The Method of Distributed Volumetric Sources for Calculating the Transient and Pseudosteady-State Productivity of Complex Well-Fracture Configurations

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Abstract

The method of Distributed Volumetric Sources (DVS) is developed to solve problems of transient and pseudo steady state fluid flow. The basic building block of the method comprises calculation of the analytical response of a rectilinear reservoir with closed outer boundaries to an instantaneous volumetric source, also shaped as a rectilinear body. The solution also provides the well-testing derivative of the response of a rectilinear reservoir with closed outer boundaries to an instantaneous source. For production engineering applications, we cast the results into a transient/pseudo-steady productivity index form. The new method is validated through comparison to results of some of the well-known well-testing solutions for simple configurations such as vertical wells with full and partial penetration, horizontal wells with uniform flux and infinite conductivity, and fractured wells with uniform flux, finite or infinite conductivity. The results show a very good agreement with the existing models. The main advantage of the new solution is its applicability over the more complex fracture/well configurations, some of which is studied in the paper.

Introduction

Over the past decades, different techniques were developed to solve single-phase, slightly-compressible flow problems in porous media where the fluid removal (injection) is from a complex well-fracture system. Most of these methods use the point source integrated over a line and/or a surface.

The major disadvantage of this method is the inherent singularity of the solution wherever the point source is placed. The method of Distributed Volumetric Sources (DVS) is developed to remove this limitation by assuming a source not in the form of a point but in the form of a rectilinear volume extended inside the surrounding rectilinear porous media. We will refer to this configuration as “box-in-box”.

The 3-D pressure response of the system to an instantaneous source can be obtained — by applying Newman’s principle — as a product of three 1-D pressure responses in each principal direction. The response of the system to a continuous source (production or injection) is obtained by numerically integrating the response of instantaneous source over time as it has been discussed, for instance, by AzarNejad et al. The aim of this work is to:

1. Develop the analytical solution of the box-in-box system to an instantaneous source.
2. Validate the applicability and demonstrate the advantage of the new solution for well testing and production forecast of simple well and fracture models through the comparison of results with well-known existing solutions. Solutions to vertical wells with full and partial penetration, horizontal wells with uniform flux, finite and infinite conductivity, and vertically fractured wells with uniform flux, finite and infinite conductivity would be studied and compared with the new DVS results.
3. Show the capabilities of the DVS method to handle more complicated well/fracture configurations. Cases with multiple transverse fractures with partial penetration and finite conductivity are presented.

DVS Model

The first step of our approach is to develop the pressure response of a rectilinear reservoir with closed boundaries to an instantaneous withdrawal from the source. The porous media is assumed to be an anisotropic, homogeneous reservoir shaped as a box. The box is oriented in line with the three principal directions of the permeability field. The source is assumed to be a smaller rectilinear box with its surfaces parallel to the reservoir boundaries. It is assumed to have the same media properties as the reservoir. Fig.1 shows the schematic of the system. The instantaneous unit withdrawal is distributed uniformly in the volume of the source. In short, we will refer to the solution as instantaneous source response of the box-in-box and will denote the response observed at a location \((x_D, y_D, z_D)\) as \(p_{ib}(box-pars;x_D, y_D, z_D, t_D)\).
DVS solution and the other solutions.

Except for the very early time observation point method” that is the observation point is obtained from the uniform flux solution with the “equivalent theoretical source box. Detailed numerical experimentation has revealed that the relation:

\[ w_x = w_y = 1.4444 r_w \]

provides the best results. Notice that on Fig. 2, the DVS solution is much nearer to the cylindrical source solution than to the line source solution.

Fully Penetrating Vertical Well

Ozkanc presented a method to obtain the performance of a horizontal well. He developed a series solution in Laplace domain for a fracture with uniform flux and augmented the solution to obtain the performance of a horizontal well. In this work we used the results given in Table 2.6.2 of Ref. (8) to compare our DVS results with. Fig. 4 shows the graphical presentation of the two solutions for an example case of uniform flux horizontal well with various horizontal penetrations. For the DVS representation of the horizontal well Eq. 3 is used to obtain the \( w_1 \) and \( w_2 \) widths of the source box.

Since the data given in Ref (8) misses the very early time period of the solution we can not see any difference between the DVS solution and the reference solution.

Uniform Flux Horizontal Well

Ozkan\(^7\) presented a method to obtain the performance of a horizontally penetrating vertical fracture with uniform flux and infinite conductivity, respectively, together with our DVS results. The DVS curves on Fig. 5 are calculated with one uniform box source, where the \( w_1 \) width is selected sufficiently small (\( 10^4 \)\( r_w \)).

