Petroleum Engineering 620 — Fluid Flow in Petroleum Reservoirs
Reservoir Flow Solutions Lecture 6 — Direct Solution of the Gas Diffusivity Equation Using Laplace Transform Methods

Cruelty is perhaps the worst kind of sin, Intellectural cruelty is certainly the worst kind of cruelty.
— G.K. Chesterton (1908)

**Topic:** Direct Solution of the Gas Diffusivity Equation Using Laplace Transform Methods

**Objectives:** (things you should know and/or be able to do)
- Be familiar with the convolution form of a non-linear partial differential equation (with a non-linear right-hand-side term), as shown below.

\[
V^2 y = \beta(y) \frac{\partial y}{\partial \tau} = \int_0^t \frac{\partial y}{\partial \tau} g(t-\tau) \, d\tau
\]

Where we assume that the \( \beta(y) \) function can be re-cast as a unique function of time (i.e., \( \beta(y) \) can be written as \( \beta(t) \)). Using \( \beta(t) \) requires assumptions as to flow regimes—we will demonstrate this assuming pseudosteady-state flow.

Taking the Laplace transform of this relation gives

\[
V^2 \hat{y}(s) = \left[ s\hat{y}(s) - y(t=0) \right] \hat{g}(s)
\]

- Be able to develop the generalized Laplace domain formulation of the non-linear radial gas diffusivity equation using the \( \beta(t) \) approach.

The real gas diffusivity equation (in radial coordinates) is given in dimensionless form by:

\[
\frac{\partial^2 P_D}{\partial r^2} + \frac{1}{r_D} \frac{\partial P_D}{\partial r_D} = \frac{\mu_c l_i}{\mu_c l_i} \frac{\partial P_D}{\partial t_D} = \beta(t_D) \frac{\partial P_D}{\partial t_D} \quad \left[ \beta(t_D) = \frac{\mu_c l_i}{\mu_c l_i} \right]
\]

where

\[
P_D = \frac{1}{141.2} q_B \mu \frac{P}{P_i - P} \quad t_D = 0.0002637 \frac{k}{\phi \mu_c l_i r_D} t \quad r_D = \frac{r}{r_w}
\]

and the pseudopressure function is given by

\[
p = \mu B_g l_i \int_{P_{base}}^P \frac{1}{\mu B_g} dp = \frac{\mu Z}{\mu Z_i} \int_{P_{base}}^P \frac{P}{\mu Z} dp
\]

- Substituting the convolution formulation into the right-hand-side of the real gas diffusivity equation gives

\[
\frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial P_D}{\partial r_D} \right] = \frac{1}{r_D} \frac{\partial P_D}{\partial r_D} = \int_0^t \frac{\partial P_D}{\partial \tau} g(t_D-\tau) \, d\tau
\]

and the Laplace transform of this relation is given by

\[
\frac{1}{r_D} \frac{d}{dr_D} \left[ r_D \frac{\partial P_D(u)}{\partial r_D} \right] = \frac{d^2 \hat{P}_D(u)}{dr_D^2} + \frac{1}{r_D} \frac{d\hat{P}_D(u)}{dr_D} = uG(u)\hat{P}_D(u)
\]

- Be familiar with and be able to develop the \( g(u) \) term. The \( g(t_D) \) term is defined by:

\[
\beta(t_D) \frac{\partial P_D}{\partial t_D} = \int_0^{t_D} \frac{\partial P_D}{\partial \tau} g(t_D-\tau) \, d\tau
\]

Note that \( g(t_D) \) (and hence \( g(u) \)) are dependent on how we choose to evaluate the \( \frac{\partial P_D}{\partial t_D} \) function. We recommend the pseudosteady-state flow condition as a starting point.

**Lecture Outline:**

- Introduction of the convolution concept
- Rewriting the partial differential equation.
- Laplace domain form of the partial differential equation.
- Application—Generalized Approach
  - Definition of \( g(t_D) \) from the convolution identity.
  - Evaluation of the \( \frac{\partial P_D}{\partial t_D} \) function for pseudosteady-state flow conditions.
  - Definition of the \( \beta(t_D) \) function (i.e., \( \beta(t_D) = \frac{\mu Z D}{(\mu Z_i) t_D} \)).
  - Definition of the \( R(t_D) \) function (i.e., \( R(t_D) = 1/\beta(t_D) = (\mu Z_i)/(\mu Z D) \)).
- Definitions of the \( g(u) \) function:

\[
g(u) = \frac{1}{u} \frac{L}{\beta(t_D)} \quad \text{or} \quad g(u) = \frac{1}{u} \frac{1}{R(t_D)}
\]

- Computational Considerations (\( R(t_D) \Rightarrow R(u) \))
  - Roumoutos and Stewart algorithm.
  - Data models for \( R(u) \).

