Estimating Relative Permeability from Production Data: A Streamline Approach
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Abstract
One of the outstanding challenges in reservoir characterization is to build high-resolution reservoir models that satisfy static as well as dynamic data. Integration of dynamic data so far has mainly focused on estimating spatial distribution of absolute permeability. Among the various properties important for simulating reservoir behavior, the relative permeability curves may be far the most poorly determined by present methods. Estimation of relative permeability simultaneously with absolute permeability is a strongly nonlinear and ill-posed estimation problem.

In this paper we present a streamline-based approach for estimating relative permeabilities from production data. The streamline approach offers two principal advantages. First, we can analytically compute the sensitivity of the production response with respect to relative permeability parameters. The approach is extremely fast and requires a single streamline simulation run. Second, we can exploit the analogy between streamlines and seismic ray tracing to develop a formalism for efficient inversion of production data. Thus, estimation of relative permeabilities is carried out in two steps: (i) matching of breakthrough or first arrival times and (ii) matching of amplitudes of the production response.

For relative permeability representations we have used the commonly used power functions and also a more flexible representation through the use of B-splines. The relative advantages of these representations are examined through inversions of water-cut data from a nine-spot pattern. Finally, we address the underlying challenges associated with the simultaneous estimation of absolute and relative permeabilities from production data. We systematically investigate the non-uniqueness associated with the inverse problem and quantitatively evaluate the role of additional data such as pressure response in addition to water-cut history at the wells.

Introduction
During reservoir characterization typically relative permeabilities are represented by assumed known functions and only the spatial distribution of absolute permeability is estimated. Relative permeability functions are generally obtained from core analysis. However, only a few small core samples are taken from the reservoir. They can hardly be assumed to represent the entire reservoir. Differences in reservoir conditions and laboratory conditions are another source of uncertainty. The cores might also have changed properties such as wettability, depending on methods for coring, cleaning etc. In cases where cores are not available, the relative permeability functions might have to be obtained by analogy. That is relative permeability functions from another similar reservoir are used. Thus, during reservoir characterization the assumption that the relative permeabilities are known functions can be a major source of weakness.

Watson et al. and Lee and Seinfeld used automatic-history matching procedures for estimating absolute permeability and relative permeability functions. They represented relative permeabilities with power functions and estimated only the exponents in the power function representation. Studies on the estimation of relative permeability from coreflood experiments have shown that the inflexibility of the power function representation is a major obstacle in determining relative permeabilities from production response. Kerig and Watson and Watson et al. represented relative permeabilities with cubic splines and B-splines respectively to estimate relative permeability from coreflood data. Yang and Watson estimated relative permeability in 2-D reservoirs using B-splines. Their approach was based on a finite-difference simulator and an adjoint state method for gradient calculations. For field-scale applications involving very large number of parameters, such an approach can be complicated and computationally burdensome.

In this paper we present a streamline-based approach to estimate relative permeabilities from production data. Streamline models are becoming increasingly popular because of their ability to perform rapid flow simulations through high-resolution reservoir models. The key underlying concept here is decoupling of the flow (pressure) and transport (saturation...
and concentration) calculations during multiphase flow simulations through introduction of a time-of-flight coordinate. This converts the multidimensional conservation equation into a series of one-dimensional equations along streamlines that can be solved very efficiently. Further details on streamline simulation can be found in the review paper by King and Datta-Gupta.

The streamline approach offers unique advantage in estimating relative permeabilities from production data. First, we can analytically compute the sensitivity of the production response with respect to relative permeability parameters. Second, for estimating relative permeability parameters, we can exploit the analogy between streamlines and seismic ray tracing to develop a formalism for efficient inversion of production data. For relative permeability representations we have used the commonly used power functions and also a more flexible representation through B-splines. The relative advantages of these representations are examined through inversions of water-cut data from a nine-spot pattern. Finally, we address the underlying challenges associated with the simultaneous estimation of absolute and relative permeabilities from production data. We systematically investigate the non-uniqueness associated with the inverse problem and quantitatively evaluate the role of additional data such as pressure response in addition to water-cut history at the wells.

Background and Approach

In our previous works we discussed an analogy between streamlines and seismic ray tracing. Using this analogy, we developed a formalism for production data integration into high-resolution reservoir models using streamline simulators. A critical aspect of our earlier work was analytic computation of the sensitivity of the production response with respect to reservoir parameters, specifically, permeability and porosity. Here we adopt a similar approach for estimating relative permeabilities using production response.

