ABSTRACT

The transient pressure behavior of a well which produces a single compressible fluid through a single-plane vertical fracture has been investigated mathematically. The fracture is assumed to possess infinite flow capacity, to be of limited radial extent, and to penetrate the producing formation completely in the vertical direction. Previous studies of vertically fractured wells have been concerned primarily with production rate performance or semisteady-state pressure behavior. This study was undertaken to ascertain the influence of vertical fractures on transient pressure tests such as pressure build-ups and flow tests.

In a vertically fractured system, flow in the region nearest the fracture is practically linear, whereas farther away from the fracture essentially radial flow prevails. Thus, transient pressure analyses based on radial flow theory are somewhat inaccurate. As fracture penetration increases radially, kh values calculated from pressure build-up and flow test curves become increasingly larger than true values. Failure to consider the effect of fracture penetration also introduces inaccuracies into the calculation of fracture length from the apparent skin factor and into the determination of average reservoir pressure. If the total length of the fracture is 20 per cent, or greater, of the drainage radius of the well, corrections must be made to pressure build-up and flow test results. Methods for correcting such results are discussed in this paper. For wells with prefracturing pressure build-up or flow test data, it is possible to estimate fracture length by comparison with postfracturing build-up or flow test results. In new wells or wells without prefracturing build-up or flow test data, fracture length must be estimated to correct the values obtained from analysis of pressure tests after fracturing. Fracturing efficiency calculations should be made whenever possible to provide an estimate of fracture length.

Tables of the dimensionless pressure drop as a function of time and fracture penetration are included in this paper. Using these values should permit analysis of other types of transient pressure behavior in vertically fractured wells.

INTRODUCTION

Hydraulic fracturing has been used quite successfully for over a decade as a completion and stimulation tech-

nique in oil and gas wells completed in low-permeability reservoirs. During this period a considerable amount of theory has evolved on the performance of hydraulically fractured reservoirs and on more efficient means of artificial fracturing. Although theory has been developed, no rigorous investigation has been made of pressure build-up and flow test behavior in such wells.

Prats et al. first discussed the performance of vertically fractured reservoirs for the case of a compressible fluid. Their work was primarily concerned with production performance at constant flowing pressure. These authors also considered large-time (semisteady-state) constant production rate behavior for vertically fractured wells; however, transient pressure behavior at constant rate was not investigated.

McGuire and Sikora and Dyes, Kemp, and Caudle employed an electrical analog to investigate the influence of artificial vertical fractures on well productivity and pressure build-up. They found that fractures which extend beyond 15 per cent of the drainage radius away from the well alter the position and slope of the straight-line portion of the build-up curve. They concluded that these effects must be considered both in the determination of the effective permeability of the formation and in any calculations of final build-up pressure. Although these authors did not undertake an exhaustive study of the influence of vertical fractures on pressure build-up performance, their limited results were quite interesting from the standpoint of the effects they demonstrated.

In a more recent paper, Scott reported the results of an investigation of the effect of vertical fractures on pressure behavior, which was conducted with a heat flow model. Scott's results appear to be consistent with those reported in Refs. 1 and 2. However, the effects of different fracture lengths on performance were not investigated.

Pressure build-ups and transient flow tests are among the most diagnostic tools available to the reservoir engineer or production engineer. Since a very high percentage of present-day well completions incorporate the hydraulic fracturing technique, a definite need exists for information concerning the effect of fractures on transient pressure performance. For these reasons we have undertaken a rigorous study of pressure build-up and flow test behavior in vertically fractured reservoirs. The objectives of this study were (1) to obtain synthetic pressure build-up and flow test curves to assess the effects of a vertical fracture, and (2) to determine the modifications which need to be made to conventional pressure build-up and flow test analysis theory for the case of vertically fractured well.

References given at end of paper.
To obtain synthetic pressure build-up and flow test curves, it was necessary to obtain a constant rate case solution to the diffusivity equation for the fractured reservoir geometry. Pressure build-up and flow test behavior were then obtained by superposition of the resulting pressure function solutions. Because of the mathematical complexities introduced by the fractured reservoir geometry, it was necessary to solve the partial differential equation by finite difference methods.

THEORY AND METHOD OF SOLUTION

A schematic view of the reservoir situation in which we are interested is depicted on Fig. 1. We assume a horizontal reservoir which is homogeneous, isotropic and completely filled with a fluid of small and constant compressibility and constant viscosity. The reservoir is initially at uniform pressure $p_i$. Gravity effects are neglected. The drainage area of the well is assumed to take the form of a square, as shown in Fig. 1. Thus, the situation is analogous to one well in a pattern of wells as would be found normally in an oil reservoir.

