) = 1$$

(Eq. 44)

for our case we have

$$\sqrt{\alpha} k_0(\alpha r) = 1$$

(Eq. 45)

or

$$x k_0(x) = 1$$

(Eq. 46)

Combining this result with Eq. 42 we obtain

$$I_0(r, \mu) = \frac{1}{\mu} k_0(\mu r^2)$$

(Eq. 47)

Eq. 47 is called the line source solution and can be inverted directly.

Eq. 47 can be reduced further to yield a logarithmic relation that is commonly referred to as the "log approximation." In order to develop this result we require an approximation for $k_0(x)$ as $x \to 0$. From Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.6.13, p. 375, we have

$$k_0(x) \approx -\ln x + \frac{1}{2} \frac{1}{x} + \frac{1}{4} \frac{1}{x^2} + \frac{1}{4} \frac{1}{x^3} + \frac{1}{4} \frac{1}{x^4} + \cdots$$

where we note that as $x \to 0$, then $x^2 \to 0$, which reduces to

$$k_0(x) \approx -\ln x$$

(Eq. 48)

or multiplying and dividing by $z$ we have

$$k_0(x) \approx \frac{1}{z} \ln \left(\frac{z}{e^{z/2}}\right)$$

(Eq. 49)

The behavior of $I_0(x)$ in the vicinity of $x \to 0$ is obtained using the series representation provided in Abramowitz and Stegun, Handbook of Mathematical Functions, Eq. 9.6.12, p. 375. This expression is

$$I_0(x) \approx 1 + \frac{1}{2} \frac{1}{x} \left[ 1 + \frac{1}{4} \frac{1}{x^2} + \frac{1}{4} \frac{1}{x^3} + \frac{1}{4} \frac{1}{x^4} + \cdots \right]$$

where as $x \to 0$ we have

$$I_0(x) \approx 1$$

(Eq. 50)
Combining Eqs. 44 and 45:

\[
\kappa_0(x) = \frac{1}{2} \ln \left( \frac{4 \mu}{\kappa_0(\mu)} \right) \quad \text{as } x \to 0
\]  

(46)

Substituting Eq. 46 into Eq. 43 we obtain:

\[
\frac{\delta_0(\mu, \tau)}{2\mu} = \frac{1}{2} \ln \left( \frac{4 \mu}{\kappa_0(\mu)} \right) \quad \text{as } \mu \to 0
\]  

(47)

or in a form more amenable to inversion we have:

\[
\frac{\delta_0(\mu, \tau)}{2\mu} = -\frac{1}{2} \ln(\mu) + \frac{1}{2} \ln \left( \frac{4 \mu}{\kappa_0(\mu)} \right)
\]  

(48)

**Outer Boundary Case 2: No-Flow outer boundary**

Combining Eqs. 37 and 16 we obtain:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) + \frac{8}{9} \frac{\kappa_2(\sqrt{\mu} r_0)}{\kappa_0(\sqrt{\mu} r_0)} = 0
\]  

(49)

Solving for the B parameter we obtain:

\[
B = \frac{A}{\kappa_0(\sqrt{\mu} r_0)}
\]  

(50)

Recalling the inner boundary condition, Eq. 38, we have:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) - B \kappa_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(51)

Combining Eqs. 50 and 38 we obtain:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) - \sqrt{\mu} \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(52)

or:

\[
A \left[ I_0(\sqrt{\mu} r_0) \kappa_0(\sqrt{\mu} r_0) - \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0) \right] = -\frac{1}{\mu}
\]  

(53)

Solving for A we have:

\[
A = \frac{\kappa_0(\sqrt{\mu} r_0)}{\sqrt{\mu} \kappa_0(\sqrt{\mu} r_0) - \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0)}
\]  

(54)

Recalling the inner boundary condition, Eq. 38, gives us:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) - B \kappa_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(55)

Combining Eqs. 49 and 17 we have:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) + 8 \kappa_2(\sqrt{\mu} r_0) = 0
\]  

(56)

Solving for the B parameter we obtain:

\[
B = \frac{A}{\kappa_0(\sqrt{\mu} r_0)}
\]  

(57)

**Outer Boundary Case 3: Constant pressure outer boundary**

Combining Eqs. 54 and 17 we have:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) + 8 \kappa_2(\sqrt{\mu} r_0) = 0
\]  

(58)

Solving for the B parameter we obtain:

\[
B = -\frac{A \sqrt{\mu} I_0(\sqrt{\mu} r_0)}{\kappa_0(\sqrt{\mu} r_0)}
\]  

(59)