Full Penetration Vertical Fracture (Uniform Flux and Infinite Conductivity Fracture)

Shown on Figs. 5-6 are the widely accepted Gringarten results for a vertically fully penetrating vertical fracture with uniform flux and with infinite conductivity, respectively.
Comparison with Selected Pseudo-steady State Productivity Index Correlations

The results of the DVS method could also be used to calculate transient as well as pseudo-steady state productivity index of a well. Method of calculation of \( J_D \) is discussed in Appendix A. To show the applicability and reliability of the results from DVS method we used the study of Chen and Asaad\(^{10} \) who proposed a simplified and full-form correlation for prediction of the pseudo-steady state productivity index of horizontal wells. The method proposed in Ref. (10) was validated with the results from example cases of Babu and Odeh\(^{11} \) and Goode and Kuchuk\(^{12} \) for the case of horizontal well. Chen and Asaad then extended their correlation to fractured wells. Here we compare our DVS results for their examples A1, B1, and B2. Table 1 shows the reservoir/ fluid properties used for the calculation and Table 2 present the results.

We can see that for the case of horizontal well an excellent agreement exists between the results of different method presented. This agreement stands still for uniform flux solution as well as infinite conductivity solution. For the case of fractured well the results are in very good agreement (within less than 1 %) for the example cases A1, and B1. However, for example case B2 the difference between Ref (10) and DVS results rises to around 10 %. We suspect that the discrepancy is due to the approximations involved in Chen and Asaad's formulation, that seem to show their fall in cases where the horizontal penetration of the fracture becomes small.

Multiple Uniform-flux Fractures Intercepting a Horizontal Well

So far we have used the box-in-box model to solve flow problems directly. In advanced applications, we need several uniform flux source boxes with their responses superposed in space.

As a simple example, we consider two small vertical fractures (1 and 2), induced transversely from a horizontal well at locations \( x_1 \) and \( x_2 \) (see Fig. 7). We represent the fractures themselves by uniform flux box sources. In addition, we assume that the total inflow from the two fractures is known, and the horizontal well receives no additional inflow. The pressure drop in the horizontal segment between the two fractures depends on the inflow from fracture 2 only. In practical terms, linear or quadratic dependence on flow rate are of interest. The first corresponds to laminar flow, the second to turbulent flow in the horizontal well. In this illustration we show only the turbulent flow case. The actual wellbore pressure is observed at location \( x_w \) and the additional pressure drop in the horizontal well (between points \( x_1 \) and \( x_w \)) depends on the total flow rate.

For every time point, we want to determine the following variables: the fractional inflow into fracture 1 (\( q_1 \)), the drawdowns at the points \( x_1 \) and \( x_2 \) (\( p_1 \) and \( p_2 \)), and the drawdown at the wellbore observation point (\( p_w \)). We have to solve the following system of equations:

\[
(p_{11} + s_1)q_1 + p_{22}(1 - q_1) = p_1
\]
\[
p_{12}q_1 + (p_{22} + s_2)(1 - q_1) = p_2
\]
\[
p_2 - p_1 = C_{21}(q_1)^2
\]
\[
p_w - p_2 = C_{w2}
\]

The first equation expresses the fact that the pressure drawdown in the center of box 1 is the effect of producing box 1 with strength \( q_1 \) and producing box 2 with strength \( 1 - q_1 \). Pressure \( p_{11} \) is calculated putting box 1 only into the reservoir and using the center of the source as observation point. Pressure \( p_{22} \) is calculated putting box 2 only into the reservoir, and using the center location of box 1 as the observation point. The skin factor, \( s_1 \), represents the effects of converging flow and near wellbore tortuosity in Fracture 1.

The meaning of the second equation should be clear from the previous discussion. The third equation expresses the fact, that \( q_1 \) fraction of the total flow is flowing through the horizontal well segment between points \( x_1 \) and \( x_2 \), and the flow is turbulent. The meaning of the coefficient \( C_{21} \) is the hypothetical pressure drop between these two points if the total flow would flow through this segment.

Similarly, the last equation is easily interpreted. The meaning of \( C_{w2} \) is the pressure drop through the segment between \( x_2 \) and \( x_w \). This pressure drop does not depend on the fractional distribution of the inflow between box 1 and box 2.

The results can be best summarized in the form of dimensionless productivity index: \( J_{D,trad} \). Therefore, our ultimate goal is to determine the time dependent productivity index \( J_{D,trad} \) of the whole well/fracture system. The solution method is presented in Appendix B. In this system of particular interest is the relative importance of (a) pressure losses due to converging flow in the fractures and (b) pressure losses in the well itself. Details of the example system are given in Table 3 and selected results are presented in Table 4.

