The Pressure Derivative Revisited — Improved Formulations and Applications
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Field Cases: Infinite-Acting Radial Flow (IARF)

Type Curve Analysis Results — SPE 11463 (Buildup Case)
(Well in an Infinite-Acting Homogeneous Reservoir)

Reservoir and Fluid Properties:
- $r_w = 0.3$ ft, $h = 100$ ft,
- $c_i = 1.1 \times 10^{-5}$ psi$^{-1}$, $\phi = 0.27$ (fraction),
- $\mu_o = 1.24$ cp, $B_o = 1.002$ RB/STB

Production Parameters:
- $q_{ref} = 9200$ STB/D, $p_{wrf(\Delta t=0)} = 1844.65$ psia

Match Results and Parameter Estimates:
- $[p_D/\Delta p]_{match} = 0.02$ psi$^{-1}$, $C_D e^{2s} = 10^6$ (dim-less)
- $[(t_d/C_D)/t]_{match} = 38$ hours$^{-1}$, $k = 399.481$ md
- $C_s = 0.25$ bbl/psi, $s = 1.91$ (dim-less)

Discussion: Unfractured oil well (SPE 11463)
- Strong wellbore storage signature ($p_{D\beta d} = 1$).
- Transition region from wellbore storage to infinite-acting radial flow.
Field Cases: **Infinite-Acting Radial Flow (IARF)**

Type Curve Analysis — SPE 12777 (Buildup Case)
(Well in an Infinite-Acting Homogeneous Reservoir)

Legend: Radial Flow Type Curve
- $p_D$ Solution
- $p_{Dd}$ Solution
- $p_{D\beta d}$ Solution

Legend:
- $p_D$ Data
- $p_{Dd}$ Data
- $p_{D\beta d}$ Data

Reservoir and Fluid Properties:
- $r_w = 0.29$ ft, $h = 107$ ft,
- $c_t = 4.2 \times 10^{-6}$ psi$^{-1}$, $\phi = 0.25$ (fraction)
- $\mu_o = 2.5$ cp, $B_o = 1.06$ RB/STB

Production Parameters:
- $q_{ref} = 174$ STB/D

Match Results and Parameter Estimates:
- $[p_D/\Delta p]_{\text{match}} = 0.018$ psi$^{-1}$, $C_{De}^{2s} = 10^{10}$ (dim-less)
- $[(t_D/C_D)/f]_{\text{match}} = 15$ hours$^{-1}$, $k = 10.95$ md
- $C_s = 0.0092$ bbl/psi, $s = 8.13$ (dim-less)

Discussion: Unfractured oil well (**SPE 12777**)

- This result is an **excellent** match of all functions.
- $\beta$-derivative function is an **excellent** diagnostic for the wellbore storage and transition flow regimes.
Field Cases: *Dual Porosity, Infinite-Acting Radial Flow*

Type Curve Analysis — SPE 13054 Well MACH X3 (Drawdown Case)
(Well in a Dual Porosity System \(p_{ss}\) — \(\omega = 1 \times 10^{-2}, \alpha = 1 \times 10^{-1}\))

Reservoir and Fluid Properties:
- \(r_w = 0.2917 \text{ ft}, h = 65 \text{ ft},\)
- \(c_t = 24.5 \times 10^{-6} \text{ psi}^{-1}, \phi = 0.048 \text{ (fraction)}\)
- \(\mu_o = 0.362 \text{ cp}, B_o = 1.8235 \text{ RB/STB}\)

Production Parameters:
- \(q_{ref} = 3224 \text{ STB/D}, p_{wf}(\Delta t=0) = 9670 \text{ psia}\)

Match Results and Parameter Estimates:
- \([p_D/\Delta p]_{match} = 0.000078 \text{ psi}^{-1}, C_D e^{2s} = 1 \text{ (dim-less)}\)
- \([((t_D/C_D)/t)]_{match} = 0.17 \text{ hours}^{-1}, k = 0.361 \text{ md}\)
- \(C_s = 0.1124 \text{ bbl/psi}, s = -4.82 \text{ (dim-less)}\)
- \(\omega = 0.01 \text{ (dim-less)}, \alpha = C_o x \lambda = 0.01 \text{ (dim-less)}\)
- \(\lambda = 6.45 \times 10^{-6} \text{ (dim-less)}\)

Discussion: Unfractured oil well in dual porosity system (SPE 13054)
- Derivative functions indicate dual porosity signature — good match.
- "Less-than-perfect" late time data match may be due to rate history effects.
Field Cases: Dual Porosity, Infinite-Acting Radial Flow

Type Curve Analysis — SPE 18160 (Buildup Case)
(Well in an Infinite-Acting Dual-Porosity Reservoir \((trn)\) — \(\omega = 0.237, \alpha = 1 \times 10^{-3}\))

Reservoir and Fluid Properties:
- \(r_w = 0.29\) ft, \(h = 7\) ft,
- \(c_t = 2 \times 10^{-5}\) psi\(^{-1}\), \(\phi = 0.05\) (fraction)
- \(\mu_o = 0.3\) cp, \(B_o = 1.5\) RB/STB
- Production Parameters:
  - \(q_{ref} = 830\) Mscf/D

Match Results and Parameter Estimates:
- \([p_D/\Delta p]_{match} = 0.09\) psi\(^{-1}\), \(C_D e^{2s} = 1\) (dim-less)
- \([(t_D/C_D)/t]_{match} = 150\) hours\(^{-1}\), \(k = 678\) md
- \(C_s = 0.0311\) bbl/psi, \(s = -1.93\) (dim-less)
- \(\alpha = 0.237\) (dim-less), \(\lambda = C_D \times \lambda = 0.001\) (dim-less)
- \(\lambda = 2.13 \times 10^{-8}\) (dim-less)

Discussion: Unfractured oil well in dual porosity system: (SPE 18160)
- Strong performance of the \(\beta\)-derivative function — particularly in the region defined by transition from wellbore storage to transient interporosity flow.
Field Cases: *Hydraulically Fractured Wells*

**Type Curve Analysis — SPE 9975 Well 5 (Buildup Case)**
(Well with Infinite Conductivity Hydraulic Fractured)

**Reservoir and Fluid Properties:**
- $r_w = 0.33$ ft, $h = 30$ ft,
- $c_t = 6.37 \times 10^{-5}$ psi$^{-1}$, $\phi = 0.05$ (fraction)
- $\mu_{gi} = 0.0297$ cp, $B_{gi} = 0.5755$ RB/Mscf

**Production Parameters:**
- $q_{ref} = 1500$ Mscf/D

**Match Results and Parameter Estimates:**
- $[pD/\Delta p]_{match} = 0.000021$ psi$^{-1}$, $C_{Dr} = 0.01$ (dim-less)
- $[(t_{Dxf}/C_{Dxf})/t]_{match} = 0.15$ hours$^{-1}$, $k = 0.0253$ md
- $C_{Df} = 1000$ (dim-less), $x_f = 279.96$ ft

**Legend:**
- Infinite Conductivity Fracture
- $p_D$ Solution
- $p_{Dd}$ Solution
- $p_{D\beta d}$ Solution
- $p_{Dd} = 1/2$
- $p_{D\beta d} = 1/2$

### Discussion: Fractured gas well: buildup test (*SPE 9975 — Well 5*)
- Wellbore storage effects ($p_{D\beta d} = 1$).
- Linear flow regime could be diagnosed clearly ($p_{D\beta d} = 1/2$) — very good match.
Field Cases: *Hydraulically Fractured Wells*

**Type Curve Analysis — SPE 9975 Well 10 (Buildup Case)
(Well with Finite Conductivity Hydraulic Fracture — $C_{fd} = 2$)

**Reservoir and Fluid Properties:**
- $r_w = 0.33 \text{ ft, } h = 27 \text{ ft,}$
- $c_t = 5.10 \times 10^{-5} \text{ psi}^{-1}$, $\phi = 0.057$ (fraction)
- $\mu_g = 0.0317 \text{ cp}$, $B_g = 0.5282 \text{ RB/Mscf}$

**Production Parameters:**
- $q_{ref} = 1300 \text{ Mscf/D}$

**Match Results and Parameter Estimates:**
- $[p_D/\Delta p]_{match} = 0.0012 \text{ psi}^{-1}$, $C_{Df} = 100$ (dim-less)
- $[(t_{Dxf}/C_D)/t]_{match} = 7.5 \text{ hours}^{-1}$, $k = 0.137 \text{ md}$
- $C_{fd} = 2$ (dim-less), $x_f = 0.732 \text{ ft}$

**Discussion:** Fractured gas well: buildup test (*SPE 9975 — Well 10*)
- $p_{D\beta_d} = 1$ indicates wellbore storage effect.
- The well is either poorly fracture-stimulated, or a "skin effect" has obscured any evidence of a fracture treatment.
Type Curve Analysis — SPE 9975 Well 12 (Buildup Case)
(Well with Infinite Conductivity Hydraulic Fracture)