The governing equations for multidimensional, multiphase incompressible flow, away from sources and sinks, can be written as,

\[ \nabla \cdot u_t = 0 \]  
(1)

\[ \phi \frac{\partial S_w}{\partial t} + u_t \cdot \nabla F_w = 0 \]  
(2)

where \( u_t \) is total velocity, \( \phi \) is porosity, \( S_w \) is water saturation and \( F_w \) is the water-cut or, fractional flow. Streamline simulators typically rely on a coordinate transformation through the use of the operator identity,

\[ u_t \cdot \nabla = \phi \frac{\partial}{\partial \tau} \]  
(3)

where \( \tau \) is the streamline time of flight or the travel time of a neutral tracer along streamlines. The multidimensional nonlinear convection equation is now reduced to a series of one-dimensional equations along streamlines,

\[ \frac{\partial S_w}{\partial t} + \frac{\partial F_w}{\partial \tau} = 0 \]  
(4)

Physically, we have moved to a coordinate system in which all streamlines are straight lines but the length of the domain, measured in units of \( \tau \), is now variable. If we assume uniform initial fluid saturation and constant injection flux, then we obtain the following self-similar solution in terms of \( \tau t \),

\[ \frac{dF_w(S_w)}{dS_w} = \frac{\tau}{t} \]  
(5)

where we have pointed out that the fractional flow is a single-valued function of water saturation. Thus, once we compute \( \tau \), Eq. 5 can be used to predict waterflood performance.

In the context of inverse modeling, we want to estimate reservoir properties using waterflood production history. Thus, we are interested in the sensitivity of the production response with respect to the streamline time of flight. Because the time of flight is a composite response, the sensitivity of the production response with respect to reservoir properties including relative permeabilities can be obtained using the chain rule of differentiation. Thus, our approach consists of the following major steps:

- **Analytical Computation of Sensitivity:** Using the streamline approach, we can derive analytic expressions for sensitivity of the time of flight with respect to relative permeabilities with a parametric and a more general non-parametric representation for the curves. The time of flight sensitivities can then be translated to water-cut sensitivities at the producing wells. This sensitivity computation involves simply evaluation of one-dimensional integrals along streamlines and requires a single simulation run.

- **Two-step Inversion:** Our approach to relative permeability estimation follows directly from seismic travel time tomography, and involves iterative linearization of the time of flight expression about a known initial model for relative permeability based on, for example, core analysis. Parameter estimation is performed in two steps. First the breakthrough or ‘first arrival’ times at all producing wells are matched. Second, the amplitudes of the water-cut response are matched at the wells.

- **Trade-off Analysis:** The non-uniqueness associated with the estimation problem is examined through the use of ‘trade-off contours’—a projection of the multidimensional objective function on to a plane defined by two varying parameters. This allows us to quantitatively evaluate the need and the role of additional data in reducing non-uniqueness during parameter estimation.
Relative Permeability Representation

We have used two different representations of relative permeabilities: parametric and non-parametric. The parametric representation assumes a priori functional form of the relative permeability curves but generally requires fewer parameters. The non-parametric representation is more versatile, does not require prior knowledge regarding the shapes of the curves but generally involves estimation of a larger number of parameters.

**Parametric Representation.** The most commonly used parametric representations for relative permeabilities are the power functions,

\[ k_{rw} = k_{rw}^o S_{nw}^{e_w}, \quad k_{ro} = k_{ro}^o S_{no}^{e_o}, \]

where \( k_{rw}^o \) is the endpoint relative permeability for water, \( S_{nw} \) is the normalized water saturation and \( e_w \) is a real number. In this representation, the unknown parameters are, \( k_{rw}^o, k_{ro}^o, e_w \) and \( e_o \). This relative permeability representation assumes a functional relationship between relative permeability and saturation. This means that if the true relative permeability curves do not follow this particular functional relationship, a significant bias error might be introduced in the estimation.

**Non-parametric Representation.** Here no assumptions are made regarding the shape or functional form of the relative permeability curves. Thus, the approach is more general and is particularly well suited for field applications. A convenient non-parametric representation of relative permeabilities can be made through the use of B-splines.\(^1\) B-spline expansions have been used to represent both relative permeability and capillary pressure functions, and is shown to successfully represent a diversity of functions.\(^12\)\(^{,}\)\(^19\)\(^,\)\(^20\) The B-spline representation for water relative permeability can be expressed as follows,

\[ k_r = \sum_{j=1}^{N} c_{w,j} B_{w,j}^o (S_{nw}), \]

where \( c_{w,j} \) is the jth B-spline coefficient for water and \( B_{w,j}^o (S_{nw}) \) is the jth B-spline of order \( r \) for water evaluated at the normalized water saturation, \( S_{nw} \). Similarly, oil relative permeabilities are represented as,

\[ k_o = \sum_{j=1}^{N} c_{o,j} B_{o,j}^o (S_{no}), \]

where \( c_{o,j} \) is the jth B-spline coefficient for oil and \( B_{o,j}^o (S_{no}) \) is the jth B-spline of order \( r \) for oil evaluated at the normalized oil saturation, \( S_{no} \).