**Reading Assignment:**

- Review attached notes.

**Exercises:** For your own practice/skills building—do NOT turn in!

From the attached notes (Appendix A - T.J. Mireles) you are to rederive the following results (show all details).

- Derive the generalized Laplace domain approach for solving the gas diffusivity equation as given in Appendix A of the Mireles thesis.
Fluid Properties: Exponential and Polynomial Functional Data Models for the $\frac{(\mu_c \varepsilon_{G})}{(\mu_{cM} \varepsilon_{M})}$ Data Function Versus Dimensionless Time.

**Fluid/Production Parameters:**
- Closed Outer Boundary, $r_D = 10^3$
- $\gamma_g = 1.0$, $T = 50^\circ F$
- $p_i = 10,000$ psi, $\Delta p = 10^2$

Legend:
- Numerical Data Function
- Analytical Exponential Model
- Analytical Polynomial Model

Fluid Properties: Numerical Model for the $\frac{(\mu_c \varepsilon_{G})}{(\mu_{cM} \varepsilon_{M})}$ Data Function Based on Roumoutos and Stewart Algorithm for Transforming Data into the Laplace Domain.

**Fluid/Production Parameters:**
- Closed Outer Boundary, $r_D = 10^3$
- $\gamma_g = 1.0$, $T = 50^\circ F$
- $p_i = 10,000$ psi, $\Delta p = 10^2$

Legend:
- Numerical Data Function
- Roumoutos & Stewart Model

Fluid Properties: $\frac{(\mu_c \varepsilon_{G})}{(\mu_c G)}$ versus $\frac{(p_i / 2)(p_i / 2)}{(p_i / 2)}$ ($\gamma_g = 0.7$ (air = 1.0), $T = 50^\circ F$).
Validation: Solution Verification for Bounded Circular Reservoir \((r_D = 10^4, q_{Dc} = 10^{-1})\).

Reservoir/Production Parameters:
\(r_D = 10^4\)  \(\quad q_{Dc} = 10^{-1}\)

Fluid Parameters:
\(p_i = 5,000\) psi  \(\quad T = 200\) °F  \(\quad \gamma_g = 0.7\)

Legend: No Flow Outer Boundary Case

- Analytical Liquid Solutions
- Numerical Gas Solutions
- Semi-Analytical \(p_D\) Gas Solution
- Semi-Analytical \(p_D'\) Gas Solution

\(q_{Dc} = \frac{B_g \mu g i f}{kh} \left( \frac{q}{P_{p,i} - P_{p,abn}} \right) \left( \ln(r_D) - 3/4 \right)\)

Region of Deviation From Liquid Solution

Bounded Circular Reservoir
Validation: Solution Verification for Bounded Circular Reservoir ($r_e D = 10^4$, $q D_c = 10^{-2}$).
Validation: Solution Verification for Bounded Circular Reservoir \( (r_eD = 10^4, q_{DC} = 10^3) \).

**Reservoir/Production Parameters:**
\( r_eD = 10 \quad q_{DC} = 10^3 \)

**Fluid Parameters:**
\( p_i = 5,000 \text{ psi} \quad T = 200^\circ F \quad \gamma_g = 0.7 \)

**Legend:**
- Analytical Liquid Solutions
- Numerical Gas Solutions
- Semi-Analytical \( p_D \) Gas Solution
- Semi-Analytical \( p_D^* \) Gas Solution

\[ q_{DC} = 141.2 \frac{B_o \mu g_l}{kh} \left[ \frac{q}{p_{p,i} - p_{p,abn}} \right] (\ln(r_eD) - 3/4) \]

<table>
<thead>
<tr>
<th>Dimensionless Pseudopressure and Pseudopressure Derivative Functions, ( p_D ) and ( p_D^* )</th>
<th>Dimensionless Time, ( t_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_D = 1 )</td>
<td>( 10^4 )</td>
</tr>
<tr>
<td>( r_D = 10 )</td>
<td>( 10^5 )</td>
</tr>
<tr>
<td>( r_D = 10^2 )</td>
<td>( 10^6 )</td>
</tr>
<tr>
<td>( r_D = 10^3 )</td>
<td>( 10^7 )</td>
</tr>
<tr>
<td>( r_D = 5 \times 10^3 )</td>
<td>( 10^8 )</td>
</tr>
<tr>
<td>( r_D = 10^4 )</td>
<td>( 10^9 )</td>
</tr>
</tbody>
</table>

Region of Deviation From Liquid Solution

Bounded Circular Reservoir
Validation: Solution Verification for Bounded Circular Reservoir—Fetkovich-Style Type Curve.

