Volume equalized constitutive equations for foamed polymer solutions

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Synopsis

In order to develop a constitutive equation for foamed polymer solutions their flow behavior was investigated in a large-scale vertical tube. Starting from theoretical considerations, two new constitutive equations are proposed. A new variable—the specific volume expansion ratio—is introduced to characterize the relative gas content of the foam. With the new variable the proposed constitutive equations take simple forms and obey the principle of volume equalizing, i.e., the corresponding Reynolds numbers possess a certain invariance property. The proposed models are fitted to the experimental data. The principle of volume equalizing is checked in light of the current knowledge on foam flow behavior.

I. INTRODUCTION

Large volumes of aqueous Hydroxypropylguar (HPG) solutions containing a small amount of foaming agent and foamed by nitrogen and/or carbon dioxide gases are used widely in the petroleum industry to fracture formations hydraulically and to transport "proppant" to fill and keep the fracture open. The design and monitoring of such an operation requires the knowledge of the rheological behavior of the foamed fluid. Typically, the actual void fraction of the gas (quality) is about 0.7 in fracturing jobs, and the fluid is termed "foam." In this work we use this terminology, although it is in order to mention that in most of the theoretical investigations the term foam is used for higher qualities, i.e., for the case of closely packed bubbles, considerably deforming during flow.

One of the most important properties of a foam is the relative gas content which can be characterized by several variables. The volume
expansion ratio, $e_m$ is the ratio of the sum of gas and liquid volumes to the liquid volume. This variable is seldom used in correlations since it goes to infinity in the case of dry foams. The variable preferred by most authors is the quality, $I$, defined as the ratio of the volume of gas to the sum of gas and liquid volumes. The quality is between zero and unity. A frequent implicit assumption when using quality-related correlations is that the pressure and hence the density of the gas phase has no direct effect on the rheological behavior.

Heller and Kuntamukkula (1987) reviewed the many efforts made to overcome the basic difficulties in applying traditional rheological concepts and measuring methods to flowing foams: (i) the structural features of the foam cannot often be neglected relative to the characteristic dimension of the viscometer device; (ii) foams are compressible and the changes induced by pressure variation are difficult to consider when evaluating measurements. These authors question most of the reported data and state that the wall stress is controlled by the thickness of a liquid film laid down by the foam along the pipe walls or against the rotating parts of the viscometer. The concept of liquid slip layer was also used by Calvert and Nezhati (1986), and found to have a larger impact on the flow rates than the "bulk rheology" of the foam which seemed to be independent of the foam quality. More than a decade ago David and Marsden (1967) arrived at a similar conclusion after correcting their measurements for compressibility and slip.

While the concept of slippage might be successful in explaining controversial data from different rheometer geometries, the predictive value of such models is very limited. The basic problem is, of course, that of finding a correlation for the slip layer thickness or the apparent slip velocity in terms of measurable macroscopic quantities. If the wall stress is to be calculated from the film thickness and film viscosity, polymer-solution liquids introduce an additional difficulty: the viscosity inside the liquid film might differ from that of the base liquid considerably because of the depletion of polymer molecules, as discussed, e.g., by Müller-Mohnsen et al. (1990).

A more design-oriented line in the literature has been focused on constructing a single medium constitutive equation for foams with the assumption of negligible slip in industrially interesting scales. In the following we will restrict our attention to this approach. Experimental work reported on aqueous foams by Blauer et al. (1974) indicated that the apparent viscosity in a pipe increases with increasing foam quality and foams of a given quality are rate thinning. This led the authors to postulate a Bingham plastic behavior with quality-dependent yield
stress and plastic viscosity. In a detailed investigation of water and HPG-solution foams Reidenbach et al. (1986) applied a Herschel-Bulkley-type equation

$$\dot{\gamma} = 0, \quad |\tau| < \tau_y, \quad (1a)$$

$$\tau = \left( \frac{\tau_y}{|\dot{\gamma}|} + K |\dot{\gamma}|^{n-1} \right) |\dot{\gamma}|, \quad |\tau| > \tau_y. \quad (1b)$$

