Boundary-Dominated Flow in Solution-Gas-Drive Reservoirs
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Summary. The performance of wells in solution-gas-drive reservoirs during the boundary-dominated flow period is examined. Both constant-wellbore-pressure and constant-oil-rate production modes in closed systems are considered. For the constant-wellbore-pressure production mode, Arps' performance-prediction equations are examined, and predictions of future performance are shown to be strong functions of well spacing, well condition, and fluid properties. The parameters b (the decline exponent) and d (the initial decline rate) in the Arps equations are expressed in terms of physical properties. The conditions under which these equations can be used are specified. An empirical procedure to predict production rates is also presented. In the case of constant-oil-rate production, an expression to correlate the pressure distribution in the reservoir is presented. The correlating function permits us to extend the definition of pseudosteady-state flow to solution-gas-drive systems. Its use also allows the simultaneous computation of average properties (pressure and saturation) during boundary-dominated flow from wellbore information.

Introduction
This work documents some theoretical results that are useful for predicting well performance from production data in solution-gas-drive reservoirs during the boundary-dominated flow period. In the process of documenting these results, we also furnish theoretical support for empirical observations that exist on this subject.

Both constant-wellbore-pressure and constant-oil-rate production modes in circular closed systems are considered. The outcomes presented are based on the theoretical results presented in Refs. 1 and 2. Specifically, the results given here follow from our ability to correlate responses of solution-gas-drive systems with the response of a slightly compressible liquid flow during the boundary-dominated flow period for constant-oil-rate and constant-wellbore-pressure production modes.

This work is divided into three parts. First, the theoretical results related to boundary-dominated flow given in Refs. 1 and 2 are outlined to establish a framework for findings presented in this paper. Second, the case of a well flowing at a constant pressure during the boundary-dominated flow period is analyzed. For this case, Fetkovich3 showed that Arps'4 empirical family of curves can be combined with the slightly compressible liquid flow solution (exponential decline response5) to obtain a family of curves that can be used to predict future performance and to estimate the reservoir PV. Refs. 1 and 6 report that during the boundary-dominated period, the rate response vs. time does not match a fixed value of the decline exponent, b, in the type curves of Ref. 3. An explanation for this observation is presented. Refs. 3 and 7 emphasize that b must be less than or equal to unity, but note that if transient data are used, the value of b in the Arps solution can be greater than unity. The development for transient flow given in Ref. 8 is used to provide a theoretical justification for this observation. An empirical procedure to predict production rates of wells produced at a constant pressure over short time spans is also presented.

Third, the situation when production is held at a constant oil rate is considered. For this case, it is known from Refs. 1, 2, and 9 that the reservoir does not achieve the pseudosteady-state condition; i.e., the derivative of pressure with respect to time is not constant and is not independent of position in the reservoir. In this work, a correlation for the pressure distribution in the reservoir during the boundary-dominated flow period is developed. This correlating function allows an extension of the pseudosteady-state concept to solution-gas-drive reservoirs. Furthermore, this function allows simultaneous computation of the values of average pressure, p, and average saturation, So, from wellbore information.

The numerical results presented in this paper were obtained with a finite-difference model described in Ref. 1. Procedures followed to ensure the accuracy of the solutions are given in Refs. 1 and 2.

Mathematical Model
A homogeneous closed cylindrical reservoir with a fully penetrating well located at its center is considered. The well is capable of producing at either a constant oil rate or a constant wellbore pressure. An annular region concentric with the wellbore, with a different permeability from the formation is used to include the effect of a skin region.10 The effects of gravity, capillary pressure, and non-Darcy flow are not considered.

Figs. 1 and 2 show the PVT properties of fluids used in this work. Fig. 3 presents relative permeability data. The data sets shown in Figs. 1 through 3 are identical to the data sets considered in Refs. 1, 2, 8, and 11 and are used here mainly to preserve continuity. The conclusions derived in this work do not depend on the specific data used in the simulations. Table 1 presents information on the range of variables examined.

Background
Following the development in Ref. 1, the dimensionless pseudopressure is defined as

\[ P_p D(r,t) = \frac{kh}{141.2q_o(t)} \left[ \int_0^r \left[ \frac{\alpha(p,S_o)}{\partial p/\partial r} \right] dr' \right] \]

Here, \( \alpha(p,S_o) = k_{ro}(S_o) / [\mu_o(p)\beta(p)] \).

An empirical procedure to predict production rates of wells producing at either a constant oil rate or a constant wellbore pressure is presented. For the constant-oil-rate production, an expression to correlate the pressure distribution in the reservoir during the boundary-dominated flow period for constant-oil-rate and constant-wellbore-pressure production modes.

\[ P_p D(r,t) = P_{PD}(t) + \left( \ln r_d - \frac{3}{4} \frac{r_d}{r_d - 1} \right) \frac{1}{k_s} \left( \frac{1}{r_d^2} - \frac{1}{r_d^2} \right) \]

for \( 1 \leq r_d \leq r_d \), where \( r_D = r_{ref} \), and

\[ P_p D(r,t) = P_{PD}(t) + \left( \ln r_d - \frac{3}{4} \frac{r_d}{r_d - 1} \right) \frac{1}{r_d^2} \left( \frac{1}{r_d^2} - \frac{1}{r_d^2} \right) \]

for \( r_d \geq r_d \). Here \( \beta_d = \) dimensionless radius of the skin zone and \( \beta_d = \) volumetric average of the pseudopressure.