B-splines are locally supported polynomials, which form the bases for polynomial splines. Choosing knot locations partitions the saturation range. The great advantage with a B-spline representation, compared to a power function representation is that any continuous function can be approximated arbitrarily well by polynomial splines, provided that sufficient number of knots are allowed.\(^1\) Unfortunately, increasing the dimension of the parameter space will cause the problem to be more ill-conditioned. An ill-conditioned problem has difficulties with stability, uniqueness and existence of solution. In the power function representation there are two parameters for each relative permeability curve. Watson et al. evaluated based on an error analysis that a B-spline expansion of dimension six is optimal for representing relative permeability in a core sample.\(^11\) For field-scale applications the relative permeability curves can take irregular shapes and consequently an even more flexible representation is necessary. The dimension of the B-spline space was therefore chosen to be eight for each relative permeability curve, and the corresponding B-spline representation is shown in Fig. 1.

**Analytic Sensitivity Calculations**

Our objective is to estimate absolute and relative permeabilities from production data, for example, water-cut history during a waterflood. In order to be able to estimate the parameters in either of the relative permeability representations, expressions for the sensitivity of the water-cut with respect to the parameters must be developed. Because we are using a streamline simulator as a forward model, parameter sensitivities can be formulated as one-dimensional integrals of analytic functions along streamlines. The computation of sensitivity for all model parameters then requires just a single streamline run.\(^5\)\(^,\)\(^15\)

The fundamental quantity in streamline simulation is the streamline time of flight defined as,

\[ \tau = \int_{\Sigma} s(x) dx, \quad (10) \]

where \( \Sigma \) defines the streamline and \( s(x) \) is the ‘slowness’, which is the reciprocal of velocity,

\[ s(x) = \frac{1}{|\nabla|} = \frac{\psi (x)}{\alpha r k (x) |\nabla P|}. \quad (11) \]

In Eq. 11, \( \psi \) is porosity, \( \alpha r \) is total mobility, \( k \) is absolute permeability and \( \nabla P \) is the pressure gradient. We can see that relative permeability parameters enter into the slowness through the total mobility term. In order to estimate the parameters of relative permeability, we must evaluate the change in slowness resulting from changes in relative permeabilities. The change in slowness in turn can be related to water-cut or fractional flow variations.

A perturbation in reservoir properties results in perturbations in slowness as follows,
\[ s(x) = s^0(x) + \delta s(x), \quad (12) \]

where \( s^0(x) \) is the slowness with the initial reservoir parameters and \( \delta s(x) \) is the change in slowness at location \( x \) due to the perturbation in reservoir properties. The corresponding change in water-cut or fractional flow can be expressed as,

\[ F_w(x,t) = F_w^0(x,t) + \delta F_w(x,t), \quad (13) \]

where \( F_w^0(x,t) \) is the fractional flow with the original set of reservoir parameters and \( \delta F_w(x,t) \) is the change in fractional flow at location \( x \) and time \( t \) due to the change in reservoir parameters.

The change in slowness with respect to total mobility is given as,

\[ \frac{\partial s(x)}{\partial \lambda_t} = -\frac{s(x)}{\lambda_t}. \quad (14) \]

We can relate this change to variation in relative permeability parameters through the use of the chain rule of differentiation. For example, for water relative permeability curves,

\[ \frac{\partial s(x)}{\partial k_w^o} = \frac{\partial s(x)}{\partial \lambda_o} \frac{\partial \lambda_o}{\partial k_w^o} = -\frac{s(x)}{\lambda_o} S_m^w \mu_w, \quad (15) \]

\[ \frac{\partial s(x)}{\partial \mu_w} = \frac{\partial s(x)}{\partial \lambda_o} \frac{\partial \lambda_o}{\partial \mu_w} = -\frac{s(x)}{\lambda_o} S_m^w \mu_w \ln S_m^w, \quad (16) \]

where \( \mu_w \) is viscosity of water. The change in slowness with respect to the power function parameters for oil has equivalent expressions, and is therefore not given.