In order to incorporate the non-Newtonian behavior of the base liquid into the constitutive equation those authors postulated that the flow behavior index $n$ was identical to that of the base liquid and the yield stress and the consistency index were quality dependent. For the yield stress the correlation was found to be independent of the fluid and the gas and was written as

$$\tau_y = C_2 \Gamma, \quad \Gamma < 0.6 \quad (2a)$$

$$\tau_y = C_3 \exp(9 \Gamma), \quad \Gamma > 0.6, \quad (2b)$$

but the consistency index was found to depend on fluid properties:

$$K = K_I \exp(C_1 \Gamma + 0.75 \Gamma^2), \quad (3)$$

where $K_I$ is the consistency index of the fluid and $C_1$ is also fluid dependent. The correlations were found to be only slightly different for nitrogen and carbon dioxide gas phases. To establish the temperature dependence of a model of such complexity is far from trivial. In such an attempt the concept of using the liquid flow behavior index was abandoned by Harris and Reidenbach (1987).

In a series of more theoretical investigations the monodisperse two-dimensional model of Princen (1983) and Prud’homme (1981) has been improved in order to relate rheological parameters to bubble size, interfacial tension, and foam quality. Khan and Armstrong (1986) and Kraynik and Hansen (1986) arrived at the conclusion that very dry foams are essentially Bingham plastics with a yield stress dominating the apparent viscosity except at extremely high shear rates. Schwartz and Princen (1987) disagree with this result and conclude that, at least for “slow fluid velocity,” the apparent viscosity exceeds that of the liquid according to the equation:

$$\eta_a = \left( \frac{\sigma}{\eta_l \dot{\gamma}} \right)^{1/3}, \quad (4)$$
where \( \sigma, \eta, \alpha, \) and \( \dot{\gamma} \) are interfacial tension, liquid viscosity, characteristic bubble size, and shear rate, respectively. Although these theories, built on a two-dimensional model, provide a considerable insight into the reasons of the peculiar behavior of dry foams, they are still far from providing constitutive equations of practical use.

In summary, large-scale experiments are necessary to add new information to the debate on the applicability of the single medium approach. The compressibility of the foam should be accounted for in two respects: first, when computing the correct superficial velocities and, second, not overlooking the additional effect on rheology of the changing quality even in a single experiment.

II. NON-NEWTONIAN FLOW WITH APPRECIABLE CHANGES OF DENSITY

The steady-flow momentum equation for one-dimensional flow in an element with cross-sectional area \( A \), perimeter \( P \), and length \( \partial l \) is (Massey, 1989):

\[
\frac{\partial p}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\tau_0}{\frac{1}{2} \rho u^2} - \frac{1}{A/P} \left( \frac{1}{2} \rho u^2 \sin \alpha \right) \partial l,
\]

where \( \rho, \rho, u, \tau_0, g, \alpha \) are pressure, density, superficial velocity, wall stress, acceleration of gravity, and angle of flow measured from the horizontal direction, respectively. The term

\[
f = \frac{\tau_0}{\frac{1}{2} \rho u^2}
\]

is the friction factor and is determined by the constitutive equation of the fluid in addition to superficial velocity, density, and geometry.

For a given incompressible fluid flowing in a pipe of constant cross section at a fixed temperature, the friction factor is fixed in any flow regime, at a given mass flowrate, whatever the constitutive equation relating shear stress to shear rate is.

If the flowing medium is compressible but Newtonian, the friction factor is still constant, since the Reynolds number determining the friction factor contains the product \( \rho u \), being invariant in a pipe of constant cross-section even if \( u \) and \( \rho \) changes. A more sophisticated two-
dimensional description of compressible flow has been suggested by Prud’homme et al. (1986), but only for ideal gas and negligible effect of gravity.

In the following we turn our attention to laminar flow of any (Newtonian or non-Newtonian) compressible medium at constant temperature in a pipe of constant cross section. The analysis is based on the friction factor concept represented by Eq. (5). A constitutive equation providing constant friction factor will be called volume equalized. In other words, a volume equalized constitutive equation has the following property: the wall stress is a homogeneous function of density of order $-1$ if temperature, mass flow rate, and cross section are fixed. Let superscript asterisk (*) denote a reference state of the flowing medium. If $p=p^*/\epsilon$, where $\epsilon$ is a positive constant, then from the continuity relationship $u=u^*\epsilon$ and the principle of volume equalizing requires $\tau_0 = \tau^*_0\epsilon$. For different constitutive equations we substitute this requirement into the expression of wall stress and find the conditions which the model parameters should obey to fulfill the principle of volume equalizing. If the above relationships are substituted into the expression of wall shear stress, for a Newtonian fluid we have

$$\tau^*_0 = \eta \left( \frac{8u^*\epsilon}{D} \right),$$

and hence the constitutive equation is volume equalized if and only if the viscosity is constant.