As seen from the results, when the pressure drop in the fracture due to flow convergence and in the wellbore due to turbulent flow are both negligible, the dimensionless productivity index of the configuration is quite high (0.9385) and Fracture 1 contributes to the total production 41 %. When the pressure losses due to flow convergence is commensurate with the pressure drop in the reservoir, the productivity of the system decreases, but the two fractures contribute to the total productions basically the same way as previously. When the convergence skin is negligible, but the wellbore pressure losses are commensurate to the reservoir pressure drop, the dimensionless productivity index is only 0.47 and this is mostly due to the fact, that Fracture 1 can contribute less to the total production (because the significant wellbore pressure loss penalizes Fracture 1 more than Fracture 2). Finally, in a realistic situation, when both the wellbore and flow convergence effects are significant, the dimensionless productivity index is relatively low (a vertical well with a vertical fracture of moderate Proppant Number less than 0.1 could perform at least as well, see Ref. (14) for details.)
Development of Solution for Finite Conductivity Source with Uni-directional Flow

As in other methods\textsuperscript{15, 16}, we calculate finite conductivity horizontal well responses by dividing the well into \( n \) segments. Each segment is represented by a uniform flux source box with well defined size and location, but unknown strength (see Fig. 8). In matrix notation the system is given by

\[
[A + C] \mathbf{q} - \mathbf{b} = 0
\]

where \( \mathbf{q} \) is the \( n \)-dimensional vector of source strengths.

The \((i,j)\)-th element of the \( A \) matrix is obtained by “activating” the \( j \)-th box source and observing the pressure at the center of the \( i \)-th box. The \((i,j)\)-th element of the \( C \) matrix describes the pressure drop in the fracture between the center of the \( i \)-th source and the wellbore reference point due to the \( j \)-th inflow. It contains as a factor — the conductivity of the well. Vector \( \mathbf{b} \) contains the unknown drawdown at the wellbore reference point that is placed at the heel of the horizontal well. The system is augmented with the requirement that the sum of strengths should be one. A robust way to obtain the strengths vector is to sum the columns of the pseudo-inverse of the matrix \([A + C]\) and then normalize it.

Once the strengths of the various segments are known, we superpose their effects.

Modeling a finite conductivity vertical fracture within the DVS framework requires a similar procedure. In fact the actual program is the same as before, only the individual source boxes are now fully penetrating vertically and the wellbore reference point is placed into the geometric center of the fracture. The \( C \) matrix contains as a factor — the conductivity of the fracture, see Appendix C for details.

In our validation example we assumed a vertically fully penetrating fracture with dimensionless fracture conductivity equal to 1.6 and horizontal penetration ratio of 0.25. It is well-known from Unified fracture design\textsuperscript{14} that the dimensionless productivity index of such a fracture is \( J_D = 0.467 \). In the DVS method the fracture has been divided into various number segments (1, 2, 4, 8, 16, and 32) and the \( J_D \) was calculated.

Table 4 shows the results. We can see that by increasing the number of segments the pseudo-steady state dimensionless productivity index tends to the known value.

Fig. 8 represents the dimensionless pressure behavior of the example case system obtained as a function of dimensionless time for a fracture system divided to 32 segments. Other curves in the figure represent the dimensionless pressure behavior calculated by the method proposed by Cinco-Ley et al.\textsuperscript{5}. Like the previous cases, very good agreement between the results can be seen. Fig. 9 represents the distribution of the source strengths along the lateral coordinate in one fracture wing. It can be seen that at early times all flow comes from the nearest segments to the well. The strength distribution becomes more uniform as the flow regime approaches pseudo-steady state. The stabilized strengths distribution in pseudo-steady state predicted by the DVS method is somewhat similar to the well-known “U-shaped” pseudo-radial distribution form, known from many studies following Cinco-Ley’s pioneering work. However, the DVS result shows less dramatic difference between the strengths of various segments. The reason is that the pseudo-steady state strengths distribution always reflects the interaction of the fracture tip with the outer boundary. The pseudo-radial calculation methods, however, neglect this effect and assume the reservoir as infinite acting. It is not rigorous to apply the pseudo-radial strength distribution (obtained neglecting the effect of interaction between fracture tip and boundary) for the pseudo-steady state case and our DVS results have definite advantage in this respect.

Fig. 10 shows the change in dimensionless productivity index for the same system as a function of dimensionless time based on the drainage area. We can see that \( J_D \) approaches the expected value of 0.467 as the system enters the pseudo-steady state flow regime and the transition to this late-time flow regime is smooth.

Also, the finite conductivity models presented in this section can be used to calculate infinite conductivity horizontal well, infinite conductivity vertical fracture, or infinite conductivity horizontal fracture by simply letting the equivalent well or fracture permeability tending to zero. We found that these more rigorous finite conductivity results are almost indistinguishable from the infinite conductivity results obtained by the “equivalent observation point” method presented earlier.

Dividing the source into \( n \) segments in the \( x \)-direction we still assume uniformity of the influx with respect to the \( z \)-direction. In other words, the flow inside the horizontal well or the vertically fully penetrating fracture is still uni-directional, the direction being the \( x \)-direction. This is an excellent approximation for the listed cases and can be applied to many other cases, such as longitudinally fractured horizontal well (or even horizontal fracture intersected by a horizontal well). The rigorous modeling of horizontal well with large \( \text{traverse} \) fractures of finite conductivity requires, however, one additional step, because the flow inside the fracture becomes two-directional. Such a development will be presented in a subsequent paper.