Fluid Parameters:
- $p_i = 5,000$ psi
- $T = 200^\circ F$
- $\gamma_g = 0.7$

Legend:
- Fetkovich-Style Type Curve—Constant Production Rate, Unfractured Well Centered in a Bounded Circular Reservoir.
- Analytical Liquid Solutions
- Numerical Gas Solutions
- Semi-Analytical $p_{PD}$ Gas Solution
- Semi-Analytical $p_{PD}'$ Gas Solution

Dimensionless Pseudopressure and Dimensionless Derivative Functions, $P_{PD}$ and $P_{PD}'$:

- $r_e = 10^4$
- $r_e = 10^3$
- $r_e = 100$
- $r_e = 10$
- $r_e = 10$

Dimensionless Decline Time, $t_{Dd}$:

- $t_{Dd} = \frac{1}{2(r_eD - 1) \ln(r_eD) - 1/2}$

Pseudopressure Derivative:

- $P_{PD} = \frac{1}{\ln(r_eD) - 1/2}$

Gas Rate:

- $q_{Dc} = 141.2 \frac{B_p \mu_g}{k h} \left[ \frac{g}{p_{PD} - p_{PD,obn}} \right] (\ln(r_eD) - 3/4)$
Validation: Solution Verification for a Bounded Circular Reservoir—Extreme Conditions \( r_e D = 10^3, \quad q_{Dc} = 10^2, \quad \gamma_g = 0.55 \).

Fluid/Reservoir/Production Parameters:
\[ \gamma_g = 0.55 \quad q_{Dc} = 10^2 \quad r_e D = 10 \]

Legend: Fetkovich-Style Type Curve—Constant Production Rate, Unfractured Well Centered in a Bounded Circular Reservoir.

Numerical Gas Solutions:
- \( p_l = 10,000 \) and \( T = 350^\circ F \)
- \( p_l = 10,000 \) and \( T = 50^\circ F \)
- \( p_l = 2,000 \) and \( T = 350^\circ F \)
- \( p_l = 2,000 \) and \( T = 50^\circ F \)

Although the use of this technique does not provide a unique correlation between \( \beta D \) and \( ppD \), the solution is valid for a wide range of conditions.

\[ t_{Dd} = \frac{1}{2 (r_e D - 1) \ln (r_e D) - 1/2} - t_{Dd} \]

\[ ppD_d = \frac{1}{\ln (r_e D) - 1/2} - ppD \]

\[ q_{Dc} = 141.2 \frac{B_g \mu_d \gamma_g}{kh} \frac{q}{\left( pp_{hi} - pp_{ab} \right)} \left( \ln (r_e D) - 3/4 \right) \]
Validation: Solution Verification for a Bounded Circular Reservoir—Extreme Conditions ($r_e = 10^3$, $q_{Dc} = 0.01$, $\gamma_g = 1.0$).

Fluid/Reservoir/Production Parameters:
- $\gamma_g = 1.0$
- $q_{Dc} = 10^2$
- $r_e = 10^3$

Legend: Fekovich-Style Type Curve --
- Constant Production Rate, Unfractured Well
- Centered in a Bounded Circular Reservoir.

Numerical Gas Solutions:
- $p_i = 10,000$ and $T = 350^\circ F$
- $p_i = 10,000$ and $T = 50^\circ F$
- $p_i = 2,000$ and $T = 350^\circ F$
- $p_i = 2,000$ and $T = 50^\circ F$

Although the use of this technique does not provide a unique correlation between $t_D$ and $p_D$, the solution is valid for a wide range of conditions.

Dimensionless Pseudopressure and $p_D$ Functions, $p_D$ and $p_D'$

Liquid Solution

$P_{PD}$ Solutions

$p_D$ Solutions

$P_{PD'}$ Solutions

$t_D = \frac{1}{2(r_e^2 - 1)} \ln \frac{1}{(r_e - 1/2)}$

$q_{Dc} = 141.2 \frac{B_g \mu_i g_i}{kh} \left[ \frac{q}{p_{p,i} - p_{p,obn}} \right] (\ln(r_e) - 3/4)$.
Validation: The Use of Exponential and Polynomial Models in the Semi-Analytical Real Gas Solution.
Derivation of an Exact Laplace Transform Formulation for the Real Gas Diffusivity Equation using a Convolution Approach


APPENDIX A
DERIVATION OF AN EXACT LAPLACE TRANSFORM FORMULATION FOR THE REAL GAS DIFFUSIVITY EQUATION USING A CONVOLUTION APPROACH FOR THE NON-LINEAR VISCOSEITY-COMRESSIBILITY PRODUCT