By using the Muskat12 material-balance equation, we show in Ref. 1 that

\[ P_{PD}(t) = \frac{kh}{141.2q_o(t)} \left[ \frac{p_l}{\alpha(p,r)} dp' \right] = 2\pi r_d \beta_d p_{PD} \]
where $\tilde{a} =$ value of function $a(p, S_o)$ at average conditions $(\bar{p}, \bar{S}_o)$ and $\bar{t}_{AD} = t_D \bar{r}_w^2 / \bar{A}$, with the dimensionless time, $t_D$, defined as

$$t_D = \frac{0.006328 k}{\mu_o \rho_o \phi \bar{r}_w^2} \int_0^{\tilde{a}} \frac{q_o(t') \bar{\lambda}_i(t')}{\bar{c}_i(t')} dt'.$$

where $\bar{\lambda}_i$ and $\bar{c}_i =$ system mobility and compressibility, respectively, corresponding to $\tilde{a}$ and $\bar{S}_o$. $\bar{\lambda}_i$ and $\bar{c}_i$ are given by

$$\bar{\lambda}_i = \frac{(k_{ro} + k_{rg})}{(\rho_o \mu_o \mu_g)} \left( \frac{\bar{S}_g}{\bar{p}} \right)$$

and $\bar{c}_i = \frac{\bar{S}_o}{B_o(\bar{p})} \frac{d B_s}{d \bar{p}} - \frac{\bar{S}_g}{B_g(\bar{p})} \frac{d B_s}{d \bar{p}} + \bar{S}_o \frac{d R_s}{d \bar{p}}.$

For the constant-oil-rate case, Eq. 5 can be simplified as

$$p_{pD}(t) = 2\pi \frac{\bar{t}_{AD}}{t_D},$$

where $\bar{t}_{AD} = \bar{t}_D \bar{r}_w^2 / \bar{A}$, with $\bar{t}_D$ defined as

$$\bar{t}_D = \frac{0.006328 k}{\mu_o \rho_o \phi \bar{r}_w^2} \int_0^{\tilde{a}} \frac{\tilde{a}(t') \lambda_i(t')}{c_i(t')} dt'.$$

Eq. 9 is an extension of the material-balance equation for single-phase liquid flow (production at a constant rate); similarly, Eq. 5 may also be considered a generalization of the material-balance equation for production at a variable rate in solution-gas-drive reservoirs.

For reference purposes, the definition of dimensionless time based on initial system properties is introduced:

$$t_D = \frac{0.006328 k \rho_o \mu_o}{\phi \bar{r}_w^2}. $$

For single-phase liquid flow during the boundary-dominated flow period, the dimensionless flow rate, $q_{pD}$, is given by

$$q_{pD}(t) = \frac{141.2 q(t) \mu B}{kh(p_1 - p_2)} = \frac{1}{D} \exp \left( -\frac{2\pi t_{AD}}{D} \right).$$
where \( D = \frac{1}{2} \left( \ln \frac{4A}{\pi C_A r_w^2} + 2s \right) \) \hspace{1cm} (13)

where \( \gamma \) is Euler’s constant and \( C_A \) is shape factor. It can be shown that the average reservoir pressure and flow rate are related by

\[
\bar{p}_D(t_D) = \frac{kh(p_i - \bar{p})}{141.2q(t)B\mu} = -D \left[ 1 - \exp \left( \frac{2\pi t_D}{D} \right) \right] \hspace{1cm} (14)
\]

Ref. 1 also shows that for solution-gas-drive systems, the following relation is valid:

\[
\bar{p}_{PD}(t_D) = -D \left[ 1 - \exp \left( \frac{2\pi t_D}{D} \right) \right] \hspace{1cm} (15)
\]

Fig. 4 exemplifies the use of the appropriate definition of dimensionless average pseudopressure, \( \bar{p}_{PD}(t) \), and dimensionless time for production at a constant wellbore pressure for both data sets used in this study. The filled-in data points correspond to Data Set 1 with \( s = 10 \) and \( r_D = 8,000 \). The open data points correspond to Data Set 2 with \( s = -2 \) and \( r_D = 2,000 \). In Fig. 4, the unbroken straight line represents Eq. 5, \( \bar{p}_{PD}(t_D) = \frac{2\pi t_D}{D} \). The circular data points represent \( \bar{p}_{PD} \) vs. \( t_D \) obtained from simulation runs. An excellent correlation with the liquid solution is obtained for all values of producing time. Similar results are obtained for the case of constant-rate production.1,2 The triangular points represent \( \bar{p}_{PD} \) values plotted as a function of \( t_D \). Agreement with Eq. 5 is good for small values of time; however, during the boundary-dominated flow period, the correlation is poor. (Similar results are also observed for constant-rate production.1,2) The unbroken curves in Fig. 4 represent the right side of Eq. 15 for two values of \( r_D \). The square data points correspond to \( \bar{p}_{PD} \) values obtained from simulation runs and plotted as a function of \( \frac{t_D}{t_D} \). The results shown here suggest that Eq. 15 will predict \( \bar{p}_{PD}(t_D) \) values with reasonable accuracy until \( \frac{t_D}{t_D} = 3 \). For larger times, Eq. 5 is a better representation of \( \bar{p}_{PD} \).

Considering Eqs. 5 and 15 and expanding the exponential function in the right side of Eq. 15, we obtain

\[
\frac{t_D}{t_AD} > \frac{t_AD}{t_AD} \hspace{1cm} (16)
\]

This inequality explains why the data points in terms of \( \frac{t_D}{t_AD} \) in Fig. 4 fall to the left of the data points in terms of \( \frac{t_AD}{t_AD} \) during boundary-dominated flow.

The results in this section form the framework for the findings presented in this work.

**Results**

As explained in the Introduction, the results of this paper are presented in two sections. We start by considering the case of constant-wellbore-pressure production mode. In this section, we establish conditions under which procedures to analyze rate data available in the literature are justified and furnish theoretical support for empirical observations existing on this subject. A procedure to predict production rates of reservoirs produced at a constant pressure is presented.

We then examine the case of production at a constant oil rate and present a correlation for the form of the pressure distribution in the reservoir. This correlation provides an extension of the pseudosteady-state concept to solution-gas-drive reservoirs.

**Constant-Pressure Production Mode. Analysis of Arps'**

**Performance-Prediction Equations.** On the basis of the success in correlating the average pseudopressure in terms of both dimensionless times \( \frac{t_D}{t_AD} \) and \( \frac{t_AD}{t_AD} \), we use these results to examine the assumptions involved in using the Arps equations for performance predictions.
Differentiating Eqs. 15 and 5 with respect to time, considering the definitions of $q_D$ (Eq. 10) and $t_D$ (Eq. 6), we obtain

$$-1 \frac{k \alpha \phi}{q_o} \frac{d}{dt} \left( \frac{t_D}{q_o} \right) = - \frac{1}{\phi} \frac{dq_o}{dt}$$

and

$$-1 \frac{k \alpha \phi}{q_o} \frac{d}{dt} \left( \frac{t_D}{q_o} \right) = - \frac{2 \pi \frac{t_D}{D}}{\phi \alpha \frac{\lambda}{c_i}}$$

respectively, with $t$ given in days.