Conclusions

The DVS method is a viable alternate to produce consistent transient and pseudo-steady state solutions of complex well/fracture systems. It provides reliable results with relatively moderate computational effort. In this paper we introduced the method and validated it with known results available for relatively simple systems. However, the real merit of the DVS approach becomes more obvious as we apply it to systems with increasing complexity, such as permeability anisotropy, partial penetration of the source in any of the principal directions and/or additional pressure losses stemming from the particular geometry of the flow path inside the sources. In such systems the ability of the method to provide a consistent transient and pseudo-steady state productivity index (without any artificial break in the transition) makes it an excellent tool for forecasting production and optimizing the completion strategy.

Acknowledgement

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Nomenclature

**Variables**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>matrix of source effects</td>
</tr>
<tr>
<td>$b$</td>
<td>drawdown vector</td>
</tr>
<tr>
<td>$C$</td>
<td>matrix of pressure drop effects</td>
</tr>
<tr>
<td>$C_{fp}$</td>
<td>dimensionless fracture conductivity</td>
</tr>
<tr>
<td>$c_t$</td>
<td>total compressibility, psi$^{-1}$</td>
</tr>
<tr>
<td>$c_{trad}$</td>
<td>conversion factor</td>
</tr>
<tr>
<td>$c_x$</td>
<td>position of the center of the source in $x$ direction, ft</td>
</tr>
<tr>
<td>$c_y$</td>
<td>position of the center of the source in $y$ direction, ft</td>
</tr>
<tr>
<td>$c_z$</td>
<td>position of the center of the source in $z$ direction, ft</td>
</tr>
<tr>
<td>$f$</td>
<td>1D solution to the flow equation</td>
</tr>
<tr>
<td>$H$</td>
<td>Heavyside unit step function</td>
</tr>
<tr>
<td>$J_D$</td>
<td>dimensionless productivity index</td>
</tr>
<tr>
<td>$J_{D,trad}$</td>
<td>traditional definition of dimensionless productivity index</td>
</tr>
<tr>
<td>$k$</td>
<td>permeability, reference permeability, md</td>
</tr>
<tr>
<td>$k_f$</td>
<td>fracture permeability</td>
</tr>
<tr>
<td>$k_x$</td>
<td>directional permeability in $x$ direction, md</td>
</tr>
<tr>
<td>$k_y$</td>
<td>directional permeability in $y$ direction, md</td>
</tr>
<tr>
<td>$k_z$</td>
<td>directional permeability in $z$ direction, md</td>
</tr>
<tr>
<td>$L$</td>
<td>reference length, ft</td>
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<tr>
<td>$p$</td>
<td>pressure, psi</td>
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<td>initial pressure, psi</td>
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<td>$p_{wf}$</td>
<td>well flowing pressure, psi</td>
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<td>$p_{D0}$</td>
<td>dimensionless pressure due to instantaneous source</td>
</tr>
<tr>
<td>$PI$</td>
<td>Productivity index, STB/d/psi</td>
</tr>
<tr>
<td>$P_{D0}$</td>
<td>dimensionless pressure due to continuous source</td>
</tr>
<tr>
<td>$q$</td>
<td>source strength, source strength vector</td>
</tr>
<tr>
<td>$r_w$</td>
<td>wellbore radius</td>
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<tr>
<td>$s$</td>
<td>Laplace parameter</td>
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<td>time</td>
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<td>integrating variable</td>
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<td>dimensionless time</td>
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<td>dimensionless time with regard to reference drainage volume</td>
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<td>$t_{D,fract}$</td>
<td>dimensionless time with regard to fracture half-length</td>
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<tr>
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<tr>
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<td>second derivative of function $u$ with respect to position</td>
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<td>$w_y$</td>
<td>source width in $y$ direction, ft</td>
</tr>
<tr>
<td>$w_z$</td>
<td>source width in $z$ direction, ft</td>
</tr>
<tr>
<td>$x_D$</td>
<td>dimensionless length in $x$ direction, $x/x_e$</td>
</tr>
<tr>
<td>$x_e$</td>
<td>length of the outer box, ft</td>
</tr>
<tr>
<td>$x_f$</td>
<td>fracture half length, ft</td>
</tr>
<tr>
<td>$x_l$</td>
<td>dimensionless starting position of the source</td>
</tr>
<tr>
<td>$x_u$</td>
<td>dimensionless ending position of the source</td>
</tr>
<tr>
<td>$y_D$</td>
<td>dimensionless width in $y$ direction, $y/y_e$</td>
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**Greek Symbols**