Development of the Convolution Formulation for a Non-Linear Partial Differential Equation

The general form of a non-linear partial differential equation is given by

\[ \nabla^2 y = \beta(t) \frac{\partial y}{\partial t} \]  

(A-1)

The key to our convolution technique is to realize that the left-hand-side (LHS) of Eq. A-1 can be transformed, but the right-hand-side (RHS) is not in a form that can be readily transformed. Specifically, we recognize that the \( \beta(t) \frac{\partial y}{\partial t} \) term is of the form

\[ \beta(t) \frac{\partial y}{\partial t} = f_1(t) f_2(t) \]  

(A-2)

From Roberts and Kaufman (Section 1-Operations, p. 4, Eq. 16), we have

\[ \mathcal{L}[f_1(t)f_2(t)] = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} f_1(\tau) f_2(u - \tau) d\tau \]  

(A-3)

where \( f_1(u) \) and \( f_2(u) \) are the Laplace transforms of \( f_1(t) \) and \( f_2(t) \), respectively. We note that Eq. A-3 is not a useful form, but from this form we can suggest a different approach—writing the \( f_1(t) f_2(t) \) product as a convolution-type formulation. This leads to

\[ f_1(t) f_2(t) = \int_0^t f_1(\tau) g(t-\tau) d\tau \]  

(A-4)

Taking the Laplace transform of Eq. A-4 gives

\[ \mathcal{L}[f_1(t)f_2(t)] = \mathcal{L}\left[ \int_0^t f_1(\tau) g(t-\tau) d\tau \right] \]

or

\[ \mathcal{L}[f_1(t)f_2(t)] = \mathcal{L}[f_1(u) g(u)] \]  

where \( g(u) = \mathcal{L}[g(t)] \).  

(A-5)

Letting \( f_1(t) = \frac{\partial y}{\partial t} \) and \( f_2(t) = \beta(t) \), then substituting into Eq. A-4 gives us

\[ \beta(t) \frac{\partial y}{\partial t} = \int_0^t \frac{\partial y}{\partial t} g(t-\tau) d\tau \]  

(A-6)

substituting Eq. A-6 into Eq. A-1 gives

\[ \nabla^2 y = \int_0^t \frac{\partial y}{\partial t} g(t-\tau) d\tau \]  

(A-7)

Of course the trick is to determine the \( g(t) \) function from the identity given by Eq. A-6, but this is a problem-specific issue and must be addressed as such. However, the relevant issue is that we can take the Laplace transform of Eq. A-7. Taking the Laplace transform of Eq. A-7 we have

\[ \nabla^2 y(u) = \left[ \mathcal{L}[y(t)] - y(t_p=0) \right] g(u) \]  

(A-8)

We note that the form given by Eq. 8 should be useful for second order, diffusion-type partial differential equations.

Application of the Convolution/Laplace Transform Approach to the Real Gas Diffusivity Equation

The dimensionless form of the real gas diffusivity equation is given as

\[ \frac{1}{r_D} \frac{\partial}{\partial r_D} \left[ r_D \frac{\partial}{\partial r_D} P_{PD} \right] = \frac{\mu c I}{\mu c I} \frac{\partial P_{PD}}{\partial D} \]

or

\[ \frac{\partial^2 P_{PD}}{\partial D^2} + \frac{1}{r_D} \frac{\partial}{\partial D} P_{PD} = \frac{\mu c I}{\mu c I} \frac{\partial P_{PD}}{\partial D} \]  

(A-9)

where \( \mu_q \) and \( c_I \) are functions of both dimensionless pseudopressure and dimensionless time.

We define the \( \beta(t_p) \) function as

\[ \beta(t_p) = \frac{\mu c I}{\mu c I} \]  

(A-10)

Substituting Eq. A-10 into Eq. A-9, we obtain

\[ \frac{\partial^2 P_{PD}}{\partial D^2} + \frac{1}{r_D} \frac{\partial}{\partial D} P_{PD} = \beta(t_p) \frac{\partial P_{PD}}{\partial D} \]  

(A-11)

Expressing the right-hand-side of Eq. A-11 in the same form as Eq. A-6 gives us

\[ \beta(t_p) \frac{\partial P_{PD}}{\partial D} = \int_0^t \frac{\partial P_{PD}}{\partial D} g(t_p-\tau) d\tau \]  

(A-12)

Substituting Eq. A-12 into Eq. A-11 yields

\[ \frac{\partial^2 P_{PD}}{\partial D^2} + \frac{1}{r_D} \frac{\partial}{\partial D} P_{PD} = \int_0^t \frac{\partial P_{PD}}{\partial D} g(t_p-\tau) d\tau \]  