Reservoir and Fluid Properties:
- $r_w = 0.33$ ft, $h = 45$ ft,
- $c_t = 4.64 \times 10^{-4}$ psi$^{-1}$, $\phi = 0.057$ (fraction)
- $\mu_{gi} = 0.0174$ cp, $B_{gi} = 1.2601$ RB/Mscf

Production Parameters:
- $q_{rel} = 325$ Mscf/D

Match Results and Parameter Estimates:
- $[p_D/\Delta p]_{match} = 0.0034$ psi$^{-1}$, $C_{Df} = 0.1$ (dim-less)
- $[(t_{Dxf}/C_{Df})/t]_{match} = 37$ hours$^{-1}$, $k = 0.076$ md
- $C_{ID} = 1000$ (dim-less), $x_f = 3.681$ ft

Legend:
- $p_D$ Solution
- $p_{Dd}$ Solution
- $p_{D\beta d}$ Solution

Discussion: Fractured gas well: buildup test (SPE 9975 — Well 12)
- Wellbore storage domination regime ($p_{D\beta d} = 1$).
- The $p_{Dd}$ and $p_{D\beta d}$ signatures during mid-to-late times confirm the well is highly stimulated.
**Field Cases: Hydraulically Fractured Wells**

**Type Curve Analysis — Well 207 (Pressure Falloff Case)**
(Well with Infinite Conductivity Hydraulic Fracture)

- **Legend:** Infinite Conductivity Fracture
  - $p_D$ Solution
  - $p_{Dd}$ Solution
  - $p_{D\beta d}$ Solution

**Match Results and Parameter Estimates:**
- $[p_D/\Delta p]_{\text{match}} = 0.009 \text{ psi}^{-1}$, $C_{Df} = 0.001$ (dim-less)
- $[(t_{Dxf}/C_{Df})/t]_{\text{match}} = 150 \text{ hours}^{-1}$, $k = 11.95 \text{ md}$
- $C_{iD} = 1000$ (dim-less), $x_f = 164.22 \text{ ft}$

**Reservoir and Fluid Properties:**
- $r_w = 0.3 \text{ ft}$, $h = 103 \text{ ft}$
- $c_t = 7.7 \times 10^{-6} \text{ psi}^{-1}$, $\phi = 0.11$ (fraction)
- $\mu_w = 0.92 \text{ cp}$, $B_w = 1 \text{ RB/STB}$

**Production Parameters:**
- $q_{\text{ref}} = 1053 \text{ STB/D}$, $p_{w_{\text{f}}(t=0)} = 3119.41 \text{ psia}$

**Discussion:** Fractured water injection well (*Samad thesis — Well 207*)

- $\beta$-derivative function confirms the existence of an infinite conductivity vertical fracture for this case ($p_{D\beta d} = 1/2$).
Field Cases: **Hydraulically Fractured Wells**

Type Curve Analysis — Well 5408 (Pressure Falloff Case)
(Well with Infinite Conductivity Hydraulic Fracture)

Legend: Infinite Conductivity Fracture
- $p_D$ Solution
- $p_{Dd}$ Solution
- $p_{D\beta d}$ Solution

Legend:
- $p_D$ Data
- $p_{Dd}$ Data
- $p_{D\beta d}$ Data

Reservoir and Fluid Properties:
- $r_w = 0.198$ ft, $h = 196$ ft,
- $c_t = 6.53 \times 10^{-6}$ psi$^{-1}$, $\phi = 0.18$ (fraction)
- $\mu_w = 0.9344$ cp, $B_w = 1.002$ RB/STB

Production Parameters:
- $q_{ref} = 350$ STB/D, $p_{wf}(\Delta t=0) = 2518.1$ psia

Match Results and Parameter Estimates:
- $[p_D/\Delta p]_{match} = 0.0045$ psi$^{-1}$, $C_{Df} = 0.1$ (dim-less)
- $[(t_{Dxf}/C_{Df})/t]_{match} = 3$ hours$^{-1}$, $k = 1.06$ md
- $C_{Df} = 1000$ (dim-less), $x_f = 29.13$ ft

Discussion: Fractured water injection well (**Samad thesis — Well 5408**)
- Wellbore storage domination ($p_{D\beta d} = 1$) and infinite-acting radial flow ($p_{Dd} = 1/2$)
  — good match with infinite conductivity fracture type curve.
Field Cases: Hydraulically Fractured Wells

Type Curve Analysis — Well 2403 (Pressure Falloff Case) (Well with Infinite Conductivity Hydraulic Fracture)

Reservoir and Fluid Properties:
- \( r_w = 0.3 \text{ ft}, h = 102 \text{ ft} \)
- \( c_t = 7.21 \times 10^{-6} \text{ psi}^{-1} \)
- \( \phi = 0.11 \) (fraction)
- \( \mu_w = 0.85 \text{ cp}, B_w = 1.002 \text{ RB/STB} \)

Production Parameters:
- \( q_{ref} = 73 \text{ STB/D}, p_{wfi(t=0)} = 2630.89 \text{ psia} \)

Match Results and Parameter Estimates:
- \( [p_D/\Delta p]_{\text{match}} = 0.18 \text{ psi}^{-1}, C_{Df} = 1 \) (dim-less)
- \( [(t_{Dxf}/C_{Df})/t]_{\text{match}} = 2 \text{ hours}^{-1}, k = 12.85 \text{ md} \)
- \( C_{Df} = 1000 \) (dim-less), \( x_f = 50.136 \text{ ft} \)
- \( p_{D\beta d} = 1 \)
- \( p_{Dd} = 1/2 \)

Legend: Data
- \( p_D \) Data
- \( p_{Dd} \) Data
- \( p_{D\beta d} \) Data

Legend: Infinite Conductivity Fracture
- \( p_D \) Solution
- \( p_{Dd} \) Solution
- \( p_{D\beta d} \) Solution

Discussion: Fractured water injection well (Samad thesis — Well 2403)
- From these data we can observe the flow regimes for wellbore storage domination (\( p_{D\beta d} = 1 \)), and the infinite-acting radial (\( p_{Dd} = 1/2 \)).
The Pressure Derivative Revisited — Improved Formulations and Applications

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Abstract

The proposed work provides a new definition of the pressure-derivative function [i.e., the b-derivative function, \( \delta p_b(t) \)], which is defined as:

\[
\delta p_b(t) = f \left( \frac{dp(t)}{dt}, \frac{dp_{b,0}}{dt}, \frac{dp_{b,0}}{dt} \right)
\]

(\( \delta p_b(t) \) is the "Bouard" well testing derivative)

This formulation is based on the "power-law" concept (i.e., the derivative of the logarithm of pressure drop with respect to the logarithm of time) — this is not a trivial definition, but rather a definition that provides a unique characteristic of the "power-law" flow regime.

The "power-law" flow regimes uniquely defined by the \( \delta p_b(t) \) function are: [i.e., a constant \( \delta p_b(t) \) behavior]

Case

\( \delta p_b(t) \)

Wellbore storage domination:

1

Reservoir boundaries (homogeneous reservoirs):

1/2

Unfractured wells (homogeneous and dual porosity reservoirs):

1/2

Fractured wells (homogeneous and dual porosity reservoirs):

1/2

Horizontal wells:

1/2

Formation linear flow:

1/2

In addition, the \( \delta p_b(t) \) function provides unique characteristic responses for cases of dual porosity (naturally fractured) reservoirs.

The \( \delta p_b(t) \) function represents a new application of the traditional pressure derivative function, the "power-law" differentiation method (i.e., computing the \( \delta p(t) \) derivative) provides an accurate and consistent method for computing the primary pressure derivative (i.e., the Cartesian derivative, \( dp(t) \)) as well as the "Bouard" well testing derivative [i.e., the "semitic" derivative, \( \delta p_b(t) \)]

The Cartesian and semitic derivatives can be extracted directly from the power-law derivative (and vice-versa) using the definition given above.

Objectives

The following objectives are proposed for this work:

1. To develop analytical solutions in dimensionless form as well as graphical presentations (type curves) of the \( \delta p_b(t) \) functions for the following cases:

- Wellbore storage domination.
- Reservoir boundaries (homogeneous reservoirs).
- Unfractured wells (homogeneous and dual porosity reservoirs).
- Fractured wells (homogeneous and dual porosity reservoirs).
- Horizontal wells (homogeneous reservoirs).

2. To demonstrate the use of \( \delta p_b(t) \) functions using type curves applied to field data cases using pressure drawdown and injection/withdrawal test data.