For a power law compressible medium with constant flow behavior index $n$ we have

$$\tau^*_0 \epsilon = k(\epsilon) \left( \frac{6n+2}{nD} \right) (u^*\epsilon)^n,$$

and therefore the principle of volume equalizing requires

$$k(\epsilon) = k^*\epsilon^{1-n}.$$

For a Bingham plastic compressible medium with constant plastic viscosity $\eta_p$ the Buckingham-Reiner equation can be written as

$$u^*\epsilon = \frac{D}{2\eta_p(\tau^*_0)^3\epsilon^3} \left[ \frac{(\tau^*_0)^4\epsilon^4}{4} - \frac{(\tau^*_0)^3\epsilon^3}{3} r_p(\epsilon) + \frac{(r_p(\epsilon))^4}{12} \right],$$

from which we have the following condition for the yield stress:
The constitutive equations, which have been proposed in the literature to describe foam behavior, do not satisfy the principle of volume equalizing.

III. VOLUME EQUALIZED CONSTITUTIVE EQUATIONS FOR FOAMS

Since the reference state is arbitrary, in the case of foams it is convenient to express all the density-dependent parameters with respect to the liquid density. For this purpose a new variable is introduced, the specific volume expansion ratio, $\varepsilon_p$, defined as the ratio of the specific volume of the foam to the specific volume of the base liquid:

$$\varepsilon_p = \frac{\varepsilon}{\rho}$$  \hspace{1cm} (12)

With the help of the specific volume expansion ratio, the volume equalized power law (VEPL) constitutive equation for foam can be given in the simple form:

$$\tau = (K\varepsilon_p^{1-n}|\dot{\gamma}|^{n-1})\dot{\gamma},$$  \hspace{1cm} (13)

where the parameters $K$ and $n$ are constant for the foams of a given gas-liquid pair at a given temperature. Obviously, Eq. (13) satisfies the condition expressed by Eq. (9).

The volume equalized Bingham plastic constitutive equation (VEBP) can be defined as:

$$\begin{align*}
\gamma &= 0, \quad |\tau| < \tau_p \varepsilon_p, & (14a) \\
\tau &= \left( \frac{r_p \varepsilon_p}{|\dot{\gamma}|} + \eta_p \right)\dot{\gamma}, \quad |\tau| > \tau_p \varepsilon_p, & (14b)
\end{align*}$$

where the parameters $r_p$ and $\eta_p$ are constant for a given gas-liquid pair at constant temperature. Equation (14) satisfies the condition formulated in Eq. (11).

There are several possibilities to define a Reynolds number based on Eq. (13), but in any case the resulting Reynolds number has the following invariance property: it is constant with respect to velocity changes caused by density changes in a pipe of constant cross section if the flow is isothermal. The practical consequence of this invariance is
that for isothermal friction pressure loss calculations a constant friction factor can be used for the whole pipe, at a given mass flowrate, even in transitional or turbulent flow regimes.

Similarly, the volume equalized Bingham-modified Reynolds number

$$ Re_{VEB} = \frac{D u_p}{\eta_p} \quad (15) $$

and the volume equalized Hedstrom number

$$ He_{VE} = \frac{D^2 \tau_p \rho}{\eta_p} \quad (16) $$

defined on the basis of Eq. (14), have the same invariance property.

An interesting graphical consequence of the volume equalizing principle is that the plot of “wall stress divided by the specific volume expansion ratio” as a function of the “shear rate divided by the specific volume expansion ratio” results in a unique curve which might serve as a replacement to the flow curve in incompressible flow.

It should be emphasized that the validity of the proposed volume equalized equations ceases when \( \varepsilon - 1 \). Therefore, we do not state, that the parameters in Eq. (13) or in Eq. (14) have any formal relation to the same parameters of the base liquid. On the contrary, it can be readily assumed, that the foam of a base liquid having power law behavior can be described well by a volume equalized Bingham plastic model and vice versa. Moreover, it is more likely, that the foam of a Newtonian base liquid also can be described by these models. Similarly, some considerations might question the validity of this or that constitutive equation in the limiting case of very dry foams. Therefore the two volume equalized models are introduced here for the quality range \( 0.2 - 0.8 \), corresponding to the experiments described below.