(A-13)

Taking the Laplace transform of Eq. A-13

\[ \frac{\partial^2 P_{PD}(u)}{\partial D^2} + \frac{1}{r_D} \frac{\partial}{\partial D} P_{PD}(u) = \left[ \mathcal{L}[P_{PD}(t)] - P_{PD}(t_p=0) \right] g(u) \]  

(A-14)

where, after applying the initial condition \( (P_{PD}(r_D, t_p=0) = 0) \), Eq. A-14 reduces to...
\[ \frac{\partial^2 p_D(u)}{\partial t_D^2} + \frac{1}{r_D} \frac{\partial p_D(u)}{\partial r_D} = u \overline{p}_D(u) g(u) \]  

(A-15)

Eq. A-15 has exactly the same form (but obviously not the same function) as the generalized formulation for dual porosity or naturally fractured reservoirs. But what is the \( g(u) \) function? Returning to the convolution identity (Eq. A-12), we have

\[ \beta(t_D) \frac{\partial p_D}{\partial t_D} = \int_0^{t_D} \beta(t_D) \frac{\partial p_D}{\partial t_D} g(t_D - \tau) d\tau \]  

(A-12)

Obviously Eq. A-12 defines the \( g(t_D) \) function, but how do we evaluate this function? We require both \( \beta(t) \) and \( \frac{\partial p_D}{\partial t_D} \), neither of which are known a priori—but perhaps we can approximate these functions, or use their known behavior for a particular flow regime (such as pseudosteady-state). This issue will be discussed and resolved in the next section.

**Modelling the Non-Linear Term**

In this section we consider how to determine the \( g(t_D) \) function. Although not obvious, we can establish a relationship for \( \frac{\partial p_D}{\partial t_D} \) using the gas material balance equation which considers behavior at average reservoir pressure conditions. While this may seem limiting, the approach is actually quite sound, and will be verified by comparison with results from numerical reservoir simulation of gas flow in porous media.

Writing the time derivative term and its chain rule expansion we have

\[ \frac{\partial p_D}{\partial t_D} = \frac{\partial p_D}{\partial p} \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial t_D} \]  

(A-16)

The appropriate identities are

\[ p_D = p_{DC} \frac{kh}{qB_g \mu_g} (p_{pi} - p_p) \]  

(A-17)

\[ p = \frac{\mu_g \xi_i}{\mu_i} \int_{p_{base}}^{p} \frac{p}{\mu_g \xi} dp \]  

(A-18)

\[ t_D = \frac{k}{\phi \mu_i c_{pi} r_w^2} \]  

(A-19)

where the \( t_{DC} \) and \( p_{DC} \) constants are given by

<table>
<thead>
<tr>
<th>Constant</th>
<th>Darcy Units</th>
<th>Field Units</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{DC} )</td>
<td>1</td>
<td>2.637x10^4 (hr) or 6.33x10^-3 (d)</td>
<td>3.557x10^6</td>
</tr>
<tr>
<td>( p_{DC} )</td>
<td>2\pi</td>
<td>7.081x10^3 or 1/141.2</td>
<td>5.356x10^4</td>
</tr>
</tbody>
</table>

And the gas material balance equation is given by

\[ \frac{\partial \bar{p}}{\partial \tau} + \frac{p_{li}}{z^i} \left[ \frac{G_p}{G} \right] = 0 \]  

(A-20)

Assuming a constant gas flow rate, \( q \), we have \( G_p = q t \), or

\[ \frac{\partial \bar{p}}{\partial \tau} + \frac{p_{li}}{z^i} \left[ \frac{q t}{G} \right] = 0 \]  

(A-21)

Using Eq. A-17, we have

\[ \frac{\partial p_D}{\partial \phi} = -p_{DC} \frac{kh}{qB_g \mu_g} \]  

(A-22)

Taking the derivative of pseudopressure with respect to pressure

\[ \frac{\partial \phi}{\partial p} = p_{li} \frac{h s_g z^2}{\mu_g} \]  

(A-23)

Then \( \frac{\partial p}{\partial z} \) is given as

\[ \frac{\partial p}{\partial z} = \frac{1}{z^2} \frac{\partial \phi}{\partial p} = \frac{1}{z^2} \frac{\partial z}{\partial \phi} = \frac{1}{z^2} \left[ \frac{1}{z} \frac{\partial z}{\partial \phi} \right] \]

where the gas compressibility, \( c_g \), is defined as

\[ c_g = \frac{1}{p} \frac{\partial z}{\partial \phi} \]  

(A-24)

which, when combined with the previous relation gives us

\[ \frac{\partial [\phi]}{\partial p} = \frac{1}{z c_g} \]  