<table>
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<th>Symbol</th>
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<tr>
<td>$\phi$</td>
<td>porosity, fraction</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Jacobi elliptic theta function</td>
</tr>
<tr>
<td>$\mu$</td>
<td>viscosity, cp</td>
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</table>

**Subscript**

<table>
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<td>$\max$</td>
<td>maximum</td>
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<tr>
<td>$\gen$</td>
<td>general</td>
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</tbody>
</table>

**References**


Appendix A
Consider a segment of the real axes between \( x = 0 \) and \( x = 1 \). We assume the diffusivity function describes the behaviour of a quantity, \( u(x,t) \) over the segment. The initial condition is zero, the boundary conditions are “zero-flux” at both \( x = 0 \) and \( x = 1 \). At time zero, an instantaneous injection uniformly distributed above the segment \((x_1, x_u)\) is introduced, where \( x_1 \) is the lower and \( x_u \) is the upper end of the source. Using the Dirac-delta, \( \delta(\tau) \) and Heaviside unit-step function, \( H(x) \), we can write the partial differential equation as:

\[
\frac{\partial u}{\partial t} + u_{x} = \delta(\tau)H(x-x_{i})-H(x-x_{\Pi})
\]

(A1)

The solution can be represented in various ways. One can use an infinite time series:

\[
u_{1} + u_{\infty} = \delta(\tau)H(x-x_{i})-H(x-x_{\Pi}) \]

\[ \times \frac{\partial u}{\partial t} + u_{x} \]

(A2)

or one can express the result in terms of Jacobi elliptic Theta function:

\[
u = \frac{1}{2(x_{u}-x_{\Pi})} \int_{-\infty}^{\infty} \theta^{2} \left( \frac{x-x_{\Pi}}{2}, t \right) + \theta^{2} \left( \frac{x+x_{\Pi}}{2}, t \right) \theta^{2} \left( \frac{x-x_{\Pi}}{2}, t \right)
\]

(A3)

or one can provide the result in Laplace transform form:

\[
u(\sigma) = \frac{1}{2(s-1)(x_{u}-x_{\Pi})} \left( \theta^{2} \left( \frac{s+x_{\Pi}}{2s}, \sigma \right) + \theta^{2} \left( \frac{s-x_{\Pi}}{2s}, \sigma \right) \right) +
\]

\[
+ (s-1) \left( \theta^{2} \left( \frac{s+x_{\Pi}}{2s}, \sigma \right) \theta^{2} \left( \frac{s-x_{\Pi}}{2s}, \sigma \right) \right) \right) \left( \theta^{2} \left( \frac{x-x_{\Pi}}{2s}, \sigma \right) \right) \]n

(A4)

Each of these representations has advantages and disadvantages. For instance, the Laplace transform solution is in closed form, but needs numerical inversion. The Theta function form needs numerical integration. The time series form, used in this work, is easy to calculate for moderate and large times, but converges slowly for short times. A numerically efficient formulation uses an additional “trick”: The infinite sum is replaced by a finite sum

\[
u = 1 + 2n_{\Pi} \sum_{n=1,2,3,4} \exp \left( -n^{2} \pi^{2} \right) \cos(n \pi x_{u}) - [\sin(n \pi x_{u}) - \sin(n \pi x_{\Pi})] / [n \pi (x_{u}-x_{\Pi})]
\]

(A5)

where

\[
\nu_{\Pi} = 1 + \text{Ceiling} \left( \frac{5.58}{\sqrt{\tau}} \right)
\]

(A6)

and

\[
u_{\Pi} = 1 + 2 \times \text{Floor} \left( \frac{n_{\Pi}}{10,000} \right)
\]

(A7)

With this modification the 1D solution is calculated with maximum a couple of thousand terms, and the result is very accurate according to the corroborate done both with multiprecision inversion of the Laplace transform form and numerical integration of the Theta function form.

To obtain the \( f(f) \) function used in Eq. 1 which takes the \( x \)-parameters: \((k_{x}, c_{x}, w_{x})\) and calculates the dimensionless response at location \(x_{D} \) at time \( t_{D} \), we write:

\[
u(x-params, x_{D}, t_{D}) = u(x_{D}, t_{D})
\]

(A8)

with the following substitutions:

\[
u_{D} = \frac{k}{k_{x} x_{D}^{2}} x_{D} = \frac{x}{x_{e}}
\]

(A9)

\[
u_{\Pi} = \frac{c_{x} - w_{x}}{x_{e}} \quad x_{e} = \frac{c_{x} + w_{x}}{x_{e}}
\]

where the reference permeability, \( k \), and the reference length, \( L \) are given by:

\[
u = \left( x_{y}, y_{z}, z_{e} \right)^{1/3} \quad k = \left( k_{y}, k_{z} \right)^{1/3}
\]

(A10)

The \( y \) and \( z \) versions are calculated similarly, substituting the \( y \) and \( z \) parameters.