A plane vertical fracture is assumed which extends over the entire vertical extent of the formation, is parallel to a drainage boundary, and is located symmetrically within the square drainage area. The effects of pressure drop within the fracture and production into the wellbore other than from the fracture are neglected. Because the fracture extends from top to bottom of the formation, and gravity effects are neglected, the problem can be represented in two-dimensional domain, as shown in Fig. 2A. Consideration of the symmetry inherent to this problem indicates that it is necessary to consider in our formulation only the symmetry element of Fig. 2A.

From the assumptions which have been made, the principle of conservation of mass, and Darcy's law, it can be shown that the pressure in the reservoir must obey the following equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{\phi c \mu}{k} \frac{\partial p}{\partial t} \quad \ldots (1)$$

When flow in the symmetry element of the reservoir is considered, it is advantageous to make all distances dimensionless with respect to the length of the side of the symmetry element $x_s$, and to define dimensionless time and dimensionless pressure drop variables as follows:

$$x_n = \frac{x}{x_s}, \quad y_n = \frac{y}{y_s}, \quad t_n = \frac{kt}{x_s^2}, \quad P = \frac{p - p_i}{\frac{q \mu}{4kh}}$$

where $x_s = y_s$.

In terms of the dimensionless variables which have been defined, Eq. 1 becomes

$$\frac{\partial^2 p}{\partial x_n^2} + \frac{\partial^2 p}{\partial y_n^2} = \frac{\partial p}{\partial t_n} \quad \ldots (2)$$

The boundary conditions which describe the physical situation of interest and for which Eq. 2 must be solved are depicted on Fig. 2B and are listed as follows:

1. At $t_n = 0, P = 0$ at all points in the reservoir (initially, reservoir pressure is $p_i$ at all points).
2. At $x_n = 0, \frac{\partial P}{\partial x_n} = 0$.
3. At $x_n = 1, \frac{\partial P}{\partial x_n} = 0$.
4. At $y_n = 1, \frac{\partial P}{\partial y_n} = 0$.
5. At $y_n = 0, x_{1a} \leq x_n \leq 1, \frac{\partial P}{\partial y_n} = 0$.
6. At $y_n = 0, 0 \leq x_n \leq x_{1a}, P = P(t_n)$.

(No flow across reservoir boundaries or lines of symmetry.)

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**Fig. 1**—Schematic View of Fractured Well and Accompanying Reservoir Drainage Volume.

**Fig. 2A**—Plane View of Fractured Reservoir Showing Position of Symmetry Element.

**Fig. 2B**—Symmetry Element from Vertically Fractured Reservoir Showing Coordinate System and Boundary Conditions.
and
\[
\left\{ \int_0^{x_p} \frac{\partial P}{\partial y} \, dx_y = -1 \right\}
\]

(The pressure in the fracture is a function of time only, and the total rate of production from one symmetry element is \(q/4\).)

Since Eq. 2 is difficult to solve analytically for the boundary conditions noted, the finite difference analog of Eq. 2 was written, and the problem was solved numerically on a computer with conventional finite difference procedures. A detailed description of the finite difference formulation and solution of this problem is presented in Appendix A.

From the computer program which was prepared, the pressure drop-time history along the fracture (which corresponds to that which would be noted in the wellbore in this case) was obtained. Also available at each time step was the pressure distribution in the reservoir. These results will be discussed in the next section.

RESULTS

VALUES OF THE DIMENSIONLESS PRESSURE DROP FUNCTION

Numerical results, in the form of dimensionless pressure drop vs dimensionless time, were obtained for six different fracture penetration ratios, \(x_f/x_e = 0.1, 0.2, 0.3, 0.5, 0.7\) and 1. Tables of the dimensionless pressure drop function for these six fracture penetration values are presented in Appendix B. Numerical results are presented in the tables for a range of \(t_e\) from \(1 \times 10^{-4}\) to 0.7. Results for intermediate fracture penetrations can be obtained by cross-plotting these results. For dimensionless time equal to or greater than 0.7, the reservoir in all cases had reached semisteady-state pressure decline. The semisteady-state pressure decline function is listed at the end of the tables in Appendix B for each separate fracture penetration.

A coordinate plot of the pressure drop function vs dimensionless time for these fracture penetrations is shown on Fig. 3. A coordinate plot of this type indicates the intermediate- to long-time behavior of the pressure drop function. Fig. 4 is a plot of pressure drop vs the logarithm of dimensionless time for the same six fracture penetrations. From it we can obtain a better idea of the early-time behavior of the pressure drop function.