(A-25)

where the reciprocal of this result is given by

\[ \frac{\partial \phi}{\partial p} = z \frac{1}{p c_g} \]  

(A-26)

Taking the derivative of the gas material balance equation (Eq. A-21) with respect to time

\[ \frac{\partial \bar{p}}{\partial t} = \frac{p_{li} q}{z^i G} \]  

(A-27)

Taking the time derivative of Eq. A-19 gives

\[ \frac{\partial t_D}{\partial t} = \frac{k}{\phi \mu_i c_{pi} r_w^2} \]  

(A-28)

The reciprocal of Eq. A-28 is

\[ \frac{\partial t_D}{\partial t} = \frac{1}{t_{DC}} \frac{\phi \mu_i c_{pi} r_w^2}{k} \]  

(A-29)

Substituting Eqs. A-22, 23, 26, 27 and 29 into Eq. A-16

\[ \frac{\partial p_D}{\partial t_D} = -p_{DC} \frac{kh}{qB_g \mu_g} \left[ \frac{\mu_g \xi_i}{\mu_i} \frac{P_{li}}{G} \right] \left[ \frac{1}{z^i} \right] \left[ \frac{1}{p c_g} \right] \left[ \frac{1}{t_{DC}} \right] \frac{\phi \mu_i c_{pi} r_w^2}{k} \]

Canceling like terms gives

\[ \frac{\partial p_D}{\partial t_D} = \frac{p_{DC} \phi \mu_i c_{pi} r_w^2}{t_{DC} GB_g \mu_g c_g} \]  

(A-28)
Where this relation is taken implicitly as a function of average reservoir pressure (recall that we use the gas material balance equation). Therefore, we must write this relation as

\[
\frac{\partial p_{D,av,g}}{\partial t_D} = \frac{\rho \theta r_w^2}{t_D} \frac{\mu g i_{ci}}{\mu_g c_i}
\]

We also assume that \( c = c_p \), which is certainly valid for the case of a dry gas reservoir.

Therefore, the final relation is given as

\[
\frac{\partial p_{D,av,g}}{\partial t_D} = \frac{\rho \theta r_w^2}{t_D} \frac{\mu g i_{ci}}{\mu_g c_i}
\] .......................... (A-30)

Substituting Eq. A-30 into Eq. A-12 gives

\[
\beta(t_D) \left[ \frac{\rho \theta r_w^2}{t_D} \frac{\mu g i_{ci}}{\mu_g c_i} \right] = \int_0^{t_D} \left[ \frac{\rho \theta r_w^2}{t_D} \frac{\mu g i_{ci}}{\mu_g c_i} \right] g(t_D - \tau) d\tau
\]

Canceling like terms gives

\[
\beta(t_D) \frac{\mu g i_{ci}}{\mu_g c_i} = \int_0^{t_D} \frac{\mu g i_{ci}}{\mu_g c_i} g(t_D - \tau) d\tau
\] .......................... (A-31)

Recalling the definition of \( \beta(t_D) \) we have

\[
\beta(t_D) = \frac{\mu c_i}{\mu g c_i}
\] .......................... (A-10)

Assuming, as we have for previous efforts, that the \( \mu_g c_i \) product is evaluated at the average pressure, \( \bar{p} \), Eq. A-10 becomes

\[
\beta(t_D) = \frac{\mu c_i}{\mu g c_i}
\] .......................... (A-32)

Substituting Eqs. A-10 and A-32 into Eq. A-31 we obtain

\[
\int_0^{t_D} \frac{1}{\beta(t_D)} g(t_D - \tau) d\tau = \frac{\beta(t_D)}{\beta(t_D)} = 1
\] .......................... (A-33)

Taking the Laplace Transform of Eq. A-33 gives

\[
\frac{1}{u} = \mathcal{L} \left( \frac{1}{\beta(t_D)} \right) g(u)
\]

Or, solving for \( g(u) \) we have

\[
g(u) = \frac{1}{u} \frac{1}{\mathcal{L} \left( \frac{1}{\beta(t_D)} \right)}
\] .......................... (A-34)

Defining a reciprocal function, \( R(t_D) \), we have

\[
R(t_D) = \frac{\mu g i_{ci}}{\mu c_i} = \frac{1}{\beta(t_D)}
\] .......................... (A-35)

Substituting Eq. A-35 into Eq. A-34 we obtain

\[
g(u) = \frac{1}{u} \frac{1}{R_D(u)}
\] .......................... (A-36)

and Eq. A-33 becomes

\[
1 = \int_0^{t_D} R_D(\tau) g(t_D - \tau) d\tau
\] .......................... (A-37)

