RCW

SOLUTIONS OF THE DIFFUSIVITY EQUATION BY THE METHOD OF LAPLACE TRANSFORMS

A Handbook for Petroleum Engineers

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This book presents the solutions of the diffusivity equation in Laplace space. Five inner boundary conditions and three outer boundary conditions are addressed. The inner boundary conditions are (1) constant rate, (2) constant rate with wellbore storage, (3) constant rate with wellbore storage and skin, (4) constant pressure, and (5) constant pressure with skin. The outer boundary conditions are (1) infinite acting, (2) constant pressure, and (3) no flow. Thus, for each coordinate system, fifteen different solutions are presented. The line source solution for radial flow and the point source solution for spherical flow are also presented, for a total of fourty-seven solutions. The early time and late time approximations to each of these solutions are also given, and inverted to real space.

RADIAL COORDINATE SYSTEM
EARLY TIME APPROXIMATIONS AT WELLBORE

	Outer Boundary Condition		
Inner Boundary	Infinite Acting	Constant Pressure	No Flow
Constant Rate	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 2\sqrt{\frac{t_D}{\pi}}$
Constant Rate Wellbore Storage	$p_D = \frac{t_D}{C_D}$	$p_D = \frac{t_D}{C_D}$	$p_D = \frac{t_D}{C_D}$
Constant Rate Wellbore Storage and Skin	$p_{wD} = \frac{t_D}{C_D}$	$p_{wD} = \frac{t_D}{C_D}$	$p_{wD} = \frac{t_D}{C_D}$
Constant Pressure	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$
Constant Pressure with Skin	$q_D = \frac{-1}{s}$ $Q_D = \frac{-t_D}{s}$	$q_D = \frac{-1}{s}$ $Q_D = \frac{-t_D}{s}$	$q_D = \frac{-1}{s}$ $Q_D = \frac{-t_D}{s}$

RADIAL COORDINATE SYSTEM
LATE TIME APPROXIMATIONS AT WELLBORE

	Outer Boundary Condition		
Inner Boundary	Infinite Acting	Constant Pressure	No Flow
Constant Rate	$p_D = \frac{1}{2}(\ln t_D + 0.80907)$	$p_D = \ln r_{eD}$	$p_D = \frac{2\,t_D}{r_{eD}^2 - 1}$
Constant Rate Wellbore Storage	$p_D = \frac{1}{2}(\ln t_D + 0.80907)$	$p_D = \ln r_{eD}$	$p_D = \frac{2\,t_D}{r_{eD}^2 - 1 + 2C_D}$
Constant Rate Wellbore Storage and Skin	$p_{wD} = \frac{1}{2}(\ln t_D + 0.80907 + 2s)$	$p_{wD} = \ln r_{eD} + s$	$p_{wD} = \frac{2\,t_D}{r_{eD}^2 - 1 + 2C_D}$
Constant Pressure	$q_D = \frac{-2}{\ln t_D + 0.80907}$ $Q_D = \frac{-2\,t_D}{\ln t_D + 0.80907}$	$q_D = \frac{-1}{\ln r_{eD}}$ $Q_D = \frac{-\,t_D}{\ln r_{eD}}$	$q_D = 0$ $Q_D = \frac{-\,(r_{eD}^2 - 1)}{2}$
Constant Pressure with Skin	$q_D = \frac{-2}{\ln t_D + 0.80907 + 2s}$ $Q_D = \frac{-2\,t_D}{\ln t_D + 0.80907 + 2s}$	$q_D = \frac{-1}{\ln r_{eD} + s}$ $Q_D = \frac{-\,t_D}{\ln r_{eD} + s}$	$q_D = 0$ $Q_D = \frac{-\,(r_{eD}^2 - 1)}{2}$

SOLUTION TO THE DIFFUSIVITY EQUATION
RADIAL COORDINATE SYSTEM

Darcy's Law
(production is negative)

$$q = -\frac{k\,A}{\mu}\,\frac{dp}{dr}$$

$$q = -\frac{2\,\pi\,k\,h}{\mu}\,r\,\frac{dp}{dr}$$

Diffusivity Equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\,\frac{\partial p}{\partial r} = \frac{\phi\,\mu\,c_t}{k}\,\frac{\partial p}{\partial t}$$

Assumptions in diffusivity equation

(1) neglect gravity effects
(2) homogeneous and isotropic reservoir
(3) constant height, porosity, and permeability
(4) constant viscosity
(5) c_t is small and constant
(6) pressure gradient squared terms are small and negligible

Initial Condition

$$p\,(r, 0) = p_i$$

Inner Boundary Condition

Line Source with Constant Rate (Infinitesimaly small r_w)

$$(r\,\frac{\partial p}{\partial r})_{r \to 0} = -\frac{q\,\mu}{2\,\pi\,k\,h}$$

Constant Rate

$$(r\,\frac{\partial p}{\partial r})_{r = r_w} = -\frac{q\,\mu}{2\,\pi\,k\,h}$$

Constant Rate with Wellbore Storage

$$q = C\,(\frac{\partial p}{\partial t})_{r = r_w} - \frac{2\,\pi\,k\,h}{\mu}\,(r\,\frac{\partial p}{\partial r})_{r = r_w}$$

Constant Rate with Wellbore Storage and Skin

$$q = C\,(\frac{dp_w}{dt}) - \frac{2\,\pi\,k\,h}{\mu}\,(r\,\frac{\partial p}{\partial r})_{r = r_w^+}$$

$$p_w = [p - s\,(r\,\frac{\partial p}{\partial r})]_{r = r_w^+}$$

Constant Pressure

$$p\ (r_{w,}\ t) = p_w = constant$$

Constant Pressure with Skin

$$p_w = [p - s\ (r\ \frac{\partial p}{\partial r})]_{r = r_w^+} = constant$$

Outer Boundary Condition

Infinite Acting

$$p\ (\infty,\ t) = p_i$$

Constant Pressure

$$p\ (r_e,\ t) = p_i$$

No Flow

$$\frac{\partial p}{\partial r}\ (\ r_e,\ t\ ) = 0$$

Define Dimensionless Variables

$$r_D = \frac{r}{r_w} \qquad r = r_D\ r_w$$

$$t_D = \frac{k\ t}{\phi\ \mu\ c_t\ r_w^2} \qquad t = \frac{\phi\ \mu\ c_t\ r_w^2\ t_D}{k}$$

For Constant Rate Inner Boundary Condition

$$p_D = \frac{2\ \pi\ k\ h\ (p - p_i)}{q\ \mu} \qquad p = p_i + \frac{p_D\ q\ \mu}{2\ \pi\ k\ h}$$

For Constant Pressure Inner Boundary Condition

$$p_D = \frac{p - p_i}{p_w - p_i} \qquad p = p_i + p_D\ (p_w - p_i)$$

Dimensionless Diffusivity Equation

Constant Rate Inner Boundary Case

$$\frac{\partial^2 (p_i + \frac{p_D\ q\ \mu}{2\ \pi\ k\ h})}{\partial (r_D\ r_w)^2} + \frac{1}{r_D\ r_w}\ \frac{\partial (p_i + \frac{p_D\ q\ \mu}{2\ \pi\ k\ h})}{\partial (r_D\ r_w)} = \frac{\phi\ \mu\ c_t}{k}\ \frac{\partial\ (p_i + \frac{p_D\ q\ \mu}{2\ \pi\ k\ h})}{\partial (\frac{\phi\ \mu\ c_t\ r_w^2\ t_D}{k})}$$

Reducing yields

$$\frac{q\ \mu}{2\ \pi\ k\ h\ r_w^2}\ \frac{\partial^2 p_D}{\partial r_D{}^2} + \frac{q\ \mu}{2\ \pi\ k\ h\ r_w^2\ r_D}\ \frac{\partial p_D}{\partial r_D} = \frac{\phi\ \mu\ c_t}{k}\ \frac{q\ \mu}{2\ \pi\ k\ h}\ \frac{k}{\phi\ \mu\ c_t\ r_w^2}\ \frac{\partial p_D}{\partial t_D}$$

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D}\ \frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D}$$

Constant Pressure Inner Boundary Case

$$\frac{\partial^2[p_i + p_D\,(p_w - p_i)]}{\partial(r_D\,r_w)^2} + \frac{1}{r_D\,r_w}\,\frac{\partial[p_i + p_D(p_w - p_i)]}{\partial(r_D\,r_w)}$$

$$= \frac{\phi\,\mu\,c_t}{k}\,\frac{\partial[p_i + p_D\,(p_w - p_i)]}{\partial(\dfrac{\phi\,\mu\,c_t\,r_w^2\,t_D}{k})}$$

$$\frac{(p_w - p_i)}{r_w^2}\,\frac{\partial^2 p_D}{\partial r_D^2} + \frac{(p_w - p_i)}{r_w^2\,r_D}\,\frac{\partial p_D}{\partial r_D} = \frac{\phi\,\mu\,c_t}{k}\,\frac{(p_w - p_i)\,k}{\phi\,\mu\,c_t\,r_w^2}\,\frac{\partial p_D}{\partial t_D}$$

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D}\,\frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D}$$

Dimensionless Initial Condition

$$p_D\,(r_D, 0) = 0$$

Dimensionless Inner Boundary Conditions

Line Source with Constant Rate

$$[r_w\,r_D\,\frac{\partial\,(p_i + \dfrac{q\,\mu\,p_D}{2\,\pi\,k\,h})}{\partial\,(r_w\,r_D)}]_{r_D \to 0} = -\,\frac{q\,\mu}{2\,\pi\,k\,h}$$

$$[r_D\,\frac{\partial p_D}{\partial r_D}]_{r_D \to 0} = -\,1$$

Constant Rate

$$[r_w\,r_D\,\frac{\partial\,(p_i + \dfrac{q\,\mu\,p_D}{2\,\pi\,k\,h})}{\partial\,(r_w\,r_D)}]_{r_D = 1} = -\,\frac{q\,\mu}{2\,\pi\,k\,h}$$

$$[\,\frac{r_w\,r_D\,q\,\mu}{r_w\,2\,\pi\,k\,h}\,\frac{\partial p_D}{\partial r_D}\,]_{r_D = 1} = -\,\frac{q\,\mu}{2\,\pi\,k\,h}$$

$$[r_D\,\frac{\partial p_D}{\partial r_D}]_{r_D = 1} = -\,1$$

Constant Rate with Wellbore Storage

$$q = C\,[\,\frac{\partial(p_i + \dfrac{q\,\mu\,p_D}{2\,\pi\,k\,h})}{\partial(\dfrac{\phi\,\mu\,c_t\,r_w^2\,t_D}{k})}\,]_{r_D = 1} - \frac{2\,\pi\,k\,h}{\mu}\,[r_w\,r_D\,\frac{\partial(p_i + \dfrac{q\,\mu\,p_D}{2\,\pi\,k\,h}\,)}{\partial(r_w\,r_D)}]_{r_D = 1}$$

$$q = \frac{C\,q\,\mu}{2\,\pi\,k\,h}\,\frac{k}{\phi\,\mu\,c_t\,r_w^2}\,(\frac{\partial p_D}{\partial t_D})_{r_D = 1} - \frac{2\,\pi\,k\,h}{\mu}\,\frac{q\,\mu}{2\,\pi\,k\,h}\,\frac{r_w}{r_w}\,(r_D\,\frac{\partial p_D}{\partial r_D})_{r_D = 1}$$

$$1 = \frac{C}{2\,\pi\,h\,\phi\,c_t\,r_w^2}\,(\frac{\partial p_D}{\partial t_D})_{r_D=1} \;-\; (r_D\,\frac{\partial p_D}{\partial r_D})_{r_D=1}$$

let

$$C_D = \frac{C}{2\,\pi\,h\,\phi\,c_t\,r_w^2}$$

$$1 = C_D\,(\frac{\partial P_D}{\partial t_D})_{r_D=1} \;-\; (r_D\,\frac{\partial P_D}{\partial r_D})_{r_D=1}$$