The very early-time behavior of the pressure drop function (usually at times of little practical significance) is essentially as though the flow were linear. At late times, after semisteady-state pressure decline is reached, the pressure drop values depend on fracture penetration. The rate of pressure decline during semisteady-state is constant and is proportional to the hydrocarbon-filled pore volume, as is the case for pure radial flow. Thus, we are assured that reservoir limit tests should give valid results in wells which are artificially fractured. The intermediate-time behavior of the pressure function, which usually gives the characteristic pressure build-up or flow test behavior, can be seen from Figs. 3 and 4 to depend greatly on fracture penetration. The deeper the fracture penetration, the nearer the performance approaches that for linear flow, which in this case would result with a completely penetrating fracture. For small fracture penetration the pressure drop performance is nearer that for radial flow, i.e., zero fracture penetration. This behavior, then, implies that the slopes of the build-up curve and the flow test curve are affected by fracture penetration. Permeability estimates as well as skin factor determinations obtained from pressure build-up and flow tests in fractured reservoirs are therefore dependent on the extent of fracture penetration. This dependence will be discussed in detail in later sections of the paper.

The program for solution of the flow equation yields information on the pressure (or pressure drop) distribution in the reservoir at any time. Fig. 5 shows the pressure drop distribution for the case of \(0.5\) fracture penetration at a dimensionless time of 0.7. The equipressure lines are elliptical except at points very near the boundary of the reservoir and in the extreme corners of the drainage area.

COMPARISON WITH PREVIOUS WORK

Previous theoretical work on the constant rate case

FIG. 3—VERTICALLY FRAC TURED RESERVOIR, DIMENSIONLESS PRESSURE DROP VS DIMENSIONLESS TIME.

FIG. 4—VERTICALLY FRAC TURED RESERVOIR, DIMENSIONLESS PRESSURE DROP VS DIMENSIONLESS TIME.
solution for the pressure behavior in a vertically fractured reservoir has been confined to the large-time (semisteady-state) analytical solution presented in Ref. 1. To check our results, we compared the semisteady-state behavior calculated from our analysis with that set forth in Ref. 1. The analytical solution is for a circular reservoir, whereas our solution is for a square reservoir in which the fracture is parallel to either of the sets of parallel sides. Thus, fracture orientation and boundary shape effects will be felt in the case of semisteady-state performance in our solution. Surprisingly, these effects are not very great. For example, for \( \frac{x_{f}}{x_{s}} = 0.5 \), the semisteady-state solution from our data is

\[
p_{s} = q_{s} \frac{a_{f}}{4kh} \left( \frac{kt}{\phi_{m}x_{s}} + 0.54 \right).
\]

Using the solution of Prats et al. (and employing a circular reservoir of equivalent pore volume and identical fracture penetration) we find

\[
p_{s} = q_{s} \frac{a_{f}}{4kh} \left( \frac{kt}{\phi_{m}x_{s}} - 0.58 \right).
\]

Thus, the semisteady-state performance obtained in our study is in good agreement with that of Ref. 1, and the small differences in results from the two studies are due to boundary shape and fracture orientation effects.

Prats' in his first investigation of vertically fractured well performance, found that for infinitely conductive fractures the behavior of a vertically fractured well could be duplicated by that of an unfractured well with wellbore radius equal to one-quarter the total length of the fracture. This relationship was also found in Ref. 1 to be true, to a good approximation, for compressible fluid in the case of semisteady-state behavior. Accordingly, we checked our results to determine whether they also agreed with this convenient approximation.

We found that the "effective wellbore radius representation" agrees with our semisteady-state results to a good approximation. For instance, for a fracture penetration of \( \frac{x_{f}}{x_{s}} = 0.5 \), we find that good agreement with our results is given by the unfractured well solution with

\[
r_{w} \approx 0.24 \left( \frac{2x_{f}}{x_{s}} \right).
\]

Prats et al. found for this case that for the circular reservoir

\[
r_{w} \approx 0.2675 \left( \frac{2x_{f}}{x_{s}} \right).
\]

The transient pressure behavior of vertically fractured wells was obtained in Scott's heat flow analog for one fracture penetration value. This model consisted of a conducting medium bounded by a circular, isothermal outer boundary. Heat was introduced to the system at constant rate through a high-conductivity metal shim which was mounted in the center of the system to simulate a fracture. The ratio of half the lateral length of the shimboundary radius in these experiments was fixed at approximately 0.182. The calculated results from our study which are nearest the results from the model study are for the case of \( \frac{x_{f}}{x_{s}} = 0.2 \). On Fig. 6 a comparison is shown between the model study results of Scott for \( \frac{x_{f}}{r} \approx 0.182 \) and our results for \( \frac{x_{f}}{x_{s}} = 0.2 \). The agreement becomes quite good if we consider possible experimental error [if \( P(t_{0}) \) values obtained from our tables by interpolation to \( \frac{x_{f}}{x_{s}} = 0.182 \) at each \( t_{0} \) value are used]. The actual difference between the data from the two sources shown on Fig. 6 is caused by the slight difference in fracture penetration.