Introduction

The well testing pressure derivative function, \( \delta p(t) \), is known to be a powerful mechanism for interpreting well test behavior — it is, in fact, perhaps the most significant single development in the history of well test analysis. The \( \delta p(t) \) function as defined by Bouard et al. [i.e., \( \delta p(t) \approx dp(t)/dt \)] provides a constant value for the case of a well producing at a constant rate in an infinite-acting homogeneous reservoir. That is, \( \delta p(t) = \text{constant} \) during infinite-acting radial flow behavior.

This single observation has made the Bouard derivative, \( \delta p(t) \), the most used diagnostic in pressure transient analysis — but what about cases where the reservoir model is not infinite-acting radial flow? Of what value then is the \( \delta p(t) \) function?

The answer is somewhat complicated in light of the fact that the Bouard derivative function has almost certainly been generated for every reservoir model in existence. Reservoir engineers have come to use the characteristic shapes in the Bouard derivative for the diagnosis and analysis of wellbore storage, boundary effects, fractured, wells, horizontal wells, and heterogeneous reservoirs. For this work, we prepare the \( \delta p(t) \) for all of those cases — but for heterogeneous reservoirs, we only consider the case of a dual porosity reservoir with pseudo-steady-state interporosity flow. The challenge is to actually define a flow regime with a particular plotting function. For example, a derivative-based plotting function that could classify a fractured well with a unique characteristic could be of significant value — as would be such functions which could be used for wellbore storage, boundary effects, horizontal wells, and heterogeneous reservoir systems.

The purpose of this work is to demonstrate that the "power-law" \( \delta p(t) \) formulation does just that — it provides a single plotting function which can be used (in isolation) as a mechanism to interpret pressure performance behavior for systems with wellbore storage, boundary effects, fractured wells, horizontal wells.

The power-law derivative function is given by:

\[
\delta p_b(t) = f \left( \frac{dp(t)}{dt}, \frac{dp_{b,0}}{dt}, \frac{dp_{b,0}}{dt} \right)
\]

where \( \delta p_b(t) \) is the "Bouard" well testing derivative.

In Appendix A, we provide the definitions of the power-law \( \delta p(t) \) derivative for various reservoir models as shown below. The graphical solution (or "type curve") for each case of interest is shown in Appendix B, and categorized as shown below.

Specific \( \delta p_b(t) \) Case

App. A

App. B

Wellbore storage dominance:

1

A-1

4,11-20

Reservoir boundaries:

- Closed reservoir:

A-2

1,2,25

- Infinite conductivity (axial, Wedd):

A-3

4

- 2-Parallel faults (large size):

A-4

1

- 3-Parallel faults (small size):

A-5

2

Fractured wells:

- Infinite conductivity vertical fracture:

A-6

2

- Finite conductivity vertical fracture:

1/2

A-7

5,9

- Horizontal well:

1/2

A-8

15

- Fractured well:

1/2

A-9

21-24

- Formation linear flow:

1/2

A-10

21-24

The origin of the \( \delta p(t) \) formulation \( \delta p(t) \) was an effort by Sowers to demonstrate that this formulation would provide a consistently better estimate of the Bouard derivative function, \( \delta p(t) \), than either the "Cartesian" or the "semitic" derivative formulations. For orientation, we present the definition of each derivative formulation below:

The "Cartesian" pressure derivative is defined as:

\[
\delta p_c(t) = f \left( \frac{dp(t)}{dt} \right)
\]

(2)

The \( \delta p(t) \) is also known as the "primary pressure derivative" [ref. 3 (Matar)].

The "semitic" or "Bouard" pressure derivative is defined as:

\[
\delta p_b(t) = f \left( \frac{dp(t)}{dt} \right)
\]

(3)

Recalling that the "primary" pressure derivative is defined as:

\[
\frac{dp(t)}{dt} = \frac{1}{\rho} \frac{d}{dt} (\rho \frac{dp(t)}{dt})
\]

(1)

Solving for the "Cartesian" or "primary pressure derivative,"

\[
\frac{dp(t)}{dt} = \frac{\delta p_c(t)}{\rho}
\]

(4)

Solving for the "semitic" or "Bouard" pressure derivative,

\[
\frac{dp(t)}{dt} = \frac{\delta p_b(t)}{\rho}
\]

(5)

Now the discussion turns to the calculation of these derivatives — what approach is best? Our options are:

1. A simple finite-difference estimate of the "Cartesian" or "primary"

2. A simple finite-difference estimate of the "semitic" or "Bouard"

3. Some type of weighted finite-difference or central difference estimation of either of the "Cartesian" or "semitic" pressure derivative functions.

This is the approach of Bouard et al. and Clark and van Gool-Bastiaan — this formulation is by far the most popular technique used to compute pressure derivative functions for the purpose of well test analysis, and will be presented in detail in the next section.

4. Other more elegant and more sophisticated approximations have been proposed for use in pressure transient (or well test) analysis, but the Bouard et al. algorithm (and its variations) continue to be the most popular approach, most likely due to the simplicity and consistency of this algorithm. To be certain, the Bouard et al. algorithm does not provide the most accurate estimate of the derivative functions, but the probity of the algorithm is very good, and the purpose of the derivative is as a diagnostic function, not a function used to provide an exact estimate.

Some of the other algorithms proposed for estimating the various pressure derivative functions are summarized below:

Moving polynomial or another type of moving regression function.

Spline approximation by Lumsden et al. is a powerful approach, but as pointed out in a general assessment of the computation of the pressure derivatives (Bouard et al.), the spline approximation requires considerable user input to obtain the "best fit" of the data, and for that reason, the method is less desirable.

Common to all Bouard et al. formulations.

Correa et al. applied a combination of power-law and logarithmic functions to represent the characteristic signal and regression analysis was used to find the best fit to the data over a specified window.

Cheng et al. utilized the fast Fourier transform and frequency-domain constraints to improve Bouard algorithm by optimizing the size of search window and they also used a Gaussian filter to denoise the pressure derivative data. This resulted in an adaptive smoothing procedure that uses recursive differentiation and integration.

Calculation of the \( \delta p_b(t) \) Derivative

To minimize the effect of truncation error, Bouard et al. introduced a weighted central-difference derivative formula:

\[
\frac{dp_b(t)}{dt} = \frac{M_L \delta p(t)_L + M_R \delta p(t)_R + M_{b,0} \delta p(t)_{b,0} - (M_L + M_R + M_{b,0}) \delta p(t)}{6(M_L + M_R + M_{b,0})}
\]

(6)

where:

- \( M_L \) = \( \frac{\delta p(t)_{L-1} - \delta p(t)_{L}}{\delta t} \)
- \( M_R \) = \( \frac{\delta p(t)_{R+1} - \delta p(t)_R}{\delta t} \)
- \( M_{b,0} \) = \( \frac{\delta p_{b,0}(t)_{L-1} - \delta p_{b,0}(t)_{L}}{\delta t} \)

The left- and right-hand subscripts represent the \( \delta p(t) \) and \( \delta p_b(t) \) respectively located a specified distance (L) from the objective point. The scale subscript represents the point of interest at which the derivative is to be computed. As for the...
L-value. Bourdet gives only general guidance as to its selection, but we have long used a formulation where $L$ is the fractional proportion of a log-cycle ($2\pi$ base). Therefore, $L=0.2$ would translate into a “sensor window” of 20 percent of a log-cycle from the point in question.

This search window approach (i.e., $L$) helps to reduce the influence of data noise on the derivative calculation. However, choosing a "small" $L$ value can cause Eq. 6a to revert to a simple central-difference between a point and its nearest neighbors, and data noise will be amplified. On the contrary, choosing a "large" $L$ value can cause Eq. 6b to provide a central-difference derivative over a very great distance — which will yield a poor estimate for the derivative, and this will tend to "smoothen" the derivative response (perhaps over-smoothing the derivative). The common range for the search window is between 10 and 50 percent of a log-cycle ($0.10 < L < 0.50$) — where we prefer a starting $L$-value of 0.2 [20 percent of a log-cycle (recall that $\log_{10}(x)$ function).

Sowers' proposed the "power-law" formulation of the weighted central difference as a method that he believed would provide a better representation of the pressure derivative than the original Bourdet formulation. In particular, Sowers provides the following definition of the power-law (or \(P^\alpha\)) derivative formulation.

\[
\frac{d^\alpha}{dt^\alpha} (\frac{dp}{dt}) = \frac{M_{\alpha-1} + M_{\alpha+1}}{M_{\alpha} + M_{\alpha+1}} \left(\frac{dp}{dt}\right) \tag{7a}
\]

where:

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6b}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6c}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6d}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6e}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6f}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6g}
\]

\[
M_{\alpha} = \frac{1}{2} \left( \frac{dp}{dt} \right) \tag{6h}
\]

Multiplying the right-hand-side of Eq. 7a by $\Delta t^\alpha$ (recall that $\Delta t^\alpha$ is the pressure drop at the point of interest), will yield the "well-testing pressure derivative" function (i.e., the typical "Bourdet" derivative definition). Sowers' provides an exhaustive evaluation of the "power-law" derivative formulation using various levels of noise in the $p$ and function and found that the power-law (or \(P^\alpha\) derivative formulation always showed improved accuracy of the well testing pressure derivative [i.e., the Bourdet derivative, $d^\alpha$/dt$^\alpha$]).