**IV. EXPERIMENTS**

The use of the volume equalized constitutive equations is illustrated here for the evaluation of large scale vertical flow experiments with foamed polymer solutions.

**A. The experimental apparatus**

The foamed fluid was pumped down a \( D = 0.0518 \text{-m-i.d.} \) pipe of length \( L = 190.7 \text{ m} \). In reverse, it ascended in the \( 189.2 \text{-m annulus of 0.0603- and 0.1007-M i.d.} \). The temperature could be considered con-
stant. The nominal flow conditions corresponding to the inlet of the descending flow varied in the following ranges. Quality: from 0.25 to 0.7; volumetric flow rates: from $2.6 \times 10^{-3}$ to $8 \times 10^{-3}$ m$^3$/s; temperature: from 14 to 22 °C; pressure: 2 MPa. The results presented here correspond to 4800 wppm (weight part per million) aqueous HPG solution, also containing 2000 wppm surfactant to assure foam stability, foamed by nitrogen gas.

A detailed description of the flow loop containing the test well is given by Spörker et al. (1991). Calibration experiments with water showed that the relative roughness of the tubing is 0.003, resulting in a limiting friction factor $f_\infty = 0.0065$. The polymer solution was prepared by batch mixing 4.8-kg hydroxypropylguar to one m$^3$ water. Experiments with the polymer solution without a gas phase were carried out at three flow rates ($2.7 \times 10^{-3}$, $5.3 \times 10^{-3}$, and $7.9 \times 10^{-3}$ m$^3$/s) and the friction factors found were $f$: 0.017, 0.0071, and 0.0044, respectively. This is with good agreement with the results of Chakrabarti et al. (1991), who showed that hydroxypropylguar solutions of higher concentration have a smooth transition from laminar flow toward's Virk's asymptote and hence exhibit a marked drag reduction effect. It was anticipated, however, that all the foam experiments would fall into the laminar flow regime, because of the higher apparent viscosity of foams.

The surfactant was mixed with the polymer solution in a continuous manner and the mixture was foamed by injection of the nitrogen gas. In order to assure a uniform texture the foam was generated in a 0.114-m-o.d., 1.2-m-long mechanical foam generator filled with curled steel chips of average width 0.02 m, length 0.5 m, and thickness 0.001 m. Visual observation confirmed the statement by Cameron and Prud'homme (1989) about the “rather narrow bubble-size distribution and small bubble size” relative to the flow passages in our experiments. Quantitative characterization of the texture was not available. The computerized data acquisition system registered the gas and liquid mass flow rates entering the system and the temperatures and pressures at the inlet and outlet of the descending and ascending stages, respectively. Since measuring and control of the gas flow rate was critical for adjusting the desired inlet qualities, a heated water bath was applied to keep the gas temperature constant in the gas flow meter. Depending on the flow rate, 3–15 min were needed to reach steady-state operation. The actual data acquisition took place with a sampling rate of 1/s over 2 min. The values were averaged to yield the representative steady-state measured pressures. Qualities, densities, and actual flow rates were computed from the mass flow rates, temperatures, and pressures. The gas
and liquid mass flow rates were adjusted to obtain the above given three overall volumetric flow rates at the inlet. More details on the experiments and additional experimental results not treated here are given by Valkó et al. (1992).

Figure 1 shows the measured outlet pressures at the bottom of the tubing, $p_2$, as a function of superficial velocity at the inlet, $u_1$, for four different nominal inlet qualities. Also shown in the figure is the nominal inlet pressure. The larger the foam quality is, the smaller is the hydrostatic head, and hence the smaller is the outlet pressure. For low flow rates this outlet pressure is still larger than the inlet pressure, but for larger flow rates the friction loss might outbalance the hydrostatic pressure increase and, hence, one would obtain lower pressures at the outlet than at the inlet. In Fig. 2 the specific volume expansion ratio is plotted as a function of quality (both at inlet and outlet conditions). Note, that there is no one-to-one correspondence between quality and specific volume expansion ratio, even at a fixed temperature, if the pressure is not constant.
B. Estimation of model parameters

Assuming a common linear velocity for both phases and using the virial equation of state truncated after the second term to describe volumetric gas behavior, the mechanical energy balance for descending flow in a vertical pipe is written as

\[
\frac{bp^3 + ap^2 + (-abc)p + (-a^2c^2)}{(2fb^3c - Dg)p^3 + 4fabc^2p^2 + 2fa^2c^2p} \, dp = -\frac{dl}{D},
\]

where the constants \( a, b, \) and \( c \) are defined in the Appendix.