PRESSURE BUILD-UP BEHAVIOR

It was possible to investigate pressure build-up behavior in vertically fractured reservoirs by using the values of the dimensionless pressure drop function listed in the tables of Appendix B and the familiar method of superposition. Pressure build-up behavior was obtained by this method with the following equation:

\[
p_{w} = p_{w} - \frac{221.8 q_{s}B}{kh} \left[ P(t_{0} + \Delta t) - P(t_{0}) \right],
\]

where

\[
t_{0} = \frac{0.000264 \cdot 2k}{\phi_{m}x_{s}^{2}}
\]

The units used in Eq. 3 are ft, cp, hr, psi, B/D and m.d.

Fig. 7 is an example of a synthetic pressure build-up curve for a vertically fractured reservoir. The assumed
reservoir parameters for this example are shown on the figure. It can be seen that this build-up curve possesses several characteristics which are not entirely unlike those usually attributed to other phenomena. For example, the rather slow rise in pressure during the early part of the build-up is characteristic of wells in which a long afterproduction period occurs. This is frequently observed in wells which produce from fairly tight reservoirs. A t20 production period occurs. This is frequently observed in older wells in tight reservoirs.

Comparison of these results with the true values given on Fig. 7 shows that the effect of the fracture is to introduce a negative "pseudo" skin factor and to cause calculation of a \( kh \) factor which is 85 per cent too great. This error also leads to determination of an erroneous average reservoir pressure.

A set of pressure build-up curves for various fracture penetrations is shown in Fig. 8. This figure illustrates the effects of fracture penetration on the slope of the build-up curve. For fractures of small penetration, the slope of the build-up curve is only slightly less than that for the unfractured (radial flow) case. However, for larger fracture penetrations, the slope of the build-up curve becomes progressively smaller. This means that in field interpretation of build-ups, the effects which are introduced by the fractured reservoir flow geometry can lead to erroneous calculation of \( kh \), skin factor, and average pressure values. The deeper the fracture penetration, the greater the over-estimation of the \( kh \) product from conventional pressure build-up analysis. This effect at least partially accounts for the appearance of "new pay" after some fracture treatments. This effect has been observed quite frequently in restimulation of older wells in tight reservoirs.

At this point some remarks on the slope of the build-up curves are in order. First, there is no truly linear section on the synthetic build-up curves. The smaller the fracture penetration, the closer the build-up curves come to having a linear section. Physically, this tendency toward linearity is interpreted as meaning that the pressure behavior is approaching that described by radial flow theory, but it never reaches the "pure radial" situation. Second, in our work on build-up analysis and on two-rate flow test analysis which appears later in this paper, we have chosen as the slope of the respective curves the maximum slope. The maximum slope was chosen because normally it is a common, easily located feature on all the curves and usually occurs within the range of shut-in time encountered in build-ups and flow tests.

We were also interested in whether the producing time before a pressure build-up is run has an effect on build-up analysis results in a vertically fractured reservoir. We found that if the well has produced long enough before the build-up to reach semisteady-state pressure decline, the build-up analysis results are not affected by producing time. This point is illustrated by the two build-up curves shown on Fig. 9. The effect of producing time here is similar to that which has been found for build-ups in unfractured wells; i.e., the effect of different producing times on the Horner-type plot [pressure vs \( \log (t + \Delta t) / \Delta t \)] is to shift the curve on the time scale and, of course, change the pressure scale. The general character and slope of the curve are preserved, however.

In the hope of finding a method for plotting the pressure build-up curve which would yield a good estimate of the reservoir parameters (regardless of fracture penetration), we also made plots of pressure vs \((\sqrt{t + \Delta t} - \sqrt{\Delta t})\). This type of pressure build-up plot is linear, with slope proportional to the \( kh \) product, in linear infinite reservoirs. A set of build-up curves of this type is presented on Fig. 10. In this case we found that analysis with the early-time slope of the curve did yield good estimates of \( kh \); however, in the field this portion of the curve would not be reliable because of afterproduction and other wellbore effects. Thus, the linear flow method of plotting the pressure build-up offers no advantage over the conventional plot.