In addition, Sowers found that the $P^\alpha$ derivative formulation was less sensitive to the $L$-value than the original Bourdet formulation — which is a product of how well the power-law relation represents the pressure drop over a specific period. Sowers did not pursue the specific application of the $P^\alpha$ derivative function ($\Delta p_\alpha(t)$) as a diagnostic plotting function, as we have done.

Type Curves Using the $\beta$ Derivative Function

Background: Without question, the Bourdet definition of the pressure derivative function is the standard for all well test analysis applications — from hard methods to sophisticated interpretation/analysis/modeling software. The advent of the $\beta$-derivative function as proposed in this paper is not expected to replace the Bourdet derivative (nor should this happen).

The $\beta$-derivative function is proposed simply to serve as a better interpretation device for certain flow regimes — in particular, those flow regimes which are represented by power-law functions (e.g., wellbore storage dominated, closed boundary effects, fractured wells, horizontal well, etc.).

In the development of the models and type curves for the $\beta$ derivative function, we reviewed numerous literature articles which proposed plotting functions based on the Bourdet pressure derivatives or related functions (e.g., the primary pressure derivative (ref. 3)). In the late 1980's the "pressure derivative ratio" was proposed (refs. 9 and 10), where this function was defined as the pressure derivative divided by the pressure drop (or $2\pi$ in radial flow applications) — this ratio was (obviously) a dimensionless quantity. In particular, the pressure derivative ratio is a parameter that is not affected by wellbore storage distortion effects. The pressure derivative ratio has found most utility in such interpretations.

In the present work, we have formulated a series of "type curves" which are presented in Appendix A, developed from the $\beta$-derivative solutions given in Appendix A.

The primary utility of the $\beta$-derivative is the resolution that this function provides for cases where the pressure drop can be represented by a power-law function — again, fractured wells, horizontal wells, and boundary-influenced (faults) and boundary-dominated (closed boundaries) are good candidates for the $\beta$-derivative.

Figure 1 — Schematic of $p_{\infty}$, and $p_{tot}$ vs. $t_0$ — Various reservoir models and well configurations with wellbore storage or skin effects

Infinite-acting radial flow — the "utility" case for the Bourdet (semilog) derivative function is not a good candidate for interpretation using the $\beta$ derivative as the radial flow regime is represented by a logarithmic approximation which can be further approximated by a power-law model.

Schematic Case. In Fig. 1 we present a schematic plot created for illustrative purposes to represent the idea of the $\beta$-derivative for several distinctly different cases. Presented are the $\beta$-derivative profiles (in schematic form) for an unfractured well (infinite-acting flow), 2 fractured cases, and a horizontal well case. We note immediately the strong character of the fractured well responses $p_{fract} = 1/2$ for the infinite conductivity fracture case and $1/4$ for the finite conductivity fracture case. Interestingly, the horizontal well case shows a $p_{fract}$ slope of approximately $1/2$, but the $p_{fract}$ function never achieves the expected $1/2$ value, perhaps due to the "fracture" reservoir configuration that was specified for this particular horizontal well case. We also note that, for all cases of boundary-dominated flow, the $p_{fract}$ function yields a constant value of unity, as expected. This observation suggests that the $p_{fract}$ function (or an auxiliary function based on the $p_{fract}$ form) may be of value for the analysis of production data. For reference, Fig. 1 is presented in a larger format in Appendix B (Fig. B-1).

Infinite-Acting Radial Flow: The $\beta$-derivative function for a single well producing at a constant flowrate in an infinite-acting homogeneous reservoir was computed using the cylindrical source solution given in ref. 11. For emphasis, we have generated the $\beta$-derivative solution (Fig. 2) with wellbore storage and skin effects, as this is the typical configuration used for well test analysis. As mentioned earlier, the $\beta$-derivative function does not demonstrate a constant behavior for the radial flow case, but as noted in Appendix A, the $\beta$-derivative function for the wellbore storage dominated flow regime yields $p_{fract} = 1$.
In Fig. 5 we present cases where \( \alpha = 1 \times 10^4 \) and \( \alpha = 1/2 \) for \( 1 \times 10^4 \) and \( \alpha = 1/2 \). As with the results for the \( p_m \) functions shown in ref. 14, these \( p_m \) functions do provide some insight into the form and character of the behavior for the case of a well producing at infinite-acting flow conditions in a dual porosity/naturally fractured reservoir system.

Hydraulically Fractured Vertical Wells: In this section we consider the case of a well with a finite conductivity vertical fracture where the \( \beta \)-derivative type curves were generated using the Cinco and Mng1 solution. In addition, we used the Ozkan solution (ref. 16) to model the case of a well with an infinite conductivity vertical fracture. The \( p_m \) functions for the case of no well storage in shown in Fig. 6. We note clear evidence of the bilinear and linear flow regimes — where these regimes appear as horizontal lines on the \( \beta \)-derivative plot (bilinear flow: \( p_m = 1/4 \), linear flow: \( p_m = 1/2 \)).

Horizontal Wells: Ozkan \( ^{1} \) created a line-source solution for modeling horizontal well performance — we used this solution to generate \( \beta \)-derivative type curves for the case of a horizontal well, where the well is vertically centered within an infinite-acting, homogeneous (and isotropic) reservoir.

In Fig. 7 we present the case of a single well with a finite conductivity vertical fracture (\( C_o = 1/10 \)) producing at a constant rate in an infinite-acting homogeneous reservoir. The well storage effects included. We observe the characteristic wellbore storage domination behavior (\( p_m = 1/4 \)), as well as the effect of bilinear (fracture and formation) flow (\( p_m = 1/4 \)). We believe that the \( p_m \) function (i.e., the \( \beta \)-derivative) will substantially improve the diagnosis of flow regimes in hydraulically fractured wells.

In Fig. 8 we present the \( p_m \) and \( p_m \) solutions for the case of a horizontal well with no well storage or skin effects, only the influence of the \( L_c \) parameter (i.e., \( L_c = 1/2 \)) included in order to illustrate the performance of horizontal wells with respect to reservoir thickness (thick reservoir (low \( L_c \)), thin reservoir (high \( L_c \))). While we do not observe any features where the \( p_m \) function is constant, we do observe a unique characteristic behavior in the \( p_m \) function, which should be of value in the diagnostic interpretation of pressure transient test data obtained from horizontal wells.

In Fig. 9 we present the \( p_m \) and \( p_m \) solutions for the case of a horizontal well with wellbore storage effects are shown in Fig. 9 (\( L_c = 10 \)).

Horizontal Wells: Ozkan \( ^{1} \) created a line-source solution for modeling horizontal well performance — we used this solution to generate \( \beta \)-derivative type curves for the case of a horizontal well, where the well is vertically centered within an infinite-acting, homogeneous (and isotropic) reservoir.

Application Procedure for \( \beta \)-Derivative Type Curves

The \( \beta \)-derivative is a ratio function — the dimensionless formulation of the \( \beta \)-derivative \( p_m \) is exactly the same function as the "data" formulation of the \( \beta \)-derivative \( \Delta p_m / \Delta t \). Therefore, when we plot the \( \Delta p_m / \Delta t \) (data) function on the grid of the \( p_m \) function (i.e., the type curve match) the y-axis functions are identical. As such, the vertical "match" is not a match at all — but rather, the model and the data functions are defined to be the same — so the vertical "match" is fixed.

At this point, the time \( \alpha \) match is the only remaining task, so the \( \Delta p_m / \Delta t \) data function is shifted on top of the \( p_m \) function, only in the horizontal direction. The time (or horizontal) match is then used to diagnose the flow regimes and provide an auxiliary match of the time axis. When the \( p_m \) function is plotted with the \( \Delta p_m / \Delta t \) and the \( p_m \) functions, we achieve a "harmony" in that the 3 functions are matched simultaneously, and one portion of the match (i.e., \( \Delta p_m / \Delta t \) and \( p_m \)) is fixed.

The procedures for type curve matching with the \( \Delta p_m / \Delta t \) data and models are essentially identical to the process given for the pressure derivative ratio functions in refs. 9 and 10. As with the "pressure derivative ratio" function (ref. 9 and 10), the \( \Delta p_m / \Delta t \) (data) function is fixed, which then fixes the \( p_m \) and the \( p_m \) functions, and only the x-axis needs to be resolved — exactly like any other type curve for that particular case. If type curves not available, and some sort of software program, model-based matching procedure is used (i.e., event/ history matching), then the \( \Delta p_m / \Delta t \) and \( p_m \) functions are matched simultaneously in the same manner that the dimensionless pressure/ derivative functions would be matched.