For the B-spline representation of relative permeabilities, we need to relate the change in slowness to the change in B-spline parameters. This can be expressed as,

\[ \frac{\partial s(x)}{\partial c_{i,j}} = \frac{\partial s(x)}{\partial \lambda_n} \frac{\partial \lambda_n}{\partial c_{i,j}} = -\frac{s(x)}{\lambda_n} B_i^n(S_m) \lambda_n \mu_w. \quad (17) \]

The change in slowness with respect to the B-spline parameters for oil has an equivalent expression, and is therefore not given.

Finally, perturbation in the fractional flow or water-cut can be related to changes in the time of flight as follows,

\[ \delta F_w = \frac{\partial F_w}{\partial \tau} \delta \tau. \quad (18a) \]

If we assume that the streamlines do not shift because of small changes in reservoir properties, then we have

\[ \delta \tau = \int_\Sigma \delta s(x) dx \quad (18b) \]

along the same trajectory \( \Sigma \). The water-cut sensitivities with respect to relative permeability parameters can now be obtained by integrating Eq. 18a over all streamlines reaching the producing wells.

**Regularization**

Inverse problems in reservoir characterization are ill-posed. This means that slight perturbations in the data may create large oscillations in the solution. In general, field production data can be noisy because of measurement errors. These errors may cause oscillations in the relative permeability estimation. Regularization is necessary to obtain physically meaningful relative permeability functions. Regularization typically entails augmenting the data-misfit with additional terms to ensure stability of the solution.\(^{21-23}\) This is particularly important for the B-spline representation of relative permeabilities because no prior assumption is made regarding the shape of the curves.

In our approach, we are using the sum of the 0th order derivative term and the second derivative term. The 0th order term will make sure that we are honoring the prior information, which can be laboratory derived relative permeability curves. The second derivative term favors solutions that are smooth. We are minimizing the second derivative of the change in estimate between two successive iterations. The second difference operator matrix, \( D \), is utilized for this purpose. The \( D \)-matrix is easily found by considering the B-splines as third order polynomials,

\[ B_j = a_{i,j} S_m^2 + b_{i,j} S_m + c_{i,j}, \quad (19) \]

where \( i \) is the interval number and \( j \) is the spline number. The second derivative of a B-spline can now be expressed as

\[ \frac{d^2}{dS_m^2} (a_{i,j} S_m^2 + b_{i,j} S_m + c_{i,j}) = 2a_{i,j} \quad (20) \]

The \( D \)-matrix is a \((N-2)\) times \(2N\) band matrix with six non-zero elements in each row. \( N \) is the dimension of the B-spline expansion used to represent the relative permeability curves. The dimension \((N-2)\) is the number of intervals in the partition of the saturation range. Each B-spline has got a mathematical expression for each interval. Third order B-splines are used, and therefore there are only three B-splines having support in each interval for each relative permeability curve. This is the reason for the band structure of the matrix. The \( D \)-matrix contains the second difference operator for both the oil and the water relative permeability. The first \( N \) columns is for the oil relative permeability, and is given as,

\[
\begin{bmatrix}
0 & \cdots & 0 & 2a_{N-2,N-2} & 2a_{N-2,N-1} & 2a_{N-2,N} \\
2a_{N-3,N-3} & 2a_{N-3,N-2} & 2a_{N-3,N-1} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 2a_{2,2} & 2a_{2,3} & 2a_{2,4} & \cdots \\
2a_{1,1} & 2a_{1,2} & 2a_{1,3} & 0 & \cdots & 0
\end{bmatrix}
\]

The last \( N \) columns of the \( D \) matrix is for the water relative permeability, and is given as,
\[ 2a_{1,1} \quad 2a_{1,2} \quad 2a_{1,3} \quad 0 \quad \ldots \quad 0 \\
0 \quad 2a_{2,2} \quad 2a_{2,3} \quad 2a_{2,4} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
2a_{N-3,N-3} \quad 2a_{N-3,N-2} \quad 2a_{N-2,N-1} \quad 0 \\
0 \quad \ldots \quad 0 \quad 2a_{N-2,N-2} \quad 2a_{N-2,N-1} \quad 2a_{N-2,N-2} \quad 2a_{N-2,N-2} \quad 2a_{N-2,N-2} \quad 2a_{N-2,N-2} \quad \cdots \]

Solutions having small values of \( \| Dy \| \) are desirable. \( \bar{Y} \) is the parameter vector containing the relative permeability parameters.

In Fig. 2 we can see three sets of relative permeability curves. The solid lines represent the true relative permeabilities. The dashed lines represent the estimated relative permeability curves when using only the 0th order term in the stabilizing functional. We can see that the dashed curves contain oscillations. The dotted curves represent the estimate obtained when using both the 0th order and the second order derivative terms. We can see that the oscillations are removed by adding the second order derivative term to the stabilizing functional.