Fig. 11 is a plot of fracture penetration vs the ratio of
true to apparent $kh$ value obtained by conventional analysis of fractured reservoir synthetic pressure build-ups. This figure shows, for instance, that for a fracture penetration of 0.5, the apparent $kh$ product as calculated from pressure build-up would be about 2.5 times the true $kh$ product.

Fig. 12 is a semilog plot of the data shown on Fig. 11. It shows that the relationship between fracture penetration and $kh$ ratio is essentially a straight line up to fracture penetrations of about 0.7. The straight-line portion of this curve can be represented to a good approximation by the relationship

$$ R = \frac{kh \text{ (true)}}{kh \text{ (apparent)}} = e^{-1.14s_{fr}}. \quad (4) $$

Eq. 4 is applicable for all normally encountered values of fracture penetration.

In field fracture treatment cases in which penetration of a fracture into adjacent, previously undrained rock can be ruled out, Eq. 4 can be used in conjunction with pressure build-up analyses before and after fracturing to estimate fracture penetration. In new wells or wells in which no build-up information prior to fracturing is available to estimate the true $kh$ value, Eq. 4 can be rearranged to estimate the true $kh$ value, provided the apparent $kh$ and an estimate of fracture penetration are available.

As discussed in the previous section, it was found in Ref. 1 and verified in our results that a well with a vertical fracture exhibits a semisteady-state pressure decline which can be approximated by that of an equivalent unfractured well with wellbore radius equal to one-fourth the length of the fracture.

It was pointed out earlier in the text that the effect of a vertical fracture (in the absence of other effects such as formation damage by fracturing fluids) is to cause calculation of a negative "pseudo" skin factor from a pressure build-up analysis. To calculate the effective wellbore radius from the skin factor, the following relationship is usually used:

$$ r_w' = \frac{r_w}{2} \quad (6) $$

is strictly valid for semisteady-state only and combine it with Eq. 5, we obtain

$$ s_{fr} = 2r_w e^{-2}. \quad (7) $$

This relationship can be used with the skin factor obtained from build-up curve analysis to estimate the length of vertical fractures. Our analyses of synthetic build-up curves show that the aforementioned formula for fracture length in the case of long fractures yields estimates of fracture lengths which are as much as 40 to 50 per cent too low.

We have calculated apparent skin factors from some synthetic build-up curves for vertically fractured reservoirs in an attempt to determine the degree of error in calculated fracture length as a function of actual fracture penetration. From this work, we obtained the curve of fracture penetration vs the ratio of calculated to true fracture length, which is shown on Fig. 13. For a fracture penetration of 0.1 or less, the calculated fracture length is within 10 per cent of the true value. To estimate the true length of a hydraulically created fracture from a pressure build-up curve, the following equation is suggested:

![Graphs and figures are not transcribed due to the nature of the content.]
\[
\log D x = 0.5 \left[ \frac{P_{e f} - P_{b r}}{\Delta P \text{ (cycle)}} + \log \left( \frac{4k}{\phi \mu} \right) \right] - 3.23, \tag{8}
\]

where \( D \) is a deviation factor obtained as a function of fracture penetration from Fig. 13. This equation is a combination of the conventional skin factor equation from pressure build-up theory, Eq. 7, and the ratio of calculated to true fracture length as presented on Fig. 13. Practical units are used in this equation, and the fracture length \( x_f \) calculated from it in feet.

We have mentioned previously that determination of the average reservoir pressure from the pressure build-up is also affected by the non-radial flow field introduced by the presence of the fracture. For example, in the pressure build-up shown on Fig. 7 we obtained an extrapolated pressure value \( p_s = 1,190 \) psi. Using the calculated permeability value and applying the Matthews-Brons method for determination of average pressure in a bounded reservoir, we calculated an average pressure for this reservoir of 952 psi. We knew, of course, that the actual average pressure was 1,047 psi. Thus, we see the effect of fracture penetration on determination of average reservoir pressure. Again, this effect becomes pronounced only for fracture penetrations of 0.1 or greater.

As has previously been noted for bounded reservoirs composed of stratified layers, we have found that a plot of \( \log (p - p(\Delta t)) vs \Delta t \) will yield a straight line during the later portion of the pressure build-up for a well in a vertically fractured reservoir. Thus, by guessing values of \( p \) and making this type plot, we can arrive at a good estimate of the average reservoir pressure by selecting the best straight line which results from our trials. This method is illustrated in Fig. 14 with the pressure build-up data of Fig. 7.

From the foregoing discussion on pressure build-up behavior, it is apparent that for fracture penetrations greater than about 0.1, the effects caused by the fracture can greatly distort build-up curve results. For fracture penetrations less than about 0.1, the pressure build-up calculations are within about 10 per cent of the true results.