Constant Rate with Wellbore Storage and Skin

$$q = C\,[\,\frac{d(p_i + \frac{q\,\mu\,p_{wD}}{2\,\pi\,k\,h})}{d(\frac{\phi\,\mu\,c_t\,r_w^2\,t_D}{k})}\,] - \frac{2\,\pi\,k\,h}{\mu}\,[r_w\,r_D\,\frac{\partial(p_i + \frac{q\,\mu\,p_D}{2\,\pi\,k\,h})}{\partial(r_w\,r_D)}\,]_{r_D=1^+}$$

$$q = \frac{C\,q\,\mu}{2\,\pi\,k\,h}\,\frac{k}{\phi\,\mu\,c_t\,r_w^2}\,(\frac{dp_{wD}}{dt_D}) - \frac{2\,\pi\,k\,h}{\mu}\,\frac{q\,\mu}{2\,\pi\,k\,h}\,\frac{r_w}{r_w}\,(r_D\,\frac{\partial p_D}{\partial r_D})_{r_D=1^+}$$

$$1 = \frac{C}{2\,\pi\,h\,\phi\,c_t\,r_w^2}\,(\frac{dp_{wD}}{dt_D}) - (r_D\,\frac{\partial p_D}{\partial r_D})_{r_D=1^+}$$

let

$$C_D = \frac{C}{2\,\pi\,h\,\phi\,c_t\,r_w^2}$$

$$1 = C_D\,(\frac{dp_{wD}}{dt_D}) - (r_D\,\frac{\partial p_D}{\partial r_D})_{r_D=1^+}$$

$$p_w = [p - s\,(r\,\frac{\partial p}{\partial r})]_{r=r_w^+}$$

$$p_i + \frac{q\,\mu\,p_{wD}}{2\,\pi\,k\,h} = [p_i + \frac{q\,\mu\,p_D}{2\,\pi\,k\,h} - s\,r_w\,r_D\,\frac{\partial(p_i + \frac{q\,\mu\,p_D}{2\,\pi\,k\,h})}{\partial(r_w\,r_D)}\,]_{r=r_w^+}$$

$$\frac{q\,\mu\,P_{wD}}{2\,\pi\,k\,h} = [\,\frac{q\,\mu\,P_D}{2\,\pi\,k\,h} - \frac{s\,r_w\,q\,\mu}{2\,\pi\,k\,h\,r_w}\,r_D\,\frac{\partial p_D}{\partial r_D}]_{r_D=1^+}$$

$$p_{wD} = [p_D - s\,r_D\,\frac{\partial p_D}{\partial r_D}]_{r_D=1^+}$$

$$\frac{dp_{wD}}{dt_D} = [\frac{\partial p_D}{\partial t_D} - s\,r_D\,\frac{\partial}{\partial t_D}(\frac{\partial p_D}{\partial r_D})]_{r_D=1^+}$$

$$1 = C_D\,[\frac{\partial p_D}{\partial t_D} - s\,r_D\,\frac{\partial}{\partial t_D}(\frac{\partial p_D}{\partial r_D})]_{r_D=1^+} - (r_D\,\frac{\partial p_D}{\partial r_D})_{r_D=1^+}$$

Constant Pressure

$$p_D\,(1,\, t_D) = 1$$

Constant Pressure with Skin

$$p_i + p_{wD}\,(p_w - p_i) = [p_i + p_D\,(p_w - p_i) - s\, r_D\, r_w \frac{\partial[p_i + p_D\,(p_w - p_i)]}{\partial(r_D\, r_w)}\;]_{r_D = 1^+}$$

$$p_{wD}\,(p_w - p_i) = [p_D\,(p_w - p_i) - \frac{s\, r_w\,(p_w - p_i)}{r_w}\; r_D \frac{\partial p_D}{\partial r_D}]_{r_D = 1^+}$$

$$p_{wD} = (p_D - s\, r_D \frac{\partial p_D}{\partial r_D})_{r_D = 1^+}$$

but $p_{wD} = 1$, so

$$1 = (p_D - s\, r_D \frac{\partial p_D}{\partial r_D})_{r_D = 1^+}$$

Dimensionless Outer Boundary Conditions

Infinite Acting

$$p_D(\infty,\, t_D) = 0$$

Constant Pressure

$$p_D\,(r_{eD},\, t_D) = 0$$

No Flow

$$\frac{\partial(p_i + \dfrac{q\,\mu\, p_D}{2\,\pi\, k\, h})}{\partial(r_w\, r_D)}\;(r_{eD},\, t_D) = 0$$

$$\frac{q\,\mu}{2\,\pi\, k\, h\, r_w}\;\frac{\partial p_D}{\partial r_D}\;(r_{eD},\, t_D) = 0$$

so

$$\frac{\partial p_D}{\partial r_D}\;(r_{eD},\, t_D) = 0$$

Laplace Transformation

Since skin factor is denoted by s, let laplace variable be z.

$$\overline{p_D}(\, r_D,\, z) = \int_0^\infty p_D(r_D,\, t_D)\; e^{-z\, t_D}\; dt_D$$

Laplace Transform of Diffusivity Equation

$$\frac{d^2\overline{p_D}}{dr_D^2} + \frac{1}{r_D}\;\frac{d\overline{p_D}}{dr_D} = z\;\overline{p_D} - p_D(0)$$

but $p_D(0) = 0$, so

$$\frac{d^2\overline{p_D}}{dr_D^2} + \frac{1}{r_D}\frac{d\overline{p_D}}{dr_D} - z\,\overline{p_D} = 0$$

This is the modified Bessel equation and the general solution is

$$\overline{p_D} = A\,K_0\,(r_D\sqrt{z}\,) + B\,I_0\,(r_D\sqrt{z}\,)$$

where A and B are constants.
The first derivative is

$$\frac{d\overline{p_D}}{dr_D} = -\,A\sqrt{z}\,K_1\,(r_D\sqrt{z}) + B\sqrt{z}\,I_1\,(r_D\sqrt{z})$$

Laplace Transform of Dimensionless Inner Boundary Conditions

Line Source with Constant Rate

$$(r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D \to 0} = -\frac{1}{z}$$

Constant Rate

$$(r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1} = -\frac{1}{z}$$

Constant Rate with Wellbore Storage

$$C_D\,[z\,\overline{p_D} - p_D(0)]_{r_D = 1} - (r_D\frac{d\overline{p_D}}{dr_D})_{r_D = 1} = \frac{1}{z}$$

$$C_D\,(z\,\overline{p_D})_{r_D = 1} - (r_D\frac{d\overline{p_D}}{dr_D})_{r_D = 1} = \frac{1}{z}$$

Constant Rate with Wellbore Storage and Skin

$$C_D\,[z\,\overline{p_D} - p_D(0) - s\,r_D\,(z\frac{d\overline{p_D}}{dr_D} - (\frac{dp_D}{dr_D})_{t_D = 0})]_{r_D = 1^+} - (r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1^+} = \frac{1}{z}$$

$$[C_D\,z\,\overline{p_D} - C_D\,s\,r_D\,z\frac{d\overline{p_D}}{dr_D} - r_D\,\frac{d\overline{p_D}}{dr_D}]_{r_D = 1^+} = \frac{1}{z}$$

$$[C_D\,z\,\overline{p_D} - r_D\,\frac{d\overline{p_D}}{dr_D}\,(C_D\,s\,z + 1)]_{r_D = 1^+} = \frac{1}{z}$$

Constant Pressure

$$\overline{p_D}\,(1, z) = \frac{1}{z}$$

Constant Pressure with Skin

$$(\overline{p_D} - s\,r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1^+} = \frac{1}{z}$$

Laplace Transformation of Dimensionless Outer Boundary Conditions

Infinite Acting

$$\overline{p_D}(\infty, z) = 0$$

Constant Pressure

$$\overline{p_D}(r_{eD}, z) = 0$$

No Flow

$$\frac{d\overline{p_D}}{dr_D}(r_{eD}, z) = 0$$

Development of Dimensionless Flowrate

Define

$$q_D = \frac{q\,\mu}{2\,\pi\,k\,h\,(p_i - p_w)}$$

Darcy's law for a radial system is

$$q(r) = \frac{-2\,\pi\,k\,h}{\mu}\left(r\,\frac{dp}{dr}\right)$$

so

$$q_D = -\frac{r}{p_i - p_w}\,\frac{d[p_i + p_D\,(p_w - p_i)]}{d(r_w\,r_D)}$$

$$q_D = -\frac{r}{r_w}\,\frac{p_w - p_i}{p_i - p_w}\,\frac{dp_D}{dr_D}$$

$$q_D = r_D\,\frac{dp_D}{dr_D}$$

Taking the laplace transform gives

$$\overline{q_D} = r_D\,\frac{d\overline{p_D}}{dr_D}$$

Development of Dimensionless Cumulative Production

Convolution Theory

$$L\left[\int_0^{t_D} F_1(t_D - \tau)\,F_2(\tau)\,d\tau\right] = f_1(z)\,f_2(z)$$

Let

$$F_1(t_D - \tau) = 1 \qquad f_1(z) = \frac{1}{z}$$

so

$$L\left[\int_0^{t_D} (1)\,F_2(\tau)\,d\tau\right] = f_1(z)\,f_2(z) = \frac{1}{z}\,f_2(z)$$

$$Q_D = \int_0^{t_D} (1)\, q_D(t_D)\, d\tau$$

$$L\,[\, Q_D \,] = \overline{Q_D} = \frac{1}{z}\, \overline{q_D}$$

Plug General Solution Into Inner Boundary Condition

Line Source with Constant Rate

$$[- A\, r_D \sqrt{z}\, K_1(r_D \sqrt{z}) + B\, r_D \sqrt{z}\, I_1\,(r_D \sqrt{z})\,]_{r_D \to 0} = \frac{-1}{z}$$

$$\mathrm{K}_1$$

$$A = \frac{1}{z}$$

Constant Rate

$$A \sqrt{z}\, K_1(\sqrt{z}) - B \sqrt{z}\, I_1(\sqrt{z}) = \frac{1}{z}$$

Constant Rate with Wellbore Storage

$$C_D\, z\, [A\, K_0(\sqrt{z}) + B\, I_0(\sqrt{z})] - [\, - A \sqrt{z}\, K_1\,(\sqrt{z}) + B \sqrt{z}\, I_1\,(\sqrt{z})] = \frac{1}{z}$$

$$A\, [C_D\, z\, K_0(\sqrt{z}) + \sqrt{z}\, K_1\,(\sqrt{z})] + B\, [C_D\, z\, I_0(\sqrt{z}) - \sqrt{z}\, I_1\,(\sqrt{z})] = \frac{1}{z}$$