Examples Using the \( \beta \)-Derivative Function

To demonstrate/validate the \( \beta \)-derivative function we present the results of 12 field examples obtained from the literature (refs. 1, 15-22). The table below provides orientation for our examples.

<table>
<thead>
<tr>
<th>Field</th>
<th>Example</th>
<th>Fmr ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>[oil]</td>
<td>Unfactored well (buildup)</td>
<td>11 18</td>
</tr>
<tr>
<td>[oil]</td>
<td>Unfactored well (buildup)</td>
<td>12 1</td>
</tr>
<tr>
<td>[oil]</td>
<td>Dial porosity (drawdown)</td>
<td>13 19</td>
</tr>
<tr>
<td>[oil]</td>
<td>Dial porosity (buildup)</td>
<td>14 20</td>
</tr>
<tr>
<td>[gas]</td>
<td>Fractured well (buildup)</td>
<td>5 15 21</td>
</tr>
<tr>
<td>[gas]</td>
<td>Fractured well (buildup)</td>
<td>6 16 21</td>
</tr>
<tr>
<td>[gas]</td>
<td>Fractured well (buildup)</td>
<td>7 17 21</td>
</tr>
<tr>
<td>[water]</td>
<td>Fractured well (fall-off)</td>
<td>8 18 22</td>
</tr>
<tr>
<td>[water]</td>
<td>Fractured well (fall-off)</td>
<td>9 19 22</td>
</tr>
<tr>
<td>[water]</td>
<td>Fractured well (fall-off)</td>
<td>10 20 22</td>
</tr>
<tr>
<td>[water]</td>
<td>Fractured well (fall-off)</td>
<td>11 21 22</td>
</tr>
<tr>
<td>[water]</td>
<td>Fractured well (fall-off)</td>
<td>12 22 22</td>
</tr>
</tbody>
</table>

In all of the example cases we were able to successfully interpret and analyze the well test data objectively by using the \( \beta \)-derivative function \( \Delta p_m / \Delta t \) in conjunction with the \( p_m \) and \( p_m \) functions. As a comment, for all of the example cases we considered, the \( \beta \)-derivative function \( \Delta p_m / \Delta t \) provided a direct analysis (i.e., the "match" was obvious using the \( \Delta p_m / \Delta t \) function — the vertical axis match was fixed, and only horizontal shifting was required). These examples and the model-based type curves validate the theory and application of the \( \beta \)-derivative function.
derivative function.

Example 1 is presented in Fig. 11 (from ref. 18) and shows the field data and model matches for the \( \Delta p(t), \Delta p(t)/t, \) and \( \Delta p(t)/t^b \) functions in dimensionless format (i.e., the \( p_{\Delta p}, p_{\Delta p}/t, \) and \( p_{\Delta p}/t^b \) "data" functions are given as symbols), along with the corresponding dimensionless solution functions (i.e., \( p_{\Delta p}, p_{\Delta p}/t, \) and \( p_{\Delta p}/t^b \) "model" functions given by the solid lines). This is the common format used to view the example cases in this work.

As noted in ref. 18, in this case wellbore storage effects are evident, and for the purpose of demonstrating a variable-rate procedure, downhole rates were measured. In Fig. 11 we note a strong wellbore storage signature, and we find that the \( p_{\Delta p} \) data function (squares) does yield the required value of unity. The \( p_{\Delta p} \) data function does not yield a quantitative interpretation — other than the wellbore storage domination region (\( p_{\Delta p} \approx 1 \)), but this function does also provide some resolution for the data in the transition region from wellbore storage and infinite-acting radial flow.

In Fig. 12 we consider the initial literature case regarding well test analysis using the Boardet pressure derivative function \( \Delta p(t)/t \) as shown in ref. 1. This is a pressure buildup test where the appropriate rate history superposition is used for the time function axes. This result is an excellent match of all functions, but in particular, the \( \Delta p(t)/t \) function is an excellent diagnostic function for the wellbore storage and transition flow regimes.

Particular to this case is the fact that the pressure buildup portion of the data was almost as long as the reported pressure drawdown portion of the data. We note this issue because we believe that in order to validate the use of the \( \Delta p(t)/t \) function, we must ensure that the analyst recognizes that this function will be affected by all of the same phenomena which affect the "Boardet" derivative function — in particular, the rate history must be accounted for, most likely using the effective time concept where a radial flow superposition function is used for the time axes.

Our next case (Example 4) also considers well performance in a dual porosity/naturally fractured reservoir (see Fig. 14). From these data, we again note a very strong performance of the \( \beta \) derivative function — particularly in the region defined by transition from wellbore storage to transient interporosity flow. Cases such as these validate the application of the \( \beta \) derivative for the interpretation of well test data obtained from dual porosity/naturally fractured reservoirs.
In Fig. 18, we present Well 207 from ref. 22, another hydraulically fractured well case. This time the well is a water injection well in an oil field, and a "falloff" test is conducted. In this case, there are no data at very early times so we cannot confirm the wellbore storage domination flow regime. However, we can use the β derivative function to confirm the existence of an infinite conductivity fracture vertical fracture. This is an important diagnostic.

Figure 19—Field example 8 type curve match — Well 207 (ref. 22 — Samadi) (pressure falloff case).

In Fig. 19, we present Well 3204 from ref. 22, where the data for this case are somewhat erratic due to acquisition at the surface (i.e., only surface pressures are used). Using the β derivative function, we can identify the wellbore storage domination regime (i.e., $p_{	ext{inf}} = 1$) and we can also reasonably confirm the existence of an infinite conductivity fracture vertical fracture ($p_{	ext{inf}} = 1/2$). The quality of these data impairs our ability to define the reservoir model uniquely, but we can presume that our assessment of the flow regimes is reasonable, based on the character of the β derivative function.

Figure 20—Field example 10 type curve match — Well 203 (ref. 22 — Samadi) (pressure falloff case).

As for characterization of the well efficiency, we can only conclude that the signature appears to be that of a well with a high conductivity vertical fracture, hence, our match using the model for a well with an infinite conductivity vertical fracture.

Figure 21—Field example 11 type curve match — Well 5408 (ref. 22 — Samadi) (pressure falloff case).

The data for Well 203, taken from ref. 22 are presented in Fig. 20. The signature given by the $p_{	ext{reg}}$, $p_{	ext{inf}}$ and $p_{	ext{inf} n}$ functions do not appear to be that of a high conductivity vertical fracture. In this case, the $p_{	ext{reg}}$, $p_{	ext{inf}}$ functions suggest a finite conductivity vertical fracture (note that these functions are less than 1/2 slope). The analysis of these data yields a fairly low estimate for the fracture conductivity (i.e., $C_{f} = 2$), where this result could suggest that the injection process is not continuing to propagate the fracture.

Figure 22—Field example 12 type curve match — Well 2403 (ref. 22 — Samadi) (pressure falloff case).

In closing this section on the example application of the β derivative function, we conclude that the β derivative can provide unique insight, particularly for pressure transient data from fractured wells. Pressure transient data which is influenced by wellbore storage, and pressure transient data (and likely production data) which are influenced by closed boundary effects. In addition, the β derivative function exhibits some diagnostic character for the pressure transient behavior of dual porosity/naturally fractured reservoir systems, although these diagnostics are less quantitative in such cases (i.e., the $p_{	ext{reg}}$, $p_{	ext{inf}}$ and $p_{	ext{inf} n}$ functions do not exhibit "constant" behavior as with other cases (e.g., wellbore storage, fracture flow regimes, and boundary-dominated flow)).

We believe that these examples confirm the utility and relevance of the β derivative function — and we expect the β derivative to find considerable practical application in the analysis/interpretation of pressure transient test data and (eventually) production data.

Summary

The primary purpose of this paper is the presentation of the new power-law or β derivative formulation, which is given by:

$$
\Delta p_{	ext{reg}}(t) = \frac{d}{dt} \left( \Delta p(t) \right) = \frac{1}{\beta} \frac{d\Delta p(t)}{dt}
$$

This function can be computed directly from data using:

- $\Delta p_{	ext{reg}}(t)$ (β derivative definition)...

The work of Sowers (ref. 2) shows that using the β derivative definition (Eq. 8) does provide a slightly more accurate derivative function than extracting the $\Delta p_{	ext{reg}}(t)$ function from the $p_{\text{reg}}(t)$ function as defined in Eq. 9. However, the benefit derived from using Eq. 8 is likely to be outweighed by the popularity (and availability) of the Boudart (or semilog) pressure derivative function $\Delta p_{\text{bl}}(t)$. In short, if a derivative computation module is being developed from nothing, Eq. 8 should be used. Otherwise, the "Boudart" derivative function $\Delta p_{\text{reg}}(t)$ should be adequate to "extract" the β derivative function $\Delta p_{\text{reg}}(t)$ via Eq. 7.