In wells which are believed to be producing through deeply penetrating fractures, an accurate pressure build-up analysis requires an estimate of the dimensionless fracture penetration \( x_f /x \). In some old wells this can be established by comparison of build-up results before and after fracturing. Fracturing efficiency calculations of the type presented in Ref. 8 should aid in estimation of fracture penetration in new wells or in wells without build-up data prior to fracturing.

It is also possible, of course, that Figs. 12 and 13 can be used to perform the build-up analysis in an iterative manner. The scheme of iteration involves assuming a fracture penetration \( x_f /x \) value and comparing the assumed value with that resulting from the pressure build-up analysis. The procedure is continued until satisfactory agreement between assumed and calculated values is obtained. Once an estimate of \( x_f /x \) is made, the apparent value of \( kh \) which is calculated from the pressure build-up curve can be corrected with Figs. 11 or 12 (or Eq. 4), and a calculation of fracture length can be made with Eq. 5.

In wells with build-ups or flow tests obtained prior to fracturing, the \( kh \) value is known, and it should be possible in many cases to obtain good estimates of fracture length by comparison of prefracturing and postfracturing build-up results. It should be kept in mind that fracture lengths estimated in this manner are based on a theory which assumes infinite flow capacity in the fracture. Thus, fracture lengths so obtained are "effective" in a sense; i.e., they are the fracture lengths for infinite capacity fractures which yield an increase in well productivity due to fracturing equivalent to that actually observed in the field.

We have not studied the manner in which finite flow capacity within the fracture modifies our results for transient pressure behavior. For fractures of nominal length (say \( x_f /x \leq 0.2 \)) the results obtained from analyses based on our methods should be satisfactory if the fracture conductivity is high. The poorest results would be in the case of extremely long fractures and relatively low fracture flow capacities. The interrelation of vertically fractured flow system geometry parameters and formation and frac-
TWO-RATE FLOW TEST BEHAVIOR

As in the case of pressure build-up behavior, \( P(t_b) \) values were superposed to obtain synthetic two-rate flow test curves for wells in vertically fractured reservoirs. The following equation was used:

\[
p_i - p_w = \frac{221.8 q_i B}{k h} \left[ P(t_b + \Delta t) - \left( 1 - \frac{q_i}{q_1} \right) P(\Delta t) \right]
\]

An example of a synthetic two-rate flow test curve for a vertically fractured reservoir obtained with the above equation is shown on Fig. 15. The assumed reservoir parameters for this example are shown on the figure, together with the results of analysis of this curve by the method of Ref. 9.

The same effects which modify pressure build-up results are also present on two-rate flow test curves. In addition to the effects of fracture penetration previously noted for pressure build-ups, the rate ratio \( q_i/q_1 \) also has an effect on the slope of the two-rate flow test curve in vertically fractured reservoirs. This effect is clearly demonstrated by the curves on Fig. 16. For instance, for a dimensionless fracture penetration of 0.1 and a rate ratio value of 0.45, the two-rate flow test curve slope is 0.9 of the slope of the corresponding pressure build-up curve. This means that the \( kh \) value calculated from the build-up curve would be closer to the true value than that calculated from the two-rate flow test curve. It is apparent, then, that a correction must be made in analyses of two-rate flow tests for the effect of the rate ratio. For fracture penetrations of 0.1 or smaller, the rate ratio effect is not great, provided the rate ratio is less than 0.5. This suggests that in most field cases it would be advisable, when two-rate flow tests are run, to employ a rate ratio \( q_i/q_1 \) of not greater than 0.5.

As in the case of the pressure build-up curve, the only means of obtaining highly accurate results from two-rate flow tests is to obtain a good estimate of the fracture penetration. If the dimensionless fracture penetration is known, a correction can be made for the fracture penetration and rate ratio effects, and a modified analysis method similar in principle to that used for pressure build-ups can be used.

DISCUSSION AND CONCLUSIONS

The results presented in this paper are strictly applicable, in a quantitative sense, only in those instances in which the fluids which flow in the formation can be considered to have a small and constant compressibility. Thus, for reservoirs in which a high gas saturation exists, the results are semi-quantitative in the usual manner.

It should also be kept in mind that the reservoir rock has been assumed to be homogeneous in nature. This, of course, will never be the case in the field. Layering and permeability distribution effects coupled with permeability anisotropy will tend to modify and/or to mask some of the effects which have been discussed in the paper. Additionally, the fracture has been assumed to be rectilinear in shape and to extend from the top to the bottom of the producing formation. However, despite the basic limitations of this single-fluid study, we believe that insight has been gained into the effects of vertical fracturing on transient pressure performance. Certainly it should be apparent that caution and good engineering judgment should be used in field interpretation of pressure build-up and flow test results obtained from vertically fractured reservoirs.