Constant Rate with Wellbore Storage and Skin

$$C_D\, z\, [A\, K_0(\sqrt{z}) + B\, I_0(\sqrt{z})] - [\, - A \sqrt{z}\, K_1\,(\sqrt{z}) + B \sqrt{z}\, I_1\,(\sqrt{z})]\, (C_D\, s\, z + 1) = \frac{1}{z}$$

$$A\, C_D\, z\, K_0(\sqrt{z}) + B\, C_D\, z\, I_0(\sqrt{z}) + A \sqrt{z}\, K_1\,(\sqrt{z})\, (C_D\, s\, z + 1) - B \sqrt{z}\, I_1\,(\sqrt{z})\, (C_D\, s\, z + 1) = \frac{1}{z}$$

$$A\, [C_D\, z\, K_0(\sqrt{z}) + \sqrt{z}\, K_1\,(\sqrt{z})\, (C_D\, s\, z + 1)] - B\, [C_D\, z\, I_0\,(\sqrt{z}) - \sqrt{z}\, I_1\,(\sqrt{z})\, (C_D\, s\, z + 1) = \frac{1}{z}$$

Constant Pressure

$$A\, K_0\,(\sqrt{z}) + B\, I_0\,(\sqrt{z}) = \frac{1}{z}$$

Constant Pressure with Skin

$$A\, K_0\,(\sqrt{z}) + B\, I_0\,(\sqrt{z}) - s\, [\, -A \sqrt{z}\, K_1\,(\sqrt{z}) + B \sqrt{z}\, I_1(\sqrt{z})] = \frac{1}{z}$$

$$A\, K_0\,(\sqrt{z}) + B\, I_0\,(\sqrt{z}) + s\, A \sqrt{z}\, K_1\,(\sqrt{z}) - s\, B \sqrt{z}\, I_1(\sqrt{z}) = \frac{1}{z}$$

$$A\, [K_0\,(\sqrt{z}) + s \sqrt{z}\, K_1\,(\sqrt{z})] + B\, [I_0\,(\sqrt{z}) - s \sqrt{z}\, I_1(\sqrt{z})] = \frac{1}{z}$$

Plug General Solution Into Outer Boundary Condition

Infinite Acting

$$A\,K_0(\infty\,\sqrt{z}) + B\,I_0(\infty\sqrt{z}) = 0$$

$$K_0(\infty) = 0 \qquad I_0(\infty) = \infty$$

$$B = 0$$

Constant Pressure

$$A\,K_0\,(r_{eD}\,\sqrt{Z}) + B\,I_0\,(r_{eD}\,\sqrt{z}) = 0$$

$$A = \;-B\,\frac{I_0\,(r_{eD}\,\sqrt{z})}{K_0(r_{eD}\,\sqrt{z})}$$

No Flow

$$-A\,\sqrt{z}\,K_1(r_{eD}\,\sqrt{z}) + B\,\sqrt{z}\,I_1\,(r_{eD}\,\sqrt{z}) = 0$$

$$A = B\,\frac{I_1(r_{eD}\,\sqrt{z})}{K_1(r_{eD}\,\sqrt{z})}$$

Early Time Approximation Functions

For all of the early time approximations, define the following functions

$$f(x) = \sqrt{\frac{\pi}{2\,x\,\sqrt{z}}}\;e^{-x\sqrt{z}}$$

$$g(x) = \frac{e^{x\sqrt{z}}}{\sqrt{2\,\pi\,x\,\sqrt{z}}}$$

Late Time Approximation Function

For all the late time approximations, define

$$h(x) = -\,[\ln(\frac{x\,\sqrt{z}}{2}) + \gamma]$$

Solution For Radial Case
Inner Boundary: Line Source at Constant Rate
Outer Boundary: Infinite Acting

From inner boundary condition

$$A = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$\overline{p_D} = \frac{K_0\,(r_D\,\sqrt{z})}{z}$$

Invert using Churchill # 116 and convolution

$$P_D = 1 * \frac{1}{2\,t_D}\,\exp(\frac{-\,r_D^2}{4\,t_D})$$

$$P_D = \int_0^{t_D} \frac{\exp(\frac{-\,r_D^2}{4\,t_D})}{2\,t_D}\,d\,t_D$$

let

$$u = \frac{r_D^2}{4\,t_D}$$

$$p_D = \frac{1}{2}\int_u^{\infty} \frac{e^{-\,u}}{u}\,du$$

$$p_D = \frac{-\,1}{2}\,E_i\,(-\,u) = \frac{-\,1}{2}\,E_i\,(\frac{-\,r_D^2}{4\,t_D})$$

Late Time Approximation

$z \to 0$

$$K_0\,(r_D\,\sqrt{z}) = -\,\ln\,(\frac{r_D\,\sqrt{z}}{2}\,) - 0.5772157$$

$$\overline{p_D} = \frac{-\,1}{z}\,[\,\ln\,(\frac{r_D\,\sqrt{z}}{2}) + 0.5772157\,]$$

$$\overline{p_D} = \frac{-\,\ln(\frac{r_D^2}{4})}{2\,z} - \frac{\ln\,z}{2\,z} - \frac{0.5772157}{z}$$

$$p_D = -\,\frac{1}{2}\,(-\,\gamma - \ln\,t_D) - \frac{1}{2}\,\ln\,\frac{r_D^2}{4} - \gamma$$

$$p_D = \frac{1}{2} \ln \frac{4\, t_D}{r_D^2} - \frac{\gamma}{2}$$

$$p_D = \frac{1}{2} [\ln \frac{t_D}{r_D^2} + \ln 4 - \gamma\,]$$

$$p_D = \frac{1}{2} [\ln \frac{t_D}{r_D^2} + 0.80907\,]$$

at $r_D = 1$

$$(p_D)_{r_D = 1} = \frac{1}{2} [\ln t_D + 0.80907\,]$$

Solution For Radial Case
Inner Boundary: Constant Rate
Outer Boundary: Infinite Acting

From inner boundary condition

$$A \sqrt{z}\, K_1(\sqrt{z}) - B \sqrt{z}\, I_1(\sqrt{z}) = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$A = \frac{1}{z^{3/2}\, K_1\,(\sqrt{z})}$$

and

$$\overline{p_D} = \frac{K_0\,(r_D \sqrt{z})}{z^{3/2}\, K_1\,(\sqrt{z})}$$

Early Time Approximation

$z \to \infty$

$$\overline{p_D} = \frac{f(r_D)}{z^{3/2} f(1)} = \frac{e^{-\sqrt{z}\,(r_D - 1)}}{\sqrt{r_D}\, z^{3/2}}$$

Invert using Churchill #85

$$p_D = \frac{1}{\sqrt{r_D}}\, [2 \sqrt{\frac{t_D}{\pi}}\, \exp(\frac{-(r_D - 1)^2}{4\, t_D}) - (r_D - 1)\, erfc(\frac{r_D - 1}{2\sqrt{t_D}}\,)]$$

at $r_D = 1$

$$(\overline{p_D})_{r_D = 1} = \frac{1}{z^{3/2}}$$

Invert using Churchill #5

$$(p_D)_{r_D = 1} = 2 \sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \to 0$

$$\overline{p_D} = \frac{h(r_D)}{\frac{z^{3/2}}{\sqrt{z}}} = \frac{-\frac{1}{2} \ln \frac{r_D^2\, z}{4} - \gamma}{z}$$

$$\overline{p_D} = -\frac{\ln z}{2z} - \frac{\ln\frac{r_D^2}{4}}{2z} - \frac{\gamma}{z}$$

Invert using Churchill #95

$$p_D = -\frac{1}{2}(-\gamma - \ln t_D) - \frac{1}{2}\ln\frac{r_D^2}{4} - \gamma$$

$$p_D = \frac{1}{2}\ln\frac{4\,t_D}{r_D^2} - \frac{\gamma}{2}$$

$$p_D = \frac{1}{2}\left[\ln\frac{t_D}{r_D^2} + \ln 4 - \gamma\right]$$

$$p_D = \frac{1}{2}\left[\ln\frac{t_D}{r_D^2} + 0.80907\right]$$

at $r_D = 1$

$$(p_D)_{r_D = 1} = \frac{1}{2}\left[\ln t_D + 0.80907\right]$$

Invert using Churchill #5

$$(p_D)_{r_D=1} = 2\sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \to 0$

$$\overline{p_D} = \frac{h(r_D) - h(r_{eD})}{z^{3/2}\left[\frac{1}{\sqrt{z}} + \frac{\sqrt{z}}{2}\,h(r_{eD})\right]}$$

$$\overline{p_D} = \frac{\ln \frac{r_{eD}}{r_D}}{z + \frac{z^2}{2}\,h(r_{eD})}$$

$$\overline{p_D} = \frac{\ln \frac{r_{eD}}{r_D}}{z}$$

Invert using Churchill #1

$$p_D = \ln \frac{r_{eD}}{r_D}$$

p_D is constant so this is steady-state.

at $r_D = 1$

$$(p_D)_{r_D=1} = \ln r_{eD}$$

Solution For Radial Case
Inner Boundary: Constant Rate
Outer Boundary: No Flow

From inner boundary condition

$$A \sqrt{z}\, K_1(\sqrt{z}) - B \sqrt{z}\, I_1(\sqrt{z}) = \frac{1}{z}$$

From outer boundary condition

$$A = B \frac{I_1(r_{eD} \sqrt{z})}{K_1(r_{eD} \sqrt{z})}$$

so

$$B = \frac{K_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,[\, I_1\,(r_{eD}\sqrt{z})\, K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\, K_1\,(r_{eD}\,\sqrt{z})]}$$

$$A = \frac{I_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,[\, I_1\,(r_{eD}\sqrt{z})\, K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\, K_1\,(r_{eD}\,\sqrt{z})]}$$

and

$$\overline{p_D} = \frac{I_1\,(\, r_{eD}\sqrt{z})\, K_0\,(r_D\sqrt{z}) + I_0\,(\, r_D\sqrt{z})\, K_1\,(\, r_{eD}\sqrt{z}\,)}{z^{3/2}\,[\, I_1\,(r_{eD}\sqrt{z})\, K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\, K_1\,(r_{eD}\,\sqrt{z})]}$$