As a caution, we conclude that the β derivative also provides a unique characterization of well test behavior in dual porosity reservoirs (although the β derivative is never constant for these cases, except for the possibility of a rare fractured or horizontal well case).

Finally, we have provided a schematic "diagnosis worksheet" for the interpretation of the β derivative function (see Appendix C).

Recommendations for Future Work

The future work on this topic should focus on the application of the β derivative concept for production data analysis.

Acknowledgements

The authors wish to acknowledge the work of Mr. Steven F. Sowers (Eocene/Vebah) — for access to his computational routines, and for his efforts to lay the groundwork for this study via his investigations of the β derivative function as a statistically enhanced formulation for computing the Boudart derivative.
Nomenclature

Variables

\( b_0 \) = Pseudosteady-state constant, dimensionless
\( B \) = FFV, REV/STB
\( c_0 \) = total system compressibility, psi\(^{-1}\)
\( C_p \) = shape factor, dimensionless
\( C_w \) = wellbore storage coefficient, bbl/psi
\( C_{g,0} \) = dim-less wellbore storage coeff. — unfractured well
\( C_{w,0} \) = dim-less wellbore storage coeff. — horizontal well
\( C_{g,0} \) = dim-less wellbore storage coeff. — fractured well
\( h \) = fracture conductivity, dimensionless
\( h_m \) = matrix height, ft
\( k \) = permeability, md
\( k_p \) = fracture permeability, md
\( k_{p0} \) = dual porosity fracture permeability, md
\( k_{m0} \) = matrix permeability, md
\( L \) = horizontal well length, ft
\( L_{d0} \) = dimensionless horizontal well length
\( L_{d0} \) = dimensionless distance from fault
\( n \) = positive integer
\( p \) = pressure, psi
\( p_{f0} \) = dimensionless pressure
\( p_{f0} \) = dimensionless pressure derivative
\( p_{f0} \) = dimensionless \( \beta \) pressure derivative
\( p_{w0} \) = initial reservoir pressure, psi
\( p_{w,0} \) = flowing pressure, psi
\( p_{w,0} \) = well flowing pressure derivative, psi
\( p_{w,0} \) = well flowing pressure derivative, dimensionless
\( p_{w,0} \) = well shut-in pressure, psi
\( p_{w,0} \) = well shut-in pressure derivative, psi
\( p_{w,0} \) = well shut-in \( \beta \) pressure derivative, dimensionless
\( q \) = flow rate, STB/Day
\( r_{o,b} \) = reservoir outer boundary radius, ft
\( r_{o,c} \) = outer reservoir boundary radius, dimensionless
\( r_w \) = wellbore radius, ft
\( r_{w,0} \) = dimensionless wellbore radius
\( r_{w,0} \) = dimensionless wellbore radius
\( s \) = time, hr
\( t_{d0} \) = dimensionless time
\( t_{d0} \) = dimensionless time with respect to drainage area
\( t_{d0} \) = dimensionless time in horizontal well case
\( t_{d0} \) = dimensionless time in fractured well case
\( z \) = distance from wellbore along fracture, ft
\( z_{d0} \) = dimensionless distance along fracture, ft
\( z_{f,r} \) = fracture length, ft
\( z \) = distance in z direction, ft
\( z_{d0} \) = dimensionless distance in z direction
\( z_{w,b} \) = well location, ft
\( z_{w,b} \) = dimensionless well location

Greek Symbols

\( \phi \) = porosity, fraction
\( \phi_f \) = fracture porosity, fraction
\( \phi_{m0} \) = matrix porosity, fraction

Subscript

\( g \) = gas
\( w \) = water
\( ws \) = wellbore storage
\( ps \) = pseudosteady-state

References

Appendix A — Table of solutions for $p_o$, $p_{ob}$, and $p_{obm}$ (conditions/flow regimes as specified).

Table A-1 — Solutions for the wellbore storage domination flow regime.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Solution Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta p_{bw}$</td>
<td>$\Delta p_{bw} = \Delta p_{ig}$</td>
</tr>
<tr>
<td>$\Delta p_{bw}$</td>
<td>$\Delta p_{bw} = \Delta p_{ig}$</td>
</tr>
<tr>
<td>$\Delta p_{bw}$</td>
<td>$\Delta p_{bw} = \Delta p_{ig}$</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $m_{bw} = \frac{1}{24} \frac{t^2}{C_o}$

(A.1.4)

Table A-2 — Solutions for a well in a finite-acting, homogeneous reservoir (closed system, any well/reservoir configuration).

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D$</td>
<td>$P_D = \frac{1}{2} \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$P_{Dm}$</td>
<td>$P_{Dm} = \frac{1}{2} \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$P_{Dm} (v P_D)$</td>
<td>$P_{Dm} (v P_D) = \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $t_{Dm} = 30 \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P = \frac{1}{M} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P_{Dm} = \frac{1}{M} \frac{12 + 1}{2} \frac{1}{C_o}$

(A.2.3)

Table A-3 — Solutions for an unfractured well in an infinite-acting, homogeneous reservoir (radial flow).

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D$</td>
<td>$P_D = \frac{1}{4} \sqrt{\frac{2}{\pi}} \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$P_{Dm}$</td>
<td>$P_{Dm} = \frac{1}{4} \sqrt{\frac{2}{\pi}} \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$P_{Dm} (v P_D)$</td>
<td>$P_{Dm} (v P_D) = \frac{1}{4} \sqrt{\frac{2}{\pi}} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $t_{Dm} = 30 \pi D_a^2 \sqrt{\frac{2}{\pi}} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P = \frac{1}{M} \sqrt{\frac{2}{\pi}} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P_{Dm} = \frac{1}{M} \sqrt{\frac{2}{\pi}} \frac{12 + 1}{2} \frac{1}{C_o}$

(A.3.3)

Table A-4 — Solutions for a single well in an infinite-acting homogeneous reservoir system with a single or multiple sealing faults.

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0 (v P_D)$</td>
<td>$p_0 (v P_D) = \frac{1}{2} \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$p_{ob}$</td>
<td>$p_{ob} = \frac{1}{2} \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
<tr>
<td>$p_{obm}$</td>
<td>$p_{obm} = \frac{1}{2} \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $t_{Dm} = 30 \pi D_a^2 \frac{1}{12} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P = \frac{1}{M} \frac{12 + 1}{2} \frac{1}{C_o}$
- $P_{Dm} = \frac{1}{M} \frac{12 + 1}{2} \frac{1}{C_o}$

(A.4.4)
Table A-3 — Solutions for a hydraulically fractured well with an infinite conductivity fracture in an infinite-acting reservoir.

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Uniform flow, } D = 0 \text{ or infinite conductivity, } \tau = 0.752) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, linear flow, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, uniform flow, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, infinite conductivity, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Uniform flow, } D = 0 \text{ or infinite conductivity, } \tau = 0.752) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, linear flow, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, uniform flow, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, infinite conductivity, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Uniform flow, } D = 0 \text{ or infinite conductivity, } \tau = 0.752) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, linear flow, } \tau = 0.2000) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Early time, uniform flow, } \tau = 0.2000) )</td>
</tr>
</tbody>
</table>

Definitions: (field units)

- $\phi = \rho g \frac{4\pi}{L^2} \frac{dP}{dt}$
- $T = \frac{1}{14.2 \rho g (u - u_0)}$
- $u = \frac{1}{u_0}$
- $\rho g = \frac{\rho g}{L^2}$
- $\phi = \frac{\phi}{u_0}$
- $\frac{dP}{dt} = \frac{dP}{dt}$

Table A-6 — Early time solutions for a hydraulically fractured well with a finite conductivity fracture — infinite-acting homogeneous reservoir (includes wellbore storage effects).

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(General solution, } \tau = 0.752) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Short-time approximation, } \tau = 0.752) )</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$\frac{dP}{dt} = \frac{2}{4\pi t} \left( \frac{L^2}{D} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2} \left( \frac{D}{L} \right)^{3/2} \left( \frac{L - r}{L} \right)^{1/2}$ ( \text{(Large-time approximation, } \tau = 0.752) )</td>
</tr>
</tbody>
</table>

Definitions: (field units)

- $\phi = \rho g \frac{4\pi}{L^2} \frac{dP}{dt}$
- $T = \frac{1}{14.2 \rho g (u - u_0)}$
- $u = \frac{1}{u_0}$
- $\rho g = \frac{\rho g}{L^2}$
- $\phi = \frac{\phi}{u_0}$
- $\frac{dP}{dt} = \frac{dP}{dt}$
Table A-7 — Solutions for an unfractured well in an infinite-acting, dual porosity (naturally fractured) reservoir system (pseudosteady-state interporosity flow model).