It has been shown and emphasized throughout this paper that fracture penetration is a critical parameter in analysis of transient pressure performance in vertically fractured reservoirs. It is important that we be able to determine with as high a degree of accuracy as possible the dimensionless fracture penetration. If we can make fracturing efficiency calculations and thereby obtain an independent estimate of fracture length, we should be able to use such information to analyze transient pressure performance in fractured reservoirs with much more confidence. It is important, then, that we continue to obtain and analyze data from fracturing operations so that fracturing efficiency calculations may be made.

The results of this paper can also be used in analysis of fractured water injection wells. In many tight reservoirs, injection wells are being operated at or above the estimated fracturing pressure of the reservoir. In other reservoirs the only apparent manner in which injectivity can be increased is by formation fracturing. The results of this report can be used to estimate the improvement in injectivity which will result by creation of a fracture of any given size.

Consequences of the manner in which the fracture is
oriented in the drainage area and the shape of the exteri­
or reservoir boundary were not considered in this study.
These effects are not believed to be great for drainage
areas of regular shapes (circular, square, hexagonal, etc.)
in which the fractured well and the fracture are centrally
located. Such effects will modify pressure build-up and
flow test curves only at very long times. In dealing wit
pressure build-up and flow test performance, we have con
sidered only the transient portion of the curves. We there­
fore believe that the results are independent of orientation
and boundary shape effects.

From the results presented in the text, the following
conclusions can be drawn:
1. In a vertically fractured system, the transient flow
regime is characterized by a region near the fracture where
the flow is practically linear and a region farther from
the fracture where the flow is essentially radial. Thus,
transient pressure analysis methods based on radial flow
theory are somewhat inaccurate.
2. The degree of inaccuracy involved in analysis of
transient flow in vertically fractured cases depends on the
magnitude of the linear flow region, or, in other words, on
the depth of penetration of the fracture into the formation.
As fracture penetration increases, $kh$ values calculated
from pressure build-up and flow test curves become in­
creasingly larger than the true values. The amount of frac­
ture penetration also affects the calculation of fracture
length and the determination of average reservoir pres­
sure.
3. For values of the dimensionless fracture penetration
less than or equal to about 0.1, conventional analysis
methods give fairly good estimates of $kh$, skin factor, and
average reservoir pressures derived from flow tests and
pressure build-ups. For values of fracture penetration
greater than 0.1, corrections should be made with the
curves in this report.
4. For deeply penetrating fractures, an estimate of frac­
ture penetration is needed if valid pressure build-up and
flow test analyses are to be obtained. In wells in which
pressure build-up or flow test data prior to fracturing are
available, fracture length can usually be found by com­
parison with postfracturing buildups or flow tests. In wells
without prefracturing build-up or flow test data, fracture
penetration can be estimated from fracturing efficiency
calculations. This points up the importance of fracturing
efficiency information.
5. In addition to the effects of fracture penetration, the
rate ratio $q_f/q_v$, which is employed in theflow test also has
an influence on flow test results. This effect becomes great­
er as fracture penetration increases. Our study suggests
that when flow tests are run, rate ratios of about 0.5 or
less should be employed in vertically fractured wells.
6. As was previously shown by Prats et al., the rate of
pressure decline after the well reaches semisteady-state is
proportional to the fluid-filled pore volume of the reser­
voir. Thus, reservoir limit tests obtained from vertically
fractured reservoirs should yield accurate estimates of the
hydrocarbon-filled pore volume.

\[
\begin{align*}
NOMENCLATURE^a \\
\text{c} & = \text{compressibility of reservoir fluid, vol/vol/psi} \\
\text{D} & = \text{ratio of calculated to true fracture length, dimensionless} \\
\text{h} & = \text{net pay thickness, ft} \\
\text{k} & = \text{permeability, md} \\
\text{p} & = \text{pressure, psi} \\
p^* & = \text{extrapolated build-up pressure, psi} \\
p & = \text{average reservoir pressure, psi} \\
R & = \text{dimensionless pressure drop} \\
q & = \text{volumetric rate of flow, B/D} \\
q_f & = \text{flow rate prior to rate change on flow test, B/D} \\
q_v & = \text{flow rate after rate change on flow test, B/D} \\
r & = \text{radius, ft or in.} \\
R & = \text{ratio of true to calculated } kh \text{ values (pressure build-up), dimensionless} \\
S & = \text{skin factor, dimensionless} \\
t & = \text{producing time, hours} \\
\Delta t & = \text{elapsed time measured from the instant of shut-in to the instant of rate change in a flow test, hours} \\
x & = \text{space variable} \\
y & = \text{space variable} \\
\phi & = \text{porosity, fraction} \\
\mu & = \text{viscosity, cp} \\
\Delta x & = \text{dimensionless distance increment in } x \text{ direction} \\
\Delta y & = \text{dimensionless distance increment in } y \text{ direction} \\
\Delta t & = \text{dimensionless time increment} \\
\end{align*}
\]