Early Time Approximation

$z \to \infty$

$$\overline{p_D} = \frac{f(r_D)\, g(r_{eD}) + f(r_{eD})\, g(r_D)}{z^{3/2}\, [f(1)\, g(r_{eD}) - f(r_{eD})\, g(1)]}$$

$$\overline{p_D} = \frac{\dfrac{1}{\sqrt{r_D\, r_{eD}}}\, \left(e^{\sqrt{z}\,(r_{eD} - r_D)} + e^{-\sqrt{z}\,(r_{eD} - r_D)}\right)}{\dfrac{z^{3/2}}{\sqrt{r_{eD}}}\, \left(e^{\sqrt{z}\,(r_{eD} - 1)} - e^{-\sqrt{z}\,(r_{eD} - 1)}\right)}$$

$$\overline{p_D} = \frac{e^{-\sqrt{z}\,(r_D - 1)}}{z^{3/2}\, \sqrt{r_D}}$$

Invert using Churchill #85

$$p_D = \frac{1}{\sqrt{r_D}}\, [2 \sqrt{\frac{t_D}{\pi}}\, \exp(\frac{-\,(r_D - 1)^2}{4\, t_D}) - (r_D - 1)\, erfc(\frac{r_D - 1}{2\, \sqrt{t_D}}\,)]$$

at $r_D = 1$

$$(\overline{p_D})_{r_D = 1} = \frac{1}{z^{3/2}}$$

Invert using Churchill #5

$$(p_D)_{r_D=1} = 2\sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \to 0$

$$\overline{p_D} = \frac{h(r_D)\,\dfrac{r_{eD}\sqrt{z}}{2} + \dfrac{1}{r_{eD}\sqrt{z}}}{z^{3/2}\,[\dfrac{1}{\sqrt{z}}\,\dfrac{r_{eD}\sqrt{z}}{2} - \dfrac{\sqrt{z}}{2}\,\dfrac{1}{r_{eD}\sqrt{z}}]}$$

$$\overline{p_D} = \frac{h(r_D)\,\dfrac{r_{eD}\,z}{2} + \dfrac{1}{r_{eD}}}{z^2\,[\dfrac{r_{eD}}{2} - \dfrac{1}{2\,r_{eD}}\ ]}$$

$$\overline{p_D} = \frac{2}{z^2\,(r_{eD}^2 - 1)}$$

Invert using Churchill #2

$$p_D = \frac{2\,t_D}{r_{eD}^2 - 1}$$

$\dfrac{dp_D}{dt_D}$ is constant so this is pseudosteady-state.

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage
Outer Boundary: Infinite Acting

From inner boundary condition

$$A\ [C_D\ z\ K_0(\sqrt{z}) + \sqrt{z}\ K_1\ (\sqrt{z})] + B\ [C_D\ z\ I_0(\sqrt{z}) - \sqrt{z}\ I_1\ (\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$A = \frac{1}{z^{3/2}\ [\ K_1\ (\sqrt{z}) + C_D\ \sqrt{z}\ K_0\ (\sqrt{z})]}$$

and

$$\overline{p_D} = \frac{K_0\ (r_D\ \sqrt{z})}{z^{3/2}\ [\ K_1\ (\sqrt{z}) + C_D\ \sqrt{z}\ K_0\ (\sqrt{z})]}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{p_D} = \frac{f(r_D)}{z^{3/2} f(1) + C_D\ z^2 f(1)}$$

$$\overline{p_D} = \frac{e^{-r_D\sqrt{z}}}{\sqrt{r_D}\ (C_D\ z^2 + z^{3/2})\ e^{-\sqrt{z}}} = \frac{e^{-\sqrt{z}(r_D - 1)}}{\sqrt{r_D}\ C_D\ z^2}$$

Invert using Churchill #83

$$p_D = \int_0^{t_D} \frac{1}{\sqrt{r_D}\ C_D}\ erfc\ (\frac{r_D - 1}{2\ \sqrt{\tau}}\ )\ d\tau$$

at $r_D = 1$

$$(\overline{p_D})_{r_D = 1} = \frac{1}{C_D\ z^2}$$

Invert using Churchill #2

$$(p_D)_{r_D = 1} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \to 0$

$$\overline{p_D} = \frac{h(r_D)}{\frac{z^{3/2}}{\sqrt{z}} + C_D\, z^2\, h(1)}$$

$$\overline{p_D} = \frac{h(r_D)}{z}$$

$$\overline{p_D} = -\frac{\ln z}{2\, z} - \frac{\ln\frac{r_D^2}{4}}{2\, z} - \frac{\gamma}{z}$$

Invert using Churchill #95

$$p_D = -\frac{1}{2}\,(-\gamma - \ln t_D) - \frac{1}{2}\,\ln\frac{r_D^2}{4} - \gamma$$

$$p_D = \frac{1}{2}\,\ln\frac{4\, t_D}{r_D^2} - \frac{\gamma}{2}$$

$$p_D = \frac{1}{2}\,[\ln\frac{t_D}{r_D^2} + \ln 4 - \gamma\,]$$

$$p_D = \frac{1}{2}\,[\ln\frac{t_D}{r_D^2} + 0.80907\,]$$

at $r_D = 1$

$$(p_D)_{r_D = 1} = \frac{1}{2}\,[\ln t_D + 0.80907\,]$$

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage
Outer Boundary: Constant Pressure

From inner boundary condition

$$A\,[C_D\,z\,K_0(\sqrt{z}) + \sqrt{z}\,K_1\,(\sqrt{z})] + B\,[C_D\,z\,I_0(\sqrt{z}) - \sqrt{z}\,I_1\,(\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$A = \; - B\,\frac{I_0\,(r_{eD}\,\sqrt{z})}{K_0(r_{eD}\,\sqrt{z})}$$

let

$$D = I_0\,(r_{eD}\sqrt{z})\,K_1\,(\sqrt{z}) + I_1\,(\sqrt{z})\,K_0\,(r_{eD}\sqrt{z})$$
$$+\,C_D\,\sqrt{z}\,[I_0\,(r_{eD}\,\sqrt{z})\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z})\,K_0(r_{eD}\,\sqrt{z})]$$

so

$$B = \frac{-\,K_0\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

$$A = \frac{I_0\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

and

$$\overline{p_D} = \frac{I_0\,(r_{eD}\sqrt{z})\,K_0\,(\,r_D\sqrt{z}) - I_0\,(r_D\sqrt{z})\,K_0\,(\,r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{p_D} = \frac{f(r_D)\,g(r_{eD}) - f(r_{eD})\,g(r_D)}{z^{3/2}\,[f(1)\,g(r_{eD}) + f(r_{eD})\,g(1)] - C_D\,z^2\,[f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)]}$$

$$\overline{p_D} = \frac{\dfrac{1}{\sqrt{r_D\,r_{eD}}}\,(e^{\sqrt{z}\,(r_{eD} - r_D\,)} - e^{-\,\sqrt{z}\,(r_{eD} - r_D)})}{\dfrac{z^{3/2}}{\sqrt{r_{eD}}}\,(e^{\sqrt{z}\,(r_{eD} - 1\,)} + e^{-\,\sqrt{z}\,(r_{eD} - 1)}) + \dfrac{C_D\,z^2}{\sqrt{r_{eD}}}(e^{\sqrt{z}\,(r_{eD} - 1)} - e^{-\,\sqrt{z}\,(r_{eD} - 1)})}$$

$$\overline{p_D} = \frac{e^{-\,\sqrt{z}\,(r_D - 1)}}{C_D\,z^2\,\sqrt{r_D}}$$

Invert using Churchill #83

$$p_D = \int_0^{t_D} \frac{1}{C_D \sqrt{r_D}} \, erfc\left(\frac{r_D - 1}{2\sqrt{\tau}}\right) d\tau$$

at $r_D = 1$

$$(p_D)_{r_D = 1} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \to 0$

$$\overline{p_D} = \frac{h(r_D) - h(r_{eD})}{z^{3/2}\left[\frac{1}{\sqrt{z}} + \frac{\sqrt{z}}{2} h(r_{eD})\right] + C_D z^2 \left[h(1) - h(r_{eD})\right]}$$

$$\overline{p_D} = \frac{\ln \frac{r_{eD}}{r_D}}{z}$$

Invert using Churchill #1

$$p_D = \ln \frac{r_{eD}}{r_D}$$

p_D is constant so this is steady-state.

at $r_D = 1$

$$(p_D)_{r_D = 1} = \ln r_{eD}$$

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage
Outer Boundary: No Flow

From inner boundary condition

$$A\,[C_D\, z\, K_0(\sqrt{z}) + \sqrt{z}\, K_1\,(\sqrt{z})] + B\,[C_D\, z\, I_0(\sqrt{z}) - \sqrt{z}\, I_1\,(\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$A = B\,\frac{I_1(r_{eD}\,\sqrt{z})}{K_1(r_{eD}\,\sqrt{z})}$$

let

$$D = I_1\,(r_{eD}\sqrt{z})\,K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\,K_1\,(r_{eD}\sqrt{z})$$
$$+\, C_D\,\sqrt{z}\,[I_0\,(\sqrt{z})\,K_1\,(r_{eD}\sqrt{z}) + I_1\,(r_{eD}\,\sqrt{z})\,K_0\,(\sqrt{z})]$$

so

$$B = \frac{K_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

$$A = \frac{I_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

and

$$\overline{p_D} = \frac{I_1\,(\,r_{eD}\sqrt{z})\,K_0\,(r_D\sqrt{z}) + I_0\,(\,r_D\sqrt{z})\,K_1\,(\,r_{eD}\sqrt{z}\,)}{z^{3/2}\,D}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{p_D} = \frac{f(r_D)\,g(r_{eD}) + f(r_{eD})\,g(r_D)}{z^{3/2}\,[f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)] + C_D\,z^2\,[f(1)\,g(r_{eD}) + f(r_{eD})\,g(1)]}$$

$$\overline{p_D} = \frac{\dfrac{1}{\sqrt{r_D\,r_{eD}}}\,(e^{\sqrt{z}\,(r_{eD} - r_D\,)} + e^{-\sqrt{z}\,(r_{eD} - r_D)})}{\dfrac{z^{3/2}}{\sqrt{r_{eD}}}\,(e^{\sqrt{z}\,(r_{eD} - 1\,)} - e^{-\sqrt{z}\,(r_{eD} - 1)}) + \dfrac{C_D\,z^2}{\sqrt{r_{eD}}}(e^{\sqrt{z}\,(r_{eD} - 1)} + e^{-\sqrt{z}\,(r_{eD} - 1)})}$$

$$\overline{p_D} = \frac{e^{-\sqrt{z}\,(r_D - 1)}}{C_D\,z^2\,\sqrt{r_D}}$$

Invert using Churchill #83

$$p_D = \int_0^{t_D} \frac{1}{C_D\,\sqrt{r_D}}\;erfc(\frac{r_D - 1}{2\,\sqrt{\tau}})\;d\tau$$

at $r_D = 1$

$$(p_D)_{r_D = 1} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{p_D} = \frac{h(r_D)\,\frac{r_{eD}\sqrt{z}}{2} + \frac{1}{r_{eD}\sqrt{z}}}{z^{3/2}\,[\frac{1}{\sqrt{z}}\,\frac{r_{eD}\sqrt{z}}{2} - \frac{\sqrt{z}}{2}\,\frac{1}{r_{eD}\sqrt{z}} + C_D\,z^2\,[\,h(1)\,\frac{r_{eD}\sqrt{z}}{2} + \frac{1}{r_{eD}\sqrt{z}}]}$$

$$\overline{p_D} = \frac{\frac{1}{r_{eD}\sqrt{z}}}{z^{3/2}\,[\frac{r_{eD}}{2} - \frac{1}{2\,r_{eD}}] + \frac{C_D\,z^2}{r_{eD}\sqrt{z}}}$$

$$\overline{p_D} = \frac{2}{z^2\,(r_{eD}^2 - 1 + 2\,C_D)}$$

Invert using Churchill #2

$$p_D = \frac{2\,t_D}{r_{eD}^2 - 1 + 2\,C_D}$$

$\frac{dp_D}{dt_D}$ is constant so this is pseudosteady-state.