Because $q_o(t) = N_{p2} / t$, Eq. 18 can be written as

$$\frac{dN_p}{d\tau} = \frac{\lambda}{c_i} \frac{\phi \alpha h}{5.614}$$

where $N_p$ is cumulative oil production. Eq. 19 shows the relative importance that parameters like relative permeability, PVT data, and PV have in the prediction of future performance.

Substituting the right sides of Eqs. 15 and 18 for the second and first expressions in the left side of Eq. 17, respectively, and simplifying, we obtain

$$\frac{dN_p}{d\tau} = \frac{2 \pi \frac{t_D}{D}}{\phi \alpha \frac{\lambda}{c_i}}$$

Eq. 20 implies that if $\frac{\lambda}{c_i}$ is approximately constant with time, then we would obtain a straight line by plotting log $q_o$ vs. time. This observation may also be expected on intuitive grounds, based on single-phase-flow theory.

We can relate Eq. 20 with the decline-curve equations of Arps$^4$ as follows. As is well known, the Arps equations can be written as

$$q_o = q_{oi} \exp(-b t), \quad \frac{dN_p}{d\tau} = \frac{\lambda}{c_i} \frac{\phi \alpha h}{5.614}$$

for exponential decline, and as

$$q_o = q_{oi}(1 + b d t)^{-1/b}, \quad \frac{dN_p}{d\tau} = \frac{\lambda}{c_i} \frac{\phi \alpha h}{5.614}$$

for hyperbolic decline. Arps' equations are applicable only for boundary-dominated flow. In these expressions, $t$ represents time since the rate was $q_{oi}$. The variables $d_i$ and $b$ in Eqs. 21 and 22 are considered to be constants and represent the nominal rate at which decline takes place and the decline exponent, respectively. For exponential decline $b = 0$, and for harmonic decline $b = 1$; in general, $b$ is in the range $0 \leq b \leq 1$.

Considering Eqs. 20 and 21, we can show that $d_i$ is given by

$$d_i = \frac{2 \pi \frac{t_D}{D}}{\phi \alpha \frac{\lambda}{c_i}}$$

Thus, we can deduce from Eq. 23 that the rate will decline in an exponential form only in the case when $\frac{\lambda}{c_i}$ is approximately constant. From Eq. 23 it is clear that $d_i$ is dependent on both rock and fluid properties of the reservoir, the physical reservoir dimensions, and the wellbore condition (see also Ref. 15).

Our simulation results indicate that the assumptions in this study, the function $c_i/\lambda$ does not remain constant with time. This point is illustrated in Fig. 5, which presents results for both data sets. The variation in $c_i/\lambda$ with time is clearly significant. These results are typical of all simulations we conducted and also appear to be typical of results in Ref. 6.

The variable $b$ can be obtained by

$$\frac{d}{dt} \left( \frac{1}{t} \ln q_o \right) = -b$$

From Eq. 20 it can be shown that

$$\frac{d}{dt} \left( \frac{1}{t} \ln q_o \right) = - \frac{D \phi A}{2 \pi \frac{t_D}{D}} \frac{\lambda}{c_i}$$

Thus, combining Eqs. 24 and 25, we obtain

$$b = \frac{D \phi A}{2 \pi \frac{t_D}{D}} \frac{1}{\lambda}$$

From Eq. 26, we observe that $b$ will be a constant as long as $c_i/\lambda_h$ varies linearly with time.

The observations regarding $d_i$ and $b$, which have not been presented before to our knowledge, demonstrate the assumptions that are inherent in using the Arps relations to analyze data or to predict future performance. Because both $d_i$ and $b$ depend on relative permeability and fluid properties, a simple material-balance equation like Muskat's$^{12}$ can be used to study the variation in $\lambda_i/c_i$ for any specific situation to determine the consequences of using the Arps equations. This observation also applies to gas reservoirs.$^{17}$ More interesting and important is that these equations clearly indicate that predictions of future performance are strong functions of well spacing, the well condition, and fluid properties, and thus, they also furnish a theoretical support for the concerns expressed, they also referred to the use of the production-rate decline curves for flow-rate predictions. For gas reservoirs, Frain and Wattenbarger$^{17}$ also observed that the rate-vs.-time plot does not match a fixed value of the decline exponent.

Fig. 6 presents a match of data obtained from three simulation runs for Data Set 1 with the Fetkovich type curve. Here $q_{Dp}$ and $t_{Dp}$ represent dimensionless decline rate and dimensionless decline time defined by Fetkovich (see Eqs. 19 through 22 of Ref. 3). The objective of this plot is to show the consequences of the variation in $c_i/\lambda_h$. For solution-gas-drive systems, Refs. 3 and 7 suggest that $b$ should be in the range $0.333 \leq b \leq 0.667$. The responses shown here fit the range $0.4 \leq b \leq 0.8$. The match shown here follows the $b = 0.7$ curve at early times. At later times, the responses cut across several curves because, for these simulations, $c_i/\lambda_h$ is not a linear function of time. As already mentioned, similar behavior is reported in Refs. 6 and 17.

The fact that the value of $b$ is not constant is most in the theoretical studies reported in the literature deserves comment. First one must consider whether one can obtain a constant value of $b$ for constant-pressure production. We found that $b$ generally is not a constant for constant-pressure production. Second, one must consider the nature of the wellbore pressure response when the production is forced to follow a specific value of $b$ in the Arps equations. Computations suggest that the production mode must be a variable-pressure/variable-rate mode if the rate is to follow a specific $b$ value. In some cases, we noted that it is possible to produce a constant value of $b$, the wellbore pressure must increase with time (assuming that the skin zone properties and drainage area remain constant with time). Third, if we assume constant-pressure production, then Eq. 26 indicates that the only possibility to obtain a constant $b$ is for the product $D \phi A (c_i/\lambda_h) / dt$ to be a constant. Because $D$ includes the skin factor, this reasoning leads to the argument that variable skin factors may account for a constant value of $b$. One obvious possibility to obtain a variable skin factor is non-Darcy flow. Fetkovich$^*$ also suggested this possibility to us. In Ref. 19 we noted that on the basis of an examination of inflow performance relations, non-Darcy flow may be the norm in solution-gas-drive reservoirs. On the basis of observations given here, it appears that non-Darcy flow may yield a constant value of $b$. This observation does not imply, however, that if field data suggest that $b$ is a constant, this result is a consequence of non-Darcy flow because there are other factors—like changes in skin zone properties and drainage area—that could cause this behavior. The above observations are important in understanding the consequences of using the Arps curves.