**Inversion**

Estimating relative permeabilities from production data entails the minimization of a penalized misfit functional as follows,

\[ J = \left\| d - S \partial Y \right\|^2 + w_1 \left\| \partial Y \right\|^2 + w_2 \left\| D \partial Y \right\|^2, \quad (21) \]

where \( d \) is the data vector, \( S \) is the sensitivity matrix, \( Y \) is the parameter vector, \( D \) is the 2nd order differential operator and \( w_1 \) and \( w_2 \) are weighting factors.

The first term on the right hand side is called the data misfit, and is expressed as follows,

\[ \left\| d - S \partial Y \right\|^2 = \sum_{i=1}^{M} \left( \partial Y_i - \sum_{j=1}^{N} S_{ij} \partial Y_j \right)^2. \quad (22) \]

This term penalizes deviations between observed and calculated water-cut or pressure.

The second term on the right hand side of Eq. 21 is the 0th order derivative in the stabilizing functional, called the prior knowledge term, and is expressed as follows,

\[ \left\| \partial Y \right\|^2 = \sum_{j=1}^{N} (\partial Y_j)^2. \quad (23) \]

This term penalizes deviations in parameter values from the initial values. This is a method of including prior knowledge, which in the case of relative permeability parameters can be relative permeability curves obtained from for example core analysis. For absolute permeability, the prior knowledge might be a reservoir model developed by geostatistical methods.

The third term on the right hand side of Eq. 21 is the second order derivative in the stabilizing functional, called the smoothing term, and is expressed as follows,

\[ \left\| D \partial Y \right\|^2 = \sum_{j=1}^{N} (\nabla \partial Y_j)^2. \quad (24) \]

This term of the objective function penalizes change in sign of the gradient of the parameter space. That is, roughness in the model is penalized.

A solution to the regularized inverse problem is obtained by solving the following augmented linear system of equations,

\[ \begin{bmatrix} S & I \end{bmatrix} \begin{bmatrix} \delta Y \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}, \quad (25) \]

where \( S \) is the sensitivity matrix, \( w_1 \) and \( w_2 \) are weighting parameters, \( I \) is the identity matrix, \( D \) is the second difference operator, and \( \bar{E} \) is the data misfit. An iterative sparse matrix solver, LSQR, is utilized to solve this augmented system efficiently.24

**Results**

In this section we discuss relative permeability estimation using water-cut data from a homogenous and a heterogeneous 9-spot pattern. First, we examine the ill-posed nature of the inverse problem by a visual examination of the objective function during minimization. This provides us with valuable insight into the non-uniqueness in the estimation problem. Second, we investigate the relative merits of the power function and B-spline representation of relative permeabilities through a series of inversion of water-cut data. Finally, the difficulties associated with the simultaneous estimation of absolute and relative permeabilities are examined and the role of additional information, for example, steady state pressure data, is quantified.

**Ill-posedness.** Before obtaining solutions to the parameter estimation problem, we first examine the nature of ill-posedness through a visual examination of the objective function during inverse modeling. Typically, we have a large number of parameters and thus, the objective function can not be visualized in its entirety. However, we can examine projections of the objective function on to planes defined by two varying parameters. The parameters can be relative permeability parameters, absolute permeability parameters or a combination. We call these projections trade-off contours.

The projections of the objective function tend to have long valleys instead of one distinct minimum.25 This means that a large number of combinations of the parameters give the same low value of the objective function. All these combinations of the parameters are therefore solutions to the problem. This means that we have problem of non-uniqueness of the solution. The non-uniqueness problem is particularly important when only water-cut data is used. We can see that the length of the valleys in the objective function decreases when prior information and smoothing are included.

We will illustrate these concepts through the use of a synthetic example that involves estimation of relative permeability parameters based on water-cut data from a 9-spot pattern with a central injector and 8 producing wells. The well configuration is shown in Fig. 3. The water-cut response is
recorded at 61 different times from the start of the waterflood up to 900 days in each of the 8 producing wells.

Figure 4 shows a projection of the objective function using the power function representation of relative permeability. We systematically vary the end point oil relative permeability and the oil relative permeability exponent over a large range and compute the objective function for each combination of parameters to generate this trade-off contour. For this figure, only the data misfit term, Eq. 22, is used in the objective function. A homogeneous permeability field is used. We can see that there is a long valley in the objective function. This means that as a result of the inversion the endpoint relative permeability of oil could take any value between 0.2 and 1.1, and the exponent of oil could take any value between 1.25 and 3.0.