Recalling the inner boundary condition, Eq. 38, gives us:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) - B \kappa_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(60)

Substituting Eq. 56 into Eq. 38 we have:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) + A \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(61)

or:

\[
A \left[ I_0(\sqrt{\mu} r_0) \kappa_0(\sqrt{\mu} r_0) + \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0) \right] = -\frac{1}{\mu}
\]  

(62)

Solving for A we have:

\[
A = \frac{-\kappa_0(\sqrt{\mu} r_0)}{\sqrt{\mu} \kappa_0(\sqrt{\mu} r_0) + \kappa_0(\sqrt{\mu} r_0) I_0(\sqrt{\mu} r_0)}
\]  

(63)

Recalling the inner boundary condition, Eq. 38, gives us:

\[
A \sqrt{\mu} I_0(\sqrt{\mu} r_0) - B \kappa_0(\sqrt{\mu} r_0) = -\frac{1}{\mu}
\]  

(64)

Substituting Eq. 57 into Eq. 38 gives:

\[
B = \frac{I_0(\sqrt{\mu} r_0)}{\kappa_0(\sqrt{\mu} r_0)}
\]  

(65)
Substituting Eqs. 57 and 58 into the general solution, Eq. 29, we obtain
\[
\rho_f(r, \mu) = \frac{k_0(u_0) I_0(u_0 \mu) - k_0(u_0 \mu) I_0(u_0)}{m \left[ u_0 k_0(u_0) I_0(u_0) + \sqrt{u_0} I_0(u_0) \right]},
\]  
(59)

As in the previous cases, we want to consider the behavior as \( \mu \to 0 \) (large \( \rho \)). As before, we have
\[
\frac{I_0(\mu)}{\mu} = 1 \quad \text{as} \quad \mu \to 0,
\]
\[
\frac{I_1(\mu)}{\mu} = 0 \quad \text{as} \quad \mu \to 0.
\]
Combining these relations with Eq. 59, we obtain
\[
\rho_f(r, \mu) = \frac{1}{m} k_0(u_0) I_0(u_0 \mu)\quad \text{as} \quad \mu \to 0.
\]  
(60)

Outer Boundary Case 4: Prescribed flux outer boundary

Combining Eqs. 57 and 18 we obtain
\[
\frac{\rho_f}{u_0} I_0(u_0 \mu) - \frac{\rho_f}{u_0} k_0(u_0 \mu) = \rho_{\text{ext}}.
\]
Recalling the inner boundary condition, Eq. 58, we have
\[
\frac{A}{\mu} I_0(u_0 \mu) - B \frac{A}{\mu} k_0(u_0 \mu) = -1.
\]
We will solve Eqs. 61 and 58 simultaneously to determine \( A \) and \( B \). The algebra becomes a bit tedious, but we will show all steps. Solving for the \( A \) parameter we divide through Eq. 61 by \( \frac{u_0}{\mu} k_0(u_0 \mu) \), then we divide through Eq. 58 by \( \frac{u_0}{\mu} k_0(u_0 \mu) \). These operations give
\[
\frac{A}{\mu} \frac{u_0}{\mu} k_0(u_0 \mu) - B = \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)}.
\]
\[
A \frac{u_0}{\mu} I_0(u_0 \mu) - B = -1.
\]
(62)

Subtracting Eq. 63 from Eq. 62 we have
\[
A \left[ \frac{u_0}{\mu} I_0(u_0 \mu) - \frac{u_0}{\mu} I_0(u_0 \mu) \right] + B \frac{u_0}{\mu} k_0(u_0 \mu) = \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)} - 1
\]
Expanding to yield a uniform denominator on both sides gives
\[
B = \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} I_0(u_0 \mu)} + \frac{1}{\mu} \frac{u_0}{\mu} k_0(u_0 \mu),
\]
(64)

Subtracting Eq. 64 from Eq. 65 we have
\[
A \left[ \frac{u_0}{\mu} I_0(u_0 \mu) - \frac{u_0}{\mu} I_0(u_0 \mu) \right] = \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)} + \frac{1}{\mu} \frac{u_0}{\mu} k_0(u_0 \mu),
\]
Expanding to yield a uniform denominator on both sides gives
\[
A \left[ \frac{u_0}{\mu} I_0(u_0 \mu) - \frac{u_0}{\mu} I_0(u_0 \mu) \right] = \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)} + \frac{1}{\mu} \frac{u_0}{\mu} k_0(u_0 \mu)
\]
Solving for \( A \) we have
\[
A = \frac{1}{\mu} \frac{u_0}{\mu} I_0(u_0 \mu) + \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)}.
\]
(65)

Comparing Eq. 67 with the result for the no-flow boundary case we recall Eq. 52
\[
B = \frac{1}{\mu} \frac{u_0}{\mu} I_0(u_0 \mu) - \frac{\rho_{\text{ext}}}{\frac{u_0}{\mu} k_0(u_0 \mu)}
\]
(66)
where we find that Eq. 67 is identical to Eq. 52 for \( q_{\text{ext}} = 0 \). Having shown this for both \( A \) and \( B \) we have verified these results.