Let us now consider the computation of PV for the results shown in Fig. 6. Fetkovich et al.$^*$ suggest that it is possible to obtain reservoir PV from the match shown in Fig. 6 by the relation

$$V_p = \frac{B_o(p)}{\bar{c}_i(p - p_w)} \frac{t_{Dp}}{q_{Dp}} \frac{t_{Dp}}{t_{Dp}}$$

where $V_p$ is in barrels and $t_{Dp}$ is the initial rate

Substituting appropriate values (\(\rho=5,344.94\) psi [36.85 MPa], \(p_{wf}=5,389\) psi [23.4 MPa], \(S_w=0.0336\), \(g_i=1,4468 \times 10^{-3}\) psi \(^{-1}\), \(21 \times 10^{-3}\) kPa \(^{-1}\)), \(q_0(t^*)=q_0(t-t^*)\) or \(q_0(t)*=q_0(t-t^*)\) is approximately warranted by rewriting Eq. 20 as

\[
q_0(t) = q_{0i} \exp(-t_{D}^2/D). \tag{28}
\]

Here, \(t_{D_0}\) must be interpreted as a pseudotime since the time that the rate was \(q_{0i}\). A similar conclusion for single-phase gas flow is presented in Ref. 18. Note that the exponential decline type can be approximately warranted by rewriting Eq. 20 as

\[
d \ln q_0(t)/t_{D_0} = -1/D. \tag{29}
\]

Thus, if we use the definition of dimensionless time \(t_{D_0}\) instead of just \(t\) in the semilogarithmic plot of \(q_0\) vs. time, we would obtain an approximately straight line for engineering purposes, as mentioned in Ref. 20.

It is important to realize that Eqs. 28 and 29 are approximate. For some simulations we conducted, these expressions were very good approximations; in other cases they were inadequate. Fig. 7 demonstrates one case where Eq. 28 does not do as well as one would hope. The circular data points are plots of rate in terms of \(t_{D_0}\) and the square data points are results expressed in terms of \(t_{D_0}\). The unbroken line is the \(b=0\) solution. The responses in terms of \(t_{D_0}\) or \(t_{D_0}\) intersect several values of \(b\). In this case, however, the application of the pseudotime concept does not resolve matters completely. Eq. 28 is extremely useful, however, if a plot of \(q_0\) vs. time intersects several values of \(b\); in such cases, extrapolations based on a plot of \(q_0\) vs. \(t_{D_0}\) would result in a much better estimate of flow rate with time.

\textbf{Application of Decline Curves to Transient Flow.} Refs. 3 and 7 note that if transient data are used to compute the value of \(b\), then such data may suggest that \(b\) is greater than unity. The basis for this observation can be seen in the following development. In Ref. 8, it is shown that during the transient flow period, the skin factor of a well produced at a constant pressure can be obtained by the relations

\[
s = \frac{1}{2} \left( \frac{-1}{d \ln q_0(t)/d t} + 1 - \ln \left( \frac{4N_p t_{D}}{t_{D}} \right) \right) \tag{30}
\]

\textbf{Application of Pseudotime Concept.} Eq. 20 suggests a procedure to take into account variations in \(\frac{q_0}{t}\). It suggests that the \(b=0\) solutions can be used for performance predictions if \(t_{D_0}\) is used. This observation follows directly from integration of Eq. 20:

\[
d \ln q_0(t)/t_{D_0} = -1/D. \tag{29}
\]

where \(N_p\) is cumulative oil production. Eq. 31 is the general form of Eq. 30.

If we combine Eq. 30 with Eq. 24, we obtain

\[
b = 2s + \ln(4t_{D}/e^2). \tag{32}
\]

This expression leads to two observations. First, \(b\) is a function of time during the transient flow period. Second, if \(b\) takes values greater than unity, then Eq. 32 yields

\[
s = 0.5[1 - \ln(4t_{D}/e^2)]. \tag{33}
\]

This inequality can be easily satisfied for large reservoirs and large \(t_{D_0}\); in fact, even for small reservoirs and small \(t_{D_0}\), the skin factor can in practice satisfy this inequality for most cases. For example, for \(t_{D_0}=10^5\), the inequality in Eq. 33 would be satisfied for \(s > -5.66\), which is a high negative skin factor. Thus, in most of the cases, \(b > 1\) if transient data are used. A similar conclusion results if Eq. 31 is used. In this case, we obtain

\[
b = \left[ 1 - \left( \frac{4N_p t_{D}}{t_{D}e^2} - 2s \right)^{-1} \right] \left( \frac{4N_p t_{D}}{t_{D}e^2} + 2s \right). \tag{34}
\]

If we assume that \(N_p = q_0\), then we obtain

\[
b = -\left[ \ln(4t_{D}/e^2) + 2s \right]/\left[ 1 - \ln(4t_{D}/e^2) - 2s \right]. \tag{35}
\]

Eq. 35 again leads to the conclusion that \(b\) is a function of time and would be greater than one in most cases when transient data are used to compute \(b\). The above development provides theoretical justification for the observations in Ref. 3 that \(b > 1\) for the Arps procedure to be applicable.