The obtained \( p_{dD}(x_{D}, y_{D}, z_{D}, t_{D}) \) multiplied by \( t_{D} \) provides the well-testing (semi-logarithmic) derivative of the dimensionless pressure at the observation point. To obtain the response to a continuous source, the integration

\[
u_{dD}(x_{D}, y_{D}, z_{D}, t_{D}) = \int_{0}^{\infty} \left( \frac{\text{d}r}{\text{d}t} \right) \text{d}x_{D}, y_{D}, z_{D}, \tau \text{d}r
\]

(A11)

is carried out numerically.

We notice that at the first time point (usually \( t_{D0} = 10^{-9} \) ) the numerical value is taken equal to the numerical value of the well testing derivative. This is why on Fig. 2 the pressure and well-testing derivative curves start from the same point.

The dimensionless time with respect to the drainage area is related to our dimensionless time through the relation:

\[
u = c_{rad} \sqrt{\frac{k_{x}}{k_{y}}}
\]

(A12)

\[
u \quad \nu_{rad} = c_{rad} \nu_{D}
\]
The dimensionless pressure, traditionally used in the petroleum industry, is related to our \( p_{ad} \) function according to the relation:

\[
p_{ad,rad} = \frac{1}{2\pi c_{trad} p_{ad}} \tag{A13}
\]

A concise way to represent the system performance is the dimensionless productivity index, \( J_D \). We calculate it from

\[
J_{D,rad} = \frac{1}{p_{ad,rad} - 2\pi t_{D,rad}} = \frac{1}{2\pi c_{trad} (p_{ad} - t_D)} \tag{A14}
\]

The dimensionless productivity index is time dependent in the transient floe regime and constant in the pseudo-steady state.

In field units, the productivity index is expressed as

\[
PI = \frac{z_s\sqrt{\frac{k_x k_y}{141.2 B \mu}}} {12112222 )1(} \tag{A15}
\]

the pressure drawdown as

\[
p_i - p_{of} = \frac{z_s\sqrt{\frac{k_x k_y}{141.2 B \mu}}} {12112222 )1(} \tag{A16}
\]

and the time as

\[
t = \frac{\phi c_{_s} \mu(x_s y_s)} {0.0002637 \sqrt{\frac{k_x k_y}{L_{rad} D}}} \tag{A17}
\]

where \( k \) is in md, \( \mu \) in cp, \( B \) in resBBL/STB, \( x_s y_s z_s \) in ft, \( c_x c_y c_z \) in STB/D, \( p_{of} \) and \( p_i \) in psi, \( t \) in hr, \( c_t \) in 1/psi, \( \phi \) is dimensionless and \( PI \) is in (STB/D/psi).

**Appendix B**

The solution of Eq. 4 is easily obtained for the fraction of the inflow into fracture 1:

\[
q_1 = \frac{2p_{21} - (p_i + s_2) - p_{22} + \sqrt{(p_i + s_2 + p_{22} - 2p_{21})^2 - 4C_{21}(p_{22} + s_2) - p_{21}^2}} {2C_{21}} \tag{A18}
\]

and for the pressure drawdown at the wellbore reference point:

\[
p_{u_i} = C_{u_2} + (p_{22} + s_2)(1 - q_1) + p_{22}q_1 \tag{A19}
\]

from which the desired productivity index is calculated using Eq. A14.

The coefficient \( p_{11}, p_{12}, p_{21} \) and \( p_{22} \) are calculated using the DVS method. Notice that their values are generated for every time point of interest in one run of the Box-inBox model.

The coefficient \( C_{w2} \) and \( C_{w21} \) represent dimensionless pressure losses. They are calculated assuming the total flow-rate in the given segment of the well, calculating the frictional pressure drop by any engineering method and then casting the pressure drop into dimensionless form in two steps: first using Eq. A16 we convert it into traditional dimensionless pressure, \( (\Delta p_{\text{rad}}) \) and then using Eq. A13, we cast it into our dimensionless pressure \( (\Delta p_D) \). The two pressure loss coefficient remains constant as far as the total production rate is fixed.

The actual values of the convergence skins \( s_1 \) and \( s_2 \) can be estimated from simple formulas. Following the recommendations in Ref (17), we can estimate the flow convergence skin factor for Fracture 1 from the additional data on fracture conductivity and wellbore radius:

\[
s_1 = \sqrt{\frac{k_x k_y h_{j1}}{k_j w_{j1}} \left[ \ln \left( \frac{h_{j1}}{2w_{j1}} \right) \right. - \frac{\pi}{2} \right] \tag{A20}
\]

However, one should be aware that \( s_1 \) and \( s_2 \) reflect the complex geometry of the well-fracture interface, often not known accurately. Twisting and turning of the fracture near the wellbore ("tortuosity") can significantly increase the skin values.