The solution of Eq. (17) is obtained by integrating it from inlet point 1 to outlet point 2. According to the volume equalizing principle the friction factor is constant and hence a closed form solution can be obtained:

\[
K_d1(p_1 - p_2) + K_d2 \ln \frac{P_1}{P_2} + K_d3 \ln \frac{P_1 - \alpha_1}{P_2 - \alpha_1} + K_d4 \ln \frac{P_1 - \alpha_2}{P_2 - \alpha_2} = \frac{L}{D},
\]
where

\[ \alpha_1 = \frac{ac}{s_d} \left[ -2fbc + (2fDg)^{0.5} \right], \]  \hspace{1cm} (19a)

\[ \alpha_2 = \frac{wC}{s_d} \left[ -2fbc - (2fDg)^{0.5} \right], \]  \hspace{1cm} (19b)

\[ K_{d1} = \frac{b}{s_d}, \]  \hspace{1cm} (19c)

\[ K_{d2} = \frac{1}{2f}, \]  \hspace{1cm} (19d)

\[ K_{d3} = \frac{bc - 2fabc/s_d + ba_1(a/b - 4fabc^2/s_d + s_d/2fb)/ac}{2(2fDg)^{0.5}}, \]  \hspace{1cm} (19e)

\[ K_{d4} = -\frac{bc - 2fabc/s_d + ba_2(a/b - 4fabc^2/s_d + s_d/2fb)/ac}{2(2fDg)^{0.5}}, \]  \hspace{1cm} (19f)

and

\[ s_d = \gamma f b^2 c^2 - Dg. \]  \hspace{1cm} (19g)

The most straightforward way to estimate the parameters of a volume equalized rheological model is the following "one-step procedure:" define computed outlet pressure as the solution of Eq. (18) for \( p_2 \), in which the friction factor is computed from the given rheological model or inlet conditions and determine the model parameters yielding the minimum sum of squares of the differences between computed and measured outlet pressures.

Since the specific volume expansion ratio is considered constant in a given cross section, the friction factors for the volume equalized constitutive equations can be determined using standard methods described by Whittaker (1985). For the volume equalized power law we have

\[ f_{VPL} = 2Kc_s \left( \frac{6n + 2}{n} \right)^n u^{n-2} \rho^{-1} D^{-n}, \]  \hspace{1cm} (20)

and for the volume equalized Bingham plastic constitutive equation the procedure to obtain the friction factor is as follows:
\[ \beta = 1 + \frac{6\eta \mu}{\tau_f \epsilon D}, \]  
\[ z = [\beta^2 + (\beta^4 - 1)^{1/2}]^{1/2}, \]  
\[ y = 2 \left( z + \frac{1}{z} \right), \]  
\[ x = 0.5 \left( y^{1/2} - \left( \frac{8\beta}{y^{1/2} - y} \right)^{1/2} \right), \]  
\[ f_{\text{VEBP}} = \frac{\tau_f \mu}{0.5u_1^2 \rho x}. \]  

Once the friction factor was known, Eq. (18) could be solved for \( p_2 \) using bisection.

The nonlinear least-squares problem was solved by a standard Gauss–Newton–Marquardt procedure.

V. RESULTS AND DISCUSSION

A. Volume equalized power law model

The parameters in Eq. (13) were estimated by the above described one-step procedure and were found to be \( K = 1.40 \) Pa s\(^n\) and \( n = 0.47 \) which are characteristic of the liquid (with given polymer concentration) and gas pair. To show both the measured and computed outlet pressures on one figure we plotted them as a function of \( u_1(1 - \Gamma_1) \), where \( u_1 \) is the actual inlet superficial velocity and \( \Gamma_1 \) is the actual inlet quality (Fig. 3). This plot shows a convincing agreement of measured and calculated results thus providing an indirect proof of the validity of the volume equalizing concept.

In an attempt to carry out a partly independent check of the validity of this model, the friction factors were determined from Eq. (18), without assuming any rheological model \textit{a priori}, but making use of the fact that the friction factor can be considered constant in one single experiment. This was easily done by solving the equation for \( f \), since both \( p_1 \) and \( p_2 \) were known. As a result, we obtained “experimental” friction factors. Figure 4 shows the experimental friction factors as a function of the volume equalized version of one of the possible power law Reynolds numbers:
As seen from the figure, the "experimental" values lie close to the theoretical relation:

$$f_{	ext{VEPL}} = \frac{2}{\text{Re}} \left( \frac{6n+2}{n} \right)^n$$

represented by the straight line in Fig. 4.