\textbf{Approximate Determination of Production Rates.} The development given above also suggests a procedure to predict flow rates over short time spans without the use of relative permeability data or decline curves. Here, we present a simple procedure to predict flow rates over short time spans that is similar to the inflow performance relationship (IPR) predictions of Standing\textsuperscript{21} and Fetkovich.\textsuperscript{22}

\textbf{Numerical Computations Suggest that for Constant-pressure Production.}

\[
\int r\left( \frac{d}{dr} (\alpha(p,S_o) \frac{\partial p}{\partial r}) \right) dr' \approx 0 \tag{36}
\]

during the boundary-dominated flow period. Intuitively, at late times, this result could be expected. Differentiating Eq. 3 with
constant-pressure production. Considering Eqs. 5 and 36, we obtain the following expression for respect to time, evaluating the resulting expression at \( t=0.4 \)

\[
\frac{d\bar{p}}{dt} = \frac{d\bar{p}_o}{dt} \left( \frac{141.2}{kh} \right) \left[ \ln \left( \frac{r_{sD}}{r_{eD}} + \frac{3}{4} \right) \right]
\]

\[
+ \left( \frac{k}{k_s} - 1 \right) \left( \frac{1}{4} \frac{r_{sD}^3 - r_{eD}^3}{r_{sD}^3 - r_{eD}^3} \right)
\]

Eq. 37 can also be written as

\[
\frac{(d\bar{q}_o/dt)_f}{(d\bar{q}_o/dt)_p} = \frac{\alpha_f(d\bar{p}/dt)_f}{\alpha_p(d\bar{p}/dt)_p},
\]

where subscripts \( f \) and \( p \) = future and present conditions, respectively.

Note that Eq. 38 resembles Standing's procedure to predict future performance given by

\[
\frac{q_{o,max,f}}{q_{o,max,p}} = \frac{\bar{p}_f \alpha_f}{\bar{p}_p \alpha_p},
\]

where \( q_{o,max} = \) maximum flow rate (rate corresponding to \( p_{op} = 0 \)). It should be noted, however, that Eq. 39 can be obtained by assuming that the function \( \alpha(p) \) varies linearly with pressure,\(^{19,23,24}\) but nothing has been assumed regarding the shape of the function \( \alpha(p) \) in the derivation of Eq. 38.

If we now assume that \( \alpha = \theta \beta^2 \), where \( \theta \) is a constant, as suggested by Fetkovich,\(^{22}\) Eq. 38 yields

\[
\frac{(d\bar{q}_o/dt)_f}{(d\bar{q}_o/dt)_p} = \frac{\bar{p}_f^2 (d\bar{p}/dt)_f}{\bar{p}_p^2 (d\bar{p}/dt)_p} = \frac{(d\bar{p}^3/dt)_f}{(d\bar{p}^3/dt)_p}.
\]

Note again that Eq. 40 resembles Fetkovich's procedure to predict future performance given by

\[
\frac{q_{o,max,f}}{q_{o,max,p}} = \frac{\bar{p}_f^3}{\bar{p}_p^3}.
\]

Fig. 8 presents the derivative of rate data with respect to time vs. average pressure for a case where Data Set 1 is used. The unbroken line represents simulator values of the function \( d\bar{q}_o/dt \) vs. \( \bar{p} \). The circular and square data points correspond to the rate derivative evaluated from Eqs. 38 and 40, respectively. The computations are done by starting with the simulation value of \( -d\bar{q}_o/dt \) at \( t=0.1 \). To evaluate the right sides of both Eqs. 38 and 40, simulation values have been used. Close agreement with actual values is obtained with both Eqs. 38 and 40. The results obtained with Eqs. 38 were slightly better than those obtained with Eq. 40. Similar results were observed in other simulations.

The main advantage of Eq. 37 is that it gives us an opportunity to predict future production rates. Eq. 37 can be rewritten

\[
\alpha \approx \frac{dq_o}{d\bar{p}} \left( \frac{141.2}{kh} \right) \left[ \ln \left( \frac{r_{sD}}{r_{eD}} + \frac{3}{4} \right) \right]
\]

\[
+ \left( \frac{k}{k_s} - 1 \right) \left( \frac{1}{4} \frac{r_{sD}^3 - r_{eD}^3}{r_{sD}^3 - r_{eD}^3} \right)
\]

Integrating Eq. 42 from \( \bar{p}_1 \) to \( \bar{p}_2 \) and to \( \bar{p}_3 \) (with \( \bar{p}_1 > \bar{p}_2 > \bar{p}_3 \)) and using the assumption \( \alpha = \theta \beta^2 \) in Eq. 42, we arrive at the result

\[
\frac{\bar{p}_3^3 - \bar{p}_1^3}{\bar{p}_2^3 - \bar{p}_1^3} = \frac{q_{o,max} - q_{o,min}}{q_{o,min} - q_{o,max}},
\]

Eq. 43 permits us to predict flow rate at a future value of \( \bar{p} \) and in this sense is similar to Eqs. 39 and 41. Once the future value of \( q_{o,max} \) is determined, an IPR curve can be developed from the relations of Vogel\(^{25}\) or Fetkovich.\(^{22}\) Note that Eq. 43 requires information at two pressure levels. In this regard, it is similar to the method of Kelkar and Cox\(^{26}\) and the pivot-point method.\(^{27}\)

Because Eq. 43 is based on the assumption that \( \alpha(\bar{p}) = \theta \beta^2 \), we would intuitively expect Eq. 43 to be applicable over short time spans. This limitation also applies to Eqs. 39 and 41 (see Ref. 19). The main advantage of Eq. 43 is that predictions can be made with rate and pressure data alone without the requirement of relative permeability data.

Figs. 9 and 10 show the use of Eq. 43 to predict flow rates. Results of flow rates vs. average reservoir pressure with both data sets are presented. The filled-in data in Fig. 9 correspond to Data Set 1, and the data in Fig. 10 to Data Set 2. The unbroken lines correspond to simulation values. The initial pair of values \( (q_o,P_i) \) was taken at the onset of the boundary-dominated flow period \( t_{1AD} = 0.078 \) (\( P = 5,662.59 \) psi [39.04 MPa]) and \( t_{1AD} = 0.093 \) (\( P = 1,438.54 \) psi [9.92 MPa]) for Data Sets 1 and 2, respectively.
Other details are given in Figs. 9 and 10. Results are normalized by the initial pair of values. The circular data points reflect the predictions made by using Eq. 43 with the second pair of data points taken at \( t_{i(D,p)} = 0.124 \) and \( t_{i(D,p)} = 0.117 \) for Data Sets 1 and 2, respectively. The square data points represent results with \( q_0 \) and \( p \) values at the same \( t_i \) as the circular points but with the other pair of \( q_0 \) and \( p \) values at a time \( t_p \) greater than that for the circular points (\( t_{i(D,p)} = 4.4095 \) for Data Set 1 and \( t_{i(D,p)} = 1.3435 \) for Data Set 2). Agreement with the numerical solution is excellent in three of the four cases considered here. For the case corresponding to the second test value of \( t_{i(D,p)} \), agreement is good only for a short time span beyond the second test values (\( t_{i(D,p)} = 0.1243, \ p = 5.638.89 \) psi [38.88 MPa]). Unfortunately, as with other empirical methods, it is not possible to deduce definite conclusions regarding the accuracy of this method. As shown in Fig. 9, if test conditions are different, results may be substantially different. The principal limitation appears to be the assumption \( \alpha(p) = \theta p^2 \). The only observation we can make is that Eq. 43 yields good results in all cases for short time spans; this limitation also applies to the methods given in Refs. 21, 22, 26, and 27. This limitation can be avoided if the method of Ref. 19 is used.