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage and Skin
Outer Boundary: Infinite Acting

From inner boundary condition

$$A\,[C_D\, z\, K_0(\sqrt{z}) + \sqrt{z}\, K_1\,(\sqrt{z})\,(C_D\, s\, z + 1)] - B\,[C_D\, z\, I_0\,(\sqrt{z}) - \sqrt{z}\, I_1\,(\sqrt{z})\,(C_D\, s\, z + 1) = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$A = \frac{1}{z^{3/2}\,[\,K_1\,(\sqrt{z}) + C_D\,\sqrt{z}\,K_0\,(\sqrt{z}) + z\,C_D\,s\,K_1\,(\sqrt{z})]}$$

and

$$\overline{p_D} = \frac{K_0\,(r_D\,\sqrt{z})}{z^{3/2}\,[\,K_1\,(\sqrt{z}) + C_D\,\sqrt{z}\,K_0\,(\sqrt{z}) + z\,C_D\,s\,K_1\,(\sqrt{z})]}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-\,K_1\,(r_D\,\sqrt{z})}{z\,[\,K_1\,(\sqrt{z}) + C_D\,\sqrt{z}\,K_0\,(\sqrt{z}) + z\,C_D\,s\,K_1\,(\sqrt{z})]}$$

$$\overline{p_{wD}} = (\overline{p_D} - s\,r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1^+}$$

$$\overline{p_{wD}} = \frac{K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})}{z^{3/2}\,[\,K_1\,(\sqrt{z}) + C_D\,\sqrt{z}\,K_0\,(\sqrt{z}) + z\,C_D\,s\,K_1\,(\sqrt{z})]}$$

Early Time Approximation

$z \longrightarrow \infty$

$$\overline{p_{wD}} = \frac{f(1) + s\,\sqrt{z}\,f(1)}{z^{3/2}\,f(1) + C_D\,z^2\,f(1) + C_D\,s\,z^{5/2}\,f(1)}$$

$$\overline{p_{wD}} = \frac{1 + s\,\sqrt{z}}{z^{3/2} + C_D\,z^2 + C_D\,s\,z^{5/2}}$$

$$\overline{p_{wD}} = \frac{s\,\sqrt{z}}{C_D\,s\,z^{5/2}} = \frac{1}{C_D\,z^2}$$

Invert using Churchill #2

$$p_{wD} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \to 0$

$$\overline{p_{wD}} = \frac{h(1) + s\sqrt{z}\,\dfrac{1}{\sqrt{z}}}{\dfrac{z^{3/2}}{\sqrt{z}} + C_D\, z^2\, h(1) + C_D\, s\, z^{5/2}\,\dfrac{1}{\sqrt{z}}}$$

$$\overline{p_{wD}} = \frac{h(1) + s}{z + C_D\, z^2\, h(1) + C_D\, s\, z^2}$$

$$\overline{p_{wD}} = \frac{h(1) + s}{z}$$

$$\overline{p_{wD}} = \frac{-\dfrac{1}{2}\ln\dfrac{z}{4} - \gamma + s}{z}$$

$$\overline{p_{wD}} = \frac{-\dfrac{1}{2}\ln z + \dfrac{1}{2}\ln 4 - \gamma + s}{z}$$

Invert using Churchill #95

$$p_{wD} = -\frac{1}{2}\left(-\gamma - \ln t_D\right) + \frac{1}{2}\ln 4 - \gamma + s$$

$$p_{wD} = \frac{1}{2}\left[\gamma + \ln t_D + \ln 4 - 2\,\gamma + 2\,s\right]$$

$$p_{wD} = \frac{1}{2}\left[\,\ln t_D + \ln 4 - \gamma + 2\,s\right]$$

$$p_{wD} = \frac{1}{2}\left[\ln t_D + 0.80907 + 2\,s\right]$$

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage and Skin
Outer Boundary: Constant Pressure

From inner boundary condition

$$A\,[C_D\,z\,K_0(\sqrt{z}) + \sqrt{z}\,K_1\,(\sqrt{z})\,(C_D\,s\,z+1)] - B\,[C_D\,z\,I_0\,(\sqrt{z}) - \sqrt{z}\,I_1\,(\sqrt{z})\,(C_D\,s\,z+1) = \frac{1}{z}$$

From outer boundary condition

$$A = -B\,\frac{I_0\,(r_{eD}\,\sqrt{z})}{K_0(r_{eD}\,\sqrt{z})}$$

let

$$\begin{aligned} D = {} & (C_D\,s\,z+1)\,[I_0\,(r_{eD}\sqrt{z})\,K_1(\sqrt{z}) + I_1\,(\sqrt{z})\,K_0\,(r_{eD}\sqrt{z})] \\ & + C_D\,\sqrt{z}\,[I_0\,(r_{eD}\,\sqrt{z})\,K_0(\sqrt{z}) - I_0\,(\sqrt{z})\,K_0\,(r_{eD}\,\sqrt{z})] \end{aligned}$$

so

$$B = \frac{-K_0\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

$$A = \frac{I_0\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

and

$$\overline{p_D} = \frac{I_0\,(r_{eD}\sqrt{z})\,K_0\,(\,r_D\sqrt{z}) - I_0\,(r_D\sqrt{z})\,K_0\,(\,r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-I_0\,(r_{eD}\,\sqrt{z})\,K_1(r_D\,\sqrt{z}) - I_1(r_D\,\sqrt{z})\,K_0\,(r_{eD}\,\sqrt{z})}{z\,D}$$

$$\overline{p_{wD}} = (\overline{p_D} - s\,r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1^+}$$

$$\overline{p_{wD}} = \frac{I_0\,(r_{eD}\,\sqrt{z})\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z})\,K_0\,(r_{eD}\,\sqrt{z}) + s\,\sqrt{z}\,[I_1\,(\sqrt{z})\,K_0\,(r_{eD}\,\sqrt{z}) + I_0\,(r_{eD}\,\sqrt{z})\,K_1\,(\sqrt{z})]}{z^{3/2}\,D}$$

Early Time Approximation

$z \longrightarrow \infty$

$$\overline{p_{wD}} = \frac{f(1)\,g(r_{eD}) - f(r_{eD})\,g(1) + s\,\sqrt{z}\,[\,f(r_{eD})\,g(1) + f(1)\,g(r_{eD})\,]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[f(r_{eD})\,g(1) + f(1)\,g(r_{eD})] + C_D\,z^2\,[f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)]}$$

$$\overline{p_{wD}} = \frac{e^{\sqrt{z}\,(r_{eD}-1)} - e^{-\sqrt{z}\,(r_{eD}-1)} + s\,\sqrt{z}\,[e^{\sqrt{z}\,(r_{eD}-1)} + e^{-\sqrt{z}\,(r_{eD}-1)}\,]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[e^{\sqrt{z}\,(r_{eD}-1)} + e^{-\sqrt{z}\,(r_{eD}-1)}\,] + C_D\,z^2\,[e^{\sqrt{z}\,(r_{eD}-1)} - e^{-\sqrt{z}\,(r_{eD}-1)}\,]}$$

$$\overline{p_{wD}} = \frac{e^{\sqrt{z}\,(r_{eD} - 1)} + s\,\sqrt{z}\,e^{\sqrt{z}\,(r_{eD} - 1)}}{(C_D\,s\,z^{5/2} + z^{3/2})\,e^{\sqrt{z}\,(r_{eD} - 1)} + C_D\,z^2\,e^{\sqrt{z}\,(r_{eD} - 1)}}$$

$$\overline{p_{wD}} = \frac{1 + s\,\sqrt{z}}{C_D\,s\,z^{5/2} + z^{3/2} + C_D\,z^2}$$

$$\overline{p_{wD}} = \frac{s\,\sqrt{z}}{C_D\,s\,z^{5/2}}$$

$$\overline{p_{wD}} = \frac{1}{C_D\,z^2}$$

Invert using Churchill #2

$$p_{wD} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{p_{wD}} = \frac{h(1) - h(r_{eD}) + s\,\sqrt{z}\,[\frac{\sqrt{z}}{2}\,h(r_{eD}) + \frac{1}{\sqrt{z}}]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[\frac{1}{\sqrt{z}} + \frac{\sqrt{z}}{2}\,h(r_{eD})] + C_D\,z^2\,[h(1) - h(r_{eD})]}$$

$$\overline{p_{wD}} = \frac{\ln r_{eD} + s\,\sqrt{z}\,\frac{1}{\sqrt{z}}}{z^{3/2}\,\frac{1}{\sqrt{z}} + C_D\,z^2\,\ln r_{eD}}$$

$$\overline{p_{wD}} = \frac{\ln r_{eD} + s}{z + C_D\,z^2\,\ln r_{eD}}$$

$$\overline{p_{wD}} = \frac{\ln r_{eD} + s}{z}$$

Invert using Churchill #1

$$p_{wD} = \ln r_{eD} + s$$

Solution For Radial Case
Inner Boundary: Constant Rate with Wellbore Storage and Skin
Outer Boundary: No Flow

From inner boundary condition

$$A\,[C_D\, z\, K_0(\sqrt{z}) + \sqrt{z}\, K_1\,(\sqrt{z})\,(C_D\, s\, z + 1)] - B\,[C_D\, z\, I_0\,(\sqrt{z}) - \sqrt{z}\, I_1\,(\sqrt{z})\,(C_D\, s\, z + 1) = \frac{1}{z}$$