It is in order to remark that Eq. (18) provides an alternative way of determining the parameters of the volume equalized power law model, which is called a "two-step procedure" of parameter estimation by Valkó et al. (1992). In the two-step method the "experimental" friction factors are first derived and then the least-squares principle is applied to Eq. (20). If this two-step procedure is used, slightly different model parameters are obtained. As usual with exponential-type models, the change in the exponent parameter is almost totally compensated by a
(more pronounced) counterchange in the pre-exponential factor (termed as "compensation effect" in some branches of science).

B. Volume equalized Bingham plastic model

The parameters in Eq. (14) were estimated by the one-step method and were found to be \( \tau_v = 7.0 \) Pa and \( \eta_p = 0.048 \) Pa s. It is natural to ask the question: which model is more adequate to the experimental results? The two different volume equalized constitutive equations yielded similar computed outlet pressures as clearly indicated by Fig. 3, where the outlet pressures computed from the volume equalized power law equation [Eq. (13)] and the ones computed from the volume equalized Bingham plastic model [Eq. (14)] are plotted as the function of the measured outlet pressure. In Fig. 6 the apparent viscosities computed from the two models according to

\[
\eta_a = \frac{\tau}{\gamma}
\]

are plotted as a function of shear rate at different expansion ratios. The ranges of shear rate and specific volume expansion ratio correspond to
the data. As obvious from the figure, the limited range of shear rates available is not sufficient to make any difference between the two models, since both the volume equalizing and the rate thinning effect can be equally well represented by them.

C. Discussion of the volume equalizing approach in view of previous experimental evidence

Since volume equalizing is a new concept, it is necessary to examine its consequences as a general principle from which appropriate constitutive equations can be derived. In an earlier excellent review of the experimental and modeling efforts Cameron and Prud’homme (1989) underlined that “currently, no universal format is available for the presentation of rheology data of foamed fracturing fluids.” Nevertheless, we will apply the volume equalized power law to several experimental facts listed in that review. Earlier interpretations of analogous experiments (Reidenbach et al., 1986; Harris, 1989) have resorted to the introduction of a variety of presumably controlling variables or phe-
nomena, such as quality, texture, shear thinning, and yield stress. These interpretations were unique to the specific experimental apparatus types used and the findings may be difficult to transfer in other situations. Below we show that there are no contradictions between our results and the previous findings.

1. Quality effects

In Fig. 7 the apparent viscosity of a 4800-wppm HPG solution foamed with nitrogen is shown as a function of quality (at room temperature and 2500 1/s—shear rate). Since the actual specific volume expansion ratio depends on pressure as well, the volume equalized power law viscosities are computed for 1 and 10 MPa. The dashed line shown on this figure is computed from the correlation of Reidenbach et al. (1986). The correlation of those authors is based on experiments conducted at pressure \( p = 6.9 \) MPa. As clearly seen from the figure, the sharp rise in the apparent viscosity at quality above 0.7 is well repre-
FIG. 7. Comparison of apparent viscosities computed from the volume equalized power law and from the Reidenbach et al. (1986) correlation. Dependence on quality.

sented by the volume equalizing model. The explanation for this phenomenon lies in the exponential behavior of the specific volume expansion ratio with increasing quality at any fixed pressure. The discrepancy of the curves at low qualities confirms again, that the volume equalized power law model with parameters obtained from medium quality experiments is not valid in the range of vanishing gas phase. At this point it is important to underline, that the idea of volume equalizing itself does not contradict the more or less experimentally, established view that for low qualities the apparent viscosity approaches the base fluid viscosity. According to the figure, the real significance of the volume equalizing principle becomes evident at higher qualities, where the use of quality as the only correlating variable may introduce large discrepancies depending on the actual pressure. In other words, while traditional correlations may give results similar to the ones provided by the volume equalized power law model, this is true only if the pressure corresponds to the one used in the experiments when deriving the correlation. If, however, extrapolation to other pressures is necessary, the two approaches behave quite differently.
2. Shear thinning