**Constant-Oil-Rate Production Mode.** In this section, we show that by correlating the pressure distribution in the reservoir during the boundary-dominated flow period, we can extend the pseudosteady-state concept to solution-gas-drive systems.

Eq. 9 can be used to write Eqs. 3 and 4 as

\[
p_{PD}(r,t) = 2 \pi \frac{t_{AD}}{t_i} \ln \frac{r^2}{r_D^2} + \frac{1}{k} \left( \frac{r_D^2 - 1}{r_D^2} \right) + \left( \ln \frac{r_D}{r_i} \right) - \frac{3}{4} + \frac{s}{r_D^2}.
\]

for \( 1 \leq r_D \leq r_{PD} \) and

\[
p_{PD}(r,t) = 2 \pi \frac{t_{AD}}{t_i} \ln r_D + \left( \ln \frac{r_D}{r_i} \right) - \frac{3}{4} + \frac{s}{r_D^2} + \left( \frac{1}{k} \left( \frac{r_D^2 - 1}{r_D^2} \right) \right).
\]

for \( r_{PD} \leq r_D \leq r_{PD} \). Eqs. 44 and 45 imply that

\[
p_{PD}(r,t) = f_1(X_{PD}), \quad \text{for} \ 1 \leq r_D \leq r_{PD}.
\]

and

\[
p_{PD}(r,t) = f_2(X_{PD}), \quad \text{for} \ 1 \leq r_D \leq r_{PD}.
\]

for \( r_{PD} \leq r_D \leq r_{PD} \).

Because pseudopressure is a function of pressure and saturation, it is possible that either one or both of these variables—pressure and/or saturation—is a function of \( X_D \) in the corresponding range of radial distance. This possibility was explored and it was found that pressure is a function of \( X_D \) for \( 1 \leq r_D \leq r_{PD} \) and \( X_D \) for \( r_{PD} \leq r_D \leq r_{PD} \). Saturation, however, is not a unique function of these variables. Fig. 11 illustrates the point that pressure is a function of \( X_D \) or \( X_{PD} \). In Fig. 11, the pressure ratio \( p(r,t) \) has been correlated for \( t_{AD} \leq 0.1 \). The data points correspond to Data Set 1 with \( s = 10 \), \( r_{PD} = 4.105 \), and \( r_{PD} = 8.000 \). The dimensionless rate, \( q_{o0} \), in Fig. 11 is defined as

\[
q_{o0} = 141.2 \left( \frac{s \cdot B_o \cdot q_o}{k \cdot r_{PD} \cdot h_p} \right).
\]

![Fig. 11—Correlation of pressure, constant-rate production.](image)

For clarity, seven locations within the skin zone and five locations outside the skin zone region have been considered. The simulation was stopped when an abandonment pressure of 245 psi [1.69 MPa] was reached, which occurred at 450 days (\( t_{AD} = 0.95 \), \( t_{AD} = 0.7 \)). The pressure responses form a single curve for the appropriate variable. These results are typical of all simulations conducted in this work. Thus, on the basis of this observation, we can write the following relations:

\[
p(p,t) = f \left( \frac{r^2}{r_D^2} \right) \exp \left[ \frac{k}{k_s} \left( \frac{r_D^2 - 1}{r_D^2} \right) \right]
\]

\[
= f_1(X_{PD}), \quad \text{for} \ 1 \leq r_D \leq r_{PD}.
\]

and

\[
p(p,t) = f_2(X_{PD}), \quad \text{for} \ 1 \leq r_D \leq r_{PD}.
\]

for \( r_{PD} \leq r_D \leq r_{PD} \). Differentiating Eq. 49 with respect to \( r \) and \( t \) and combining the resulting equations, we obtain the following expression for the pressure gradient:

\[
\frac{\partial p}{\partial r} = \left( \frac{1}{r} \right) - \frac{r}{r_D^2} \frac{k}{k_s} \frac{\partial p}{\partial r} \left( -2 \pi \left( 0.006328 \right) \frac{k}{\phi A} \right).
\]

for \( 1 \leq r_D \leq r_{PD} \). If we differentiate Eq. 9 with respect to time by using the definitions of \( B_{PD} \) and \( \frac{t_{AD}}{t_i} \) for constant rate, we obtain the following expression for \( \alpha \):

\[
\alpha = (-5.614 q_o X_o \phi A h) / \partial (\phi A h dt).
\]

Muskat's material-balance equation is given by

\[
\frac{\partial \rho}{\partial \phi} = (\beta \partial \rho / \partial \phi),
\]

where \( \beta = S_o / B_o \). In Refs. 1 and 2, it is shown that \( \beta(r,t) = \beta(p, S_o) \), where an overbar indicates volumetric-average values. Denoting \( \beta(p, S_o) \) by \( \bar{\beta} \), we have

\[
-5.614 q_o / \phi A h = \partial \rho / \partial t,
\]

where \( q_o = \) production rate. The production rate for \( s = 0 \) is given by

\[
q_o = k_s h \left( \frac{r}{r_w} \right).
\]

where \( k_s = \) skin-zone permeability. Substituting the right side of Eq. 51 into Eq. 55 for \( dp/\partial r \), neglecting \( (r_w/r) \), and using Eq.
Eq. 56, of course, is also valid for \( s = 0 \). The importance of this expression to solution-gas-drive systems is that it is a generalization of the pseudosteady-state concept at the positions \( r_i \) and \( r_f \). The question that needs to be addressed at this point is whether the generalization of the pseudosteady-state concept is valid at other positions in the reservoir. To answer this question, let us consider for simplicity the region outside the skin zone. Differentiating Eq. 50 with respect to \( r \) and \( t \) and combining the resulting equations, we obtain

\[
\frac{\partial p}{\partial r} = \left( \frac{1}{r} - \frac{1}{r^2} \right) \frac{\partial p}{\partial t} \left( -2 \pi 0.006328 \frac{k}{\phi A} \frac{\lambda_i}{c_i} \right). \tag{57}
\]