In Fig. 5, regularization is added to the objective function used in Fig. 4. We assume that we have some knowledge of relative permeabilities, for example, from laboratory experiments. The length of the valley is now reduced. This reduces the number of possible combinations of the two parameters, endpoint relative permeability of oil and exponent of oil. On top of the trade-off contour is shown the result of 20 inversions. The inversion results are indicated as crosses and show a considerable spread around the minimum of the objective function.

Figure 6 shows the same case as Fig. 4, but now with the heterogeneous permeability field shown in Fig. 3. We can see that with a heterogeneous permeability field, the estimation problem becomes even more ill-posed. The endpoint relative permeability of oil can now take any value between 0.17 and 1.1, and the exponent of oil can take values between 1.07 and 3.0.

In Fig. 7 prior knowledge of the relative permeability parameters is added to the objective function from Fig. 6. We can see that this reduces the ill-posedness of the estimation problem. On top of the trade-off contour we have shown results from 20 inversions. We can see that the results are spread out along the long valley in the objective function.

Figure 8 shows the trade-off contour made by varying two B-spline coefficients in a 8 dimensional B-spline expansion of water. The objective function consists of only the data misfit term, which is the sum of the squared differences between observed and calculated water-cut responses. In this case a homogeneous permeability field is used. We can see multiple long valleys in the objective function. In addition to long valleys, we can see local minima. Local minima are isolated sets of relative permeability parameters, which fit the water-cut data.

In Fig. 9, regularization is added to the objective function, and the multiple long valleys are reduced to one region. Estimates obtained from 20 inversions are superimposed on the trade-off contour as crosses. Unlike for the power function representation, we can see that these 20 estimates are clustered around the minimum of the objective function. With the regularization included, the parameter estimation problem seems more well-posed. Furthermore, the parameters in the B-spline representation appear to be better constrained than those of the power function representation.

Figure 10 shows the same case as Fig. 8, but the heterogeneous permeability field from Fig. 3 is used. In this case, we can see multiple long valleys in the objective function, together with some local minima. This means that it is impossible to get reliable estimates of relative permeability parameters based on only water-cut information.

In Fig. 11, prior knowledge and smoothing are added to the objective function from Fig. 10. We can again see that including prior information to the objective function reduces the ill-posedness of the problem. On top of the trade-off contour, we can see results from 20 inversions. The inversion results are clustered around the minimum in the objective function.

Relative Permeability Estimation. In this section, some synthetic examples of relative permeability estimation will be presented. A set of relative permeability parameters is selected as the true parameter set both for the power function and the B-spline representation of relative permeability. The selected parameter values for power functions are shown in Table 1 and for B-splines in Table 2. The forward model is run with the true relative permeability parameters as input in order to generate synthetic water-cut responses in the eight producing wells of a 9-spot pattern. Then, these synthetic water-cut responses are used as data for the inversion, and the relative permeability parameters are estimated. The results are then compared to the known true set of parameters.

In Fig. 12, the relative permeability curves are represented by power functions. A homogeneous permeability field is used. The true relative permeability curves are shown as black solid curves. The ten other relative permeability curves shown are results from inversions with different initial guesses for the relative permeability parameters. We can see that there is a substantial range of endpoint relative permeability values for water in the inversion results. The water-cut responses are most sensitive to the endpoint relative permeability of water causing it to change in order to make up for other parameters differing from their true values. This will be avoided by a B-spline representation of relative permeability because of the B-splines’ local support. This means that changing the endpoint relative permeability of water will change the relative permeability curve only in the region of support for the last B-spline.

Figure 13 shows the results of inversions with a B-spline representation of relative permeability for the water-cut data used in Fig. 12. We can see that the results from the inversions are much closer when using a B-spline representation of relative permeability than when using a power function representation.

In Fig. 14, results from inversions using a power function representation of relative permeability are again shown, but in this case the heterogeneous permeability field shown in Fig. 3 is used. We can see that the spread in the estimates is significantly larger in this case compared to the homogeneous permeability case. This shows that in field situations when we
are dealing with highly heterogeneous permeability fields, representing relative permeability with power functions can lead to substantial errors in the reservoir model.

Figure 15 shows estimates of relative permeabilities obtained by inversions with a B-spline representation of relative permeability and a heterogeneous permeability field. Even with a heterogeneous permeability field, there are hardly any spread in the estimates of relative permeability obtained using a B-spline representation.