**Intermediate Summary**

**General Non-Linear pde: Standard Form**

\[
\frac{\partial^2 p_{D}}{\partial t^2} + \frac{2}{t_D} \frac{\partial p_{D}}{\partial t_D} = \beta(t_D) \frac{\partial p_{D}}{\partial t_D}
\] .......................... (A-11)

where

\[
\beta(t_D) = \frac{\mu c_i}{\mu g c_i}
\] .......................... (A-10)

and

\[
R(t_D) = \frac{\mu g c_i}{\mu c_i} = \frac{1}{\beta(t_D)}
\] .......................... (A-35)

**General Non-Linear pde: Convolution Form**

\[
\frac{\partial^2 p_{D}(u)}{\partial t^2} + \frac{1}{t_D} \frac{\partial p_{D}(u)}{\partial t_D} = \int_0^{t_D} \frac{\partial p_{D}(u)}{\partial t_D} g(t_D - \tau) d\tau
\] .......................... (A-13)

where it’s Laplace transform is given by

\[
\frac{\partial^2 p_{D}(u)}{\partial t^2} + \frac{1}{t_D} \frac{\partial p_{D}(u)}{\partial t_D} = \int_0^{t_D} \frac{\partial p_{D}(u)}{\partial t_D} g(t_D - \tau) d\tau
\] .......................... (A-15)

**Convolution Identity:**

\[
\beta(t_D) \frac{\partial p_{D}}{\partial t_D} = \int_0^{t_D} \frac{\partial p_{D}}{\partial t_D} g(t_D - \tau) d\tau
\] .......................... (A-12)

\[g(u) Functions: (Based on Average Reservoir Pressure)

\[
g(u) = \frac{1}{u} \frac{1}{\mathcal{L} \left( \frac{1}{\beta(t_D)} \right)}
\] .......................... (A-34)

or in the real domain, the convolution identity for the \( g(t_D) \) function is given by

\[
\int_0^{t_D} \frac{1}{\beta(t_D)} g(t_D - \tau) d\tau = \frac{\beta(t_D)}{\beta(t_D)} = 1
\] .......................... (A-33)

The alternative case is to use the \( R(t_D) \) formulation which is given by

\[
R_D(t_D) = \frac{\mu g i_{ci}}{\mu c_i}
\] .......................... (A-38)

where the \( g(u) \) function is given by

\[
g(u) = \frac{1}{u} \frac{1}{R_D(u)}
\] .......................... (A-36)

and the convolution identity for the \( g(t_D) \) function is given as
Application Approach

Given the general Laplace transform result based on the convolution approach, Eq. A-15, we have

$$\frac{1}{T_d} \int_0^{T_d} r(t) g(t-\tau) \, d\tau = u g(u) \tilde{P}_{D}(u)$$

(A-15)

where as in the case of a naturally fractured reservoir, the \( u g(u) \) function is substituted for all \( u \) terms, except for the \( 1/u \) term in front of a given solution in the Laplace domain.

Specifically, Eq. A-15 implies that

$$\tilde{p}_{D, gw}(u) = g(u) \tilde{p}_{D}(u)$$

(A-39)

where \( \tilde{p}_{D}(u) \) is the "fluid" (or linear) solution. Given that we obtain the \( g(u) \) function from the identity

$$g(u) = \frac{1}{u} \frac{1}{\tilde{R}_p(u)}$$

(A-36)

We must develop strategies to obtain the \( \tilde{R}_p(t_d) \) function. Recall that \( \tilde{R}_p(t_d) \) is given by

$$\tilde{R}_p(t_d) = \frac{\mu_i \epsilon_i}{\mu_k \varepsilon_i}$$

(A-38)

Recalling the gas material balance equation for a constant flowrate (Eq. A-21) we have

$$\frac{\tilde{p}}{\tilde{z}} = \frac{P_i}{z_i} \left[ 1 - \frac{q t}{G} \right]$$

(A-21)

Substituting Eq. A-19 into Eq. A-21 gives

$$\tilde{p}/\tilde{z} = \frac{P_i}{z_i} \left[ 1 - \frac{G}{G} \frac{q t}{\frac{\mu_i \epsilon_i}{k \mu_k \varepsilon_i}} \right]$$

(A-40)

Using Eqs. A-38 and 40 and a table of gas properties (\( \mu_i, c_i, \) and \( z \)), and performing a table look-up of \( \tilde{R}_p(t_d) = (\mu_i \epsilon_i)/(\mu_k \varepsilon_i) \) based on \( \tilde{p}/\tilde{z} \) gives us the table below