The oil rate at any position \( r (r_i < r < r_f) \) is

\[
q_o(r) = \frac{k}{141.2} \frac{\partial p}{\partial r} \tag{58}
\]

Substituting the right side of Eq. 57 into the right side of Eq. 58 using Eq. 52, we obtain

\[
q_o(r) = \frac{\alpha}{\alpha_o (r_{ws})} \left( 1 - \frac{r^2}{r_{e}^2} \right) \frac{\partial p}{\partial t} \tag{59}
\]

Refs. 1 and 2 show that during the boundary-dominated flow period, the following expression is valid:

\[
q_o(r) = q_o(r_{ws}) \left( 1 - \frac{r^2}{r_{e}^2} \right). \tag{60}
\]

Thus, from Eqs. 59 and 60 it follows that

\[
\bar{\alpha} \frac{\partial p}{\partial t} = \alpha \frac{\partial p}{\partial t} \tag{61}
\]

for any position \( r (r_i < r < r_f) \). The same result can be obtained for positions inside the skin zone. Thus, Eq. 61 represents a generalization of the pseudosteady-state concept to solution-gas-drive systems for all positions in the reservoir.

Eq. 61 also provides an insight into the behavior of solution-gas-drive systems produced at a constant oil rate. Differentiating Eqs. 3 and 4 with respect to time and considering the definitions of pseudopressure and average pseudopressure given by Eqs. 1 and 5, respectively, we obtain

\[
\frac{\partial p}{\partial t} = \int r \left[ \frac{\partial}{\partial t} \left[ \alpha (p, S_o, \frac{\partial p}{\partial r}) \right] \right] dr' = \frac{\alpha}{\partial r} \cdot \alpha \frac{\partial p}{\partial t}. \tag{62}
\]

Comparing Eq. 61 with Eq. 62, we can conclude that the integral term in Eq. 62 can be considered to be negligible when compared with the other terms. This observation has been confirmed by numerical results. In fact, it has been found that the integral in Eq. 62 may even be considered negligible toward the end of the transient flow period. The same result is also true for constant-wellbore-pressure production for the boundary-dominated flow (Eq. 36) and during the transient flow period, as reported by Camacho-V. Fig. 12 is a plot of the variation in \( \alpha \frac{\partial p}{\partial t} \) with distance \( r_{D} \). The ordinate is normalized in terms of \( k / \phi A \) i.e., \( \alpha (\partial p / \partial t) \). Our objective was to test the reasonableness of the approximation given by Eq. 61 during the boundary-dominated flow period. Results for the transient period are also presented (filled-in data points). This case represents a severe test of our development because the value of \( s \) is large (\( k / \phi A = 8.08 \)). For the case \( t_{D, D} = 0.089 \), the variation in \( \alpha \frac{\partial p}{\partial t} \) is significant, although for small distances \( (r_{D} < 1,000) \), the variation in \( \alpha \frac{\partial p}{\partial t} \) is negligible. This behavior during transient flow can be expected by noting that during this flow period, the pseudopressure (Eq. 1 with \( \rho = p \)) may be expressed in terms of the exponential integral function. Thus, if the ratio \( (\partial p / \partial t_{D}) \) is big enough for the semilogarithmic approximation to be valid \( (\partial p / \partial t_{D} > 10) \), the derivative of the pseudopressure with respect to time would be independent of \( r \). For the case presented in Fig. 12 at \( r_{D} = 1,000 \) and \( t_{D, D} = 0.089 \), the ratio \( \partial p / \partial t_{D} = 17.89 \), which explains the constant behavior of \( \alpha \frac{\partial p}{\partial t} \) for \( r_{D} < 1,000 \). For the boundary-dominated flow period, the overall variation is less than 25%. Outside the skin zone, the maximum variation is less than 19%. The variation decreases with time.

The validity of the expressions derived above may also be proved by simultaneously solving Eqs. 54 and 56 to compute \( \bar{\rho} \) and \( S_o \) and comparing the results with the \( \bar{\rho} \) and \( S_o \) values obtained by solving known material-balance equations like

\[
\frac{S_o}{B_o} = \frac{5.615}{\phi A h} N_p + \left( \frac{S_o}{B_o} \right)_{i} \tag{63}
\]

and

\[
\frac{S_e + R_s S_o}{B_o} = - \frac{5.615}{\phi A h} G_p + \left( \frac{S_e + R_s S_o}{B_o} \right)_{i} \tag{64}
\]

where \( N_p \) and \( G_p \) = cumulative oil and gas productions respectively. Note that the system of equations 63 and 64 is independent of relative permeability data; also these equations are valid for all times.

The data required to solve Eqs. 54 and 56 simultaneously are an initial pair of \( \bar{\rho} \) and \( S_o \) values, relative permeability data, an estimate of PV, and the value of the function \( \alpha(r) \). Fig. 13 shows computations of average pressure and saturation for Data Set 2. The value of \( r_{D} \) is 100 and the skin factor is zero. The unbroken and dashed lines correspond to simulator values of \( S_o \) and \( \bar{\rho} \), respectively. These lines also represent the solution of Eqs. 63 and 64. The circular and square data points correspond to \( S_o \) and \( \bar{\rho} \).
respectively, both computed by solving Eqs. 54 and 56 simultaneously with the fourth-order Runge-Kutta procedure. Values of $p_{wf}$ and $\alpha_{wf}$ from simulation runs were used. The maximum relative differences in the values of $\bar{S}_g$ and $\bar{p}$ are 1.1 and 4.5%, respectively. Similar results are obtained in other cases. These results also indicate that the approximation given by Eq. 56 is reasonable. These computations suggest that it is possible to compute $\bar{p}$ and $\bar{S}_g$ from wellbore data (pressure and gas and oil rates).