Figure 16 shows a different synthetic example of B-spline representation of relative permeability with a homogeneous permeability field. The true relative permeability curves in this example change curvature between concave and convex at a few saturation values. This is possible in field simulations when the relative permeabilities are pseudo functions. That is, the relative permeabilities are not purely functions of saturation. They are affected by other factors, like the degree of heterogeneity etc. In this case, the true relative permeability curves are not smooth power functions. This means that with a power function representation of relative permeability, we would be unsuccessful in estimating these relative permeability curves. We can see that with the B-spline representation of relative permeability, the ten estimates obtained by inversions are almost identical to each other and the black dashed relative permeability curves representing the true relative permeabilities.

Simultaneous Estimation of Absolute and Relative Permeability. In this section we study the inverse problem posed by the simultaneous estimation of absolute and relative permeability. This is a far more challenging estimation problem compared to the estimation of either spatial distribution of absolute permeability or relative permeability parameters individually. Absolute and relative permeabilities appear as a product in the flow equations. It is therefore difficult to isolate the effect of each component. We show that water-cut data together with regularization of absolute and relative permeability parameters are not sufficient to derive reliable estimates of both absolute and relative permeabilities. This becomes apparent when we examine trade-off contours where the relative permeability parameters are varied together with the absolute permeability.

In generating these trade-off contours we have used a 21x21x1 grid for permeability. Again, we are using water-cut data from a 9-spot pattern. This includes water-cut responses recorded in 8 producing wells at 61 times from the start of a water-flood up to 900 days. In addition, we will examine the role of additional data, for example, steady pressure in 6 observing wells and prior knowledge of parameters. The prior knowledge of relative permeability parameters might be from laboratory experiments and the prior knowledge of absolute permeability might be a model generated by geostatistical methods.

For generating the trade-off contours, we have varied the absolute permeability of grid block (6,6,1) together with the 6th B-spline coefficient for water. The true value of permeability for block (6,6,1) is given in Table 3 both for a homogeneous case and also for the heterogeneous permeability field shown in Fig. 3.

Figure 17 shows a projection of an objective function consisting of only the misfit between observed and calculated water-cut responses. In this case a homogeneous permeability field is used. We can see that there is a long valley in the objective function indicating the highly ill-posed nature of the problem. The absolute permeability of block (6,6,1) can take any value between 0.01 mD and 20.0 mD. The 6th B-spline coefficient of water relative permeability can take any value between 0.27 and 0.63.

In Fig. 18, prior knowledge of both the absolute and relative permeability parameters is added to the objective function from Fig. 17. We can see that this constrains the absolute permeability. With the addition of regularization, the absolute permeability of block (6,6,1) can take values between 3.0 mD and 8.0 mD. The 6th B-spline coefficient of water can take values between 0.4 and 0.55. The minimum is now much better defined.

Figure 19 shows a projection of an objective function consisting of the misfit between observed and calculated water-cut responses and the misfit between observed and calculated steady pressures. This is also for a homogeneous permeability field. We can see that the pressure information is valuable in terms of constraining the absolute permeability. We can recall from Fig. 17 that with only the water-cut information, absolute permeability of block (6,6,1) can take any value between 0.01 mD and 20.0 mD. With pressure information included, the absolute permeability of block (6,6,1) can only take values in the range 5.0 mD to 7.5 mD.

In Fig. 20 prior knowledge of both the absolute and relative permeability parameters and smoothing is added to the objective function from Fig. 19. We can now see that we have a well-defined minimum in the objective function. This means that simultaneous estimation of absolute and relative permeabilities using water-cut and pressure-information together with regularization should give reliable estimates.

Figure 21 shows a projection generated by using the heterogeneous permeability field shown in Fig. 3. The objective function consists of the misfit between observed and calculated water-cut responses. We can again see that using only water-cut data does not constrain the absolute permeability. The absolute permeability of block (6,6,1) can take any value between 0.01 mD and 20.0 mD.

In Fig. 22 prior knowledge of the absolute and relative permeability parameters and smoothing are added to the objective function from Fig. 21. We can see that the range of possible values for the 6th B-spline coefficient is lowered. However the ill-posedness of the estimation problem is only slightly reduced.

Figure 23 shows a projection of an objective function consisting of the misfit between calculated and observed water-cut responses and calculated and observed steady pressures. This is also for a heterogeneous permeability field. We can see that the addition of steady pressure data significantly constrains the absolute permeability. Now the absolute permeability of block (6,6,1) can take values in the
range 9.0 mD to 12.5 mD while previously the absolute permeability of block (6,6,1) could take any value between 0.01 mD and 20.0 mD.

In Fig. 24 prior knowledge of the absolute and relative permeability parameters is added to the objective function from Fig. 23. We can now see that there is a well-defined minimum. This means that by use of pressure and multiphase production history together with regularization, it is possible to simultaneously obtain reliable estimates of both absolute and relative permeability parameters, even in the presence of spatial variation in permeability.