From outer boundary condition

$$A = B\,\frac{I_1(r_{eD}\,\sqrt{z})}{K_1(r_{eD}\,\sqrt{z})}$$

let

$$D = (C_D\, s\, z + 1)\,[I_1\,(r_{eD}\sqrt{z})\,K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\,K_1\,(r_{eD}\sqrt{z})]$$
$$+ C_D\,\sqrt{z}\,[I_0\,(\sqrt{z})\,K_1\,(r_{eD}\,\sqrt{z}) + I_1\,(r_{eD}\,\sqrt{z})\,K_0\,(\sqrt{z})]$$

so

$$B = \frac{K_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$
$$A = \frac{I_1\,(r_{eD}\sqrt{z})}{z^{3/2}\,D}$$

and

$$\overline{p_D} = \frac{I_1\,(\,r_{eD}\sqrt{z})\,K_0\,(r_D\sqrt{z}) + I_0\,(\,r_D\sqrt{z})\,K_1\,(\,r_{eD}\sqrt{z}\,)}{z^{3/2}\,D}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-\,I_1\,(r_{eD}\,\sqrt{z})\,K_1(r_D\,\sqrt{z}) + I_1(r_D\,\sqrt{z})\,K_1\,(r_{eD}\,\sqrt{z})}{z\,D}$$

$$\overline{p_{wD}} = (\overline{p_D} - s\,r_D\,\frac{d\overline{p_D}}{dr_D})_{r_D = 1^+}$$

$$\overline{p_{wD}} = \frac{I_1\,(r_{eD}\sqrt{z})\,K_0\,(\sqrt{z}) + I_0\,(\sqrt{z})\,K_1\,(r_{eD}\sqrt{z}) + s\,\sqrt{z}\,[I_1\,(r_{eD}\,\sqrt{z})\,K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\,K_1\,(r_{eD}\,\sqrt{z})]}{z^{3/2}\,D}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{p_{wD}} = \frac{f(1)\,g(r_{eD}) + f(r_{eD})\,g(1) + s\,\sqrt{z}\,[\,f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)] + C_D\,z^2\,[f(1)\,g(r_{eD}) + f(r_{eD})\,g(1)]}$$

$$\overline{p_{wD}} = \frac{e^{\sqrt{z}\,(r_{eD}-1)} + e^{-\sqrt{z}\,(r_{eD}-1)} + s\,\sqrt{z}\,[e^{\sqrt{z}\,(r_{eD}-1)} - e^{-\sqrt{z}\,(r_{eD}-1)}]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[e^{\sqrt{z}\,(r_{eD}-1)} - e^{-\sqrt{z}\,(r_{eD}-1)}] + C_D\,z^2\,[e^{\sqrt{z}\,(r_{eD}-1)} + e^{-\sqrt{z}\,(r_{eD}-1)}]}$$

$$\overline{p_{wD}} = \frac{e^{\sqrt{z}\,(r_{eD}-1)} + s\,\sqrt{z}\,e^{\sqrt{z}\,(r_{eD}-1)}}{(C_D\,s\,z^{5/2} + z^{3/2})\,e^{\sqrt{z}\,(r_{eD}-1)} + C_D\,z^2\,e^{\sqrt{z}\,(r_{eD}-1)}}$$

$$\overline{p_{wD}} = \frac{1 + s\,\sqrt{z}}{C_D\,s\,z^{5/2} + z^{3/2} + C_D\,z^2}$$

$$\overline{p_{wD}} = \frac{s\,\sqrt{z}}{C_D\,s\,z^{5/2}}$$

$$\overline{p_{wD}} = \frac{1}{C_D\,z^2}$$

Invert using Churchill #2

$$p_{wD} = \frac{t_D}{C_D}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{p_{wD}} = \frac{\frac{-\,r_{eD}\,\sqrt{z}}{2}\,h(1) - \frac{1}{r_{eD}\,\sqrt{z}} + s\,\sqrt{z}\,[\frac{-1}{\sqrt{z}}\,\frac{r_{eD}\,\sqrt{z}}{2} + \frac{1}{r_{eD}\,\sqrt{z}}\,\frac{\sqrt{z}}{2}]}{(C_D\,s\,z^{5/2} + z^{3/2})\,[\frac{1}{r_{eD}\,\sqrt{z}}\,\frac{\sqrt{z}}{2} - \frac{1}{\sqrt{z}}\,\frac{r_{eD}\,\sqrt{z}}{2}] + C_D\,z^2\,[\frac{-\,r_{eD}\,\sqrt{z}}{2}\,h(1) - \frac{1}{r_{eD}\,\sqrt{z}}}$$

$$\overline{p_{wD}} = \frac{\frac{1}{r_{eD}\,\sqrt{z}}}{z^{3/2}\,(\frac{r_{eD}}{2} - \frac{1}{2\,r_{eD}}) + \frac{C_D\,z^2}{r_{eD}\,\sqrt{z}}}$$

$$\overline{p_{wD}} = \frac{1}{z^2\,[\frac{1}{2}\,(r_{eD}^2 - 1) + C_D]}$$

Invert using Churchill #2

$$p_{wD} = \frac{2\,t_D}{r_{eD}^2 - 1 + 2\,C_D}$$

Solution For Radial Case
Inner Boundary: Constant Pressure
Outer Boundary: Infinite Acting

From inner boundary condition

$$A\,K_0\,(\sqrt{z}) + B\,I_0\,(\sqrt{z}) = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$A = \frac{1}{z\,K_0\,(\sqrt{z})}$$

and

$$\overline{p_D} = \frac{K_0\,(\,r_D\sqrt{z}\,)}{z\,K_0\,(\sqrt{z})}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-\,K_1\,(r_D\,\sqrt{z})}{\sqrt{z}\,K_0\,(\sqrt{z})}$$

$$\overline{q_D} = \frac{-\,r_D\,K_1\,(r_D\,\sqrt{z})}{\sqrt{z}\,K_0\,(\sqrt{z})}$$

$$\overline{Q_D} = \frac{-\,r_D\,K_1\,(r_D\,\sqrt{z})}{z^{3/2}\,K_0\,(\sqrt{z})}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{q_D} = \frac{-\,r_D\,f(r_D)}{\sqrt{z}\,f(1)} = \frac{-\,r_D\,e^{-\sqrt{z}\,(r_D-1)}}{\sqrt{z}\,\sqrt{r_D}}$$

$$\overline{q_D} = \frac{-\,\sqrt{r_D}\,e^{-\sqrt{z}\,(r_D-1)}}{\sqrt{z}}$$

$$\overline{Q_D} = -\,\frac{\sqrt{r_D}\,e^{-\sqrt{z}\,(r_D-1)}}{z^{3/2}}$$

Invert using Churchill #84

$$q_D = -\sqrt{\frac{r_D}{\pi\,t_D}}\,\exp\!\left(\frac{-\,(r_D-1)^2}{4\,t_D}\right)$$

Invert using Churchill #85

$$Q_D = -\sqrt{r_D}\,[2\sqrt{\frac{t_D}{\pi}}\,\exp(\frac{-(r_D-1)^2}{4\,t_D}) - (r_D-1)\,erfc(\frac{r_D-1}{2\sqrt{t_D}})]$$

at $r_D = 1$

$$(q_D)_{r_D=1} = -\frac{1}{\sqrt{\pi\,t_D}}$$

$$(Q_D)_{r_D=1} = -2\sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \to 0$

$$(q_D)_{r_D=1} = \frac{2}{\ln t_D + 0.80907}$$

$$(Q_D)_{r_D=1} = \frac{2\,t_D}{\ln t_D + 0.80907}$$

Solution For Radial Case
Inner Boundary: Constant Pressure
Outer Boundary: Constant Pressure

From inner boundary condition

$$A\,K_0\,(\sqrt{z}) + B\,I_0\,(\sqrt{z}) = \frac{1}{z}$$

From outer boundary condition

$$A = -B\,\frac{I_0\,(r_{eD}\,\sqrt{z})}{K_0(r_{eD}\,\sqrt{z})}$$

so

$$B = \frac{-K_0\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

$$A = \frac{I_0\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

and

$$\overline{p_D} = \frac{I_0\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\,r_D\sqrt{z}\,) - I_0\,(\,r_D\sqrt{z}\,)\,K_0\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-I_0\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,) - I_1\,(\,r_D\sqrt{z}\,)\,K_0\,(\,r_{eD}\sqrt{z}\,)}{\sqrt{z}\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

$$\overline{q_D} = \frac{-r_D\,[I_0\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,) + I_1\,(\,r_D\sqrt{z}\,)\,K_0\,(\,r_{eD}\sqrt{z}\,)]}{\sqrt{z}\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

$$\overline{Q_D} = \frac{-r_D\,[I_0\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,) + I_1\,(\,r_D\sqrt{z}\,)\,K_0\,(\,r_{eD}\sqrt{z}\,)]}{z^{3/2}\,[\,I_0\,(r_{eD}\,\sqrt{z}\,)\,K_0\,(\sqrt{z}) - I_0\,(\sqrt{z}\,)\,K_0\,(r_{eD}\,\sqrt{z}\,)]}$$

Early Time Approximation

$z \longrightarrow \infty$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\;\frac{f(r_{eD})\,g(r_D) + f(r_D)\,g(r_{eD})}{f(r_{eD})\,g(1) - f(1)\,g(r_{eD})}$$

$$\overline{q_D} = \frac{\dfrac{r_D}{\sqrt{z}\,\sqrt{r_D\,r_{eD}}}\,\left(e^{\sqrt{z}\,(r_{eD}-r_D)} + e^{-\sqrt{z}\,(r_{eD}-r_D)}\right)}{\dfrac{1}{\sqrt{r_{eD}}}\,\left(-e^{\sqrt{z}\,(r_{eD}-1)} + e^{-\sqrt{z}\,(r_{eD}-1)}\right)}$$

$$\overline{q_D} = -\sqrt{\frac{r_D}{z}}\;e^{-\sqrt{z}\,(r_D-1)}$$

$$\overline{Q_D} = -\frac{\sqrt{r_D}}{z^{3/2}}\, e^{-\sqrt{z}\,(r_D - 1)}$$

Invert using Churchill #84

$$q_D = -\sqrt{\frac{r_D}{\pi\, t_D}}\, \exp(\frac{-(r_D - 1)^2}{4\, t_D})$$

Invert using Churchill #85

$$Q_D = -\sqrt{r_D}\, [2\sqrt{\frac{t_D}{\pi}}\, \exp(\frac{-(r_D - 1)^2}{4\, t_D}) - (r_D - 1)\, erfc(\frac{r_D - 1}{2\sqrt{t_D}})]$$

at $r_D = 1$

$$(q_D)_{r_D = 1} = \frac{-1}{\sqrt{\pi\, t_D}}$$

$$(Q_D)_{r_D = 1} = -2\sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\, \frac{h(r_{eD})\, \frac{r_D\sqrt{z}}{2} + \frac{1}{r_D\sqrt{z}}}{h(r_{eD}) - h(1)}$$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\, \frac{\frac{1}{r_D\sqrt{z}}}{-\ln r_{eD}}$$

$$\overline{q_D} = \frac{-1}{z \ln r_{eD}}$$

$$\overline{Q_D} = \frac{-1}{z^2 \ln r_{eD}}$$

Invert using Churchill #1

$$q_D = \frac{-1}{\ln r_{eD}}$$

Invert using Churchill #2

$$Q_D = \frac{-t_D}{\ln r_{eD}}$$

Solution For Radial Case
Inner Boundary: Constant Pressure
Outer Boundary: No Flow