Conclusions

In the past 20 years, it has been shown that decline-curve analysis involves the same basic principles as well-test analysis. Unfortunately, like well-test analysis, extensions to multiphase-flow conditions have been based on analogy and empiricism. Most disconcerting is that it is normally impossible to reproduce Arps' decline curves with numerical models, although these models reproduce the behavior of wells for other conditions very well. This fact suggests that an important gap exists in our understanding of the physical mechanisms that influence well behavior in solution-gas-drive systems. This study attempted to provide a general framework for studying well performance for multiphase flow in general and solution-gas drive in particular and fills an important gap by highlighting the basic assumptions in the use of Arps' equations. It is surprising that the basis for the Arps equations has not been examined until now. This rigorous examination permits us to evaluate the consequences of predictions based on the Arps equations and also provides us with an improved analysis procedure for predicting well performance by use of Arps' exponential decline curve. In addition, the role of various variables—such as well spacing, well condition, and fluid and formation properties (see Conclusions 1 and 2)—on performance predictions have been examined. Besides examining decline-curve analysis, a simple expression for predicting flow rates over short-time spans has been presented. The advantage of the method is that relative permeability and fluid-property data are not needed. This empirical procedure is particularly suited for predicting future IPR's.

1. The variable $d_i$ in Arps’ equations depends on the function $\lambda_i/\tilde{e}_i$. This function must be approximately constant with time to obtain an exponential decline in rate.
2. The function $\tilde{e}_i/\lambda_i$ must vary linearly with time for the decline exponent, $b_i$, in the Arps equations to be independent of time.
3. Exponential decline behavior can be used to approximate production rate if a pseudotime, $\bar{t}$, based on $c_i$ and $\lambda_i$, is used.
4. If transient data are used with Arps' decline curves, theoretical justification exists for the conclusion that $b$ will be a function of time and will take values greater than unity in most of the cases.
5. For constant-oil-rate production, the pressure distribution in the reservoir can be correlated as a function of distance and time.
6. The concept of pseudosteady-state can be extended to solution-gas-drive systems. The product $d\rho/dt$ is approximately independent of distance during boundary-dominated flow.

Nomenclature

- $A$ = drainage area, ft$^2$ [m$^2$]
- $b$ = decline exponent in Arps equations
- $B_p$ = gas FVF, RB/scf [res m$^3$/std m$^3$]
- $B_o$ = oil FVF, RB/STB [res m$^3$/stock-tank m$^3$]
- $c_T$ = total system compressibility at initial conditions, psi$^{-1}$ [kPa$^{-1}$]
- $\bar{c}_i$ = total system compressibility at average conditions (Eq. 8), psi$^{-1}$ [kPa$^{-1}$]
- $C_D$ = shape factor
- $\bar{d}_i$ = nominal rate at which decline takes place, parameter in Arps equations
- $D$ = dimensionless constant (Eq. 13)
- $G_p$ = cumulative gas production, Mscf [std m$^3$]
- $h$ = formation thickness, ft [m]
- $k_a$ = absolute permeability, md
- $k_{rg}$ = relative permeability to gas, fraction
- $k_{ro}$ = relative permeability to oil, fraction
- $n$ = exponent of backpressure curve
- $N_p$ = cumulative oil production, STB [stock-tank m$^3$]
- $p$ = pressure, psi [kPa]
- $P_i$ = initial pressure, psi [kPa]
- $p_{wf}$ = wellbore flowing pressure, psi [kPa]
- $\bar{p}$ = average reservoir pressure, psi [kPa]
- $q_{ID}$ = decline curve dimensionless rate defined in Ref. 3
- $q_g$ = gas flow rate, Mscf/D [std m$^3$/d]
- $q_o$ = oil flow rate, STB/D [stock-tank m$^3$/d]
- $q_{ID}$ = dimensionless rate (Eq. 48)
- $r$ = distance, ft [m]
- $r_D$ = dimensionless distance
- $r_e$ = external drainage radius, ft [m]
- $r_s$ = radius of altered-permeability zone, ft [m]
- $r_w$ = wellbore radius, ft [m]
- $r_i$ = radius where $p(r)=p$, ft [m]
- $R$ = producing GOR, scf/STB [std m$^3$/stock-tank m$^3$]
- $R_s$ = solution GOR, scf/STB [std m$^3$/stock-tank m$^3$]
- $s_R$ = mechanical skin factor
- $S_g$ = gas saturation
- $S_i$ = critical gas saturation
- $S_o$ = oil saturation
- $S_{or}$ = residual oil saturation
- $S_{wi}$ = initial and immobile water saturation
- $S_w$ = volumetric average of gas saturation
- $S_o$ = volumetric average of oil saturation
- $t$ = time, hours or days
- $\bar{t}$ = pseudotime corresponding to $\bar{t}_D$, hr-psi/cp
- $t^*$ = time at which rate is $q_{ID}$
- $t_{ID}$ = decline curve dimensionless time defined in Ref. 3
- $t_{AD}$ = dimensionless time based on $A$ and initial conditions
- $\bar{t}_{ID}$ = dimensionless time based on initial conditions (Eq. 11)
- $\bar{t}_D$ = dimensionless time (Eq. 10)
- $\bar{t}_D$ = dimensionless time (Eq. 6)
- $\bar{t}_p$ = pore volume, bbl [m$^3$]
- $X_D$ = correlating variable outside skin zone during boundary-dominated flow and constant-rate production (Eq. 47)
- $X_{ID}$ = correlating variable inside skin zone during boundary-dominated flow and constant-rate production (Eq. 46)
- $\alpha$ = function of pressure and saturation (Eq. 2)
- $\beta$ = function of pressure and saturation (Eq. 53)
- $\gamma$ = Euler’s constant, 0.57721...
- $\lambda_i$ = total mobility at initial conditions, cp$^{-1}$ [mPa·s]$^{-1}$
- $\bar{\lambda}_i$ = total mobility at average conditions (Eq. 7), cp$^{-1}$ [mPa·s]$^{-1}$
- $\mu_g$ = gas viscosity, cp [mPa·s]
- $\mu_o$ = oil viscosity, cp [mPa·s]
- $\phi$ = porosity

Subscripts

- $D$ = dimensionless
- $e$ = external
- $i$ = initial conditions
- $s$ = property of skin region or shut-in conditions
- $w$ = wellbore

Acknowledgment

Computing time was provided by the U. of Tulsa.

References