**Appendix C**

Each row of the \( C \) matrix represents the effect of the source strengths on the additional pressure drop in the fracture from the center of a source to the wellbore reference point.

For example, for the finite conductivity vertical fracture represented by \( n = 4 \) source boxes first we construct a \( D \) matrix:

\[
D = \begin{bmatrix}
11 & 3 & 0 & 0 \\
\frac{8}{3} & \frac{8}{3} & 0 & 0 \\
\frac{7}{8} & \frac{7}{8} & 0 & 0 \\
0 & 0 & \frac{3}{3} & 1 \\
0 & 0 & \frac{3}{3} & \frac{11}{8}
\end{bmatrix}
\]

The two zeros present in the first row of matrix \( D \) means, that strengths \( q_3 \) and \( q_4 \) do not contribute to the frictional pressure losses in the fracture from point 1 to the wellbore reference point (because \( q_3 \) and \( q_4 \) flows in the other fracture wing.)

The next step is to calculate the arithmetic average of the elements of the matrix (denoted by \( d \)), finally we calculate the \( C \) matrix:

\[
C = \frac{k_{j_{gen}} \left(2w_{j}\right)^2 \left(2w_{j}\right)} {k Lx_{j}} \tag{A21}
\]

The above \( C \) matrix contains the generalized dimensionless fracture conductivity:

\[
C_{j_{gen}} = \frac{k_{j_{gen}} \left(2w_{j}\right)^2 \left(2w_{j}\right)} {k Lx_{j}} \tag{A22}
\]

where the term \( k_{j_{gen}} \left(2w_{j}\right)^2 \left(2w_{j}\right) \) can be considered as a generalized conductivity expressed with the cross sectional area available for flow in the fracture.
<table>
<thead>
<tr>
<th>Reservoir/Fluid Properties</th>
<th>Unit</th>
<th>A1</th>
<th>B1</th>
<th>B2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drainage Length in x, 2x_e</td>
<td>ft</td>
<td>2640</td>
<td>2000</td>
<td>4000</td>
</tr>
<tr>
<td>Drainage Length in y, 2y_e</td>
<td>ft</td>
<td>2640</td>
<td>4000</td>
<td>2000</td>
</tr>
<tr>
<td>Formation thickness, h</td>
<td>ft</td>
<td>45</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>Horizontal well length, L_h</td>
<td>ft</td>
<td>1475</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>Well location in x, x_w</td>
<td>ft</td>
<td>1320</td>
<td>1250</td>
<td>1500</td>
</tr>
<tr>
<td>Well location in y, y_w</td>
<td>ft</td>
<td>1320</td>
<td>3000</td>
<td>1000</td>
</tr>
<tr>
<td>Well location in z, z_w</td>
<td>ft</td>
<td>22.5</td>
<td>50</td>
<td>150</td>
</tr>
<tr>
<td>Wellbore radius, r_w</td>
<td>ft</td>
<td>0.33</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Formation volume Factor, B</td>
<td>rb/stb</td>
<td>1.05</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Fluid viscosity, μ</td>
<td>cp</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Horizontal permeability in x, k_x</td>
<td>md</td>
<td>350</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>Vertical permeability in y, k_y</td>
<td>md</td>
<td>70</td>
<td>50</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 1- Reservoir/Fluid Properties Used for Example Cases of Chen-Assad^{10} (Comparison of Different Pseudosteady State Productivity Index Correlations)

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
<th>PI (stb/d/psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A1</td>
</tr>
<tr>
<td>Horizontal Well</td>
<td>Babu-Odeh^{11}</td>
<td>11.40</td>
</tr>
<tr>
<td></td>
<td>Chen-Assad^{10} Simp.</td>
<td>11.45</td>
</tr>
<tr>
<td></td>
<td>Chen-Assad^{10} Full</td>
<td>11.44</td>
</tr>
<tr>
<td></td>
<td>DVS</td>
<td>11.44</td>
</tr>
<tr>
<td></td>
<td>Goode-Kuchuk^{12}</td>
<td>13.00</td>
</tr>
<tr>
<td></td>
<td>Chen-Assad^{10} Simp.</td>
<td>12.99</td>
</tr>
<tr>
<td></td>
<td>Chen-Assad^{10} Full</td>
<td>12.99</td>
</tr>
<tr>
<td></td>
<td>DVS</td>
<td>12.75</td>
</tr>
<tr>
<td>Fractured Well</td>
<td>Chen-Assad^{10} Simp.</td>
<td>14.10</td>
</tr>
<tr>
<td></td>
<td>DVS</td>
<td>14.07</td>
</tr>
<tr>
<td></td>
<td>Chen-Assad^{10} Simp.</td>
<td>16.51</td>
</tr>
<tr>
<td></td>
<td>DVS</td>
<td>16.10</td>
</tr>
</tbody>
</table>