As mentioned above, we do not see evidence to prefer other two or three parameter rheological models to the power law because the shear rate available is always limited. Cameron and Prud’homme also stated, that “there is a considerable difficulty in estimating yield stress by extrapolating from moderate shear rates.” In Fig. 8 the apparent viscosity of the above described foam at 0.7 quality is shown as computed from the volume equalized power law for pressures 1 and 10 MPa, respectively. It is interesting to compare these results with a more complicated model incorporating (quality dependent) yield stress and consistency. The dashed line shown is computed from the Reidenbach et al. (1986) correlation, derived from the experiments of those authors at pressure $p=6.9 \text{ MPa}$. As seen from the figure, the volume equalizing power law does not contradict that best available representation of fracturing foam rheology. Moreover, the results are near each other, in spite of the fact that the underlying rheological model is much simpler in our case.
3. Texture effects

According to Reidenbach et al. (1986) and Harris (1989), the variation of texture with time ("equilibration" in their experiments) is less significant at higher shear rates and hence we did not consider texture as an independent variable in our data treatment. Rather it is supposed that texture is determined by the gas content and shear rate. If there are any changes in texture, they are reflected in the overall shear rate behavior of the constitutive equation. Harris (1989) observed a small decrease in viscosity of 0.7 quality aqueous/nitrogen foams when pressure increases from 1.4 to 11 MPa. While that author and Cameron and Prud'homme (1989) both attribute this fact to the finer texture of the foam at higher pressure, the volume equalizing principle provides a more straightforward explanation (see Fig. 8) without referring to the texture at all.

4. Gas composition effects

Reidenbach et al. (1986) found that the apparent viscosity of a nitrogen foam is always larger than the one of a carbon dioxide foam at the same quality (if pressure and temperature are fixed). The volume equalized principle can predict this experimental fact. If we use the reasonable assumption, that the volume equalized power law parameters are the same for both gases (Valkó et al., 1992), we can compute the ratio of the two apparent viscosities for a given quality, pressure, and shear rate. For example, assuming the temperature and pressure used in the Reidenbach et al. (1986) experiments and selecting typical quality and shear rate values: \( \Gamma = 0.7 \) and \( \dot{\gamma} = 1000 \) 1/s we obtain

\[
\frac{\eta_{a,N_2}}{\eta_{a,CO_2}} = \left( \frac{\epsilon_{N_2}}{\epsilon_{CO_2}} \right)^{-0.47} \approx \left( \frac{2.82}{2.05} \right)^{0.53} \approx 1.18. \tag{25a}
\]

Applying the two separate correlations derived by Reidenbach et al. (1986) for nitrogen and carbon dioxide foams, the same ratio is

\[
\frac{\eta_{a,N_2}}{\eta_{a,CO_2}} \approx 0.0397 \text{ Pa s} \approx 0.0500 \text{ Pa s} \approx 1.19. \tag{25b}
\]

As seen from the nearly identical numbers, the volume equalizing principle has an appreciable predictive power.
VI. CONCLUSIONS

A new class of constitutive equations has been proposed for describing the flow behavior of non-Newtonian compressible fluids. The new class is characterized by an invariance property called here volume equalizing. Since the Newtonian flow of compressible fluids obeys the volume equalizing principle, the newly introduced class of constitutive equations can be considered a more natural generalization of the Newtonian constitutive equation than previously proposed equations missing this property. In particular, two equations have been introduced for foams: the volume equalized power law and the volume equalized Bingham plastic. Both constitutive equations have a simple form if written in terms of a newly introduced variable called the specific volume expansion ratio. Large-scale experiments with foamed polymer solutions having inlet and outlet qualities in the range of 0.2–0.8 indicate a good agreement between data and model predictions for both models. The practical significance of the newly introduced models are twofold. First, the invariance of the friction factor eases the computation of friction losses in isothermal pipe flow. Second, the very simple form and small number of parameters of the volume equalized models make the parameter estimation much easier than for other published models. Comparison of the volume equalizing principle with previously published experimental results showed no contradictions. Some experimental facts, previously attributed to texture or composition effects, can be easily explained in the framework of the volume equalizing principle.

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APPENDIX: MECHANICAL ENERGY BALANCE FOR VERTICAL DESCENDING FLOW OF FOAMS

The virial equation of state truncated after the second term (Reid et al., 1987) is used for representing the volumetric behavior of the gas phase:

\[ \frac{1}{\rho_g} \frac{RT}{M_g} \left( \frac{1}{P + B'} \right) \]
where $B'$ is the modified second virial coefficient of the gas and $R,T,M_g$, and $p$ are the universal gas constant, temperature, molar mass of gas, and the pressure, respectively.