From inner boundary condition

$$A\,K_0\,(\sqrt{z}) + B\,I_0\,(\sqrt{z}) = \frac{1}{z}$$

From outer boundary condition

$$A = B\,\frac{I_1(r_{eD}\,\sqrt{z})}{K_1(r_{eD}\,\sqrt{z})}$$

so

$$B = \frac{K_1\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

$$A = \frac{I_1\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

and

$$\overline{p_D} = \frac{I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\,r_D\sqrt{z}\,) + I_0\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,)}{z\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)}{\sqrt{z}\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

$$\overline{q_D} = \frac{r_D\,[I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)]}{\sqrt{z}\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

$$\overline{Q_D} = \frac{r_D\,[I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)]}{z^{3/2}\,[\,I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)]}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\,\frac{f(r_{eD})\,g(r_D) - f(r_D)\,g(r_{eD})}{f(1)\,g(r_{eD}) + f(r_{eD})\,g(1)}$$

$$\overline{q_D} = \frac{\sqrt{r_D}}{z}\,\frac{(-\,e^{\sqrt{z}\,(r_{eD} - r_D\,)} + e^{-\,\sqrt{z}\,(r_{eD} - r_D)})}{(e^{\sqrt{z}\,(r_{eD} - 1\,)} + e^{-\,\sqrt{z}\,(r_{eD} - 1)})}$$

$$\overline{q_D} = -\sqrt{\frac{r_D}{z}}\,e^{-\,\sqrt{z}\,(r_D - 1)}$$

$$\overline{Q_D} = -\sqrt{\frac{r_D}{z^{3/2}}}\, e^{-\sqrt{z}\,(r_D - 1)}$$

Invert using Churchill #84

$$q_D = -\sqrt{\frac{r_D}{\pi\, t_D}}\, \exp\left(\frac{-(r_D - 1)^2}{4\, t_D}\right)$$

Invert using Churchill #85

$$Q_D = -\sqrt{r_D}\,\left[2\sqrt{\frac{t_D}{\pi}}\, \exp\left(\frac{-(r_D - 1)^2}{4\, t_D}\right) - (r_D - 1)\, erfc\left(\frac{r_D - 1}{2\sqrt{t_D}}\right)\right]$$

at $r_D = 1$

$$(q_D)_{r_D = 1} = \frac{-1}{\sqrt{\pi\, t_D}}$$

$$(Q_D)_{r_D = 1} = -2\sqrt{\frac{t_D}{\pi}}$$

Late Time Approximation

$z \to 0$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\, \frac{\frac{1}{r_{eD}\sqrt{z}}\,\frac{r_D\sqrt{z}}{2} - \frac{1}{r_D\sqrt{z}}\,\frac{r_{eD}\sqrt{z}}{2}}{h(1)\,\frac{r_{eD}\sqrt{z}}{2} + \frac{1}{r_{eD}\sqrt{z}}}$$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\, \frac{\frac{r_D}{2\, r_{eD}} - \frac{r_{eD}}{2\, r_D}}{\frac{1}{r_{eD}\sqrt{z}}}$$

$$\overline{q_D} = \frac{\frac{r_D^2}{2\, r_{eD}} - \frac{r_{eD}}{2}}{\frac{1}{r_{eD}}}$$

$$\overline{q_D} = -\frac{1}{2}\,(r_{eD}^2 - r_D^2)$$

$$\overline{Q_D} = -\frac{1}{2\, z}\,(r_{eD}^2 - r_D^2)$$

Invert using Churchill #1

$$q_D = 0$$

Invert using Churchill #1

$$Q_D = -\frac{1}{2}\,(r_{eD}^2 - r_D^2)$$

For $r_D = 1$

$$(Q_D)_{r_D = 1} = -\frac{1}{2}\,(r_{eD}^2 - 1)$$

Solution For Radial Case
Inner Boundary: Constant Pressure with Skin
Outer Boundary: Infinite Acting

From inner boundary condition

$$A\,[K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})] + B\,[I_0\,(\sqrt{z}) - s\,\sqrt{z}\,I_1(\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$B = 0$$

so

$$A = \frac{1}{z\,[\,K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})]}$$

and

$$\overline{p_D} = \frac{K_0\,(\,r_D\sqrt{z}\,)}{z\,[\,K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})]}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-\,K_1\,(r_D\,\sqrt{z})}{\sqrt{z}\,[\,K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})]}$$

$$\overline{q_D} = \frac{-\,r_D\,K_1\,(r_D\,\sqrt{z})}{\sqrt{z}\,[\,K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})]}$$

$$\overline{Q_D} = \frac{-\,r_D\,K_1\,(r_D\,\sqrt{z})}{z^{3/2}\,[\,K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})]}$$

Early Time Approximation

$z \longrightarrow \infty$

$$\overline{q_D} = \frac{-\,r_D\,f(r_D)}{\sqrt{z}\,[f(1) + s\,\sqrt{z}\,f(1)]} = \frac{-\sqrt{r_D}\,e^{-\sqrt{z}\,(r_D - 1)}}{\sqrt{z}\,[1 + s\,\sqrt{z}]}$$

$$\overline{q_D} = \frac{-\sqrt{r_D}\,e^{-\sqrt{z}\,(r_D - 1)}}{s\,z}$$

Invert using Churchill #83

$$q_D = -\,\frac{\sqrt{r_D}}{s}\,erfc(\frac{r_D - 1}{2\,\sqrt{t_D}}\,)$$

at $r_D = 1$

$$(q_D)_{r_D = 1} = -\,\frac{1}{s}$$

$$Q_D = \int_0^{t_D} q_D(\tau)\,d\tau$$

$$(Q_D)_{r_D=1} = \int_0^{t_D} \frac{-1}{s}\, d\tau$$

$$(Q_D)_{r_D=1} = \frac{-1}{s} \int_0^{t_D} d\tau$$

$$(Q_D)_{r_D=1} = \frac{-t_D}{s}$$

Late Time Approximation

$z \to 0$

$$(q_D)_{r_D=1} = \frac{-2}{\ln t_D + 0.80907 + 2\, s}$$

$$(Q_D)_{r_D=1} = \frac{-2\, t_D}{\ln t_D + 0.80907 + 2\, s}$$

Solution For Radial Case
Inner Boundary: Constant Pressure with Skin
Outer Boundary: Constant Pressure

From inner boundary condition

$$A\,[K_0(\sqrt{z}) + s\sqrt{z}\,K_1(\sqrt{z})] + B\,[I_0(\sqrt{z}) - s\sqrt{z}\,I_1(\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$A = -B\,\frac{I_0(r_{eD}\sqrt{z})}{K_0(r_{eD}\sqrt{z})}$$

let

$$D = I_0(r_{eD}\sqrt{z})\,K_0(\sqrt{z}) - I_0(\sqrt{z})\,K_0(r_{eD}\sqrt{z})$$
$$+\ s\sqrt{z}\,[I_0(r_{eD}\sqrt{z})\,K_1(\sqrt{z}) + I_1(\sqrt{z})\,K_0(r_{eD}\sqrt{z})]$$

so

$$B = \frac{-K_0(r_{eD}\sqrt{z})}{z\,D}$$
$$A = \frac{I_0(r_{eD}\sqrt{z})}{z\,D}$$

and

$$\overline{p_D} = \frac{I_0(r_{eD}\sqrt{z})\,K_0(r_D\sqrt{z}) - I_0(r_D\sqrt{z})\,K_0(r_{eD}\sqrt{z})}{z\,D}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{-I_0(r_{eD}\sqrt{z})\,K_1(r_D\sqrt{z}) - I_1(r_D\sqrt{z})\,K_0(r_{eD}\sqrt{z})}{\sqrt{z}\,D}$$

$$\overline{q_D} = \frac{-r_D\,[I_0(r_{eD}\sqrt{z})\,K_1(r_D\sqrt{z}) + I_1(r_D\sqrt{z})\,K_0(r_{eD}\sqrt{z})]}{\sqrt{z}\,D}$$

$$\overline{Q_D} = \frac{-r_D\,[I_0(r_{eD}\sqrt{z})\,K_1(r_D\sqrt{z}) + I_1(r_D\sqrt{z})\,K_0(r_{eD}\sqrt{z})]}{z^{3/2}\,D}$$

Early Time Approximation

$z \rightarrow \infty$

$$\overline{q_D} = \frac{-r_D}{\sqrt{z}}\,\frac{f(r_{eD})\,g(r_D) + f(r_D)\,g(r_{eD})}{f(1)\,g(r_{eD}) - f(r_{eD})\,g(1) + s\sqrt{z}\,[f(1)\,g(r_{eD}) + f(r_{eD})\,g(1)]}$$

$$\overline{q_D} = -\frac{\sqrt{r_D}}{z}\,\frac{(e^{\sqrt{z}(r_{eD}-r_D)} + e^{-\sqrt{z}(r_{eD}-r_D)})}{(e^{\sqrt{z}(r_{eD}-1)} - e^{-\sqrt{z}(r_{eD}-1)}) + s\sqrt{z}\,[(e^{\sqrt{z}(r_{eD}-1)} + e^{-\sqrt{z}(r_{eD}-1)}]}$$

$$\overline{q_D} = -\sqrt{\frac{r_D}{z}}\,\frac{e^{-\sqrt{z}(r_D-1)}}{1 + s\sqrt{z}}$$

$$\overline{q_D} = -\frac{\sqrt{r_D}}{s\,z}\,e^{-\sqrt{z}\,(r_D - 1)}$$

Invert using Churchill #83

$$q_D = -\frac{\sqrt{r_D}}{s}\,erfc\left(\frac{r_D - 1}{2\sqrt{t_D}}\right)$$

at $r_D = 1$

$$(q_D)_{r_D = 1} = \frac{-1}{s}$$

$$Q_D = \int_0^{t_D} q_D(\tau)\,d\tau$$

$$(Q_D)_{r_D = 1} = \int_0^{t_D} \frac{-1}{s}\,d\tau$$

$$(Q_D)_{r_D = 1} = \frac{-1}{s}\int_0^{t_D} d\tau$$

$$(Q_D)_{r_D = 1} = \frac{-t_D}{s}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{q_D} = \frac{-r_D}{\sqrt{z}}\;\frac{h(r_{eD})\,\frac{r_D\sqrt{z}}{2} + \frac{1}{r_D\sqrt{z}}}{h(1) - h(r_{eD}) + s\sqrt{z}\,[\frac{1}{\sqrt{z}} + \frac{\sqrt{z}}{2}\,h(r_{eD})]}$$

$$\overline{q_D} = \frac{-r_D}{\sqrt{z}}\;\frac{\frac{1}{r_D\sqrt{z}}}{\ln r_{eD} + s}$$

$$\overline{q_D} = \frac{-1}{z\,[\ln r_{eD} + s]}$$

$$\overline{Q_D} = \frac{-1}{z^2\,[\ln r_{eD} + s]}$$

Invert using Churchill #1

$$q_D = \frac{-1}{\ln r_{eD} + s}$$

Invert using Churchill #2

$$Q_D = \frac{-t_D}{\ln r_{eD} + s}$$

Solution For Radial Case
Inner Boundary: Constant Pressure with Skin
Outer Boundary: No Flow