Table 2- Comparison of DVS Results with Various Pseudo-steady State Productivity Index Correlations
Reservoir parameters

- \( x_e = 1866.7 \text{ ft} \)
- \( y_e = 1866.7 \text{ ft} \)
- \( z_e = 100 \text{ ft} \)
- \( k_x = 0.1 \text{ md} \)
- \( k_y = 0.1 \text{ md} \)
- \( k_z = 0.01 \text{ md} \)

Fracture No1 parameters

- center location: \( x_1 = 1555.6 \text{ ft} \)
- \( y: \text{ middle} \)
- \( z: \text{ middle} \)
- \( y\text{-half length}: y_{h1} = 250 \text{ ft} \)
- \( \text{height}: h_{f1} = 75 \text{ ft} \)

DVS representation:
- \( c_x = 1555.6 \text{ ft}, c_y = 933.4 \text{ ft}, c_z = 50 \text{ ft} \)
- \( w_x = 1 \text{ in.}, w_y = 250 \text{ ft}, w_z = 37.5 \text{ ft} \)

Fracture No 2 parameters

- center location: \( x_2 = 933.4 \text{ ft} \)
- \( y: \text{ middle} \)
- \( z: \text{ middle} \)
- \( y\text{-half length}: y_{h2} = 250 \text{ ft} \)
- \( \text{height}: h_{f2} = 75 \text{ ft} \)

DVS representation:
- \( c_x = 933.4 \text{ ft}, c_y = 933.4 \text{ ft}, c_z = 50 \text{ ft} \)
- \( w_x = 1 \text{ in.}, w_y = 250 \text{ ft}, w_z = 37.5 \text{ ft} \)

### Table 3- Horizontal Well with Two Transverse Fractures, Reservoir and Fracture parameters

<table>
<thead>
<tr>
<th>( C_{21} )</th>
<th>( C_{w2} )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( q_1 )</th>
<th>( J_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.4120</td>
<td>0.9385</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.4387</td>
<td>0.6344</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.3804</td>
<td>0.4706</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.4128</td>
<td>0.3772</td>
</tr>
</tbody>
</table>

### Table 4- The Relative Importance of Well Pressure Loss and Flow Convergence Skin in the Two Transverse Fracture System

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J_D )</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5628</td>
<td>Uniform flux</td>
</tr>
<tr>
<td>2</td>
<td>0.4393</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.4503</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.4614</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.4659</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.4676</td>
<td></td>
</tr>
</tbody>
</table>

### Table 5- Stabilized Dimensionless Productivity Index of the Finite Conductivity Vertical Fracture (Vertically Fully Penetrating, \( I_c = 0.25, C_m = 1.6 \)) Calculated with Various Fracture Discretization Schemes
Figure 1- Schematic of the Box-in-Box Model

Figure 2- Comparison of the DVS Results With Line Source and Cylindrical Source Solutions: Vertical Well in a Bounded Reservoir (In the Box-in-Box Model $w_z = z_e/2$, $w_x = w_y = 0.477$ ft, and the actual value of $z_e$ is irrelevant)

Calculation Parameters:
- $A = 40$ Acres
- $r_o = 0.33$ ft
- $x_e = y_e = 1320$ ft
Figure 3- Comparison of DVS Results with Yildiz’s Model for a Partially-Penetrating Vertical Well
(In the Box-in-Box Model $h$, $h_p$, and $h_b$ are represented by $z_e$, $2w_z$, and $c_z-w_z$ respectively)

Figure 4- Comparison of DVS Results with Ozkan’s and Gringarten’s Solutions for a Horizontal Well in a Bounded Reservoir, Uniform Flux Solution (In the Box-in-Box Model $x_{ed}=x/w_x$, $y_{ed}=y/w_x$, and $L_D=z/w_x$)
Figure 5: Comparison of DVS Results with Gringarten’s Solution for a Vertically Fractured Well in a Bounded Reservoir; Uniform Flux Solution ($x_{id} = x_i/w$ in the Box-in-Box Model).

Figure 6: Comparison of DVS Results with Gringarten’s Solution for a Vertically Fractured Well in a Bounded Reservoir; Infinite Conductivity Solution. ($x_{id} = x_i/w$ in the Box-in-Box Model).
Figure 7- Schematics of a System of Two Transverse Fractures Intersected by a Horizontal Well

Figure 8- Comparison of DVS with Cinco-Ley et al. Results for Finite Conductivity Fracture
Figure 9- Distribution of the Contribution of Different Segments to Total Production at Various Times, Finite Conductivity Vertical Fracture of Full Vertical Penetration ($C_m = 1.6$ and $I_x = 0.25$)

Figure 10- Transient and Pseudo-steady State Productivity Index of the Finite Conductivity Fracture of Full Vertical Penetration ($C_m = 1.6$ and $I_x = 0.25$)