Conclusions
1. We have presented analytic expressions for the sensitivity of the production response with respect to relative permeability parameters using streamline simulators. The sensitivity computation is extremely efficient and requires a single streamline simulation.
2. We have developed a formalism for estimating relative permeabilities from production data. Both power function and an alternative B-spline representation for relative permeabilities have been considered.
3. B-spline representations of relative permeability are more flexible than power function representations leading to a reduction in bias error.
4. The local support of the B-splines makes it easier to identify the relative permeability parameters in a B-spline representation than in a power function representation.
5. During simultaneous estimation of absolute permeability and relative permeability, the main source of non-uniqueness is the non-linear trade-off between parameters. Another source of non-uniqueness is the presence of isolated and distinct local minima.
6. Both the estimation of relative permeability and the simultaneous estimation of absolute and relative permeability are strongly ill-posed when only production data is used. This might cause meaningless solutions.
7. Prior knowledge reduces the non-uniqueness of both the estimation problems.
8. Pressure information is necessary in addition to water-cut and regularization in order to simultaneously estimate absolute and relative permeabilities.

Nomenclature
\[ k_{rw} = \text{water relative permeability} \]
\[ k_{ro} = \text{endpoint water relative permeability} \]
\[ N = \text{B-spline expansion dimension} \]
\[ M = \text{number of data} \]
\[ p = \text{pressure} \]
\[ r = \text{order of B-spline} \]
\[ s = \text{slowness} \]
\[ S = \text{sensitivity matrix} \]
\[ S_{nn} = \text{normalized oil saturation} \]
\[ S_{nw} = \text{normalized water saturation} \]
\[ S_w = \text{water saturation} \]
\[ t = \text{time} \]
\[ u_t = \text{total Darcy velocity} \]
\[ v = \text{interstitial velocity} \]
\[ w_1 = \text{weighting factor} \]
\[ w_2 = \text{weighting factor} \]
\[ x = \text{location} \]
\[ Y = \text{parameter vector} \]
\[ e = \text{data misfit} \]
\[ \phi = \text{porosity} \]
\[ \lambda_t = \text{total mobility} \]
\[ \tau = \text{time of flight} \]
\[ \mu = \text{viscosity} \]
\[ \mu_o = \text{water viscosity} \]

Acknowledgements
We would like to acknowledge the financial support of the industrial sponsors of the Joint Industry Project on integrated reservoir description and partial financial support from the Norwegian Research Council.

References
Table 1. True Parameters for Power Function Representation

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Table 2. True Parameters for B-spline Representation

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Table 3. True Absolute Permeability

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<td>(6,6,1)</td>
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Fig. 6 Trade-off contour using water-cut, power function representation and heterogeneous permeability field.

Fig. 7 Trade-off contour using water-cut and regularization, power function representation and heterogeneous permeability field. The crosses indicate inversion results.

Fig. 8 Trade-off contour using water-cut, B-spline representation and homogeneous permeability field.

Fig. 9 Trade-off contour using water-cut and regularization, B-spline representation and homogeneous permeability field. The crosses indicate inversion results.

Fig. 10 Trade-off contour using water-cut, B-spline representation and heterogeneous permeability field.

Fig. 11 Trade-off contour using water-cut and regularization, B-spline representation and heterogeneous permeability field. The crosses indicate inversion results.
Fig. 12 True and estimated relative permeability curves, power function representation, homogeneous permeability field.

Fig. 13 True and estimated relative permeability curves, B-spline representation, homogeneous permeability field.

Fig. 14 True and estimated relative permeability curves, power function representation, heterogeneous permeability field.

Fig. 15 True and estimated relative permeability curves, B-spline representation, heterogeneous permeability field.

Fig. 16 True and estimated relative permeability curves, B-spline representation, homogeneous permeability field.

Fig. 17 Trade-off contour using water-cut, homogeneous permeability field.
Fig. 18 Trade-off contour using water-cut and regularization, homogeneous permeability field.

Fig. 19 Trade-off contour using water-cut and steady pressures, homogeneous permeability field.

Fig. 20 Trade-off contour using water-cut, steady pressures and regularization, homogeneous permeability field.

Fig. 21 Trade-off contour using water-cut, heterogeneous permeability field.

Fig. 22 Trade-off contour using water-cut and regularization, heterogeneous permeability field.

Fig. 23 Trade-off contour using water-cut and steady pressures, heterogeneous permeability field.
Fig. 24 Trade-off contour using water-cut, steady pressures and regularization, heterogeneous permeability field.