<table>
<thead>
<tr>
<th>( t ) (user)</th>
<th>( t_d ) (user or Eq. A-19)</th>
<th>( \tilde{p}/\tilde{z} ) (Eq. A-21 or Eq. A-40)</th>
<th>( \tilde{R}_p(t_d) = \frac{\mu_i \epsilon_i}{\mu_k \varepsilon_i} ) (Eq. A-38)</th>
</tr>
</thead>
<tbody>
<tr>
<td>xxx</td>
<td>xxx</td>
<td>xxx</td>
<td>xxx</td>
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<td>xxx</td>
<td>xxx</td>
<td>xxx</td>
</tr>
</tbody>
</table>

This approach should be used as a table look-up—e.g., specify \( t_d \) and return with \( \tilde{R}_p(t_d) \). Using this approach requires some type of algorithm or function to take the \( \tilde{R}_p(t_d) \) data into the Laplace domain. This will be discussed in the next section.

Roumboustos and Stewart Algorithm for \( \tilde{R}_p(t_d) \)

The most consistent approach for modelling the \( \tilde{R}_p(t_d) \) function in the Laplace domain is to actually bring the data \( \tilde{R}_p(t_d) \) versus \( t_d \) table into the Laplace domain. We choose to use the Roumboustos and Stewart algorithm as it is simple, straightforward to program, and very accurate. The Roumboustos and Stewart algorithm considers the data to be connected by piecewise linear functions (i.e., individual data points are connected with straight-line segments).

For a given table of data (i.e., \( f(t) \) versus \( t \)) the Roumboustos and Stewart algorithm is given as

$$\tilde{f}(u) = \frac{1}{u^2} m_1 (1-e^{-u t_1}) + \frac{1}{u^2} \sum_{i=2}^{n-1} m_i (e^{-u t_{i-1}} - e^{-u t_i}) + \frac{1}{u^2} m_n e^{-u t_{n-1}}$$

(A-41)

where the slope terms (\( m_i \)'s) are taken as backward differences given by

$$m_i = \frac{\tilde{f}(u) - \tilde{f}(u - t_i)}{t_i - t_{i-1}}$$

(A-42)

Curve Fit Approach for \( \tilde{R}_p(t_d) \)

Alternatively, we could curve fit \( \tilde{R}_p(t_d) \) using a model in terms \( t_d \). We are reluctant to recommend this as a general application due to the variations in gas properties that could yield an \( \tilde{R}_p(t_d) \) trend that is not well fitted by a specific model. Such an event would yield poor results and possibly bias the application in general. The better approach is to take the \( \tilde{R}_p(t_d) \) data directly into the Laplace domain using an algorithm such as the Roumboustos and Stewart method.

Still, the application of data models for the \( \tilde{R}_p(t_d) \) is worthwhile, especially in the possible development of closed form real domain solutions—although this concept in not investigated in this thesis. Experience and intuition show that single-term exponential as well as cubic and higher polynomial functions are well suited to match the behavior of the \( \tilde{R}_p(t_d) \) function. These functions are shown below.

The polynomial form of the \( \tilde{R}_p(t_d) \) trend is given as

$$\tilde{R}_p(t_d) = 1 + a_{1} t_d + a_{2} t_d^2 + a_{3} t_d^3 + a_{4} t_d^4 + ... + a_{n} t_d^n$$

(A-43)

Taking the Laplace transform of Eq. 41 gives

$$\tilde{R}_p(u) = \frac{1}{u} + \frac{a_1}{u^2} + 2 \frac{a_2}{u^3} + 3 \frac{a_3}{u^4} + 4 \frac{a_4}{u^5} + ... + n \frac{a_n}{u^{n+1}}$$

(A-44)
Recalling the \( g(u) \) identity, Eq. A-36, we have
\[
g(u) = \frac{1}{u} \frac{1}{\overline{R_p(u)}}
\]  
(A-36)

Substituting Eq. A-42 into Eq. A-36, we obtain
\[
g(u) = \frac{1}{1 + \frac{a_1}{u} + 2\frac{a_2}{u^2} + 3\frac{a_3}{u^3} + 4\frac{a_4}{u^4} + \ldots + n\frac{a_n}{u^n}}
\]  
(A-43)

The single-term exponential form of the \( R_p(t_p) \)-t trend is given as
\[
R_p(t_p) = a_0 \exp(-a_1 t_p)
\]  
(A-44)

Taking the Laplace transform of this expression is
\[
\overline{R_p(u)} = a_0 \frac{1}{u + a_1}
\]  
(A-45)

Substituting Eq. A-45 into Eq. A-36 we have
\[
g(u) = \frac{1}{a_0} \left( 1 + \frac{a_1}{u} \right)
\]  
(A-46)