**SUBSCRIPTS**

\[
\begin{align*}
D & = \text{dimensionless quantity} \\
e & = \text{drainage boundary} \\
w & = \text{well} \\
f & = \text{fracture} \\
l & = \text{initial reservoir conditions} \\
1, 2 & = \text{employed to indicate space and time position in finite difference formulation} \\
\end{align*}
\]

REFERENCES


**APPENDIX A**

**FINITE DIFFERENCE SOLUTION OF THE CONSTANT RATE, VERTICAL FRACTURE PROBLEM**

To solve Eq. 2 of the text for the boundary conditions
discussed under “Theory and Method of Solution”, the
problem domain was represented by a square mesh
($\Delta x = \Delta y$).

To formulate a finite difference approximation to Eq. 2,
that is,

\[
\begin{align*}
p^* & = \text{extrapolated build-up pressure, psi} \\
p & = \text{average reservoir pressure, psi} \\
R & = \text{dimensionless pressure drop} \\
q & = \text{volumetric rate of flow, B/D} \\
q_f & = \text{flow rate prior to rate change on flow test, B/D} \\
q_v & = \text{flow rate after rate change on flow test, B/D} \\
r & = \text{radius, ft or in.} \\
R & = \text{ratio of true to calculated } kh \text{ values (pressure build-up), dimensionless} \\
S & = \text{skin factor, dimensionless} \\
t & = \text{producing time, hours} \\
\Delta t & = \text{elapsed time measured from the instant of shut-in to the instant of rate change in a flow test, hours} \\
x & = \text{space variable} \\
y & = \text{space variable} \\
\phi & = \text{porosity, fraction} \\
\mu & = \text{viscosity, cp} \\
\Delta x & = \text{dimensionless distance increment in } x \text{ direction} \\
\Delta y & = \text{dimensionless distance increment in } y \text{ direction} \\
\Delta t & = \text{dimensionless time increment} \\
\end{align*}
\]

**SUBSCRIPTS**

\[
\begin{align*}
D & = \text{dimensionless quantity} \\
e & = \text{drainage boundary} \\
w & = \text{well} \\
f & = \text{fracture} \\
l & = \text{initial reservoir conditions} \\
1, 2 & = \text{employed to indicate space and time position in finite difference formulation} \\
\end{align*}
\]

### TABLE 1—DIMENSIONLESS PRESSURE DROP FUNCTION

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- $fHll i$
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we center the space derivatives about the mesh point \( i, j, k \) and the time derivative about point \( i, j, k + \frac{1}{2} \). This scheme, which is indicated on Fig. 17, is the standard explicit formulation. Thus, in finite difference form Eq. 2 becomes

\[
\frac{\partial P}{\partial x_0} + \frac{\partial^2 P}{\partial y_0^2} = \frac{\partial P}{\partial t}
\]

Eq. A-1 was used for all interior grid points (points not on the boundaries).

At all boundary points not on the fracture, Eq. A-1 must be modified to include the appropriate boundary condition. This gives

\[
P_{st, i, k} = P_{i-1, j, k}; \quad x_0 = 0 \text{ and } x_0 = 1
\]

\[
P_{st, i, k} = P_{i+1, j, k}; \quad y_0 = 0 \text{ and } y_0 = 1
\]

(Appendix B)

### TABLES OF THE DIMENSIONLESS PRESSURE DROP FUNCTION

In Table 1 values of the dimensionless pressure drop function \( P \) are listed as a function of dimensionless time \( t_n \) for fracture penetration values of 0.1, 0.2, 0.3, 0.5, 0.7 and 1.

For engineering usage of these tables with the usual system of practical units (i.e., psi, cp, ft, B/D, md, hr), the following equations are valid:

\[
\rho_t - \rho_{ef} = \frac{221.8qB}{kh} P \left( t_n \frac{x_j}{x_r} \right)
\]

\[
t_n = \frac{0.000264kt}{\Phi \mu x_r^2}
\]

Values of \( P \) for fracture penetrations not listed can be found at any particular \( t_n \) value by a simple cross-plot of \( P \) vs \( x_j/x_r \).