From inner boundary condition

$$A\,[K_0\,(\sqrt{z}) + s\,\sqrt{z}\,K_1\,(\sqrt{z})] + B\,[I_0\,(\sqrt{z}) - s\,\sqrt{z}\,I_1(\sqrt{z})] = \frac{1}{z}$$

From outer boundary condition

$$A = B\,\frac{I_1(r_{eD}\,\sqrt{z})}{K_1(r_{eD}\,\sqrt{z})}$$

let

$$D = I_0\,(\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) + I_1\,(\,r_{eD}\sqrt{z}\,)\,K_0\,(\sqrt{z}\,)$$
$$+\,s\,\sqrt{z}\,[I_1\,(r_{eD}\sqrt{z})K_1\,(\sqrt{z}) - I_1\,(\sqrt{z})\,K_1\,(r_{eD}\,\sqrt{z})]$$

so

$$B = \frac{K_1\,(\,r_{eD}\sqrt{z}\,)}{z\,D}$$

$$A = \frac{I_1\,(\,r_{eD}\sqrt{z}\,)}{zD}$$

and

$$\overline{p_D} = \frac{I_1\,(\,r_{eD}\sqrt{z}\,)K_0(r_D\,\sqrt{z}) + I_0\,(r_D\,\sqrt{z})\,K_1\,(r_{eD}\,\sqrt{z})}{z\,D}$$

$$\frac{d\overline{p_D}}{dr_D} = \frac{I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)}{\sqrt{z}\,D}$$

$$\overline{q_D} = \frac{r_D\,[I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)]}{\sqrt{z}\,D}$$

$$\overline{Q_D} = \frac{r_D\,[I_1\,(\,r_D\sqrt{z}\,)\,K_1\,(\,r_{eD}\sqrt{z}\,) - I_1\,(\,r_{eD}\sqrt{z}\,)\,K_1\,(\,r_D\sqrt{z}\,)]}{z^{3/2}\,D}$$

Early Time Approximation

$z \to \infty$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\,\frac{f(r_{eD})\,g(r_D) - f(r_D)\,g(r_{eD})}{f(1)\,g(r_{eD}) + f(r_{eD})\,g(1) + s\,\sqrt{z}\,[f(1)\,g(r_{eD}) - f(r_{eD})\,g(1)]}$$

$$\overline{q_D} = \frac{\sqrt{r_D}}{z} \frac{(-e^{\sqrt{z}\,(r_{eD} - r_D)} + e^{-\sqrt{z}\,(r_{eD} - r_D)})}{(e^{\sqrt{z}\,(r_{eD} - 1)} + e^{-\sqrt{z}\,(r_{eD} - 1)}) + s\sqrt{z}\,[(e^{\sqrt{z}\,(r_{eD} - 1)} - e^{-\sqrt{z}\,(r_{eD} - 1)}]}$$

$$\overline{q_D} = -\sqrt{\frac{r_D}{z}}\,\frac{e^{-\sqrt{z}\,(r_D - 1)}}{1 + s\sqrt{z}}$$

$$\overline{q_D} = -\frac{\sqrt{r_D}}{s\,z}\,e^{-\sqrt{z}\,(r_D - 1)}$$

Invert using Churchill #83

$$q_D = -\frac{\sqrt{r_D}}{s}\,erfc(\frac{r_D - 1}{2\sqrt{t_D}})$$

at $r_D = 1$

$$(q_D)_{r_D = 1} = \frac{-1}{s}$$

$$Q_D = \int_0^{t_D} q_D(\tau)\,d\tau$$

$$(Q_D)_{r_D = 1} = \int_0^{t_D} \frac{-1}{s}\,d\tau$$

$$(Q_D)_{r_D = 1} = \frac{-1}{s}\int_0^{t_D} d\tau$$

$$(Q_D)_{r_D = 1} = \frac{-t_D}{s}$$

Late Time Approximation

$z \rightarrow 0$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\,\frac{\frac{1}{r_{eD}\sqrt{z}}\,\frac{r_D\sqrt{z}}{2} - \frac{1}{r_D\sqrt{z}}\,\frac{r_{eD}\sqrt{z}}{2}}{h(1)\,\frac{r_{eD}\sqrt{z}}{2} + \frac{1}{r_{eD}\sqrt{z}} + s\sqrt{z}\,[\frac{r_{eD}\sqrt{z}}{2}\,\frac{1}{\sqrt{z}} - \frac{\sqrt{z}}{2}\,\frac{1}{r_{eD}\sqrt{z}}]}$$

$$\overline{q_D} = \frac{r_D}{\sqrt{z}}\,\frac{\frac{r_D}{2\,r_{eD}} - \frac{r_{eD}}{2\,r_D}}{\frac{1}{r_{eD}\sqrt{z}} + \frac{r_{eD}\sqrt{z}}{2}\,h(1) + s\sqrt{z}\,[\frac{r_{eD}}{2} - \frac{1}{2\,r_{eD}}]}$$

$$\overline{q_D} = \frac{\frac{r_D^2}{2\,r_{eD}} - \frac{r_{eD}}{2}}{\frac{1}{r_{eD}}}$$

$$\overline{q_D} = -\frac{1}{2}\left(r_{eD}^2 - r_D^2\right)$$

$$\overline{Q_D} = -\frac{1}{2\,z}\left(r_{eD}^2 - r_D^2\right)$$

Invert using Churchill #1

$$q_D = 0$$

Invert using Churchill #1

$$Q_D = -\frac{1}{2}\left(r_{eD}^2 - r_D^2\right)$$

For $r_D = 1$

$$(Q_D)_{r_D = 1} = -\frac{1}{2}\left(r_{eD}^2 - 1\right)$$

LINEAR COORDINATE SYSTEM EARLY TIME APPROXIMATIONS AT WELLBORE

	Outer Boundary Condition		
Inner Boundary	Infinite Acting	Constant Pressure	No Flow
Constant Rate	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 2\sqrt{\frac{t_D}{\pi}}$
Constant Rate Wellbore Storage	$p_D = \frac{t_D}{C_D}$	$p_D = \frac{t_D}{C_D}$	$p_D = \frac{t_D}{C_D}$
Constant Rate Wellbore Storage and Skin	$p_{wD} = \frac{t_D}{C_D}$	$p_{wD} = \frac{t_D}{C_D}$	$p_{wD} = \frac{t_D}{C_D}$
Constant Pressure	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = \frac{-1}{\sqrt{\pi t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$
Constant Pressure with Skin	$q_D = \frac{-1}{s_D}$ $Q_D = \frac{-t_D}{s_D}$	$q_D = \frac{-1}{s_D}$ $Q_D = \frac{-t_D}{s_D}$	$q_D = \frac{-1}{s_D}$ $Q_D = \frac{-t_D}{s_D}$

LINEAR COORDINATE SYSTEM
LATE TIME APPROXIMATIONS AT WELLBORE

	Outer Boundary Condition		
Inner Boundary	Infinite Acting	Constant Pressure	No Flow
Constant Rate	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 1$	$p_D = t_D + \frac{1}{3}$
Constant Rate Wellbore Storage	$p_D = 2\sqrt{\frac{t_D}{\pi}}$	$p_D = 1$	$p_D = \frac{t_D}{C_D + 1}$
Constant Rate Wellbore Storage and Skin	$p_{wD} = 2\sqrt{\frac{t_D}{\pi}}$	$p_{wD} = 1$	$p_{wD} = \frac{t_D}{C_D + 1}$
Constant Pressure	$q_D = \frac{-1}{\sqrt{\pi\, t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = -1$ $Q_D = -(t_D + \frac{1}{3})$	$q_D = 0$ $Q_D = -1$
Constant Pressure with Skin	$q_D = \frac{-1}{\sqrt{\pi\, t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = \frac{-1}{\sqrt{\pi\, t_D}}$ $Q_D = -2\sqrt{\frac{t_D}{\pi}}$	$q_D = 0$ $Q_D = -1$

SOLUTION TO THE DIFFUSIVITY EQUATION
LINEAR COORDINATE SYSTEM

Darcy's Law

(production is negative)

$$q = -\frac{k\,A}{\mu}\,\frac{dp}{dx}$$

Diffusivity Equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{\phi\,\mu\,c_t}{k}\,\frac{\partial p}{\partial t}$$

Assumptions in diffusivity equation

(1) neglect gravity effects
(2) homogeneous and isotropic reservoir
(3) constant height, porosity, and permeability
(4) constant viscosity
(5) c_t is small and constant
(6) pressure gradient squared terms are small and negligible

Initial Condition

$$p\,(x, 0) = p_i$$

Inner Boundary Condition

Constant Rate

$$\left(\frac{\partial p}{\partial x}\right)_{x=0} = -\frac{q\,\mu}{k\,A}$$

Constant Rate with Wellbore Storage

$$q = C\left(\frac{\partial p}{\partial t}\right)_{x=0} - \frac{k\,A}{\mu}\left(\frac{\partial p}{\partial x}\right)_{x=0}$$

Constant Rate with Wellbore Storage and Skin

$$q = C\left(\frac{dp_w}{dt}\right) - \frac{k\,A}{\mu}\left(\frac{\partial p}{\partial x}\right)_{x=0^+}$$

$$p_w = [p - s \, (\frac{\partial p}{\partial x})]_{x=0^+}$$

Constant Pressure

$$p \, (0, t) = p_w = constant$$

Constant Pressure with Skin

$$p_w = [p - s \, (\frac{\partial p}{\partial x})]_{x=0^+} = constant$$

Outer Boundary Condition

Infinite Acting

$$p \, (\infty, t) = p_i$$

Constant Pressure

$$p \, (L, t) = p_i$$

No Flow

$$\frac{\partial p}{\partial x} \, (L, t) = 0$$

Define Dimensionless Variables

$$x_D = \frac{x}{L} \qquad x = x_D \, L$$

$$s_D = \frac{s}{L} \qquad s = s_D \, L$$

$$t_D = \frac{k \, t}{\phi \, \mu \, c_t \, L^2} \qquad t = \frac{\phi \, \mu \, c_t \, L^2 \, t_D}{k}$$

For Constant Rate Inner Boundary Condition

$$p_D = \frac{k \, A \, (p - p_i)}{q \, \mu \, L} \qquad p = p_i + \frac{p_D \, q \, \mu \, L}{k \, A}$$