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{c_D} \right) \right] \left[ \frac{1}{c_D} \left( \frac{1}{c_D} \right) \right] \left( \text{Logarithmic approximation} \right)$ (A.7.1)</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Logarithmic approximation} \right)$ (A.7.2)</td>
</tr>
<tr>
<td>$P_D$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left( \text{Logarithmic approximation} \right)$ (A.7.3)</td>
</tr>
<tr>
<td>$P_D(\theta_1/\theta_2)$</td>
<td>$P_D(\theta_1/\theta_2) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Logarithmic approximation} \right)$ (A.7.4)</td>
</tr>
<tr>
<td>$P_D(t) \propto$</td>
<td>$\propto \left( \frac{1}{c_D} \right) \left( \text{Logarithmic approximation} \right)$ (A.7.5)</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $t_D = 2.43 \times 10^{-4} \frac{m}{\mu \eta \phi_i (k_{Dw} \phi_i + k_{Dm} \phi_m)} (A.7.6)$
- $P_S = 1.14 \frac{GPa}{Pa} (A.7.7)$
- $p_{inj} = p_{out} = p_{inj} (A.7.8)$
- $B_S = 1.2 \frac{m^3}{GPa} (A.7.9)$
- $C_0^2 = \frac{\theta_D}{\theta_D (A.7.10)}$

Table A-8 — Solutions for a hydraulically fractured well in an infinite-acting, dual porosity (naturally fractured) reservoir system (pseudosteady-state interporosity flow model).

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, early time} \right)$ (A.8.1)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, intermediate time} \right)$ (A.8.2)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, early time} \right)$ (A.8.3)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, intermediate time} \right)$ (A.8.4)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, early time} \right)$ (A.8.5)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, intermediate time} \right)$ (A.8.6)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, early time} \right)$ (A.8.7)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, intermediate time} \right)$ (A.8.8)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, early time} \right)$ (A.8.9)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Fracture storage dominated flow period, intermediate time} \right)$ (A.8.10)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, early time} \right)$ (A.8.11)</td>
</tr>
<tr>
<td>$P_D(t)$</td>
<td>$P_D(t) = \frac{1}{2} \left[ \frac{1}{c_D} \right] \left[ \frac{1}{c_D} \right] \left( \text{Total system compressibility dominated flow period, intermediate time} \right)$ (A.8.12)</td>
</tr>
</tbody>
</table>

Definitions (field units):

- $t_D = 2.43 \times 10^{-4} \frac{m}{\mu \eta \phi_i (k_{Dw} \phi_i + k_{Dm} \phi_m)} (A.8.13)$
- $P_S = 1.14 \frac{GPa}{Pa} (A.8.14)$
- $p_{inj} = p_{out} = p_{inj} (A.8.15)$
- $B_S = 1.2 \frac{m^3}{GPa} (A.8.16)$
- $C_0^2 = \frac{\theta_D}{\theta_D (A.8.17)}$
- $C_0^2 = \frac{\theta_D}{\theta_D (A.8.18)}$
### Table A-9 — Solutions for an infinite conductivity horizontal well in an infinite-acting, homogeneous (and isotropic) reservoir system.

<table>
<thead>
<tr>
<th>Description</th>
<th>Relation</th>
</tr>
</thead>
</table>
| \(P_D\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.1) |
| \(P_o\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.2) |
| \(P_{h_i}\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.3) |
| \(P_{h_i}\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.4) |
| \(P_{h_i}\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.5) |
| \(P_{h_i}\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.6) |
| \(P_{h_i}\) | \[
\frac{1}{k_h} \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] \] (A.9.7) |

### Definitions: (field units)

\[
P = \sum_{n=1}^{N} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{r} \cdot \left( \frac{r_D^2}{R_n^2} - \frac{r_n^2}{R_n^2} \right) \cdot \exp \left( \frac{-r_D^2}{4R_n^2} \right) \cdot \left( \frac{r_n^2}{R_n^2} \right) \cdot \left( \frac{1}{\theta_{n+1}} \right) \cdot \left( \frac{1}{\theta_n} \right) \] (A.9.10)

\[
t = \frac{1}{10} \cdot \frac{I_D}{\phi_h / h} \] (A.9.11)

\[
P_o = \frac{1}{141.2} \cdot \frac{Q}{r_D^2} \] (A.9.12)

\[
t_p = \frac{1}{\frac{Q}{r_D^2} - \frac{P_o}{r_D^2}} \] (A.9.13)

\[
C_P = t_p \cdot r_D^2 \] (A.9.14)

\[
C_D = \frac{K}{\mu_o / \rho_o} \] (A.9.15)

\[
L_D = \frac{1}{24} \] (A.9.16)

\[
\rho_o (g/cm^3) \] (A.9.17)

\[
\mu_o (cP) \] (A.9.18)
## Appendix C — Diagnostic worksheet — A Summary of schematic well test responses for the $\beta$-derivative formulation.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\log(p_D)$</th>
<th>$\log(p_{D,0})$</th>
<th>$\log([\rho])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal well in an infinite reservoir</td>
<td>![Graph]</td>
<td>![Graph]</td>
<td>![Graph]</td>
</tr>
<tr>
<td>Fractured well in an infinite reservoir</td>
<td>![Graph]</td>
<td>![Graph]</td>
<td>![Graph]</td>
</tr>
<tr>
<td>Fractured well in an infinite reservoir</td>
<td>![Graph]</td>
<td>![Graph]</td>
<td>![Graph]</td>
</tr>
<tr>
<td>Fractured well in an infinite reservoir</td>
<td>![Graph]</td>
<td>![Graph]</td>
<td>![Graph]</td>
</tr>
</tbody>
</table>

**Figure C.1** — Summary of schematic well test responses for the $\beta$-derivative formulation.
Use of Pressure Derivative in Well-Test Interpretation

Dominique Bourdet, * SPE, J.A. Ayoub, SPE, and Y.M. Pirard, ** SPE, Flopetrol-Johnston Schlumberger
Log-Log Plot (Full Test History — $\Delta t_e$ Format-Includes Rate History)
(Pressure and Pressure Derivative Functions — Bourdet Example — SPE 12777)

Effective Shut-In Time, $\Delta t_e = \Delta t/(1+\Delta t/t_p)$, hours ($t_p = 15.33$ hr)

Legend: Bourdet Example ($\Delta t$ Format)

- $\Delta p$, Pressure drop (psi)
- $\Delta p'$, Pressure drop derivative (psi)
Effective Time Semilog Plot (Radial Flow)
(Bourdet Example — SPE 12777)

Wellbore Pressure Drop \( \Delta p = p_w - p_wf \), psi

Legend: Buildup 2

Effective Shut-In Time, \( \Delta t_e = \Delta t / (1 + \Delta t / t_p) \), hr \( (t_p = 15.33 \text{ hr}) \)
Muskat Cartesian (Late Time) Plot (Average Reservoir Pressure)

(Bourdet Example — SPE 12777)

Wellbore Pressure Drop ($\Delta P = \Delta P_{ws} - P_{wf}$ at $\Delta t = 0$), psi

Pressure Derivative, $d\Delta P/d\Delta t$, psi/hr

Legend: Buildup 2

If you use this plot, you must explain rationale and results.
Use of Pressure Derivative in Well Test Interpretation

Dominique Bourdet,* SPE, J.A. Ayoub, SPE, and Y.M. Pirard,** SPE, RicepetroleumJohnsonSchumberger

Summary. A well-test interpretation method based on the analysis of the time rate of pressure change and the actual pressure response is discussed. A differential algorithm is proposed, and several field examples illustrate how the method analyzes the response, making interpretation of well tests easier and more accurate.

Introduction

Two new strategies for the use of pressure data recorded during a well test have been used for many years to evaluate reservoir characteristics. Static reservoir pressure, measured in shot-in-wells, is used to predict reservoirs in place through material balance calculations. Transient-pressure analysis provides a description of the reservoir flowing behavior. Many methods have been proposed for interpretation of transient tests, but the best known and most widely used is Horn-er’s. More recently, type curves, which indicate the pressure response of flowing wells under a variety of well and reservoir configu-

rentations, were introduced. 13 Comparison of transient-pressure measurements with type curves provides the only reliable means for identifying that portion of the pressure data that can be analyzed by conventional straight-line analysis methods.

Recently, the quality of well test interpretation has improved considerably because of the availability of accurate pressure data (from electronic pressure gauges) and the development of new soft-

ware for computer-aided analysis. As an increasing number of theo-

retical interpretation tools that allow a more detailed definition of the flow behavior in the producing formation are now in use.

Surprisingly, the commonly used analysis techniques have not fully developed to the point where they are in practice and in interpre-
tation models, making the interpretation procedure complicated and tedious, and leaving the reservoirs as a black box with various reasons in apparent or complete, difficult to distinguish, variables for the prediction of reservoir parameters. The technique of least squares on a pressure- vs.
time graph is a “trick”—convenient for hand analysis but ignoring powerful competing facilities that are available. Fur-

thermore, some of the previously available methods fail to use all the data available and can result in significant errors.