In order to determine the particular solution for this case, we substitute Eqs. 64 and 67 into the general solution (Eq. 27). This gives

\[
\frac{q_0}{\rho_0} = \frac{1}{m} \frac{k_0 (\sigma_k R_0) S_i (\sigma_k R_0) + S_0 (\sigma_k R_0) k_1 (\sigma_k R_0)}{\sqrt{2} k_1 (\sigma_k R_0) S_i (\sigma_k R_0) - \sqrt{2} S_1 (\sigma_k R_0) k_1 (\sigma_k R_0)}
\]

\[
+ \frac{1}{m} \frac{\tilde{q}_{\text{ext}} (\sigma_k R_0)}{\sqrt{2} \rho_0} \frac{k_0 (\sigma_k R_0) \sqrt{2} S_1 (\sigma_k R_0) + S_0 (\sigma_k R_0) \sqrt{2} k_1 (\sigma_k R_0)}{\sqrt{2} k_1 (\sigma_k R_0) S_i (\sigma_k R_0) - \sqrt{2} S_1 (\sigma_k R_0) k_1 (\sigma_k R_0)}
\]

(48)

where the first part of Eq. 68 is exactly Eq. 55, the solution for the no-flow boundary case (i.e., \( \tilde{q}_{\text{ext}} = 0 \)). Note that \( \tilde{q}_{\text{ext}} = \frac{1}{2} (\tilde{q}_{\text{ext}} (\sigma_k R_0)) \).

As with the previous cases we can consider the behavior of Eq. 68 as \( \mu \to 0 \). As we saw before:

\[
\sqrt{2} k_1 (\sigma_k R_0) = 1 \quad \text{as} \quad \mu \to 0
\]

\[
\sqrt{2} S_1 (\sigma_k R_0) = 0 \quad \text{as} \quad \mu \to 0
\]

Combining these relations with Eq. 68 gives

\[
\frac{q_0}{\rho_0} = \frac{1}{m} \frac{k_0 (\sigma_k R_0)}{S_i (\sigma_k R_0)} + \frac{1}{m} \frac{S_0 (\sigma_k R_0)}{S_i (\sigma_k R_0) k_1 (\sigma_k R_0)}
\]

\[
+ \frac{1}{m} \frac{\tilde{q}_{\text{ext}} (\sigma_k R_0)}{\sqrt{2} \rho_0} \frac{k_0 (\sigma_k R_0) \sqrt{2} S_1 (\sigma_k R_0) + S_0 (\sigma_k R_0) \sqrt{2} k_1 (\sigma_k R_0)}{\sqrt{2} k_1 (\sigma_k R_0) S_i (\sigma_k R_0) - \sqrt{2} S_1 (\sigma_k R_0) k_1 (\sigma_k R_0)}
\]

(49)

The second term in Eq. 68 (or 69) should not be arbitrarily reduced without a comprehensive study of the interplay of individual terms. For example, reduction using the behavior of \( S_1 (\sigma_k R_0) \) and \( k_1 (\sigma_k R_0) \) as \( x \to 0 \) yields \( S_i (\sigma_k R_0) / S_0 (\sigma_k R_0) \) which tends to \( 1 / 0 = \infty \) as \( \mu \to 0 \). Considerable care must be exercised when making such reductions.
Log-log Plot: Constant Well Rate Solutions for a Bounded Circular Reservoir: Laplace Transform Solutions—Radial Flow Case (SPE 25479)

Legend: Error Analysis for the Laplace Transform Solution Approximations:
- "Cylindrical Source" Formulation
- "Line Source" Formulation
- First Approximate Solution is an Expansion of the "Line Source" Form
- Second Approximation is a Truncated Form of the First Approximation

No Flow Outer Boundary Cases

Constant Pressure Outer Boundary Cases Shown as Straight Lines where during Steady-State: $p_D(u) = \ln(r_{eD})/u$

Transient Radial Flow

Dimensionless Laplace Transform Parameter, $u$

($\leftarrow$ Time is increasing to the left in this rendering)