Introducing the following notation:

$$w_g = \frac{m_g}{m_g + m_l},$$

$$a = \frac{RT}{M_g},$$

$$b = w_g \frac{RTB'}{M_g} + \left(1 - w_g\right) \frac{1}{\rho_g},$$

$$c = \frac{4(m_g + m_l)}{D^2 \pi},$$

where $w_g$ is the mass fraction of the gas phase, $m_g$ is the gas mass rate, $m_l$ is the mass rate of the incompressible liquid, and $D$ is the pipe diameter, the specific volume of the foam is expressed as

$$\frac{1 - a}{\rho} + b$$

Consequently the superficial velocity is given by

$$u = -a + \frac{c}{p}$$

and—since $a$, $b$, and $c$ are constants—we have

$$du = -ac dp$$

Neglecting the effect of radial distribution of the velocity and inserting Eqs. (A1)–(A3) and $A/P = D/4$ into Eq. (5) we obtain Eq. (17).

**NOMENCLATURE**

- $a$ (m$^2$/s$^2$) constant coefficient
- $b$ (m$^3$/kg) constant coefficient
- $B$ (m$^3$/mol) second virial coefficient
- $B'$ (1/Pa) modified 2nd virial coefficient $B' = B/(RT)$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>c[kg/(s m²)]</td>
<td>constant coefficient</td>
</tr>
<tr>
<td>D(m)</td>
<td>pipe diameter</td>
</tr>
<tr>
<td>f</td>
<td>Fanning friction factor</td>
</tr>
<tr>
<td>g(m/s²)</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>K(Pa s⁵)</td>
<td>power law consistency index</td>
</tr>
<tr>
<td>K₃₁(1/Pa)</td>
<td>coefficient</td>
</tr>
<tr>
<td>K₃₂</td>
<td>coefficient</td>
</tr>
<tr>
<td>K₃₃</td>
<td>coefficient</td>
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<tr>
<td>K₃₄</td>
<td>coefficient</td>
</tr>
<tr>
<td>l(m)</td>
<td>length coordinate</td>
</tr>
<tr>
<td>L(m)</td>
<td>length</td>
</tr>
<tr>
<td>Mₐ(kg/mol)</td>
<td>molar mass of the gas phase</td>
</tr>
<tr>
<td>mₚ, mₐ (kg/s)</td>
<td>mass rates of gas and liquid, respectively</td>
</tr>
<tr>
<td>n</td>
<td>power law flow behavior index</td>
</tr>
<tr>
<td>p(Pa)</td>
<td>pressure</td>
</tr>
<tr>
<td>R[J/(mol K)]</td>
<td>universal gas constant</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>s₉(m²/s²)</td>
<td>coefficient</td>
</tr>
<tr>
<td>T(K)</td>
<td>temperature</td>
</tr>
<tr>
<td>u(m/s)</td>
<td>superficial velocity</td>
</tr>
<tr>
<td>vₐ, vₗ (m³/kg)</td>
<td>specific volume of foam and liquid, respectively</td>
</tr>
<tr>
<td>w₉</td>
<td>mass fraction of gas in foam</td>
</tr>
<tr>
<td>α</td>
<td>angle</td>
</tr>
<tr>
<td>ρₐ(Pa)</td>
<td>coefficients</td>
</tr>
<tr>
<td>Γ</td>
<td>quality of foam</td>
</tr>
<tr>
<td>γ(1/s)</td>
<td>shear rate</td>
</tr>
<tr>
<td>τ(Pa)</td>
<td>shear stress</td>
</tr>
<tr>
<td>τₚ(Pa)</td>
<td>yield stress</td>
</tr>
<tr>
<td>τₚₚ(Pa)</td>
<td>wall shear stress</td>
</tr>
<tr>
<td>εₚ</td>
<td>specific volume expansion ratio</td>
</tr>
<tr>
<td>τₚ(Pa)</td>
<td>plastic viscosity</td>
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<tr>
<td>τₚₚ(Pa)</td>
<td>apparent viscosity</td>
</tr>
<tr>
<td>ρ, ρₗ(kg/m³)</td>
<td>density of foam and liquid, respectively</td>
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</tbody>
</table>

**References**