We have proposed an interpretation based on the analysis of the derivative of pressure with respect to the appropriate time function. This parameter is also an index of the rate of storage function and, since the derivative has a physical meaning, it can be related to the characteristics of the pressure "wave" at the time of the test.

The approach was adopted in the manner of the Horner method to analyze the global response with improved definition.

Use of derivative of pressure vs. time is mathematically satisfying because the parameter is independent of the nonlinear function of the flow equation, and, unlike the pressure derivative, it has the same units as time.

Our limitation of the pressure derivative is in the difficulty of collecting measurable pressure transient data. Accurate and frequent pressure measurements are required. However, pressure monitoring and the computer processor technologies now avail-

able at well sites allow pressure-derivative analysis.

The resulting derivative data for a homogeneous reservoir and compared with conventional interpretation tech-
niques, has provided a new tool for testing, evaluating, and predicting the occurrence of actual pressure data are discussed. Application of the derivative analysis to heter-

ogenous formations reveals the good definition obtained with derivative plots, and the distinction between currently used interpretation models is clearly shown.

Transient Pressure Analysis Applied to Homogeneous Reservoirs

Conventional well-test interpretation has focused on the homogene-

ous reservoir solution. The corresponding pressure-analysis methods have been discussed extensively in the literature and are currently used.

Two complementary approaches are used for transient-pressure analysis: (1) a global approach is used to diagnose the pressure behavior and to identify the various characteristic flow regimes, and (2) specialized analyses, valid only for specific flow regimes, are performed on selected portions of the pressure data. Results of anal-

yses with both approaches must be consistent.

Diagnosis of pressure behavior is performed by type-curve analy-

A conventional well test consists of a downward flowing fluid from a reservoir into a well As fluid is produced, the reservoir expands and the pressure decreases. The flow behavior is governed by a single parameter, the fluid pressure drop, which is a function of time and is related to the reservoir properties. The analysis of the test is based on the assumption that the reservoir is homogeneous and that the boundary conditions are in place. The analysis is then performed by comparing the experimental data with a type curve, which is a graphical representation of the flow behavior of a reservoir under similar conditions. The type curve is obtained by simulating the reservoir response for a range of reservoir and well conditions. The analysis involves comparing the experimental data with the type curve to determine the reservoir properties, such as the initial reservoir pressure, the fluid flow rate, and the reservoir permeability. The analysis is performed using a numerical reservoir simulator, which simulates the flow of fluid in the reservoir under different conditions. The simulator is calibrated using the experimental data, and the reservoir properties are determined by minimizing the difference between the simulated and the experimental data. The analysis is performed iteratively, and the reservoir properties are refined until the simulated data match the experimental data. The analysis is performed for each well in the reservoir, and the results are used to map the reservoir and to optimize the production strategy.
Differential Algorithm

The main concern when actual data are being differentiated is to improve the signal-to-noise ratio. Some noise will always be present because of gauge resolution, electronic circuitry, vibrations, etc. Differentiation is difficult, if not inconclusive, for the relatively high noise level associated with a low sampling rate. This is frequently the case with mechanical gauges, which also produce noise on both pressure and time axes.

Several approaches for differentiating data have been tried (Appendix B). Because the correct result is not known when working with actual data, modified type curves were used to evaluate the different methods. A random noise ratio proportional to and independent of the amplitude of the pressure signal was added to the type curve, and the number of points generating the type curve was reduced to a minimum sample size.

Preferred Algorithm. The algorithm presented here is simple, and it is not subject to see-through interpretation. This differentiation algorithm reproduces the test curve over the complete time interval better than others. It uses one point before and one point after the point of interest, i.e., calculates the corresponding derivative, and places their weighted means at the point considered (Fig. 9).

\[ \frac{dP}{dT} = \frac{1}{2} \left( \sum_{j=0}^{n} \frac{P_{j}}{T_{j}} - \sum_{j=0}^{n} \frac{P_{j+1}}{T_{j+1}} \right) \]

where \( n \) is the number of points before and after the point of interest.

By plotting the data in logarithmic form (Fig. 11), a straight line is often obtained that is a proportion of the slope of this straight line. The slope of the straight line is the initial pressure derivative, and the x-axis intercept is the initial reservoir pressure.

When the complete recommended procedure is used, the distortions produced by the differentiation algorithm presented here are practically independent of the point density is the curve. The same effect is expected to be produced on both data and theoretical curves, as opposed to the algorithm that uses all the points present in a given time interval for smoothing.

Some of the irregularities observed in the derivative behavior were found to be part of the reservoir response. For example, oscillations of pressure caused by fluid effects are emphasized by the derivative at late time, when the signal is barely changing. Another advantage is that the derivative still gives results when the last flowing pressure is missing, as when the gauge is new or has a time in it due to causes of changing wellbore storage. The x-axis offset point is not produced to produce the derivative curve, but rather, that enough data are available, the unique match is possible and the skin is accessible. The derivative plots also tend to compensate for starting-time errors encountered when data are not accurate enough compared with the pressure-gauge sampling frequency. In addition, for gas wells, the horizontal part of the derivative of the potential does not replace the calculation of an integrated by that of a product.

\[ \text{dev/ind (mbar)} = \frac{dP}{dT} \left( \frac{dP}{dT} \right) \]

Application to Heterogeneous Reservoir Bathing

Recent theoretical developments and related publications demonstrate a general oil industry interest in the behavior of heterogeneous formations. In fact, it is our experience, based on a very large number of tests, that in some areas, up to 30 or 40% of the tests show a heterogeneous behavior. This is evident when high-pressure data, high-definition analysis techniques such as plots of the derivative of pressure, and computer-aided interpretation are used. The combined recent progress in data acquisition, data processing, and computing techniques offers new prospects for the interpretation of well-test data. More information is pulled out of the well during today's data interpretation model should highlight the multifaceted data available for analysis.

Fig. 11 presents a typical drawdown-log-log plot of \( dP/dt \) vs. \( P \) for a heterogeneous reservoir, as shown in the previous examples. The heterogeneous nature of the behavior is even more evident in the pressure response.

The well-log storage rate is always the first flow regime to appear.

1. Distinct late-time and reservoir response to the late-time response may follow. Such behavior may be a result of the effects of a fractured well, a partially penetrating well, a fractured channel, or a multilayered reservoir.

2. After the production time, the system starts to exhibit a radial flow, which may be due to an inherent heterogeneity system composed of all producing elements.

3. Beyond these effects may occur at late time.


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Tom BLASINGAME | t-blasingame@tamu.edu | Texas A&M U.
TABLE 2—PRESSURE CHANGE vs. ELAPSED TIME

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Fig. 14—Effect of block geometry on double-porosity response (transient blocks to flow rate).

Conclusions

Transient test interpretation techniques have been reduced to the identification of a few characteristic times that can be easily calculated. The pressure response to a flow rate step change is not influenced by the presence of other flow rates if the pressure response is not influenced by the presence of other flow rates. The transient test interpretation techniques have been reduced to the identification of a few characteristic times that can be easily calculated. The pressure response to a flow rate step change is not influenced by the presence of other flow rates if the pressure response is not influenced by the presence of other flow rates. The transient test interpretation techniques have been reduced to the identification of a few characteristic times that can be easily calculated. The pressure response to a flow rate step change is not influenced by the presence of other flow rates if the pressure response is not influenced by the presence of other flow rates.
Appendix C—Results of Analysis of Data, Table 2

Data are matched against the type curve for a well with wellbore storage and skin in a reservoir with homogeneous behavior and pseudo-steady state interference flow. The match parameters are defined as $C_{p} = 7.011 \times 10^{-1}$, $a = 4.015$, $\alpha = 4.5 \times 10^{-4}$, $n = 4$, pressure match ($\rho_{w} = 9.72 \times 10^{-1}$ psia $-1.6 \times 10^{-1}$ kPa $^{-1}$), and time match ($\rho_{w} = 7.50 \times 10^{-1}$ kPa $^{-1}$).

It follows that $k = \frac{141.3 \times \phi_{w} \rho_{w} g_{w} \Delta h_{n}}{141.3 \Delta h_{n}}$, $J = 141.3 \phi_{w} \rho_{w} g_{w} \Delta h_{n}$, $C = \frac{9.050 \times 9.72 \times 10^{-1}}{9.72 \times 10^{-1}}$, $a = 0.0952 \times \phi_{w} \rho_{w} g_{w} \Delta h_{n}$, $\alpha = 3.675 \times 10^{-4}$, and $\alpha = 3.9 \times 10^{-3}$.

$\rho_{w} = 7.83 \times 10^{5}$ psia $^{-1}$. The excess reservoir pressure was evaluated at $p = 7.83 \times 10^{5}$ psia $^{-1}$.


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